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On a conjecture of Chen and Yui: Resultants and discriminants

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Abstract. In [5], Chen and Yui conjectured that Gross–Zagier type formulas may also exist for Thompson series. In this work, we verify Chen and Yui's conjecture for the cases for Thompson series $j_p(\tau)$ for $\Gamma_0(p)$ for p prime, and equivalently establish formulas for the prime decomposition of the resultants of two ring class polynomials associated to $j_p(\tau)$ and imaginary quadratic fields and the prime decomposition of the discriminant of a ring class polynomial associated to $j_p(\tau)$ and an imaginary quadratic field. Our method for tackling Chen and Yui's conjecture on resultants can be used to give a different proof to a recent result of Yang and Yin. In addition, as an implication, we verify a conjecture recently raised by Yang, Yin, and Yu.

1 Introduction

Denote by $j(\tau)$ the well-known Klein's modular j-invariant. The value of $j(\tau)$ at an imaginary quadratic point of negative fundamental discriminant $-d$, known as singular modulus, is one of the most important objects in algebraic number theory, as it provides us with explicit constructions of the Hilbert class field over $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ as well as its associated Hilbert class polynomial $H_{\mathcal{K}}(x)$ (see, e.g., [7]). As a result, it helps with characterizing the representability of primes by a binary quadratic form (see, e.g., [7]). These very deep results have always been motivating many mathematicians to study various properties of singular moduli. For example, in their seminal work [9], Gross and Zagier established a remarkable formula for describing the prime decomposition of the rational norm of the difference of two singular moduli, which are equivalent to the resultant of two Hilbert class polynomials associated to $j(\tau)$ over different imaginary quadratic fields, and can be equivalently reformulated as

$$\log|\mathrm{result}(H_{\mathcal{K}_1}(x), H_{\mathcal{K}_2}(x))| = \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})} \log|j(\tau_{Q_1}) - j(\tau_{Q_2})|$$

$$= -\frac{1}{2} \sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}}\ \sum_{n \mid \frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n, \tag{1.1}$$

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where result$(p(x), q(x))$ denotes the resultant of polynomials $p(x)$ and $q(x)$, $-d_1$ and $-d_2$ are two coprime negative fundamental discriminants, $\mathcal{K}_i = \mathbb{Q}(\sqrt{-d_i})$, $\mathcal{Q}_d$ denotes the set of positive definite quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$, $\tau_Q = \frac{-b+\sqrt{-d}}{2a}$ is the unique imaginary quadratic point defined in the upper half plane induced by the quadratic form $Q = aX^2 + bXY + cY^2 \in \mathcal{Q}_d$, and for primes l with $\left(\frac{d_1 d_2}{l}\right) \neq -1$, $\varepsilon(l)$ is defined by

$$\varepsilon(l) = \begin{cases} \left(\dfrac{-d_1}{l}\right) & \text{if } (l, d_1) = 1, \\ \left(\dfrac{-d_2}{l}\right) & \text{if } (l, d_2) = 1, \end{cases} \tag{1.2}$$

and is extended completely multiplicatively to n. Such a formula also indicates an upper bound, $\frac{d_1 d_2}{4}$, for the prime factors of the resultant of two Hilbert class polynomials associated to $j(\tau)$, and is now called the Gross–Zagier CM value formula.

In view of the resultant interpretation, Gross and Zagier also considered the discriminant of a Hilbert class polynomial $H_{\mathcal{K}}(x)$ associated to the j-invariant and an imaginary quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{-p})$ with $p > 3$ a prime number and congruent to 3 modulo 4, which can be regarded as the complementary case for the "rational norm" of two singular moduli associated to imaginary quadratic points of the *same* fundamental discriminant, and they showed that [9, Corollary 4.8]

(1.3)

$$\log|\mathrm{disc}(H_{\mathcal{K}}(x))| = \frac{1}{2} \sum_{q \text{ inert}} \sum_{1 \le n \le p-1} \sum_{\substack{[\mathfrak{a}] \in \mathrm{Cl}_{\mathcal{K}} \\ [\mathfrak{a}] \neq [\mathcal{O}_{\mathcal{K}}]}} R_{\mathfrak{a}}(n) \sum_{k \ge 1} R\left(\frac{p-n}{q^k}\right) \log q + \frac{h_{\mathcal{K}} - 1}{2} \log p,$$

where disc$(p(x))$ denotes the discriminant of a polynomial $p(x)$,

$$R_{\mathfrak{a}}(n) = \left\{ x \in \mathfrak{a} \,\middle|\, \frac{\mathrm{N}(x)}{\mathrm{N}(\mathfrak{a})} = n \right\} \quad \text{and} \quad R(n) = \frac{1}{2} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_{\mathcal{K}}} R_{\mathfrak{a}}(n),$$

$\mathrm{Cl}_{\mathcal{K}}$ denotes the class group of $\mathcal{K}$, and $h_{\mathcal{K}}$ denotes the class number of $\mathcal{K}$. Note that by the definition of $R_{\mathfrak{a}}(n)$, the counting function $R(n)$ is defaulted to be zero for n nonintegral so that the summation over $k \ge 1$ is finite. We now call such a compact and interesting formula the Gross–Zagier discriminant formula.

On the other hand, representation-theoretically, the modular j-invariant is also known as the Thompson series of $\mathrm{SL}_2(\mathbb{Z})$, which indeed belongs to the family of Thompson series of discrete groups of moonshine [6] (also called the Hauptmoduln for discrete groups of moonshine in the language of modular forms). Such a fact also inspires mathematicians in related areas to consider and investigate the values of Thompson series of discrete groups of moonshine at imaginary quadratic points now called singular values. For example, Chen and Yui [5] showed that the singular value of a Thompson series $j_N(\tau)$ of level N at an imaginary quadratic point of fundamental discriminant $-d$ generates the ring class field of conductor N over $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$, and $j(\tau_Q)$'s as Q ranges over $\mathcal{Q}_d(N)/\Gamma_0(N)$ are exactly the associated Galois conjugates over $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$, where $\mathcal{Q}_d(N)$ denotes the set of positive definite quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $(a, N) = 1$, and thus the so-called ring class

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polynomial $H_{\mathcal{K},N}(x)$ of conductor N associated with $j_N(\tau)$ and $\mathcal{K}$ is exactly

$$H_{\mathcal{K},N}(x) = \prod_{[Q]\in\mathcal{Q}_d(N)/\Gamma_0(N)} (x - j_N(\tau_Q)). \tag{1.4}$$

At the end of [5], they equivalently computed a large number of examples of

$$\log|\mathrm{result}(H_{\mathcal{K}_1,N}(x), H_{\mathcal{K}_2,N}(x))| \quad \text{and} \quad \log|\mathrm{disc}(H_{\mathcal{K},N}(x))|, \tag{1.5}$$

and conjectured that Gross–Zagier type formulas should exist for (1.5). In the present work, we verify Chen and Yui's conjecture for the cases for $\Gamma_0(p)$, namely for

$$j_p(\tau) = \left(\frac{\eta(\tau)}{\eta(p\tau)}\right)^{\frac{24}{p-1}} \tag{1.6}$$

for $p \in \{2,3,5,7,13\}$, where $\eta(\tau)$ is the Dedekind eta function, and establish explicit Gross–Zagier type formulas for both $\log|\mathrm{result}(H_{\mathcal{K}_1,p}(x), H_{\mathcal{K}_2,p}(x))|$ and $\log|\mathrm{disc}(H_{\mathcal{K},p}(x))|$.

The first main result of this work is summarized as follows, which verifies Chen and Yui's conjecture on $\log|\mathrm{result}(H_{\mathcal{K}_1,p}(x), H_{\mathcal{K}_2,p}(x))|$.

Theorem 1.1 *Let $p \in \{2,3,5,7,13\}$. Let $-d_1, -d_2$ be two coprime fundamental discriminants, write $\mathcal{K}_i = \mathbb{Q}(\sqrt{-d_i})$, and let $H_{\mathcal{K}_i,p}(x)$ be defined by (1.4). In addition, denote by $h_{\mathcal{K}_i}$ the class number of $\mathcal{K}_i$ divided by half the size of its group of units, and denote by $\chi_{d_i}(\cdot)$ the quadratic character $\left(\frac{-d_i}{\cdot}\right)$ associated to $\mathcal{K}_i$. Also, let $\varepsilon(\cdot)$ be defined by (1.2). Then for $\chi_{d_1}(p) = -1$ or 1, one has that*

$$\begin{aligned}
&\log|\mathit{result}(H_{\mathcal{K}_1,p}(x), H_{\mathcal{K}_2,p}(x))| \\
&= -\frac{(p-1-\chi_{d_1}(p)-\chi_{d_2}(p))}{2} \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\left|\frac{d_1d_2-x^2}{4}\right.} \varepsilon(n)\log n \\
&\quad - \frac{(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_2}(p))}{4} \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod{4p}}} \sum_{n\left|\frac{d_1d_2-x^2}{4p}\right.} \varepsilon(n)\log n \\
&\quad + 3\frac{(2p-1-\chi_{d_1}(p))(2p-1-\chi_{d_2}(p))}{p-1} h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log p.
\end{aligned}$$

Remark 1.2 Since $-d_1, -d_2$ are coprime, by symmetry, the case for $\chi_{d_1}(p) = 0$, i.e., $p|d_1$, is equivalent to the case for $\chi_{d_2}(p) = 0$, and that is why $\chi_{d_1}(p)$ is only assumed to be -1 or 1.

Remark 1.3 Recall that for $p = 2$, $\chi_d(2) = \left(\frac{-d}{2}\right)$ is defined to be $0, 1$, or -1 depending on whether $2|d$, $-d \equiv 1 \pmod 8$ or $-d \equiv 5 \pmod 8$.

In the following, we give some examples computed using Theorem 1.1. They all match with the logarithm of the numeric given in [5, Appendix 5, p. 315], in which

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Chen and Yui indeed computed

$$\prod_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \prod_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} |j_p(\tau_{Q_1}) - j_p(\tau_{Q_2})|.$$

Example 1.4

(1) Take $-d_1 = -3$, $-d_2 = -11$, and $p = 2$ for which $\left(\frac{-d_1}{p}\right) = \left(\frac{-d_2}{p}\right) = -1$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-3}),2}(x), H_{\mathbb{Q}(\sqrt{-11}),2}(x))|\\&= -\frac{2+1}{2}(-10\log 2) + 12\times\frac{1}{3}\times 4\log 2\\&= \log(2^{31}).\end{aligned}$$

(2) Take $-d_1 = -3$, $-d_2 = -7$, and $p = 2$ for which $\left(\frac{-d_1}{p}\right) = -1$ and $\left(\frac{-d_2}{p}\right) = 1$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-3}),2}(x), H_{\mathbb{Q}(\sqrt{-7}),2}(x))|\\&= -\frac{2-1}{2}(-2\log 3 - 2\log 5) + 12\times\frac{1}{3}\times 2\log 2\\&= \log(2^{8}\cdot 3\cdot 5).\end{aligned}$$

(3) Take $-d_1 = -3$, $-d_2 = -4$, and $p = 2$ for which $\left(\frac{-d_1}{p}\right) = -1$ and $p|d_2$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-3}),2}(x), H_{\mathbb{Q}(\sqrt{-4}),2}(x))|\\&= -\frac{2}{2}(-2\log 2 - \log 3) + 6\log 2\\&= \log(2^{8}\cdot 3).\end{aligned}$$

(4) Take $-d_1 = -7$, $-d_2 = -11$, and $p = 2$ for which $\left(\frac{-d_1}{p}\right) = \left(\frac{-d_2}{p}\right) = 1$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-7}),2}(x), H_{\mathbb{Q}(\sqrt{-11}),2}(x))|\\&= -\frac{(2-1)}{2}(-2\log 7 - 2\log 13 - 2\log 17 - 2\log 19) + 12\times 2\log 2\\&= \log(2^{24}\cdot 7\cdot 13\cdot 17\cdot 19).\end{aligned}$$

(5) Take $-d_1 = -7$, $-d_2 = -4$, and $p = 2$ for which $\left(\frac{-d_1}{p}\right) = 1$ and $p|d_2$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-7}),2}(x), H_{\mathbb{Q}(\sqrt{-4}),2}(x))|\\&= 0 - \frac{2}{2}(-2\log 3) + 3\times 2\times 3\times\frac{1}{2}\log 2\\&= \log(2^{9}3^{2}).\end{aligned}$$

(6) Take $-d_1 = -11$, $-d_2 = -8$, and $p = 7$ for which $\left(\frac{-d_1}{p}\right) = -1$ and $\left(\frac{-d_2}{p}\right) = -1$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-11}),7}(x), H_{\mathbb{Q}(\sqrt{-8}),7}(x))|\\&= -\frac{8}{2}(-12\log 2 - 4\log 7 - 2\log 13) + 2\times 7^2\log 7\\&= \log(2^{48}7^{114}13^8).\end{aligned}$$

(7) Take $-d_1 = -4$, $-d_2 = -3$, and $p = 7$ for which $\left(\frac{-d_1}{p}\right) = -1$ and $\left(\frac{-d_2}{p}\right) = 1$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-4}),7}(x), H_{\mathbb{Q}(\sqrt{-3}),7}(x))|\\&= -\frac{7-1}{2}(-2\log 2 - \log 3) + 12\times\frac{1}{2}\times\frac{1}{3}\times 7\log 7\\&= \log(2^6 3^3 7^{14}).\end{aligned}$$

(8) Take $-d_1 = -11$, $-d_2 = -7$, and $p = 7$ for which $\left(\frac{-d_1}{p}\right) = -1$ and $p|d_2$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-11}),7}(x), H_{\mathbb{Q}(\sqrt{-7}),7}(x))|\\&= -\frac{7}{2}(-2\log 7 - 2\log 13 - 2\log 17 - 2\log 19) + 7\times 13\log 7\\&= \log(7^{98}13^7 17^7 19^7).\end{aligned}$$

(9) Take $-d_1 = -3$, $-d_2 = -7$, and $p = 7$ for which $\left(\frac{-d_1}{p}\right) = 1$ and $p|d_2$. Then Theorem 1.1 yields that

$$\begin{aligned}&\log|\mathrm{result}(H_{\mathbb{Q}(\sqrt{-3}),7}(x), H_{\mathbb{Q}(\sqrt{-7}),7}(x))|\\&= -\frac{5}{2}(-2\log 3 - 2\log 5) - 0 + 6\times\frac{1}{3}\times 13\log 7\\&= \log(3^5 5^5 7^{26}).\end{aligned}$$

The following corollary follows immediately from Theorem 1.1.

Corollary 1.5 *Let $p \in \{2,3,5,7,13\}$. Let $-d_1, -d_2$ be two coprime fundamental discriminants, write $\mathcal{K}_i = \mathbb{Q}(\sqrt{-d_i})$, and let $H_{\mathcal{K}_i,p}(x)$ be defined by (1.4). Then, any prime factor of the resultant of $H_{\mathcal{K}_1,p}(x)$ and $H_{\mathcal{K}_2,p}(x)$ is either p or bounded by $\frac{d_1 d_2}{4}$.*

The second main result of the present work is stated as follows, which verifies Chen and Yui's conjecture on $\log\left|\mathrm{disc}\left(H_{\mathcal{K},p}(x)\right)\right|$.

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Theorem 1.6 *Let $p \in \{3,5,7,13\}$. Let $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of odd discriminant $-d < -3$, and let $H_{\mathcal{K},p}(x)$ be defined by (1.4). Then one has that*

$$\log\left|\operatorname{disc}\left(H_{\mathcal{K},p}(x)\right)\right| = -\frac{(p-\chi_d(p))h_{\mathcal{K}}}{4} \sum_{\substack{[\mathfrak{a}]\in Cl_{\mathcal{K}}(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}} \times\left[\sum_{l=0}^{pd-1}\sum_{X,Y=-\infty}^{\infty}\kappa\left(1-\frac{d(2AX+BYp)^2+(dYp-2Al)^2}{4Ad}, \frac{Bl}{d}f_1^{(\mathfrak{a})}-\frac{2Al}{d}f_2^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})}\right)\right.$$
$$\left.+\frac{24}{p-1}\sum_{\substack{1\le k\le p-1\\ 0\le l\le pd-1\\ Ck+l\equiv 0 \pmod{pd}}}\kappa\left(0, kf_2^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})}\right)\right],$$

where $\mathfrak{a} = [A, \frac{B+\sqrt{-d}}{2}]$ *with* $-d = B^2-4AC$, $f_1^{(\mathfrak{a})} = \begin{pmatrix}-1 & B\\ 0 & A\end{pmatrix}$, $f_2^{(\mathfrak{a})} = \begin{pmatrix}0 & C\\ 1 & 0\end{pmatrix}$, $L_{p,-}^{(\mathfrak{a})}$ *denotes the lattice* $\mathbb{Z}f_1^{(\mathfrak{a})}+\mathbb{Z}pf_2^{(\mathfrak{a})}$, *and* $\kappa(m,\varphi)$ *is defined in* Definition 6.1 *and can be explicitly computed for* $(p,d)=1$ *via* Theorem 8.1 *given in* Section 8.

Remark 1.7 One will see in Definition 6.1 that $\kappa(m,\varphi)$ is defined to be 0 for $m < 0$. Also, it is clear that there are only finitely many integer pairs (X,Y) such that $1-\frac{d(2AX+BYp)^2+(dYp-2Al)^2}{4Ad} \ge 0$. Therefore, the inner sum over X, Y is actually a finite sum.

Remark 1.8 As one can see from the proof of Theorem 1.6 given in Section 7.1, the case for $p=2$ can also be treated similarly, and one can obtain that

$$\log\left|\operatorname{disc}\left(H_{\mathcal{K},2}(x)\right)\right| = -\frac{(2-\chi_d(2))h_{\mathcal{K}}}{4} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(2)\\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}} \times\left[\sum_{l=0}^{d-1}\sum_{X,Y=-\infty}^{\infty}\kappa\left(1-\frac{d(AX+BY)^2+(dY-Al)^2}{Ad}, \frac{Bl}{d}f_1^{(\mathfrak{a})}-\frac{2Al}{d}f_2^{(\mathfrak{a})}+L_{2,-}^{(\mathfrak{a})}\right)\right.$$
$$\left.+24\sum_{\substack{0\le l\le d-1\\ C+l\equiv 0 \pmod{d}}}\kappa\left(0, \frac{B(C+l)}{d}f_1^{(\mathfrak{a})}-\frac{2A(C+l)-d}{d}f_2^{(\mathfrak{a})}+L_{2,-}^{(\mathfrak{a})}\right)\right].$$

And similarly, one has to compute the local Whittaker functions related to the finite place $p=2$, whose computations can also be done with the formulas given in

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[13, Theorem 4.4]. For example, for d odd, one may show that

$$W_2(s, m, \varphi_-) = 1 + \sum_{2 \le k \le a+3} \left(\frac{(-1)^{\frac{\varepsilon_{\varphi_-}(k)-1}{2}} \varepsilon_{\varphi_-}(k)}{2} \right) \times \psi_2\left(m2^{-k} - \frac{1+d}{8A}\right) \mathrm{Char}(4\mathbb{Z}_2)\left(m2^{3-k} - \frac{1+d}{A}\right) 2^{-ks},$$

where $a = \mathrm{ord}_2(m)$,

$$\varepsilon_{\varphi_-}(k) = \begin{cases} 1 & \text{if } k \ge 1 \text{ odd} \\ d & \text{if } k \ge 2 \text{ even,} \end{cases}$$

and $\psi_2(\cdot)$ is the standard additive character defined on $\mathbb{Q}_2$.

Similar to Corollary 1.5, we obtain an upper bound for the prime factors of the discriminant of the ring class polynomial $H_{\mathcal{K},p}(x)$ associated to $j_p(\tau)$ and an imaginary quadratic field $\mathcal{K}$ of odd discriminant from Theorems 1.6 and 8.1, which also verify Chen and Yui's another conjecture associated with $j_p(\tau)$ [5, Remark 5.3 (1)].

Corollary 1.9 *Let $p \in \{2, 3, 5, 7, 13\}$. Let $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of odd discriminant $-d < -3$, and let $H_{\mathcal{K},p}(x)$ be defined by* (1.4). *Then $l \le \max\{d, p\}$ for any prime l dividing disc$(H_{\mathcal{K},p}(x))$.*

As we will employ different approaches to prove Theorems 1.1 and 1.6, respectively, in the following, we briefly discuss our methodologies for both cases. Regarding the proof of Theorem 1.1, by (1.4), one can see that

$$\log|\mathrm{result}(H_{\mathcal{K}_1,N}(x), H_{\mathcal{K}_2,N}(x))| = \frac{1}{2} \sum_{[Q_1] \in \mathcal{Q}_{d_1}(N)/\Gamma_0(N)} \sum_{[Q_2] \in \mathcal{Q}_{d_2}(N)/\Gamma_0(N)} \log|j_N(\tau_{Q_1}) - j_N(\tau_{Q_2})|^2,$$

and thereby to compute the resultant on the left is equivalent to computing the double sum on the right. For $N = 1$, they are exactly the central objects studied in [9] by Gross and Zagier. In [9], Gross and Zagier gave an analytic proof to (1.1), in which they first related the automorphic form $\log|j(z_1) - j(z_2)|^2$ to an automorphic Green function, and then converted the calculations of the double sum into the computations of the average value of the automorphic Green function over a 0-cycle of the underlying modular surface, which could be accomplished by computing the derivative of the Fourier coefficients of certain nonholomorphic Hilbert Eisenstein series of weight 1 via the so-called holomorphic projection.

In [10] and [18], Gross and Zagier, and the author independently established similar relationship between $\log|j_p(z_1) - j_p(z_2)|^2$ and automorphic Green functions for $\Gamma_0(p)$. These results indicate that one may follow Gross and Zagier's analytic proof to compute the associated double sum as above. As such, to prove Theorem 1.1, we follow closely Gross and Zagier's analytic proof together with taking advantages of some results of Gross, Kohnen, and Zagier, and extending a key lemma of Gross and Zagier to the cases for $\Gamma_0(p)$.

For the proof of Theorem 1.6, similar to the cases for resultants, by (1.4), one can see that

$$\log|\mathrm{disc}(H_{\mathcal{K},N}(x))| = -\frac{1}{4} \sum_{\substack{[Q],[Q'] \in \mathcal{Q}_d(N)/\Gamma_0(N) \\ [Q] \neq [Q']}} -2\log|j_N(\tau_Q) - j_N(\tau_{Q'})|^2,$$

and thus the computations of the left-hand side can be boiled down to that of the double sum on the right. In particular, by the well-known isomorphism $\mathcal{Q}_d(N)/\Gamma_0(N) \cong \mathrm{Cl}_{\mathcal{K}}(N) = I_{\mathcal{K}}(N)/P_{\mathcal{K},\mathbb{Z}}(N)$, where $I_{\mathcal{K}}(N)$ is the multiplicative group generated by the $\mathcal{O}_{\mathcal{K}}$-ideals with norm prime to N, and $P_{\mathcal{K},\mathbb{Z}}(N)$ is the subgroup of $I_{\mathcal{K}}(N)$ generated by principal $\mathcal{O}_{\mathcal{K}}$-ideals $\alpha\mathcal{O}_{\mathcal{K}}$ with $\alpha \equiv a \pmod{N\mathcal{O}_{\mathcal{K}}}$ for some integer a coprime to N, one can rewrite the double sum in the language of $\mathcal{O}_{\mathcal{K}}$-ideals as

$$\sum_{\substack{[Q],[Q'] \in \mathcal{Q}_d(N)/\Gamma_0(N) \\ [Q] \neq [Q']}} \log|j_N(\tau_Q) - j_N(\tau_{Q'})|^2 = \sum_{\substack{[\mathfrak{c}],[\mathfrak{c}'] \in \mathrm{Cl}_{\mathcal{K}}(N) \\ [\mathfrak{c}] \neq [\mathfrak{c}']}} \log|j_N(\tau_{\mathfrak{c}}) - j_N(\tau_{\mathfrak{c}'})|^2,$$

where $\tau_{\mathfrak{c}}$ denotes the imaginary quadratic point associated to the integral ideal $\mathfrak{c} = [a, \frac{b+\sqrt{-d}}{2}]$, i.e., $\tau_{\mathfrak{c}} = \frac{b+\sqrt{-d}}{2a}$. One will see in Section 6.3 that the ideal interpretation is more intrinsic when we realize the index set of points as a small CM 0-cycle than the quadratic form interpretation. For $N = 1$, they are the quantities that the Gross–Zagier discriminant formula (1.3) delicately describes. The original proof of (1.3) given by Gross and Zagier [9] relies on deep connections between the modular j-invariant and elliptic curves. In recent work [19], the author of the present work gave a different proof to (1.3) using the theory of Borcherds lifts. Roughly speaking, as it has been shown that the automorphic form $-2\log|j(z_1) - j(z_2)|^2$ is a Borcherds lift, the double sum can be viewed as the average value of the associated Borcherds lift over a 0-cycle. Once we realize the 0-cycle as a so-called small CM 0-cycle, we may compute the average value using Schofer's celebrated small CM value formula.

In [21], the author of the present work had shown that the automorphic forms $-2\log|j_p(z_1) - j_p(z_2)|^2$ are all Borcherds lifts, and these ultimately motivate us to follow the same idea stated above to extend the Gross–Zagier discriminant formula to the cases for $j_p(\tau)$. In the proof of Theorem 1.6, we realize the index set of the double sum as a small CM 0-cycle that allows us to apply Schofer's small CM value formula to the associated Borcherds lifts. After that, we explicitly compute the relevant lattices and local Whittaker functions that will lead the formula to a concrete form.

Based on the discrepancies between these methodologies, this work is organized as follows. Sections 2–4 are devoted to proving Theorem 1.1 and its relevant consequences. We first state and prove several preliminary results, and give the proof of Theorem 1.1 in Section 2. In Sections 3 and 4, we, respectively, show how our method can be used to give a different proof to a recent result of Yang and Yin [16], which relies heavily on the theory of Borcherds lifts (see, e.g., [1, 2]), and to verify a conjecture recently raised by Yang, Yin, and Yu [17].

After Section 4, the remainder consisting of Sections 5–8 is devoted to proving Theorem 1.6 and Corollary 1.9. We first briefly review the main concepts of the theory of Borcherds lifts in the adelic setting, and indicate how the discriminant of the ring class polynomial $H_{\mathcal{K},p}(x)$ associated to the j-invariant and an imaginary quadratic field $\mathcal{K}$ is related to the values of a Borcherds lift in Section 5. In Section 6, we review Schofer's small CM value formula [14] and derive a preliminary version of Theorem 1.6. In Sections 7 and 8, we complete the full picture of Theorem 1.6 by carrying out the computations of relevant lattices and stating explicit formulas that can be used to compute the quantity $\kappa(m,\varphi)$, respectively. At the end, we conclude by computing $\log|\mathrm{disc}(H_{\mathbb{Q}(\sqrt{-7}),3}(x))|$ using Theorem 1.6, which recovers the numeric obtained by Chen and Yui [5, Appendix 4, p. 306].

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We first state and prove several preliminary results that play crucial roles in the proof. The proof of Theorem 1.1 is given at the end of the section.

The following lemma is equivalently given by Gross, Kohnen, and Zagier [8, Proposition 2, p. 531] together with the fact that the modular curves associated to $\Gamma_0(p)$ for $p \in \{2,3,5,7,13\}$ are all of genus zero so that the associated spaces of weight 2 cusp forms are trivial.

Lemma 2.1 (Gross, Kohnen, and Zagier) *Let $p \in \{2,3,5,7,13\}$. Let $-d_1, -d_2$ be two coprime negative fundamental discriminants, write $\mathcal{K}_i = \mathbb{Q}(\sqrt{-d_i})$, and define $h_{\mathcal{K}_i}$ to be the class number of $\mathcal{K}_i$ divided by half the size of its group of units. Let $\varepsilon(n)$ be defined by (1.2). Then one has that*

$$\lim_{s\to 1}\left(-2\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}}\ \sum_{\ell \mid \frac{n^2-d_1d_2}{4}}\varepsilon(\ell)Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right)\right.$$
$$+4\pi\left(h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)}+h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\right)$$
$$\left.-4\pi h_{\mathcal{K}_1}h_{\mathcal{K}_2}\left(\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}+\frac{6}{\pi}\right)\right)$$
$$=-\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}}\ \sum_{n\mid\frac{d_1d_2-x^2}{4}}\varepsilon(n)\log n,$$

where

$$Q_{s-1}(t)=\int_0^\infty\left(t+\sqrt{t^2-1}\cosh v\right)^{-s}dv$$

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defined for $\mathrm{Re}(s) > 0$ *and* $t > 1$, $\zeta_{\mathcal{K}_i}(s)$ *is the Dedekind zeta function associated to* $\mathcal{K}_i$, $\zeta(s)$ *is the usual Riemann zeta function, and* $\Gamma(s)$ *is the usual Gamma function.*

Moreover, assume that $\left(\frac{-d_i}{p}\right) \neq -1$, *and take* $\beta_i \in \mathbb{Z}/2p\mathbb{Z}$ *such that* $-d_i \equiv \beta_i^2 \pmod{4p}$. *Then*

$$\lim_{s\to 1}\left(-\sum_{\substack{n\in\mathbb{Z}\\ |n|>\sqrt{d_1d_2}\\ n\equiv\beta_1\beta_2 \pmod{2p}}}\ \sum_{\ell\left|\frac{n^2-d_1d_2}{4p}\right.} \varepsilon(\ell)Q_{s-1}\left(\frac{|n|}{\sqrt{d_1d_2}}\right) + \frac{4\pi}{p+1}\left(h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)} + h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\right) - \frac{4\pi h_{\mathcal{K}_1}h_{\mathcal{K}_2}}{p+1}\left(\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} + \frac{6}{\pi}\right)\right)$$
$$= -\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x\equiv\beta_1\beta_2 \pmod{2p}}}\ \sum_{n\left|\frac{d_1d_2-x^2}{4p}\right.} \varepsilon(n)\log n + \frac{6h_{\mathcal{K}_1}h_{\mathcal{K}_2}(p-1)}{(p+1)^2}\log p.$$

In particular, if $\chi_{d_1}(p) = 1$ *and* $\chi_{d_2}(p) \neq -1$, *one can rewrite the external sum on the right-hand side using Legendre symbols to get rid of* β_i, *and obtain*

$$\lim_{s\to 1}\left(-\sum_{\substack{n\in\mathbb{Z}\\ |n|>\sqrt{d_1d_2}\\ n\equiv\beta_1\beta_2 \pmod{2p}}}\ \sum_{\ell\left|\frac{n^2-d_1d_2}{4p}\right.} \varepsilon(\ell)Q_{s-1}\left(\frac{|n|}{\sqrt{d_1d_2}}\right) + \frac{4\pi}{p+1}\left(h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)} + h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\right) - \frac{4\pi h_{\mathcal{K}_1}h_{\mathcal{K}_2}}{p+1}\left(\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} + \frac{6}{\pi}\right)\right)$$
$$= -\frac{2-\chi_{d_2}(p)}{2}\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod{4p}}}\ \sum_{n\left|\frac{d_1d_2-x^2}{4p}\right.} \varepsilon(n)\log n + \frac{6h_{\mathcal{K}_1}h_{\mathcal{K}_2}(p-1)}{(p+1)^2}\log p.$$

Proof These closely follow from [8, Proposition 2, p. 531] specialized to $m = 1$ and $N = p$ together with the fact [8, Equation (10)] that the spaces of cusp forms of weight 2 for $\Gamma_0(p)$ for $p \in \{2, 3, 5, 7, 13\}$ are all trivial, so that $a_1 = 0$ in [8, Proposition 2, p. 531].

The last equality follows from simple relations between the sets

$$S_1 := \{x \in \mathbb{Z} : x^2 < d_1 d_2,\ x \equiv \beta_1\beta_2 \pmod{2p}\}$$

and

$$S_2 := \{x \in \mathbb{Z} : x^2 < d_1 d_2,\ x^2 \equiv d_1 d_2 \pmod{4p}\},$$

that is,

$$S_2 = \begin{cases} S_1 \sqcup (-S_1) & \text{if } \chi_{d_2}(p) = 1, \\ S_1 & \text{if } \chi_{d_2}(p) = 0. \end{cases} \qquad \blacksquare$$

The next lemma can be found in [8, Corollary, p. 516].

Lemma 2.2 (Gross, Kohnen, and Zagier) *Let $-d_1, -d_2$ be two coprime negative fundamental discriminants, and let $\varepsilon(n)$ be defined by* (1.2)*. Let $\mathcal{Q}_{d_i,N}$ denote the set of positive definite quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d_i$ with $N|a$, and let $\Gamma_0(N)$ act simultaneously on $\mathcal{Q}_{d_1,N} \times \mathcal{Q}_{d_2,N}$. Define $B_\Delta(Q_1, Q_2) = b_1b_2 - 2a_1c_2 - 2a_2c_1$ for $Q_1 = a_1X^2 + b_1XY + c_1Y^2$ and $Q_2 = a_2X^2 + b_2XY + c_2Y^2$, and denote by $t(N)$ the number of prime factors of N. Then the following identity holds.*

$$\frac{1}{2}\left|\{(Q_1, Q_2) \in (\mathcal{Q}_{d_1,N} \times \mathcal{Q}_{d_2,N})/\Gamma_0(N) | B_\Delta(Q_1, Q_2) = -n\}\right| = 2^{t(N)} \sum_{\ell \left| \frac{n^2 - d_1 d_2}{4N}\right.} \varepsilon(\ell).$$

The following lemma can be viewed as an extension of Lemma 2.2 in the sense of binary quadratic forms.

Lemma 2.3 *Let p be a prime. Let $-d_1, -d_2$ be two coprime negative fundamental discriminants, and let $\chi_{d_i}(\cdot) = \left(\frac{-d_i}{\cdot}\right)$ be the quadratic character associated to $\mathbb{Q}(\sqrt{-d_i})$. Denote by $\mathcal{Q}_{d_i}(p)$ the set of positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d_i$ with $(a, p) = 1$. Let $B_\Delta(Q_1, Q_2)$ be defined as in* Lemma 2.2*, and let $\rho_p(n)$ be the counting function defined by*

$$\rho_p(n) = \frac{1}{2}\left|\{(Q_1, Q_2) \in (\mathcal{Q}_{d_1}(p) \times \mathcal{Q}_{d_2}(p))/\Gamma_0(p) | B_\Delta(Q_1, Q_2) = -n\}\right|.$$

Then, for $\chi_{d_1}(p) = -1$ or 1, one has that

$$\rho_p(n) = (p - 1 - \chi_{d_1}(p) - \chi_{d_2}(p)) \sum_{\ell \left| \frac{n^2 - d_1 d_2}{4}\right.} \varepsilon(\ell) + (1 + \chi_{d_1}(p)) \sum_{\ell \left| \frac{n^2 - d_1 d_2}{4p}\right.} \varepsilon(\ell).$$

Remark 2.4 As explained in Remark 1.2, the case for $\chi_{d_1}(p) = 0$ is excluded by symmetry.

Proof Denote by $\mathcal{Q}_{d,p,\beta}$ the set of positive definite quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $p|a$ and $b \equiv \beta \pmod{2p}$. Then one first notes that for

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$i \in \{1,2\}$, the set of positive definite binary quadratic forms of discriminant $-d_i$,

$$\mathcal{Q}_{d_i} = \mathcal{Q}_{d_i}(p) \sqcup \mathcal{Q}_{d_i,p},$$

where

$$\mathcal{Q}_{d_i,p} = \begin{cases} \varnothing & \text{if } \chi_{d_i}(p) = -1, \\ \mathcal{Q}_{d_i,p,\beta_i} \sqcup \mathcal{Q}_{d_i,p,2p-\beta_i} & \text{if } \chi_{d_i}(p) = 1, \\ \mathcal{Q}_{d_i,p,\beta_i} & \text{if } \chi_{d_i}(p) = 0, \end{cases}$$

and that $\Gamma_0(p)$ acts independently on $\mathcal{Q}_{d_i}(p)$, $\mathcal{Q}_{d_i,p,\beta_i}$ and $\mathcal{Q}_{d_i,p,2p-\beta_i}$, and thus

$$\begin{aligned} \rho_p(n) = {} & \frac{1}{2}\left|\{(Q_1,Q_2) \in (\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2})/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & - \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & - \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1,p} \times \mathcal{Q}_{d_2}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & + \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1,p} \times \mathcal{Q}_{d_2,p}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right|. \end{aligned}$$

Then clearly, the modular group $SL_2(\mathbb{Z})$ has a well-defined group action on $\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2}$, and thus by Lemma 2.2 and the assumption that d_1, d_2 are coprime, one can see that

$$\begin{aligned} & \frac{1}{2}\left|\{(Q_1,Q_2) \in (\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2})/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & = [SL_2(\mathbb{Z}) : \Gamma_0(p)]\frac{1}{2}\left|\{(Q_1,Q_2) \in (\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2})/SL_2(\mathbb{Z}) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & = (p+1) \sum_{\ell \left| \frac{n^2-d_1d_2}{4}\right.} \varepsilon(\ell). \end{aligned}$$

For the quantity

$$\frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right|,$$

one first notes by the definition of $\mathcal{Q}_{d_i,p}$ that

$$\begin{aligned} & \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| \\ & = \begin{cases} 0 & \text{if } \chi_{d_2}(p) = -1, \\ \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p,\beta_2}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| & \text{if } \chi_{d_2}(p) = 1, \\ +\frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p,2p-\beta_2}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| & \\ \frac{1}{2}\left|\{(Q_1,Q_2) \in \left(\mathcal{Q}_{d_1} \times \mathcal{Q}_{d_2,p,\beta_2}\right)/\Gamma_0(p) | B_\Delta(Q_1,Q_2) = -n\}\right| & \text{if } \chi_{d_2}(p) = 0. \end{cases} \end{aligned}$$

Now by the fact $(\Gamma_0(p) \times \Gamma_0(p))/\Gamma_0(p) = \Gamma_0(p) \times \{I\}$, one may deduce that

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$$
\begin{aligned}
&\left|\{(Q_1,Q_2)\in\left(\mathcal{Q}_{d_1}\times\mathcal{Q}_{d_2,p,\beta_2}\right)/\Gamma_0(p)\,|\,B_\Delta(Q_1,Q_2)=-n\}\right|\\
&\quad=\sum_{[Q_2]\in\mathcal{Q}_{d_2,p,\beta_2}/\Gamma_0(p)}\ \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(p)}\ \sum_{\gamma\in\Gamma_0(p)}|\{(\gamma\cdot Q_1,Q_2)|\,B_\Delta(\gamma\cdot Q_1,Q_2)=-n\}|\\
&\quad=\sum_{[Q_2]\in\mathcal{Q}_{d_2,p,\beta_2}/\Gamma_0(p)}\ \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\mathrm{SL}_2(\mathbb{Z})}\ \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}|\{(\gamma\cdot Q_1,Q_2)|\,B_\Delta(\gamma\cdot Q_1,Q_2)=-n\}|.
\end{aligned}
$$

By [8, Proposition, p. 505], it is known that

$$
\mathcal{Q}_{d_2,p,\beta_2}/\Gamma_0(p)\cong\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})
$$

via $[Q_2]\to[Q_2]$. Therefore, one can deduce that

$$
\begin{aligned}
&\frac{1}{2}\left|\{(Q_1,Q_2)\in\left(\mathcal{Q}_{d_1}\times\mathcal{Q}_{d_2,p,\beta_2}\right)/\Gamma_0(p)\,|\,B_\Delta(Q_1,Q_2)=-n\}\right|\\
&\quad=\frac{1}{2}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\mathrm{SL}_2(\mathbb{Z})}\ \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\,\mathrm{SL}_2(\mathbb{Z})}\ \sum_{\gamma\in\mathrm{SL}_2(\mathbb{Z})}|\{(\gamma\cdot Q_1,Q_2)|\,B_\Delta(\gamma\cdot Q_1,Q_2)=-n\}|\\
&\quad=\frac{1}{2}|\{(Q_1,Q_2)\in(\mathcal{Q}_{d_1}\times\mathcal{Q}_{d_2})/\mathrm{SL}_2(\mathbb{Z})|\,B_\Delta(Q_1,Q_2)=-n\}|\\
&\quad=\sum_{\ell\left|\frac{n^2-d_1d_2}{4}\right.}\varepsilon(\ell).
\end{aligned}
$$

Therefore, incorporating the values of $\chi_{d_2}(p)$, one indeed has that

$$
\frac{1}{2}\left|\{(Q_1,Q_2)\in\left(\mathcal{Q}_{d_1}\times\mathcal{Q}_{d_2,p}\right)/\Gamma_0(p)\,|\,B_\Delta(Q_1,Q_2)=-n\}\right|=(1+\chi_{d_2}(p))\sum_{\ell\left|\frac{n^2-d_1d_2}{4}\right.}\varepsilon(\ell).
$$

Similarly, one also has that

$$
\frac{1}{2}\left|\{(Q_1,Q_2)\in\left(\mathcal{Q}_{d_1,p}\times\mathcal{Q}_{d_2}\right)/\Gamma_0(p)\,|\,B_\Delta(Q_1,Q_2)=-n\}\right|=(1+\chi_{d_1}(p))\sum_{\ell\left|\frac{n^2-d_1d_2}{4}\right.}\varepsilon(\ell).
$$

Finally, by Lemma 2.2, one has that

$$
\frac{1}{2}\left|\{(Q_1,Q_2)\in\left(\mathcal{Q}_{d_1,p}\times\mathcal{Q}_{d_2,p}\right)/\Gamma_0(p)\,|\,B_\Delta(Q_1,Q_2)=-n\}\right|=(1+\chi_{d_1}(p))\sum_{\ell\left|\frac{n^2-d_1d_2}{4p}\right.}\varepsilon(\ell),
$$

and putting all of the above together, one obtains the desired formula for $\rho_p(n)$. ■

Remark 2.5 For d divisible by p, one should notice that $\mathcal{Q}_{d,p,\beta}=\mathcal{Q}_{d,p,2p-\beta}$ since β is either 0 or p.

The proof of the following lemma is straightforward, and is left to the reader.

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Lemma 2.6 *Suppose that* $-d_1 \equiv \beta_1^2 \pmod{4p}$ *and* $-d_2 \equiv \beta_2^2 \pmod{4p}$ *for some fixed* $\beta_1, \beta_2 \in \mathbb{Z}/2p\mathbb{Z}$. *Then one has that*

$$\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}} \sum_{\ell \left| \frac{n^2-d_1d_2}{4p}\right.} \varepsilon(\ell) Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right) = \frac{(1+\chi_{d_2}(p))}{2} \sum_{\substack{n\in\mathbb{Z}\\ |n|>\sqrt{d_1d_2}\\ n\equiv\beta_1\beta_2 \pmod{2p}}} \sum_{\ell \left| \frac{n^2-d_1d_2}{4p}\right.} \varepsilon(\ell) Q_{s-1}\left(\frac{|n|}{\sqrt{d_1d_2}}\right).$$

Proofs of the following lemma can be found in [10, Proposition 2.22] and [18, Proposition 2.1] specializing $\varphi_{\infty,\infty}(s)$ to the cases for $\Gamma_0(p)$ by [11, p. 163].

Lemma 2.7 *Let* $p \in \{2,3,5,7,13\}$, *and let* $j_p(\tau)$ *be defined by* (1.6). *Let* $G_p(z_1,z_2;s)$ *be the automorphic Green function associated to* $\Gamma_0(p)$ *defined for* $z_1 \neq \Gamma_0(p)z_2$ *by*

$$G_p(z_1,z_2;s) = \sum_{\gamma\in\Gamma_0(p)} g_s(z_1,\gamma\cdot z_2),$$

where

$$g_s(z_1,z_2) = -2Q_{s-1}\left(1+\frac{|z_1-z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}\right),$$

and $Q_{s-1}(z)$ *is defined as in* Lemma 2.1. *Let* $\tilde{E}_p(\tau;s)$ *be the weight 0 nonholomorphic Eisenstein series associated to the cusp* $i\infty$ *defined by*

$$\tilde{E}_p(\tau;s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mathrm{Im}(\gamma\cdot\tau)^s \tag{2.1}$$

with $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, -I \right\rangle$. *Then one has that*

$$\log|j_p(z_1)-j_p(z_2)|^2 = \lim_{s\to 1}\Bigg(G_p(z_1,z_2;s)+4\pi\tilde{E}_p(z_1;s)+4\pi\tilde{E}_p(z_2;s) - 4\pi^{\frac{3}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{p-1}{p^{2s}-1} - \frac{24}{p+1}\Bigg).$$

Remark 2.8 Note that in [10, Proposition 2.22], the limit given on the right-hand side is represented by the so-called Archimedean height of $X_0(p) := \Gamma_0(p)\backslash(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q}))$ which is actually the same as $\log|j_p(z_1)-j_p(z_2)|^2$ for $p \in \{2,3,5,7,13\}$ by [10, Condition (2.3), p. 237] since for such cases, $j_p(\tau)$ are uniformizers for $X_0(p)$.

Proofs of the following lemmas can be found in [20, Corollaries 3.1 and 3.2], respectively.

Lemma 2.9 *Let p be a prime, and let $\tilde{E}_p(\tau; s)$ be defined by (2.1). Let $-d$ be a negative fundamental discriminant, and denote by $\mathcal{Q}_d(p)$ the set of positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $(a, p) = 1$. Write $\tau_Q = \frac{-b+\sqrt{-d}}{2a}$ for the unique imaginary quadratic point defined in the upper half plane induced by the quadratic form $Q = aX^2 + bXY + cY^2$. Then one has that*

$$\sum_{[Q]\in\mathcal{Q}_d(p)/\Gamma_0(p)} \tilde{E}_p(\tau_Q; s) = 2^{-s} d^{\frac{s}{2}} \frac{(1 - \chi_d(p)p^{-s})}{(1 + p^{-s})} \frac{\zeta_{\mathcal{K}}(s)}{\zeta(2s)},$$

where $\chi_d(p)$ is quadratic character associated to $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$, $\zeta_{\mathcal{K}}(s)$ is the Dedekind zeta function associated to $\mathcal{K}$, and $\zeta(s)$ is the Riemann zeta function.

Remark 2.10 Note that in [20, Corollary 3.1] we consider

$$E_p(z, s) = \sum_{(m,n)\neq(0,0)} \frac{\mathbf{1}_p(n)\mathrm{Im}(z)^s}{|mpz + n|^{2s}},$$

where $\mathbf{1}_p(\cdot)$ denotes the principal character modulo p, which is related to $\tilde{E}_p(z, s)$ via

$$E_p(z, s) = 2(1 - p^{-2s})\zeta(2s)\tilde{E}_p(z, s).$$

See also [20, Equation (2.1)]. This is where the denominator $\zeta(2s)$ comes from.

Lemma 2.11 *Let $-d$ be a negative fundamental discriminant, denote by $\mathcal{Q}_d(p)$ the set of positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $(a, p) = 1$, and let $\chi_d(\cdot)$ be the quadratic character associated to $\mathbb{Q}(\sqrt{-d})$. Write $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$, and define $h_{\mathcal{K}}$ to be the class number of $\mathcal{K}$ divided by half the size of its group of units. Then one has that*

$$|\mathcal{Q}_d(p)/\Gamma_0(p)| = (p - \chi_d(p))h_{\mathcal{K}}.$$

Remark 2.12 Indeed it is shown in [20, Corollary 3.2] that

$$|\mathcal{Q}_d(p)/\Gamma_0(p)| = (p - \chi_d(p)) \times \frac{|\mathcal{O}_{\mathcal{K}}(p)^\times|}{|\mathcal{O}_{\mathcal{K}}^\times|} \times (\text{class number of } \mathcal{K}),$$

where $\mathcal{O}_{\mathcal{K}}$ and $\mathcal{O}_{\mathcal{K}}(p)$ are the maximal order and the order of index p of $\mathcal{K}$, respectively. It is clear that the product $\frac{|\mathcal{O}_{\mathcal{K}}(p)^\times|}{|\mathcal{O}_{\mathcal{K}}^\times|} \times (\text{class number of } \mathcal{K})$ is the same as our weighted class number $h_{\mathcal{K}}$.

With the aid of the preliminary results stated above, we are now ready for

Proof of Theorem 1.1 First write $h_{\mathcal{K}_i}(p)$ for the cardinality $|\mathcal{Q}_{d_i}(p)/\Gamma_0(p)|$. By Lemma 2.7 and the same argument as that given in [9], one has that

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(2.2)
$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \log|j_p(\tau_{Q_1})-j_p(\tau_{Q_2})|^2\\
&=\lim_{s\to1}\left(-2\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}} \rho_p(n)Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right)\right.\\
&\quad+4\pi h_{\mathcal{K}_2}(p)\sum_{[Q]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)} \tilde{E}_p(\tau_Q;s)\\
&\quad+4\pi h_{\mathcal{K}_1}(p)\sum_{[Q]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \tilde{E}_p(\tau_Q;s)\\
&\quad-4h_{\mathcal{K}_1}(p)h_{\mathcal{K}_2}(p)\pi^{\frac32}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{p-1}{p^{2s}-1}\\
&\quad\left.-h_{\mathcal{K}_1}(p)h_{\mathcal{K}_2}(p)\frac{24}{p+1}\right).
\end{aligned}$$

By Lemmas 2.3, 2.6, 2.9, and 2.11, one can further deduce from (2.2) that

(2.3)
$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(p)/\Gamma_0(p)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(p)/\Gamma_0(p)} \log|j_p(\tau_{Q_1})-j_p(\tau_{Q_2})|^2\\
&=\lim_{s\to1}\left((p-1-\chi_{d_1}(p)-\chi_{d_2}(p))\left(-2\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}}\ \sum_{\ell\mid\frac{n^2-d_1d_2}{4}} \varepsilon(\ell)Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right)\right)\right.\\
&\quad+(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))\left(-\sum_{\substack{n\in\mathbb{Z}\\ |n|>\sqrt{d_1d_2}\\ n\equiv\beta_1\beta_2 \pmod{2p}}}\ \sum_{\ell\mid\frac{n^2-d_1d_2}{4p}} \varepsilon(\ell)Q_{s-1}\left(\frac{|n|}{\sqrt{d_1d_2}}\right)\right)\\
&\quad+4\pi(p-\chi_{d_2}(p))h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{(1-\chi_{d_1}(p)p^{-s})}{(1+p^{-s})}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)}\\
&\quad+4\pi(p-\chi_{d_1}(p))h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{(1-\chi_{d_2}(p)p^{-s})}{(1+p^{-s})}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\\
&\quad-4h_{\mathcal{K}_1}h_{\mathcal{K}_2}\pi^{\frac32}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{(p-\chi_{d_1}(p))(p-\chi_{d_2}(p))(p-1)}{p^{2s}-1}\\
&\quad\left.-h_{\mathcal{K}_1}h_{\mathcal{K}_2}\frac{24(p-\chi_{d_1}(p))(p-\chi_{d_2}(p))}{p+1}\right).
\end{aligned}$$

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By the Laurent expansions of $\frac{1-\chi_d(p)p^{-s}}{1+p^{-s}}$ and $\frac{1}{p^{2s}-1}$ at $s=1$,

$$\frac{1-\chi_d(p)p^{-s}}{1+p^{-s}} = \frac{p-\chi_d(p)}{p+1} + \frac{p(1+\chi_d(p))\log p}{(p+1)^2}(s-1) + O((s-1)^2)$$

and

$$\frac{1}{p^{2s}-1} = \frac{1}{p^2-1} - 2\frac{p^2\log p}{(p^2-1)^2}(s-1) + O((s-1)^2),$$

and Lemma 2.1, one can easily simplify (2.3) and show that

$$\begin{aligned}
&\sum_{[Q_1]\in\Omega_{d_1}(p)/\Gamma_0(p)} \sum_{[Q_2]\in\Omega_{d_2}(p)/\Gamma_0(p)} \log|j_p(\tau_{Q_1}) - j_p(\tau_{Q_2})|^2 \\
&= -(p-1-\chi_{d_1}(p)-\chi_{d_2}(p)) \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\mid\frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n \\
&\quad - \frac{(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_2}(p))}{2} \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod{4p}}} \sum_{n\mid\frac{d_1d_2-x^2}{4p}} \varepsilon(n)\log n \\
&\quad + \frac{6(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(p-1)h_{\mathcal{K}_1}h_{\mathcal{K}_2}}{(p+1)^2}\log p \\
&\quad + \frac{p(p-\chi_{d_2}(p))(1+\chi_{d_1}(p))\log p}{(p+1)^2}\lim_{s\to1}\left(4\pi h_{\mathcal{K}_1}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)}(s-1)\right) \\
&\quad + \frac{p(p-\chi_{d_1}(p))(1+\chi_{d_2}(p))\log p}{(p+1)^2}\lim_{s\to1}\left(4\pi h_{\mathcal{K}_2}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}(s-1)\right) \\
&\quad + 2\frac{p^2(p-\chi_{d_1}(p))(p-\chi_{d_2}(p))\log p}{(p+1)^2(p-1)}\lim_{s\to1}\left(4h_{\mathcal{K}_1}h_{\mathcal{K}_2}\pi^{\frac{3}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}(s-1)\right) \\
&= -(p-1-\chi_{d_1}(p)-\chi_{d_2}(p)) \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\mid\frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n \\
&\quad - \frac{(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_2}(p))}{2} \sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod{4p}}} \sum_{n\mid\frac{d_1d_2-x^2}{4p}} \varepsilon(n)\log n \\
&\quad + \frac{6(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(p-1)h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log p}{(p+1)^2} \\
&\quad + \frac{12p(p-\chi_{d_2}(p))(1+\chi_{d_1}(p))h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log p}{(p+1)^2} \\
&\quad + \frac{12p(p-\chi_{d_1}(p))(1+\chi_{d_2}(p))h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log p}{(p+1)^2}
\end{aligned}$$

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$$
\begin{aligned}
&+2\frac{p^2(p-\chi_{d_1}(p))(p-\chi_{d_2}(p))\log p}{(p+1)^2(p-1)}12h_{\mathcal{K}_1}h_{\mathcal{K}_2}\\
&=-(p-1-\chi_{d_1}(p)-\chi_{d_2}(p))\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}}\ \sum_{n\left|\frac{d_1d_2-x^2}{4}\right.}\varepsilon(n)\log n\\
&\quad-\frac{(1+\chi_{d_1}(p))(1+\chi_{d_2}(p))(2-\chi_{d_2}(p))}{2}\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod{4p}}}\ \sum_{n\left|\frac{d_1d_2-x^2}{4p}\right.}\varepsilon(n)\log n\\
&\quad+6\frac{(2p-1-\chi_{d_1}(p))(2p-1-\chi_{d_2}(p))}{p-1}h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log p.
\end{aligned}
$$

Finally, dividing both sides by 2, one obtains the desired formula. ■

Remark 2.13 Theorem 1.1 considers the logarithm of the resultant, while in the proof above, we evaluate the logarithm of its square, that is, twice the logarithm of the resultant.

3 Remark I: a result of Yang and Yin

In this section, we show how to employ our method to give a different proof to a recent result of Yang and Yin [16].

Let $f(\tau)$ be the 24th power of the Weber function defined by

$$
f(\tau)=2^{12}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}=\frac{2^{12}}{j_2(\tau)}. \tag{3.1}
$$

In [16], Yang and Yin employed the theory of Borcherds lifts and the so-called big CM value formula [3] to prove the following theorem.

Theorem 3.1 (Yang and Yin) *Let $-d_1$ and $-d_2$ be two coprime negative fundamental discriminants for which $\chi_{d_1}(2)=\chi_{d_2}(2)=1$, i.e., $-d_1\equiv -d_2\equiv 1 \pmod 8$, and denote by $\mathcal{Q}_{d_i}(2)$ the set of positive definite binary quadratic forms $aX^2+bXY+cY^2$ of discriminant $-d$ with $(a,2)=1$. Then one has that*

$$
\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}(2)/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}(2)/\Gamma_0(2)}\log|f(\tau_{Q_1})-f(\tau_{Q_2})|^2\\
&=\sum_{\substack{t=\frac{x+\sqrt{d_1d_2}}{2}\\ x^2\equiv d_1d_2 \pmod{16}}}\ \sum_{\substack{\mathfrak{p}\subset\mathcal{O}_F\\ inert\ in\ E}}\frac{1+ord_{\mathfrak{p}}(t\mathfrak{q}_t^{-2})}{2}\tilde{\rho}(t\mathfrak{q}_t^{-2}\mathfrak{p}^{-1})\log N(\mathfrak{p}),
\end{aligned}
$$

where $F=\mathbb{Q}(\sqrt{d_1d_2})$, $E=\mathbb{Q}(\sqrt{-d_1},\sqrt{-d_2})$, $\mathfrak{q}_t$ is the unique prime ideal of F above 2 such that $ord_{\mathfrak{q}_t}(t\mathcal{O}_F)\geq 1$, and $\tilde{\rho}$ is a counting function defined for integral ideals $\mathfrak{a}\subset\mathcal{O}_F$ by

$$
\tilde{\rho}(\mathfrak{a})=\left|\left\{\mathfrak{U}\subset\mathcal{O}_E \mid N_{E/F}(\mathfrak{U})=\mathfrak{a}\right\}\right|.
$$

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It is worthwhile to remark that Yang and Yin's method [16] may give another proof to our Theorem 1.1, and one may find some related work and ideas in [21].

By [16, Remark 4.1], one can easily see that it is equivalent to the following reformulation.

Theorem 3.2 *Let $-d_1$ and $-d_2$ be two coprime negative fundamental discriminants for which $\chi_{d_1}(2) = \chi_{d_2}(2) = 1$, i.e., $-d_1 \equiv -d_2 \equiv 1 \pmod 8$, and denote by $\mathcal{Q}_{d_i}(2)$ the set of positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $(a, 2) = 1$. Let $\varepsilon(n)$ be defined by (1.2). Then one has that*

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(2)/\Gamma_0(2)} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(2)/\Gamma_0(2)} \log|f(\tau_{Q_1}) - f(\tau_{Q_2})|^2 = -\sum_{\substack{x\in\mathbb{Z} \\ x^2<d_1d_2 \\ x^2\equiv d_1d_2 \pmod{16}}} \sum_{n \mid \frac{d_1d_2-x^2}{16}} \varepsilon(n)\log n.$$

The proof of Theorem 3.2 is similar to that of Theorem 1.1, and follows from Lemma 2.3 and the following four lemmas.

Lemma 3.3 *Let $G_2(z_1, z_2; s)$ be defined as in* Lemma 2.7. *Let $\tilde{E}_2^{(0)}(\tau; s)$ be the weight 0 nonholomorphic Eisenstein series associated to the cusp 0 defined by*

$$\tilde{E}_2^{(0)}(\tau; s) = \sum_{\gamma\in\Gamma_\infty\backslash\sigma\Gamma_0(2)\sigma^{-1}} \operatorname{Im}(\gamma\sigma\cdot\tau)^s$$

with $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -I \right\rangle$ *and* $\sigma = \begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$. *Then one has that*

$$\log|f(z_1) - f(z_2)|^2 = \lim_{s\to 1}\Bigg(G_2(z_1, z_2; s) + 4\pi\tilde{E}_2^{(0)}(z_1; s) + 4\pi\tilde{E}_2^{(0)}(z_2; s) - 4\pi^{\frac{3}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{1}{2^{2s}-1} - 8\Bigg).$$

Proof The proof is essentially the same as that of Lemma 2.7 given in [18]. We omit the details. ∎

Remark 3.4 By the definition of $f(\tau)$, one can see that $f(\tau)$ has only a simple pole at the cusp 0 of the modular curve $X_0(2) = \Gamma_0(2)\backslash(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q}))$, and that is why we consider the weight 0 nonholomorphic Eisenstein series associated to the cusp 0.

Lemma 3.5 *Let $-d$ be a negative fundamental discriminant, write $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$, and let $\mathcal{Q}_d(2)$ be the set of positive definite binary quadratic forms $aX^2 + bXY + cY^2$ of discriminant $-d$ with $(a, 2) = 1$. Then one has that*

$$\sum_{[Q]\in\mathcal{Q}_d(2)/\Gamma_0(2)} \tilde{E}_2^{(0)}(\tau_Q; s) = 2^{-s}d^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}}(s)}{\zeta(2s)}\frac{2^{1-2s}}{1+2^{-s}}.$$

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Proof By the definition of $\tilde{E}_2^{(0)}(z;s)$ and [11, p. 47], one can easily deduce that

$$\tilde{E}_2^{(0)}(\tau;s) = 2^{-s}\left(\sum_{(m,n)=1}{}' \frac{\mathrm{Im}(\tau)^s}{|m\tau+n|^{2s}} - \sum_{(m,n)=1}{}' \frac{\mathbf{1}_2(n)\mathrm{Im}(\tau)^s}{|m(2\tau)+n|^{2s}}\right)$$
$$= 2^{-s-1}\frac{1}{\zeta(2s)}\left(\sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{Im}(\tau)^s}{|m\tau+n|^{2s}} - \frac{1}{1-2^{-2s}}\sum_{m,n\in\mathbb{Z}}{}' \frac{\mathbf{1}_2(n)\mathrm{Im}(\tau)^s}{|m(2\tau)+n|^{2s}}\right),$$

where $\mathbf{1}_2(\cdot)$ denotes the principal character modulo 2. Then, by the proof of [20, Section 3.1], one can easily see that

$$\tilde{E}_2^{(0)}(\tau_Q;s)$$
$$= 2^{-s-1}\frac{1}{\zeta(2s)}\left(2^{-s+1}d^{\frac{s}{2}}\sum_{\substack{\mathcal{B}\in[\mathcal{A}]\\ integral}} \frac{1}{\mathrm{N}(\mathcal{B})^s} - \frac{1}{1-2^{-2s}}2^{-s+1}d^{\frac{s}{2}}\sum_{\substack{\mathfrak{b}\in[\mathfrak{a}\mathcal{O}_{\mathcal{K}}]\\ integral}} \frac{\mathbf{1}_2(\mathrm{N}(\mathfrak{b}))}{\mathrm{N}(\mathfrak{b})^s}\right),$$

where $\mathcal{A} = [a, a\tau_Q]$, $[\mathcal{A}]$ denotes the $\mathcal{O}_{\mathcal{K}}$-ideal class associated to $\mathcal{A}$, $\mathfrak{a} = [a, a(2\tau_Q)]$, and $[\mathfrak{a}\mathcal{O}_{\mathcal{K}}]$ denotes the $\mathcal{O}_{\mathcal{K}}$-ideal class of conductor 2 associated to $\mathfrak{a}\mathcal{O}_{\mathcal{K}}$, i.e., prime-to-2 ideal class with respect to prime-to-2 principal ideals subgroup. By the assumption $\chi_d(2) = 1$, one can note that $h_{\mathcal{K}}(2)$ coincides with the ideal class number of $\mathcal{K}$, and thus the ideal class group of $\mathcal{K}$ is isomorphic to the ring class group of conductor 2 of $\mathcal{K}$. Therefore, one can deduce that

$$\sum_{[Q]\in\mathcal{Q}_d(2)/\Gamma_0(2)} \tilde{E}_2^{(0)}(\tau_Q;s) = 2^{-2s}d^{\frac{s}{2}}\frac{1}{\zeta(2s)}\left(\zeta_{\mathcal{K}}(s) - \frac{1}{1-2^{-2s}}\sum_{\mathcal{B}\subset\mathcal{O}_{\mathcal{K}}} \frac{\mathbf{1}_2(\mathrm{N}(\mathcal{B}))}{\mathrm{N}(\mathcal{B})^s}\right)$$
$$= 2^{-2s}d^{\frac{s}{2}}\frac{1}{\zeta(2s)}\left(\zeta_{\mathcal{K}}(s) - \frac{(1-2^{-s})(1-\chi_d(2)2^{-s})}{1-2^{-2s}}\zeta_{\mathcal{K}}(s)\right)$$
$$= 2^{-2s}d^{\frac{s}{2}}\frac{1}{\zeta(2s)}\left(\zeta_{\mathcal{K}}(s) - \frac{(1-2^{-s})^2}{1-2^{-2s}}\zeta_{\mathcal{K}}(s)\right)$$
$$= 2^{-s}d^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}}(s)}{\zeta(2s)}\frac{2^{1-2s}}{1+2^{-s}}$$

as desired. ■

Lemma 3.6 *Let $\varepsilon(n)$ be defined by (1.2) associated with coprime negative fundamental discriminants $-d_1, -d_2$ for which $\varepsilon(2) = 1$. Then one has that for any positive integer k such that $4k = d_1d_2 - x^2$ for some integer x with $x^2 < d_1d_2$,*

$$\sum_{\substack{n|k\\ n\ odd}} \varepsilon(n) = 0.$$

Proof By the assumption on k and the definition of $\varepsilon(\cdot)$, one can tell that $\varepsilon(k) = -1$ and k is not a square as d_1 and d_2 are coprime, and thus one can first deduce that

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$$\sum_{n|k}\varepsilon(n) = \sum_{\substack{n|k\\ n<\sqrt{k}}}\left(\varepsilon(n)+\varepsilon\left(\frac{k}{n}\right)\right)$$
$$= \sum_{\substack{n|k\\ n<\sqrt{k}}}(\varepsilon(n)-\varepsilon(n))$$
$$= 0.$$

On the other hand, by the assumption $\varepsilon(2)=1$, one can note that

$$\sum_{n|k}\varepsilon(n) = \sum_{j=0}^{\mathrm{ord}_2(k)}\sum_{2^j|k}\sum_{\substack{n|k\\ n\ odd}}\varepsilon(2^j n)$$
$$= \sum_{j=0}^{\mathrm{ord}_2(k)}\sum_{2^j|k}\sum_{\substack{n|k\\ n\ odd}}\varepsilon(n)$$
$$= (\mathrm{ord}_2(k)+1)\sum_{\substack{n|k\\ n\ odd}}\varepsilon(n),$$

and this together with the identity above implies that

$$\sum_{\substack{n|k\\ n\ odd}}\varepsilon(n) = 0.$$

∎

Lemma 3.7 *Let $-d_1$ and $-d_2$ be two coprime negative fundamental discriminants for which $\chi_{d_1}(2)=\chi_{d_2}(2)=1$, i.e., $-d_1\equiv -d_2\equiv 1 \pmod 8$, and let $\varepsilon(n)$ be defined by (1.2). Then one has that*

$$\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}}\ \sum_{n\left|\frac{d_1d_2-x^2}{4}\right.}\varepsilon(n)\log n - 2\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 8}}\ \sum_{n\left|\frac{d_1d_2-x^2}{8}\right.}\varepsilon(n)\log n$$
$$= -\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod{16}}}\ \sum_{n\left|\frac{d_1d_2-x^2}{16}\right.}\varepsilon(n)\log n$$

Proof First of all, by the assumption $\varepsilon(2)=1$ and the proof of Lemma 3.6, one recalls that for any positive integer k such that $\varepsilon(k)=-1$,

$$\sum_{n|k}\varepsilon(n)\log n = (\mathrm{ord}_2(k)+1)\sum_{\substack{n|k\\ n\ odd}}\varepsilon(n)\log n.$$

This simply leads to

$$\sum_{n|2k}\varepsilon(n)\log n - 2\sum_{n|k}\varepsilon(n)\log n = -\mathrm{ord}_2(k)\sum_{\substack{n|k\\ n\ odd}}\varepsilon(n)\log n$$
$$= -\sum_{n|\frac{k}{2}}\varepsilon(n)\log n, \tag{3.2}$$

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where the last equality makes sense since when k is odd, the sum on the right-hand side is over an empty set, which is defaulted to be 0. Now setting $k = \frac{d_1d_2-x^2}{8}$ and summing both sides of (3.2) over all integers x such that $x^2 < d_1d_2$, one obtains the desired identity. ■

We now present a different proof to Theorem 3.2.

Proof of Theorem 3.2 Write $h_{\mathcal{K}_i}(2)$ for $|\mathcal{Q}_{d_i}(2)/\Gamma_0(2)|$. By Lemmas 2.1, 2.3, 2.11, 3.3, 3.5, and 3.7, one can deduce that

$$\sum_{[Q_1]\in\mathcal{Q}_{d_1}(2)/\Gamma_0(2)} \sum_{[Q_2]\in\mathcal{Q}_{d_2}(2)/\Gamma_0(2)} \log|f(\tau_{Q_1}) - f(\tau_{Q_2})|^2$$

$$= \lim_{s\to1}\left(-2\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}} \rho_2(n)Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right) + 4\pi h_{\mathcal{K}_2}(2)\sum_{[Q]\in\mathcal{Q}_{d_1}(2)/\Gamma_0(2)} \tilde{E}_2^{(0)}(\tau_Q;s) + 4\pi h_{\mathcal{K}_1}(2)\sum_{[Q]\in\mathcal{Q}_{d_2}(2)/\Gamma_0(2)} \tilde{E}_2^{(0)}(\tau_Q;s) - h_{\mathcal{K}_1}(2)h_{\mathcal{K}_2}(2)\left(4\pi^{\frac{3}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{1}{2^{2s}-1}+8\right)\right)$$

$$= \lim_{s\to1}\left(-\left(-2\sum_{\substack{n\in\mathbb{Z}\\ n>\sqrt{d_1d_2}\\ n^2\equiv d_1d_2 \pmod 4}}\sum_{\ell\mid\frac{n^2-d_1d_2}{4}} \varepsilon(\ell)Q_{s-1}\left(\frac{n}{\sqrt{d_1d_2}}\right)\right) + 4\left(-\sum_{\substack{n\in\mathbb{Z}\\ |n|>\sqrt{d_1d_2}\\ n\equiv\beta_1\beta_2 \pmod 4}}\sum_{\ell\mid\frac{n^2-d_1d_2}{8}} \varepsilon(\ell)Q_{s-1}\left(\frac{|n|}{\sqrt{d_1d_2}}\right)\right) + 4\pi h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)}\frac{2^{1-2s}}{1+2^{-s}} + 4\pi h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\frac{2^{1-2s}}{1+2^{-s}} - 4h_{\mathcal{K}_1}h_{\mathcal{K}_2}\pi^{\frac{3}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{1}{2^{2s}-1} - 8h_{\mathcal{K}_1}h_{\mathcal{K}_2}\right)$$

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$$= \sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\mid\frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n - 2\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 8}} \sum_{n\mid\frac{d_1d_2-x^2}{8}} \varepsilon(n)\log n$$

$$-\frac{5}{9}\log 2\lim_{s\to1}\left(4\pi h_{\mathcal{K}_2}2^{-s}d_1^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_1}(s)}{\zeta(2s)}+4\pi h_{\mathcal{K}_1}2^{-s}d_2^{\frac{s}{2}}\frac{\zeta_{\mathcal{K}_2}(s)}{\zeta(2s)}\right)(s-1)$$

$$+\frac{8}{9}\log 2h_{\mathcal{K}_1}h_{\mathcal{K}_2}4\pi\lim_{s\to1}\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}(s-1)+\frac{24h_{\mathcal{K}_1}h_{\mathcal{K}_2}}{9}\log 2$$

$$= \sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\mid\frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n - 2\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 8}} \sum_{n\mid\frac{d_1d_2-x^2}{8}} \varepsilon(n)\log n$$

$$+h_{\mathcal{K}_1}h_{\mathcal{K}_2}\left(-\frac{120}{9}\log 2+\frac{96}{9}\log 2+\frac{24}{9}\log 2\right)$$

$$= \sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}} \sum_{n\mid\frac{d_1d_2-x^2}{4}} \varepsilon(n)\log n - 2\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 8}} \sum_{n\mid\frac{d_1d_2-x^2}{8}} \varepsilon(n)\log n$$

$$= -\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod{16}}} \sum_{n\mid\frac{d_1d_2-x^2}{16}} \varepsilon(n)\log n.$$

■

4 Remark II: a conjecture of Yang, Yin, and Yu

In this section, we give an affirmative answer to a conjecture recently raised by Yang, Yin, and Yu [17].

Let $\lambda(\tau)$ be the so-called modular λ-invariant, the modular parametrization from the modular curve $X(2) := \Gamma(2)\backslash\left(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})\right)$ to the isomorphism classes of the elliptic curves in Legendre form over $\mathbb{C}$, defined by

$$\lambda(\tau) = -\frac{1}{16}q^{-\frac{1}{2}}\prod_{n=1}^{\infty}\left(\frac{1-q^{n-\frac{1}{2}}}{1+q^n}\right)^8.$$

In a recent paper [17], Yang, Yin, and Yu showed that for an negative fundamental discriminant $-d\equiv 5 \pmod 8$, i.e., $\chi_d(2)=-1$, the Galois conjugates of $\lambda\left(\frac{-d+\sqrt{-d}}{2}\right)$ over $\mathbb{Q}$ are exactly $\lambda(\tau_Q)$ as Q ranges over $\mathcal{Q}_d/\Gamma(2)=\mathcal{Q}_d(2)/\Gamma(2)$, and at the end of [17], they conjectured that

Conjecture 4.1 (Yang, Yin and Yu) *The rational norm of* $\lambda\left(\frac{-d_1+\sqrt{-d_1}}{2}\right)-\lambda\left(\frac{-d_2+\sqrt{-d_2}}{2}\right)$ *is a sixth power for any coprime fundamental discriminants* $-d_1,-d_2<-3$ *for which* $-d_1,-d_2\equiv 5 \pmod 8$, *i.e.,*

$$\prod_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma(2)}\ \prod_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma(2)}|\lambda(\tau_{Q_1})-\lambda(\tau_{Q_2})|=m^6$$

for some $m\in\mathbb{Z}$.

We now conclude this part by showing how Theorem 1.1 implies that

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Theorem 4.2 Conjecture 4.1 *is true.*

We start with the following observations.

Lemma 4.3 *Let $j_2(\tau)$ be defined by (1.6). Then the following identities hold.*

$$(16\lambda(\tau)-8)^2 = j_2(\tau)+64, \tag{4.1}$$

$$16\lambda(\tau+1)-8 = -(16\lambda(\tau)-8). \tag{4.2}$$

Proof For (4.1), one first notes that $j_2(\tau)$ has only a simple pole at the cusp $i\infty$ of $X_0(2) = \Gamma_0(2)\backslash(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q}))$. Since $\Gamma_0(2)/\pm\Gamma(2) = \{I, T\}$, where $T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ fixes the cusp $i\infty$, then the cusp $i\infty$ totally ramifies in $X(2)$ of index 2, and thus, as a modular function for $\Gamma(2)$, $j_2(\tau)$ has only a double pole at the cusp $i\infty$. Since $\lambda(\tau)$ has only a simple pole at the cusp $i\infty$ of $X(2)$, then $j_2(\tau)$ must be a quadratic polynomial in $\lambda(\tau)$, whose coefficients can be easily determined by the principal parts of their Fourier expansions at the cusp $i\infty$.

For (4.2), since $\lambda(\tau+1) = \lambda(T\cdot\tau)$, T stabilizes the cusp $i\infty$, and $\Gamma(2)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, then $\lambda(\tau+1)$ is also a modular function for $\Gamma(2)$ with only a simple pole at the cusp $i\infty$, and thus it must be a linear polynomial in $\lambda(\tau)$. Comparing the principal parts of their Fourier expansions at the cusp $i\infty$, one obtains the desired relation. ∎

Proof of Theorem 4.2 By (4.1) and (4.2) together with the facts that $\Gamma_0(2) = \Gamma(2)(\{I\}\cup\{T\})$, where $T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, and $|\mathcal{Q}_{d_i}/\Gamma(2)| = 2|\mathcal{Q}_{d_i}(2)/\Gamma_0(2)| = 6h_{\mathcal{K}_i}$ by Lemma 2.11, one can deduce that

$$\begin{aligned}
&\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma(2)} \log|\lambda(\tau_{Q_1})-\lambda(\tau_{Q_2})| \\
&\quad= \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma(2)} \log|(16\lambda(\tau_{Q_1})-8)-(16\lambda(\tau_{Q_2})-8)| \\
&\qquad -144h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2 \\
&\quad= 2\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)} \log|(16\lambda(\tau_{Q_1})-8)-(16\lambda(\tau_{Q_2})-8)| \\
&\qquad + \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)} \log|(16\lambda(\tau_{Q_1}+1)-8)-(16\lambda(\tau_{Q_2})-8)| \\
&\qquad + \sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)} \log|(16\lambda(\tau_{Q_1})-8)-(16\lambda(\tau_{Q_2}+1)-8)| \\
&\qquad -144h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2 \\
&\quad= 2\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)} \log|(16\lambda(\tau_{Q_1})-8)-(16\lambda(\tau_{Q_2})-8)| \\
&\qquad + 2\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\ \sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)} \log|(16\lambda(\tau_{Q_1})-8)+(16\lambda(\tau_{Q_2})-8)|
\end{aligned}$$

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$$
\begin{aligned}
&-144h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2\\
&=2\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)}\log\left|(16\lambda(\tau_{Q_1})-8)^2-(16\lambda(\tau_{Q_2})-8)^2\right|\\
&\quad-144h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2\\
&=\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma_0(2)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma_0(2)}\log\left|j_2(\tau_{Q_1})-j_2(\tau_{Q_2})\right|^2-144h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2\\
&=-3\sum_{\substack{x\in\mathbb{Z}\\ x^2<d_1d_2\\ x^2\equiv d_1d_2 \pmod 4}}\sum_{n\left|\frac{d_1d_2-x^2}{4}\right.}\varepsilon(n)\log n-48h_{\mathcal{K}_1}h_{\mathcal{K}_2}\log 2,
\end{aligned}
$$

where the last line follows from Theorem 1.1 specialized to $\chi_{d_1}(2)=\chi_{d_2}(2)=-1$ under the conditions $-d_1,-d_2\equiv 5 \pmod 8$. Clearly, since d_1d_2 is odd, the multipliers of $\log n$ in the double sum must be all even as $x=0$ is not included in the outer sum and both x and $-x$ for which $x\neq 0$ and $x^2\equiv d_1d_2 \pmod 4$ are included therein. Therefore, the multipliers of $\log n$ involved in

$$
\sum_{[Q_1]\in\mathcal{Q}_{d_1}/\Gamma(2)}\sum_{[Q_2]\in\mathcal{Q}_{d_2}/\Gamma(2)}\log|\lambda(\tau_{Q_1})-\lambda(\tau_{Q_2})|
$$

are all multiples of 6, and hence Conjecture 4.1 holds. ∎

5 Adelic formulation of Borcherds lifts

In this section, we briefly review the theory of Borcherds lifts in the adelic setting [12], and show that the discriminant of the class polynomial $H_{\mathcal{K},p}(x)$ defined by (1.4) can be expressed in terms of the CM values of a Borcherds lift, which are explicitly computable via Schofer's small CM value formula summarized in Section 6.

5.1 Rational quadratic space

Let V be a vector space over $\mathbb{Q}$ with the quadratic form Q of signature $(n,2)$. For a $\mathbb{Q}$-algebra F, we write $V(F)=V\otimes_{\mathbb{Q}}F$. Let $\mathbb{D}$ denote the Grassmannian of oriented negative 2-planes of $V(\mathbb{R})$. Then $\mathbb{D}$ is a symmetric space for $O(n,2)$ and has a Hermitian structure. It can be viewed as an open subset $\mathcal{Q}_-$ of a quadric in $\mathbb{P}^1(V)(\mathbb{C})$. Explicitly,

$$
\mathbb{D}\cong\mathcal{Q}_-:=\{w\in\mathbb{P}^1(V)(\mathbb{C})\,|\,(w,w)=0,\ (w,\bar{w})<0\}/\mathbb{C}^\times,
$$

where the isomorphism is given by $[x,-y]\to x+iy$ for a properly oriented basis $[x,-y]$, and this gives a complex structure on $\mathbb{D}$. Let $H=\mathrm{GSpin}(V)$ be the general spin group of V. Let $\mathbb{A}$ be the adele ring over $\mathbb{Q}$ and $\mathbb{A}_f$ be the associated finite adele ring. Assume K to be an open compact subgroup of $H(\mathbb{A}_f)$ such that $H(\mathbb{A})=H(\mathbb{Q})H(\mathbb{R})^+K$, where $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$. Define

$$
X_K:=H(\mathbb{Q})\backslash\left(\mathbb{D}\times H(\mathbb{A}_f)/K\right).
$$

Note that the assumption that $H(\mathbb{A})=H(\mathbb{Q})H(\mathbb{R})^+K$ implies that the space X_K has exactly one connected component, and thus is connected. This is the set of complex

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points of a quasi-projective variety rational over $\mathbb{Q}$, and if $\Gamma_K = H(\mathbb{Q}) \cap H(\mathbb{R})^+ K$, then

$$X_K \cong \Gamma_K \backslash \mathbb{D}^+$$

via $[z, h] \to [\gamma^{-1} z]$, where $\mathbb{D}^+ \subset \mathbb{D}$ is the subset of positively oriented negative 2-planes, and $h = \gamma k$ for some $\gamma \in H(\mathbb{Q})^+$ and $k \in K$ by the assumption.

One can view $\{w \in \mathbb{P}^1(V)(\mathbb{C}) | (w, w) = 0, (w, \bar{w}) < 0\}$ as a line bundle $\mathcal{L}$ over $\mathbb{D}$, and it descends to a line bundle $\mathcal{L}_K$ over X_K. Assume that

$$V(\mathbb{R}) = V_0 + \mathbb{R}e + \mathbb{R}f$$

where e and f are such that $Q(e) = Q(f) = 0$ and $(e, f) = 1$. Then the signature of V_0 is $(n-1, 1)$ and for the disjoint union of two negative cones

$$\mathcal{C} = \{y \in V_0 | Q(y) < 0\},$$

we have

$$\mathbb{D} \cong \mathcal{H} := \{z \in V_0(\mathbb{C}) | \operatorname{Im}(z) \in \mathcal{C}\}.$$

The isomorphism is given by $z \to w(z) := e - Q(z)f + z$ composed with projection to $\mathcal{Q}_-$. The map $z \to w(z)$ can be viewed as a holomorphic section of the line bundle $\mathcal{L}$.

Example 5.1 Let $V = M_2(\mathbb{Q})$ be the set of 2×2 matrices over $\mathbb{Q}$ with the quadratic form $Q(x) = \det(x)$ of signature $(2, 2)$. Then one can check that $\mathbb{D} \cong \mathbb{H}^2 \cup \overline{\mathbb{H}}^2$ via $z = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix} \to w(z) = (z_1, z_2)$ and [15, p. 137]

$$H = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 | \det(g_1) = \det(g_2)\},$$

acting on V via conjugation $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. Now take

$$K_p = \left\{(g_1, g_2) \in H(\hat{\mathbb{Z}}) | g_i \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod p\right\}.$$

Then one may see that $H(\mathbb{Q}) \cap H(\mathbb{R})^+ K_p = \Gamma_0(p) \times \Gamma_0(p)$ and $X_{K_p} \cong Y_0(p) \times Y_0(p)$, where $Y_0(p) = \Gamma_0(p) \backslash \mathbb{H}$ the open modular curve for $\Gamma_0(p)$.

Definition 5.1 A modular form on $\mathbb{D} \times H(\mathbb{A}_f)$ of weight k is a function $f : \mathbb{D} \times H(\mathbb{A}_f) \to \mathbb{C}$ meromorphic on $\mathbb{D}$ such that

(1) $f(z, hk) = f(z, h)$ for all $k \in K$,
(2) $f(\gamma z, \gamma h) = j(\gamma, z)^k f(z, h)$ for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is the automorphy factor induced by the isomorphism w.

5.2 Borcherds lifts

For $z \in \mathbb{D}$, let $\mathrm{pr}_z : V(\mathbb{R}) \to z$ be the projection map, and for $x \in V(\mathbb{R})$, let $R(x, z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x))$. Then we define

$$(x, x)_z = (x, x) + 2R(x, z),$$

and our Gaussian for V is the function

$$\varphi_\infty(x,z) = e^{-\pi(x,x)_z}.$$

For $\tau \in \mathbb{H}$ with $\tau = u + iv$, let

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix},$$

and $g'_\tau = (g_\tau, 1) \in \mathrm{Mp}_2(\mathbb{R})$. Let $l = \frac{n}{2} - 1$, $G = \mathrm{SL}_2$, and ω be the Weil representation of the metaplectic cover $G'_{\mathbb{A}}$ of $G_{\mathbb{A}}$ on $\mathcal{S}(V(\mathbb{A}_f))$, the Schwartz space of $V(\mathbb{A}_f)$, where we write $G_{\mathbb{A}}$ for $G(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$. Then, for the linear action of $H(\mathbb{A}_f)$, we write $\omega(h)\varphi(x) = \omega(h^{-1}x)$ for $\varphi \in \mathcal{S}(V(\mathbb{A}_f))$. For $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$, we have the linear functional on $\mathcal{S}(V(\mathbb{A}_f))$ given by

$$\varphi \to \theta(\tau, z, h; \varphi) := v^{-\frac{1}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)\left(\varphi_\infty(\cdot, z) \otimes \omega(h)\varphi\right)(x).$$

Let L be a lattice of V, and let L' be the dual lattice of L defined by

$$L' = \{x \in V \mid (x, L) \subset \mathbb{Z}\}.$$

Let $\mathcal{S}_L$ be the subspace of $\mathcal{S}(V(\mathbb{A}_f))$ consisting of functions with support in $\hat{L}'$ and constant on cosets of $\hat{L}$, where $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then

$$\mathcal{S}_L = \bigoplus_{\eta \in L'/L} \mathbb{C}\phi_\eta, \qquad \phi_\eta = \mathrm{Char}(\eta + \hat{L}).$$

Let $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ be the full inverse image of $\mathrm{SL}_2(\mathbb{Z}) \subset G(\mathbb{R})$ in $\mathrm{Mp}_2(\mathbb{R})$.

Definition 5.2 Let ω_L be the Weil representation associated to $\mathcal{S}_L$. A function $\vec{F} : \mathbb{H} \to \mathcal{S}_L$ is a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ and type ω_L for Γ' if

(1) $\vec{F}(\gamma'\tau) = (c\tau + d)^{1-\frac{n}{2}}\omega_L(\gamma')\vec{F}(\tau)$ for all $\gamma' \in \Gamma'$,

(2) $\vec{F}(\tau)$ has a Fourier expansion

$$\vec{F}(\tau) = \sum_{\eta \in L'/L} \sum_{\substack{m \in Q(\eta)+\mathbb{Z} \\ m \gg -\infty}} c(m, \eta) q^m \phi_\eta,$$

where the condition $m \equiv Q(\eta) \pmod{\mathbb{Z}}$ follows from the transformation law for $T' \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For the theta function

$$\theta(\tau, z, h) = \sum_{\mu \in L'/L} \theta(\tau, z, h; \phi_\mu),$$

we can pair it with $\vec{F}(\tau)$ by the following $\mathbb{C}$-bilinear pairing

$$\langle \vec{F}(\tau), \theta(\tau, z, h) \rangle = \sum_{\mu \in L'/L} \sum_{m \in Q(\mu)+\mathbb{Z}} c(m, \mu) q^m \theta(\tau, z, h; \phi_\mu).$$

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Using this pairing, we define a regularized integral as in [2] called a Borcherds lift,

$$\Phi(z,h;\vec{F}) := \underset{s=0}{\mathrm{CT}}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle\vec{F}(\tau),\theta(\tau,z,h)\rangle v^{-2-s}dudv\right\},$$

where $\underset{s=0}{\mathrm{CT}}$ denotes the constant term in the Laurent expansion at $s=0$ of

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle\vec{F}(\tau),\theta(\tau,z,h)\rangle v^{-2-s}dudv,$$

$\mathcal{F}_t$ is the truncated fundamental domain defined by

$$\mathcal{F}_t := \{\tau\in\mathcal{F}|\,\mathrm{Im}(\tau)\le t\},$$

and $\mathcal{F}$ is the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Then the following theorem is the core result of the theory of Borcherds lifts, which reveal the connection between Borcherds lifts and modular forms on $\mathrm{O}(n,2)$.

Theorem 5.2 (Borcherds) *There is a meromorphic modular form $\Psi(z,h;\vec{F})$ of weight $\frac{1}{2}c(0,0)$ on $\mathbb{D}\times H(\mathbb{A}_f)$ such that*

$$\Phi(z,h;\vec{F}) = -2\log|\Psi(z,h;\vec{F})|^2|y|^{c(0,0)} - c(0,0)\left(\log(2\pi)+\Gamma'(1)\right), \tag{5.1}$$

where $y=\mathrm{Im}(z)$. Such a meromorphic modular form is called the Borcherds form arising from the Borcherds lift of a modular form $\vec{F}$.

A proof of the following theorem can be found in [21, Theorem 1.4].

Theorem 5.3 *Let $p\in\{2,3,5,7,13\}$, and let $j_p(\tau)$ be defined by (1.6). Under the identification $X_{K_p}\cong Y_0(p)\times Y_0(p)$ shown in* Example 5.1, *the automorphic Green function $-2\log|j_p(z_1)-j_p(z_2)|^2$ is a Borcherds lift associated to the lattice $L_p=\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ p\mathbb{Z} & \mathbb{Z}\end{pmatrix}$. Precisely,*

$$-2\log|j_p(z_1)-j_p(z_2)|^2 = \Phi(z,1;\vec{F}_p),$$

where

$$\vec{F}_p=\vec{F}_p(\tau)=\sum_{\mu\in L'_p/L_p}\ \sum_{\substack{m\in Q(\mu)+\mathbb{Z}\\ m\gg-\infty}} c(m,\mu)q^m$$

is a weakly holomorphic modular form of weight 0 and type ω_{L_p} with

$$L'_p/L_p=\left\{\mu_{l,k}=\begin{pmatrix}0 & l/p\\ k & 0\end{pmatrix}\right\}_{0\le l,k\le p-1}$$

for which $c(-1,\mu_{0,0})=1$, $c(m,\mu)=0$ for $m<0$ and $\mu\ne\mu_{0,0}$, $c(0,\mu_{0,k})=\frac{24}{p-1}$ for $1\le k\le p-1$, and otherwise, $c(0,\mu_{l,k})=0$.

Throughout the remainder of the work, we write $\Phi(z;\vec{F}_p)$ for $\Phi(z,1;\vec{F}_p)$.

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5.3 The discriminant of $H_{\mathcal{K},p}(x)$

As before, $H_{\mathcal{K},p}(x)$ denotes the ring class polynomial associated to $j_p(\tau)$ and an imaginary quadratic field $\mathcal{K}$ defined by (1.4). It is known [5] that $\{j_p(\tau_{\mathfrak{c}})\}_{[\mathfrak{c}]\in \mathrm{Cl}_{\mathcal{K}}(p)}$, where $\tau_{\mathfrak{c}}$ denotes the imaginary quadratic point associated to the integral ideal $\mathfrak{c} = [a, \frac{b+\sqrt{-d}}{2}]$, i.e., $\tau_{\mathfrak{c}} = \frac{b+\sqrt{-d}}{2a}$, are all the Galois conjugates of $H_{\mathcal{K},p}(x)$, then one can easily see that by the group structure of $\mathrm{Cl}_{\mathcal{K}}(p)$,

$$\begin{aligned} \log|\mathrm{disc}(H_{\mathcal{K},p}(x))| &= \sum_{\substack{[\mathfrak{c}],[\mathfrak{c}']\in \mathrm{Cl}_{\mathcal{K}}(p)\\ [\mathfrak{c}]\neq[\mathfrak{c}']}} \log|j_p(\tau_{\mathfrak{c}}) - j_p(\tau_{\mathfrak{c}'})| \\ &= \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}} \sum_{[\mathfrak{b}]\in \mathrm{Cl}_{\mathcal{K}}(p)} \log|j_p(\tau_{\mathfrak{a}\mathfrak{b}}) - j_p(\tau_{\mathfrak{b}})| \\ &= -\frac{1}{4} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}} \sum_{[\mathfrak{b}]\in \mathrm{Cl}_{\mathcal{K}}(p)} -2\log|j_p(\tau_{\mathfrak{a}\mathfrak{b}}) - j_p(\tau_{\mathfrak{b}})|^2 \\ &= -\frac{1}{8} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(p)\\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}} \sum_{z\in Z_p(U_{\mathfrak{a}})} \Phi(z;\vec{F}_p), \end{aligned} \tag{5.2}$$

where $Z_p(U_{\mathfrak{a}})$ is a 0-cycle of X_{K_p} that is identified with

$$\sum_{[\mathfrak{b}]\in \mathrm{Cl}_{\mathcal{K}}(p)} \{(\tau_{\mathfrak{a}\mathfrak{b}}, \tau_{\mathfrak{b}})\} + \sum_{[\mathfrak{b}]\in \mathrm{Cl}_{\mathcal{K}}(p)} \{(\tau_{\bar{\mathfrak{a}}\mathfrak{b}}, \tau_{\mathfrak{b}})\} \tag{5.3}$$

under the identification $X_{K_p} \cong Y_0(p) \times Y_0(p)$ indicated in Example 5.1, and (5.2) follows from Theorem 5.3. Now the whole problem is boiled down to evaluating the average value of the Borcherds lift $\Phi(z;\vec{F}_p)$ over the 0-cycle $Z_p(U_{\mathfrak{a}})$, and this can be done with the aid of Schofer's celebrated work, which is briefly reviewed in the next section.

6 Small CM value formula

In this section, we briefly review Schofer's small CM value formula [14] (see also Bruinier and Yang's work [4], which extends Schofer's work to harmonic weak Maass forms and has deep applications in the Gross–Zagier arithmetic formula [10]) and certain related key concepts, such as small CM 0-cycles and Fourier coefficients of an incoherent Eisenstein series of weight 1.

6.1 Small CM 0-cycles

Assume that we have a rational splitting $V = V_+ \oplus U$, where V_+ is of signature $(n, 0)$ and U is of signature $(0, 2)$. Then U gives rise to a two-point subset $\mathbb{D}_0$ of $\mathbb{D}$. Let $T = \mathrm{GSpin}(U)$ and let $K(T) = K \cap T(\mathbb{A}_f)$. Then there is an embedding $T \hookrightarrow H$ and we have a so-called small CM 0-cycle of X_K as $U \cong \mathcal{K}$ [15, Chapter 5] as quadratic spaces for some imaginary quadratic field $\mathcal{K}$,

$$Z(U)_K := T(\mathbb{Q})\backslash\big(\mathbb{D}_0 \times T(\mathbb{A}_f)/K(T)\big) \hookrightarrow X_K,$$

which is a 0-cycle of X_K.

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6.2 Eisenstein series for SL_2 and Schofer's small CM value formula

In this subsection, we will state Schofer's small CM value formula that is deeply connected to certain Eisenstein series for SL_2, and thus prior to that, we first briefly review the concepts of these Eisenstein series. One may find more details in [13, Section 2]. Assume that V is of signature $(n, 2)$ with $n \geq 1$. Inside of $G_{\mathbb{A}}$, we have the subgroups

$$N_{\mathbb{A}} := \{n(b) | \, b \in \mathbb{A}\}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

and

$$M_{\mathbb{A}} := \{m(a) | \, a \in \mathbb{A}^\times\}, \quad m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Define the quadratic character $\chi = \chi_V$ of $\mathbb{A}^\times/\mathbb{Q}^\times$ via the global Hilbert symbol by

$$\chi(x) = \left((-1)^{\frac{n(n-1)}{2}} \det(V), x \right),$$

where $\det(V) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the determinant of the matrix for the quadratic form Q on V. For $s \in \mathbb{C}$, let $I(s, \chi)$ be the principal series representation of $G_{\mathbb{A}}$. This space consists of smooth functions $\Phi(g, s)$ on $G_{\mathbb{A}} \times \mathbb{C}$ such that

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g, s).$$

A section $\Phi(g, s) \in I(s, \chi)$ is called standard if its restriction to $K_\infty K_f$, where $K_\infty = SO(2)$ and $K_f = \mathrm{SL}_2(\hat{\mathbb{Z}})$, is independent of s. We let $P = MN$ and define the Eisenstein series associated to a standard section $\Phi(g, s)$ by

$$E(g, z; \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi(\gamma g, s).$$

This series converges for $\Re(s) > 1$ and has a meromorphic continuation to the whole s-plane. In addition, the Archimedean component $\Phi^l_\infty(g, s) : G(\mathbb{R}) \times \mathbb{C} \to \mathbb{C}$ of $\Phi(g, s)$ is taken as follows. For $l \in \mathbb{Z}$, let $\tilde{\chi}_l$ be the character of K_∞ defined by

$$\tilde{\chi}_l(k_\theta) = e^{il\theta}, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

Then let $\Phi^l_\infty(g, s) : G(\mathbb{R}) \times \mathbb{C} \to \mathbb{C}$ be the normalized eigenfunction of weight l for K_∞, i.e.,

$$\Phi^l_\infty(gk, s) = \tilde{\chi}_l(k)\Phi^l_\infty(g, s).$$

Moreover, we have a $G_{\mathbb{A}}$-intertwining map

$$\lambda = \lambda_V : \mathcal{S}(V(\mathbb{A})) \to I\left(\frac{n}{2}, \chi\right)$$

such that $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$, and the function $\lambda(\varphi)$ has a unique extension to a standard section $\Phi(g, s) \in I(s, \chi)$ such that $\Phi(g, \frac{n}{2}) = \lambda(\varphi)$. Now, for any $\varphi \in \mathcal{S}(V(\mathbb{A}_f))$, there is an associated standard section in $I(s, \chi)$ given by $\Phi(g, s) =$

$\Phi_\infty^l(g,s)\otimes\lambda(\varphi)$, and by strong approximation, the Eisenstein series $E(g,s;\Phi)$ associated to such a section Φ is determined by the Eisenstein series

$$E(\tau,s;\varphi,l) := v^{-\frac{l}{2}}E(g_\tau,s;\Phi_\infty^l\otimes\lambda(\varphi)),$$

which is a nonholomorphic modular form of weight l on $\mathbb{H}$.

Definition 6.1 Consider $V=U$ of signature $(0,2)$ and view $U\cong\mathbb{Q}(\sqrt{-d})$ of discriminant $-d<0$. Let χ_d be the quadratic character of $\mathbb{Q}_{\mathbb{A}}^\times$ defined via the global Hilbert symbol by $\chi_d(x)=(-d,x)_{\mathbb{A}}$. For $\varphi\in\mathcal{S}(U(\mathbb{A}_f))$, let $A_m(v,s,\varphi)$ be the mth Fourier coefficient of the associated Eisenstein series $E(\tau,s;\varphi,1)$, i.e.,

$$E(\tau,s;\varphi,1)=\sum_{m\in\mathbb{Q}}A_m(v,s,\varphi)q^m.$$

Then define

$$\kappa(m,\varphi)=\begin{cases}\lim\limits_{v\to\infty}A_m'(v,0,\varphi) & \text{if } m>0,\\ \lim\limits_{v\to\infty}(A_0'(v,0,\varphi)-\varphi(0)\log v) & \text{if } m=0,\\ 0 & \text{if } m<0,\end{cases}$$

where $A_m'(v,s,\varphi)$ is the derivative of $A_m(v,s,\varphi)$ with respect to s.

Remark 6.1 It is known (see, e.g., [13, Section 7]) that $A_m(v,s,\varphi)$ is holomorphic and vanishes at $s=0$, and thus it is natural to consider $A_m'(v,0,\varphi)$.

In his brilliant work [14, Corollary 3.5] (see also [4, Theorem 4.7]), Schofer established the following formula expressing the average value of a Borcherds lift over a small CM 0-cycle in terms of $\kappa(m,\varphi)$, which is now known as Schofer's small CM value formula.

Theorem 6.2 (Schofer) *Let $\vec{F}:\mathbb{H}\to\mathcal{S}_L$ be a weakly holomorphic modular form for ρ_L of weight $1-\frac{n}{2}$ with Fourier expansion*

$$\vec{F}(\tau)=\sum_{\eta\in L'/L}\sum_{m\in Q(\eta)+\mathbb{Z}}c(m,\eta)q^m\phi_\eta$$

and $c(0,0)=0$. For $V=V_+\oplus U$, where $V_+=U^\perp$ of signature $(n,0)$, write L_+ and L_- for $L\cap V_+$ and $L\cap U$, respectively. Let $pr_\pm$ denote the projections of V onto V_+ and U, respectively, and write $x_\pm$ for $pr_\pm(x)$ for $x\in V$. Then

$$\sum_{z\in Z(U)_K}\Phi(z;\vec{F})$$
$$=\frac{4}{vol(K(T))}\sum_{\lambda\in L'/(L_++L_-)}\sum_{m\geq 0}c(-m,\lambda+L)\sum_{\ell\in\lambda_++L_+}\kappa(m-Q(\ell),\lambda_-+L_-).$$

Here, by abuse of notation, we mean $\kappa\left(m,\mathrm{Char}(\lambda_-+L_-\otimes_{\mathbb{Z}}\hat{\mathbb{Z}})\right)$ by $\kappa(m,\lambda_-+L_-)$. Also, note that here we consider $\Phi(z;\vec{F})$ instead of $\log|\Psi(z;\vec{F})|$, which are related to (5.1), and thus we have a multiplier of $\frac{4}{vol(K(T))}$ instead of $-\frac{2}{vol(K(T))}$ on the right.

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6.3 Interpretation of $Z_p(U_{\mathfrak{a}})$ as a small CM 0-cycle

In this subsection, we show how the 0-cycle $Z_p(U_{\mathfrak{a}})$ defined by (5.3) can be interpreted as a small CM 0-cycle of the two-dimensional variety X_{K_p} considered in Example 5.1. Recall that $V = \mathrm{M}_2(\mathbb{Q})$, $Q(x) = \det(x)$,

$$H = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 |\ \det(g_1) = \det(g_2)\},$$

and

$$K_p = \left\{(g_1, g_2) \in H |\ g_i \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p}\right\}.$$

Take $[\mathfrak{a}] \in \mathrm{Cl}_{\mathcal{K}}(p) - \{[\mathcal{O}_{\mathcal{K}}]\}$ with $\mathfrak{a} = [A, \frac{B+\sqrt{-d}}{2}]$, where $A > 0$ and $-d = (B^2 - 4AC)$ for some $0 < C \in \mathbb{Z}$. By the assumption that $-d$ is odd (square-free) and fundamental, and the well-known isomorphism $\mathrm{Cl}_{\mathcal{K}}(p) \cong \mathcal{Q}_d/\Gamma_0(p)$, where $\mathcal{Q}_d$ denotes the set of primitive positive definite quadratic forms of discriminant $-d$, one can always make a choice of $\mathfrak{a}$ for which $(A, dp) = 1$. Now we can view $\mathcal{K} = \mathbb{Q}(\sqrt{-d}) = \mathbb{Q}A + \mathbb{Q}\frac{B+\sqrt{-d}}{2}$ as a rational quadratic space of signature $(2,0)$ with quadratic form $Q_{\mathfrak{a}}(x) = \frac{\mathrm{N}(x)}{\mathrm{N}(\mathfrak{a})} = \frac{\mathrm{N}(x)}{A}$. Then one can identify $(\mathcal{K}, Q_{\mathfrak{a}})$ with a $(2,0)$ subspace $(V_{\mathfrak{a}} = \mathbb{Q}e_1^{(\mathfrak{a})} + \mathbb{Q}e_2^{(\mathfrak{a})}, \det(x))$ of V, where

$$e_1^{(\mathfrak{a})} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad e_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ -1 & B \end{pmatrix},$$

via

$$xA + y\frac{B+\sqrt{-d}}{2} \to x\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} + y\begin{pmatrix} 0 & C \\ -1 & B \end{pmatrix}.$$

Then its orthogonal complement $U_{\mathfrak{a}} = V_{\mathfrak{a}}^{\perp} = \mathbb{Q}f_1^{(\mathfrak{a})} + \mathbb{Q}f_2^{(\mathfrak{a})}$, where

$$f_1^{(\mathfrak{a})} = \begin{pmatrix} -1 & B \\ 0 & A \end{pmatrix} \quad \text{and} \quad f_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ 1 & 0 \end{pmatrix},$$

turns out to be a $(0,2)$ subspace of V, which can be identified with $(\mathcal{K} = \mathbb{Q}A + \mathbb{Q}\frac{B+\sqrt{-d}}{2}, -\frac{\mathrm{N}(x)}{A})$, and gives rise to a two-point subset $\{z_0^{\pm}\}$ of $\mathbb{D}$. Under the above identification given in Example 5.1, one sees that $z_0^{\pm}$ are of the form

$$z_0^{\pm} = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix} \quad \text{with } \pm\Im(z_i) > 0.$$

Then one must have $(z_0^{\pm}, e_1^{(\mathfrak{a})}) = (z_0^{\pm}, e_2^{(\mathfrak{a})}) = 0$, which imply that

$$(z_1, z_2)^{\pm} = \left(\frac{B \pm \sqrt{-d}}{2A}, \frac{B \pm \sqrt{-d}}{2}\right) = (\tau_{\mathfrak{a}}, \tau_{\mathcal{O}_{\mathcal{K}}}) \cup (\tau_{\overline{\mathfrak{a}}}, \tau_{\overline{\mathcal{O}}_{\mathcal{K}}}).$$

By the isomorphism $\mathcal{Q}_d(p)/\Gamma_0(p) \cong \mathrm{Cl}_{\mathcal{K}}(p)$, the $\mathcal{O}_{\mathcal{K}}$-ideal class $[\mathfrak{a}]$ defines a point on $Y_0(p) \times Y_0(p)$ as above, and thus it is independent of the choice of the representative $\mathfrak{a}$.

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Now, by [15, Lemma 25.2], one sees that $T_{\mathfrak{a}} = \mathrm{GSpin}(U_{\mathfrak{a}}) \cong \mathcal{K}^\times = \mathbb{Q}(\sqrt{-d})^\times$ via

$$x + yf_1^{(\mathfrak{a})} \otimes f_2^{(\mathfrak{a})} \to x + y\frac{-B+\sqrt{-d}}{2}.$$

In addition, according to [15, Section 25.3], one can easily work out the embedding of $T_{\mathfrak{a}}$ into $H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 | \det(g_1) = \det(g_2)\}$, which yields that

$$x + yf_1^{(\mathfrak{a})} \otimes f_2^{(\mathfrak{a})} \to \left(\begin{pmatrix} x - By & Cy \\ -Ay & x \end{pmatrix}, \begin{pmatrix} x - By & ACy \\ -y & x \end{pmatrix}\right).$$

Under the embedding and the isomorphism $T_{\mathfrak{a}} \cong \mathcal{K}^\times$, one can easily see by the definition of K_p that $K_p(T_{\mathfrak{a}}) = T_{\mathfrak{a}}(\mathbb{A}_f) \cap K_p \cong (\hat{\mathbb{Z}} + \hat{\mathbb{Z}}p\frac{-B+\sqrt{-d}}{2})^\times = \hat{\mathcal{O}}_{\mathcal{K}}(p)^\times = (\hat{\mathbb{Z}} + p\hat{\mathcal{O}}_{\mathcal{K}})^\times$, and thus one has that

$$T_{\mathfrak{a}}(\mathbb{Q})\backslash T_{\mathfrak{a}}(\mathbb{A}_f)/K_p(T_{\mathfrak{a}}) \cong \mathcal{K}^\times\backslash\mathcal{K}_f^\times/\hat{\mathcal{O}}_{\mathcal{K}}(p)^\times \cong \mathrm{Cl}_{\mathcal{K}}(p).$$

Furthermore, by the fact shown above and Shimura's reciprocity law, one can identify the small CM 0-cycle $T_{\mathfrak{a}}(\mathbb{Q})\backslash\left(\{z_0^\pm\} \times T_{\mathfrak{a}}(\mathbb{A}_f)/K_p(T_{\mathfrak{a}})\right)$ associated to $\mathfrak{a}$ with

$$\sum_{\mathfrak{b}\in\mathrm{Cl}_{\mathcal{K}}(p)} \{(\tau_{\mathfrak{a}\mathfrak{b}}, \tau_{\mathfrak{b}})\} + \sum_{\mathfrak{b}\in\mathrm{Cl}_{\mathcal{K}}(p)} \{(\tau_{\bar{\mathfrak{a}}\mathfrak{b}}, \tau_{\mathfrak{b}})\},$$

which is exactly $Z_p(U_{\mathfrak{a}})$ defined as in (5.3). Finally, by Lemma 2.11, one has that

$$\deg(Z_p(U_{\mathfrak{a}})) = 2|\mathrm{Cl}_{\mathcal{K}}(p)| = 2(p - \chi_d(p))h_{\mathcal{K}}.$$

The interpretation of $Z_p(U_{\mathfrak{a}})$ as a small CM 0-cycle above together with (5.2), Theorems 5.3 and 6.2 and [4, Lemma 4.4] yield the following proposition.

Proposition 6.3 *Let $p \in \{2, 3, 5, 7, 13\}$. Let $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of odd discriminant $-d < -3$, and denote by $h_{\mathcal{K}}$ the class number of $\mathcal{K}$, and by $\chi_d(\cdot)$ the quadratic character associated to $\mathcal{K}$. Let $H_{\mathcal{K},p}(x)$ be defined by (1.4). Then one has that*

(6.1)

$$\log|disc(H_{\mathcal{K},p}(x))| = -\frac{(p-\chi_d(p))h_{\mathcal{K}}}{4} \sum_{\substack{[\mathfrak{a}]\in Cl_{\mathcal{K}}(p) \\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}}$$

$$\times \left[\sum_{\lambda\in L_p/(L_{p,+}^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})})} \sum_{x\in\lambda_+ + L_{p,+}^{(\mathfrak{a})}} \kappa(1-\det(x), \lambda_- + L_{p,-}^{(\mathfrak{a})}) \right.$$

$$\left. + \frac{24}{p-1}\sum_{k=1}^{p-1}\left(\sum_{\lambda\in L_p/(L_{p,+}^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})})} \sum_{x\in\mu_{0,k,+}+\lambda_{l,+}+L_{p,+}^{(\mathfrak{a})}} \kappa(-\det(x), \mu_{0,k,-} + \lambda_{l,-} + L_{p,-}^{(\mathfrak{a})}) \right)\right],$$

where $L_p = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $L_{p,+}^{(\mathfrak{a})} = L_p \cap V_{\mathfrak{a}} = \mathbb{Z}e_1^{(\mathfrak{a})} + \mathbb{Z}pe_2^{(\mathfrak{a})}$, *and* $L_{p,-}^{(\mathfrak{a})} = L_p \cap U_{\mathfrak{a}} = \mathbb{Z}f_1^{(\mathfrak{a})} + \mathbb{Z}pf_2^{(\mathfrak{a})}$.

7 Lattice computations

In this section, we explicitly compute the representatives λ that are involved in (6.1), as well as its projections $\lambda_\pm$ to complete the presentation of Theorem 1.6.

7.1

Suppose that p is an odd prime. Recall that since $-d$ is an odd fundamental discriminant, by the isomorphism $\mathrm{Cl}_{\mathcal{K}}(p) \cong \mathcal{Q}_d(p)/\Gamma_0(p)$, one can easily see that for each ideal class $[\mathfrak{a}]$, there is a representative $\mathfrak{a} = [A, \frac{B+\sqrt{-d}}{2}]$ for which $(A, dp) = 1$. As shown in Section 6.3, under the isomorphisms of quadratic spaces, $(V_{\mathfrak{a}}, \det(x)) \cong (\mathcal{K}, \frac{\mathrm{N}(x)}{A})$ and $(U_{\mathfrak{a}}, \det(x)) \cong (\mathcal{K}, -\frac{\mathrm{N}(x)}{A})$, one has that $L_{p,+}^{(\mathfrak{a})} \cong \mathfrak{a}(p) = [A, p\frac{B+\sqrt{-d}}{2}]$ and $L_{p,-}^{(\mathfrak{a})} \cong \mathfrak{a}(p) = [A, p\frac{B+\sqrt{-d}}{2}]$ as lattices of the quadratic spaces $V_{\mathfrak{a}}$ and $U_{\mathfrak{a}}$, respectively. It is known that $\mathfrak{a}(p)' = \frac{1}{p\sqrt{-d}}\mathfrak{a}(p)$, and one can easily show that $\mathfrak{a}(p)'/\mathfrak{a}(p) = \{\frac{An}{p\sqrt{-d}} + \mathfrak{a}(p)\}$ for $n \in \mathbb{Z}/p^2d\mathbb{Z}$. Then, under the isomorphism $\mathcal{K} \cong V_{\mathfrak{a}}$, one has

$$\frac{An}{p\sqrt{-d}} + \mathfrak{a}(p) \to \frac{Bn}{pd}e_1^{(\mathfrak{a})} - \frac{2An}{pd}e_2^{(\mathfrak{a})} + L_{p,+}^{(\mathfrak{a})} \in L_{p,+}^{(\mathfrak{a})'}/L_{p,+}^{(\mathfrak{a})}. \tag{7.1}$$

Similarly, under the isomorphism $\mathcal{K} \cong U_{\mathfrak{a}}$, one has

$$\frac{An}{p\sqrt{-d}} + \mathfrak{a}(p) \to \frac{Bn}{pd}f_1^{(\mathfrak{a})} - \frac{2An}{pd}f_2^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})} \in L_{p,-}^{(\mathfrak{a})'}/L_{p,-}^{(\mathfrak{a})}.$$

Therefore,

$$\frac{Bn_1}{pd}e_1^{(\mathfrak{a})} - \frac{2An_1}{pd}e_2^{(\mathfrak{a})} + \frac{Bn_2}{pd}f_1^{(\mathfrak{a})} - \frac{2An_2}{pd}f_2^{(\mathfrak{a})} + L_{p,+}^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})} \in L_{p,+}^{(\mathfrak{a})'}/L_{p,+}^{(\mathfrak{a})} \oplus L_{p,-}^{(\mathfrak{a})'}/L_{p,-}^{(\mathfrak{a})},$$

where

$$\begin{aligned}
&\frac{Bn_1}{pd}e_1^{(\mathfrak{a})} - \frac{2An_1}{pd}e_2^{(\mathfrak{a})} + \frac{Bn_2}{pd}f_1^{(\mathfrak{a})} - \frac{2An_2}{pd}f_2^{(\mathfrak{a})} \\
&= \begin{pmatrix} -\frac{B(n_2-n_1)}{pd} & -\frac{2ACn_1 + (2AC - B^2)n_2}{pd} \\ -\frac{2A(n_2-n_1)}{pd} & -\frac{AB(n_1-n_2)}{pd} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{B(n_2-n_1)}{pd} & -\left(\frac{2AC(n_1-n_2)}{pd} + \frac{n_2}{p}\right) \\ -\frac{2A(n_2-n_1)}{pd} & -\frac{AB(n_1-n_2)}{pd} \end{pmatrix}.
\end{aligned}$$

Since $L_p/(L_{p,+}^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})}) \subset L_{p,+}^{(\mathfrak{a})'}/L_+^{(\mathfrak{a})} \oplus L_{p,-}^{(\mathfrak{a})'}/L_-^{(\mathfrak{a})}$ and $(2A, dp) = 1$, then

$$\frac{Bn_1}{pd}e_1^{(\mathfrak{a})} - \frac{2An_1}{pd}e_2^{(\mathfrak{a})} + \frac{Bn_2}{pd}f_1^{(\mathfrak{a})} - \frac{2An_2}{pd}f_2^{(\mathfrak{a})}$$

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$$= \begin{pmatrix} -\frac{B(n_2-n_1)}{pd} & -\left(\frac{2AC(n_1-n_2)}{pd}+\frac{n_2}{p}\right) \\ -\frac{2A(n_2-n_1)}{pd} & -\frac{AB(n_1-n_2)}{pd} \end{pmatrix} \in L_p = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

if and only if $n_1 \equiv n_2 \pmod{p^2 d}$ and $n_2 \equiv 0 \pmod{p}$, and thus in terms of $e_1^{(\mathfrak{a})}, e_2^{(\mathfrak{a})}, f_1^{(\mathfrak{a})}$, and $f_2^{(\mathfrak{a})}$, elements $\mathfrak{l}$ of $L_p/(L_{p,+}^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})})$ are of the form

$$\mathfrak{l} = \frac{B(pl+p^2dm)}{pd}e_1^{(\mathfrak{a})} - \frac{2A(pl+p^2dm)}{pd}e_2^{(\mathfrak{a})} + \frac{B(pl)}{pd}f_1^{(\mathfrak{a})} - \frac{2A(pl)}{pd}f_2^{(\mathfrak{a})} \pmod{(L_{p,+}^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})})},$$

where $l, m \in \mathbb{Z}$, and thus explicitly,

$$L_p/(L_{p,+}^{(\mathfrak{a})}+L_{p,-}^{(\mathfrak{a})}) = \left\{ \frac{Bl}{d}e_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pe_2^{(\mathfrak{a})}) + \frac{Bl}{d}f_1^{(\mathfrak{a})} - \frac{2Al}{pd}(pf_2^{(\mathfrak{a})}) + L_{p,+}^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})} \,\middle|\, 0 \le l \le pd-1 \right\},$$

since $(2A, pd) = 1$, $L_{p,+}^{(\mathfrak{a})} = \mathbb{Z}e_1^{(\mathfrak{a})} + \mathbb{Z}(pe_2^{(\mathfrak{a})})$, and $L_{p,-}^{(\mathfrak{a})} = \mathbb{Z}f_1^{(\mathfrak{a})} + \mathbb{Z}(pf_2^{(\mathfrak{a})})$. Therefore, the projections $\lambda_\pm = \lambda_{l,\pm}$ are given by

$$\lambda_{l,+} = \frac{Bl}{d}e_1^{(\mathfrak{a})} - \frac{2Al}{d}e_2^{(\mathfrak{a})}, \quad \lambda_{l,-} = \frac{Bl}{d}f_1^{(\mathfrak{a})} - \frac{2Al}{d}f_2^{(\mathfrak{a})}. \tag{7.2}$$

In addition, one can also compute and show that

$$\mu_{0,k,+} = \frac{BCk}{d}e_1^{(\mathfrak{a})} - \frac{2ACk}{d}e_2^{(\mathfrak{a})}, \quad \mu_{0,k,-} = \frac{BCk}{d}f_1^{(\mathfrak{a})} - \frac{(2AC-d)k}{d}f_2^{(\mathfrak{a})},$$

and thus

$$\begin{aligned} \mu_{0,k,+} + \lambda_{l,+} &= \frac{B(Ck+l)}{d}e_1^{(\mathfrak{a})} - \frac{2A(Ck+l)}{d}e_2^{(\mathfrak{a})}, \\ \mu_{0,k,-} + \lambda_{l,-} &= \frac{B(Ck+l)}{d}f_1^{(\mathfrak{a})} - \frac{2A(Ck+l)-dk}{d}f_2^{(\mathfrak{a})}. \end{aligned} \tag{7.3}$$

Note that since $\kappa(m,\varphi) = 0$ for $m < 0$, to have $\kappa(-\det(x), \mu_{0,k,-} + \lambda_{l,-} + L_{p,-}^{(\mathfrak{a})}) \neq 0$ for $x \in \mu_{0,k,+} + \lambda_{l,+} + L_{p,+}^{(\mathfrak{a})}$, one must have $\mu_{0,k,+} + \lambda_{l,+} \in L_{p,+}^{(\mathfrak{a})}$ which implies that $pd|(Ck+l)$. Therefore, writing $Xe_1^{(\mathfrak{a})} + Ype_2^{(\mathfrak{a})}$ for elements of $L_{p,+}^{(\mathfrak{a})}$, any $x \in \lambda_{l.+} + L_{p,+}^{(\mathfrak{a})}$ is of the form $\left(\frac{Bl}{d}+X\right)e_1^{(\mathfrak{a})} + \left(-\frac{2Al}{d}+Yp\right)e_2^{(\mathfrak{a})}$, and using the relation $B^2 - 4AC = -d$, the formula (6.1) can be written explicitly and simplified as

$$\log\left|\operatorname{disc}\left(H_{\mathcal{K},p}(x)\right)\right| = -\frac{(p-\chi_d(p))h_{\mathcal{K}}}{4} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(p) \\ [\mathfrak{a}]\neq[\mathcal{O}_{\mathcal{K}}]}}$$

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$$\times \left[\sum_{l=0}^{pd-1} \sum_{X,Y=-\infty}^{\infty} \kappa\left(1 - \frac{d(2AX+BYp)^2 + (dYp-2Al)^2}{4Ad}, \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{d} f_2^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})}\right) \right.$$

$$\left. + \frac{24}{p-1} \sum_{\substack{1\le k\le p-1 \\ 0\le l\le pd-1 \\ Ck+l\equiv 0 \pmod{pd}}} \kappa\left(0, \frac{B(Ck+l)}{d} f_1^{(\mathfrak{a})} - \frac{2A(Ck+l)-dk}{d} f_2^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})}\right) \right]$$

$$= -\frac{(p-\chi_d(p))h_{\mathcal{K}}}{4} \sum_{\substack{[\mathfrak{a}]\in \mathrm{Cl}_{\mathcal{K}}(p) \\ [\mathfrak{a}]\ne[\mathcal{O}_{\mathcal{K}}]}}$$

$$\times \left[\sum_{l=0}^{pd-1} \sum_{X,Y=-\infty}^{\infty} \kappa\left(1 - \frac{d(2AX+BYp)^2 + (dYp-2Al)^2}{4Ad}, \frac{Bl}{d} f_1^{(\mathfrak{a})} - \frac{2Al}{d} f_2^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})}\right) \right.$$

$$\left. + \frac{24}{p-1} \sum_{\substack{1\le k\le p-1 \\ 0\le l\le pd-1 \\ Ck+l\equiv 0 \pmod{pd}}} \kappa\left(0, k f_2^{(\mathfrak{a})} + L_{p,-}^{(\mathfrak{a})}\right) \right].$$

We are now ready for

Proof of Theorem 1.6 This follows from Theorem 5.3, Proposition 6.3, and the lattice computations given above. ∎

and

Proof of Corollary 1.9 By Theorems 1.6 and 8.1 and Remark 1.8, for a prime l dividing $\mathrm{disc}(H_{\mathcal{K},p}(x))$, one can see that either $l|d$, $l \le md$, or $l = p$, where

$$md = \left(1 - \frac{d(2AX+BYp)^2 + (dYp-2Al)^2}{4Ad}\right) d$$
$$= d - \frac{d(2AX+BYp)^2 + (dYp-2Al)^2}{4A} \le d.$$

Therefore, any prime factor of $\mathrm{disc}(H_{\mathcal{K},p}(x))$ is bounded by $\max\{d,p\}$. ∎

Remark 7.1 Note by the identification $L_{p,+}^{(\mathfrak{a})} \cong \mathfrak{a}(p)$ and (7.2) that $d - md$ is actually the product of the lattice norm of an element of $\frac{Al}{\sqrt{-d}} + \mathfrak{a}(p)$ and d, and thus it is integral, and so is md.

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8 Formulas for $A_m(v, s, \varphi(l))$ and a computational example

In this section, we assume that $(p, d) = 1$. Taking $\mathfrak{a} = [A, \frac{B+\sqrt{-d}}{2}]$ an $\mathcal{O}_{\mathcal{K}}$-ideal with norm prime p for $1 \le l \le pd - 1$, and writing $\varphi(l) = \mathrm{Char}\left(\frac{Al}{\sqrt{-d}} + L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\right) = \prod_{q<\infty} \varphi(l)_q$, where $L_{p,-}^{(\mathfrak{a})} = \mathfrak{a}(p) = [A, p\frac{B+\sqrt{-d}}{2}]$ and $\varphi(l)_q = \mathrm{Char}\left(\frac{Al}{\sqrt{-d}} + L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \mathbb{Z}_q\right)$, we state below explicit formulas for the Fourier coefficients $A_m(v, s, \varphi(l))$ defined in Definition 6.1 specialized to the quadratic space $\left(U = \mathcal{K}, Q_{\mathfrak{a}}(x) = -\frac{\mathrm{N}(x)}{A}\right)$ under the assumption that $(p, d) = 1$, via which one can concretely compute the corresponding $\kappa\left(m, \frac{Al}{\sqrt{-d}} + L_{p,-}^{(\mathfrak{a})}\right)$. It is known (see, e.g., [13, Section 2]) that $A_m(v, s, \varphi(l))$ can be represented by a product of the so-called local Whittaker functions $W_q(s, m, \varphi(l)_q)$, and thus it suffices to compute $W_q(s, m, \varphi(l)_q)$. When $p = 1$ and $-d < -4$ odd, the formulas for $W_q(s, m, \varphi(l)_q)$ had been computed by Schofer [14]. Under the assumption $(p, d) = 1$, for $p \ge 3$, one only needs to work out the local Whittaker function $W_p(s, m, \varphi(l)_p)$ at the place p as $L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \mathbb{Z}_q = L_{1,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \mathbb{Z}_q$ for $q \ne p$. The computations wholly rely on [13, Theorems 4.3 and 4.4], and one can find the details in [19, Theorem 5.2]. In the following, we define some notation, state the formula for $W_p(s, m, \varphi(l)_p)$, and sequentially state the formula for $A_m(v, s, \varphi(l))$ in Theorem 8.1.

As discussed in Subsection 6.3, $L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p f_1^{(\mathfrak{a})} + \mathbb{Z}_p p f_2^{(\mathfrak{a})}$ with

$$f_1^{(\mathfrak{a})} = \begin{pmatrix} -1 & B \\ 0 & A \end{pmatrix}, \quad f_2^{(\mathfrak{a})} = \begin{pmatrix} 0 & C \\ 1 & 0 \end{pmatrix}.$$

The Gram matrix of $L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with respect to $Q_{\mathfrak{a}}(\cdot)$ is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivalent to $S = \mathrm{diag}(2\varepsilon_1, 2\varepsilon_2 p^2)$ with $\varepsilon_1 = -A$ and $\varepsilon_2 = -\frac{d}{4A}$, and we define

$$t_{\varphi(l)} = \begin{cases} m & \text{if } p|l, \\ m - \frac{4A^2l^2\varepsilon_2}{d^2} & \text{if } p \nmid l, \end{cases}$$

and

$$a = a(l) = \mathrm{ord}_p(t_{\varphi(l)}).$$

Then the formula for $W_p(s, m, \varphi(l)_p)$ is given by

(1) for $p|l$,

$$W_p(s, m, \varphi(l)_p)$$
$$= \begin{cases} 1 + \left(\frac{\varepsilon_1 m}{p}\right) p^{-s} & \text{if } a = 0, \\ 1 + (p-1)\begin{pmatrix} p^{-2s} + \chi_d(p)p^{-3s} + \cdots \\ +\chi_d(p)p^{-(a-1)s} + p^{-as} \end{pmatrix} - \chi_d(p)p^{-(a+1)s} & \text{if } a \ge 2 \text{ even}, \\ 1 + (p-1)\begin{pmatrix} p^{-2s} + \chi_d(p)p^{-3s} + \cdots \\ +p^{-(a-1)s} + \chi_d(p)p^{-as} \end{pmatrix} - p^{-(a+1)s} & \text{if } a \ge 1 \text{ odd}, \\ 1 + (p-1)\sum_{n=2}^{\infty} \chi_d(p)^n p^{-ns} & \text{if } a = \infty, \end{cases}$$

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(2) for $p \nmid l$,

$$W_p(s, m, \varphi(l)_p) = \begin{cases} 1 + \left(\dfrac{\varepsilon_1 t_{\varphi(l)}}{p}\right) p^{-s} & \text{if } a = 0, \\ 1 & \text{if} a \geq 1. \end{cases}$$

To state the formula for $A_m(v, s, \varphi(l))$, we also need the following components. Let

$$\Lambda(s; \chi_d) = d^{\frac{s}{2}} \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s; \chi_d)$$

be the completed Hecke L-function associated to the quadratic character χ_d known to satisfy that

$$\Lambda(1; \chi_d) = h_{\mathcal{K}},$$

where $h_{\mathcal{K}}$ is the weighted class number of $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ defined as in Theorem 1.1, and let $L_p(s; \chi_d)$ be the local part of $\Lambda(s; \chi_d)$ at the finite place p defined by

$$L_p(s; \chi_d) = \frac{1}{1 - \chi_d(p) p^{-s}}.$$

Define the local part of the divisor function of weight $-s$ associated to the quadratic character χ_d by

$$\sigma_{-s,q}(m, \chi_d) = \frac{1 - (\chi_d(q) q^{-s})^{\mathrm{ord}_q(m)+1}}{1 - \chi_d(q) q^{-s}},$$

which is the local Whittaker function component of $A_m(v, s, \varphi(l))$ at the place $q \nmid pd$, and write $\Psi_{-1}(s, 4\pi m v)$ (see [13] for its precise definition) for the Archimedean component of $A_m(v, s, \varphi(l))$, which satisfies that $\Psi_{-1}(0, 4\pi m v) = 1$. Now we have

Theorem 8.1 *For a prime q, write $|N|_q$ for the q-adic norm of integer N. Then one has that for $m > 0$ such that $m \in -\frac{Al^2}{d} + \mathbb{Z}$,*

$$\Lambda(s+1; \chi_d) A_m(v, s, \varphi(l)) = -2 \prod_{q \nmid pd} \sigma_{-s,q}(m, \chi_d)$$
$$\times \prod_{\substack{q \mid d \\ \mathrm{ord}_q(md) > 0}} (1 + (-d, -m)_q |md|_q^{\frac{s}{2}}) L_p(s+1; \chi_d) \frac{1}{p} W_p(s, m, \varphi(l)_p)$$
$$\times (2m\sqrt{d\pi v})^s \Psi_{-1}(s, 4\pi m v),$$

and for $m = 0$,

$$A_0(v, s, \varphi(l)) = v^{\frac{s}{2}} \varphi(l)(0) - \sqrt{\pi} v^{-\frac{s}{2}} d^{-\frac{1}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \frac{L(s; \chi_d)}{L(s+1; \chi_d)} \frac{L_p(s+1; \chi_d)}{L_p(s; \chi_d)} \frac{1}{p}$$
$$\times W_p(s, 0, \varphi(l)_p).$$

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Moreover, let $\mu_{0,k,-} + \lambda_{l,-}$ *be defined by* (7.3)*, and write* $\xi(k,l) = \mathrm{Char}(\mu_{0,k,-} + \lambda_{l,-} + L_{p,-}^{(\mathfrak{a})} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$*. Then*

$$A_0(v,s,\xi(k,l)) = -\sqrt{\pi} v^{-\frac{s}{2}} d^{-\frac{1}{2}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \frac{L(s;\chi_d)}{L(s+1;\chi_d)} \frac{L_p(s+1;\chi_d)}{L_p(s;\chi_d)} \frac{1}{p} (1+\chi_d(p)p^{-s}).$$

Remark 8.2 Recall that $\kappa(m,\varphi)$ is defined via $A'_m(v,0,\varphi)$. By the product formula for $A_m(v,s,\varphi)$ and the fact that $A_m(v,0,\varphi) = 0$, one may see that the value of $\kappa(m,\varphi)$ is given by some multiple of $\log q$ for exactly one prime q whose associated product component of $A_m(v,s,\varphi)$ is the only one vanishing at $s = 0$, i.e., if we write $A_m(v,s,\varphi) = \prod_{\ell \le \infty} W_\ell(s,m,\varphi_\ell)$, then

$$A'_m(v,0,\varphi) = W'_q(0,m,\varphi_q) \prod_{\substack{\ell \le \infty \\ \ell \ne q}} W_\ell(s,m,\varphi_\ell),$$

where $W'_q(0,m,\varphi)$ induces $\log q$, and $W_\ell(0,m,\varphi_\ell)$ is simply given in terms of multiples of $\mathrm{ord}_\ell(m)$ or the class number of $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$. One can further see that q is either an inert prime, i.e., $\chi_d(q) = -1$, apart from p such that $\mathrm{ord}_q(m)$ is odd, a ramified prime such that $(-d,-m)_q = -1$ and $\mathrm{ord}_q(md) > 0$, or the odd prime p.

Example 8.3 Take $-d = -7$ and $p = 3$. Then $\mathrm{Cl}_{\mathcal{K}}(3) = \{[\mathcal{O}_{\mathcal{K}}], [\mathfrak{a}], [\bar{\mathfrak{a}}], [\mathfrak{b}]\}$, where $\mathfrak{a} = [2, \frac{-1+\sqrt{-7}}{2}]$ and $\mathfrak{b} = [4, \frac{-3+\sqrt{-7}}{2}]$. It is easy to note that $Z_p(U_{\mathfrak{a}}) = Z_p(U_{\bar{\mathfrak{a}}})$. Then, by Theorem 1.6, one first has that

$$\begin{aligned}
&\log\left|\mathrm{disc}\left(H_{\mathbb{Q}(\sqrt{-7}),3}(x)\right)\right| \\
&= -2\Bigg(\kappa\left(1, L_{p,-}^{(\mathfrak{a})}\right) + \sum_{l=1,20} \kappa\left(\frac{5}{7}, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) + \sum_{l=4,17} \kappa\left(\frac{3}{7}, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) \\
&\quad + 2\sum_{l=10,11} \kappa\left(\frac{3}{7}, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) + \sum_{l=6,15} \kappa\left(\frac{5}{7}, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) \\
&\quad + \sum_{l=5,16} \kappa\left(\frac{6}{7}, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) + \sum_{l=7,14} \kappa\left(0, -\frac{l}{7} f_1^{(\mathfrak{a})} - \frac{4l}{7} f_2^{(\mathfrak{a})}\right) \\
&\quad + 12\kappa\left(0, 2f_2^{(\mathfrak{a})}\right) + 12\kappa\left(0, f_2^{(\mathfrak{a})}\right)\Bigg) \\
&\quad - \Bigg(\kappa\left(1, L_{p,-}^{(\mathfrak{b})}\right) + \sum_{l=1,8,13,20} \kappa\left(\frac{3}{7}, -\frac{3l}{7} f_1^{(\mathfrak{b})} - \frac{8l}{7} f_2^{(\mathfrak{b})}\right) \\
&\quad + \sum_{l=2,5,16,19} \kappa\left(\frac{5}{7}, -\frac{3l}{7} f_1^{(\mathfrak{b})} - \frac{8l}{7} f_2^{(\mathfrak{b})}\right) \\
&\quad + \sum_{l=3,18} \kappa\left(\frac{6}{7}, -\frac{3l}{7} f_1^{(\mathfrak{b})} - \frac{8l}{7} f_2^{(\mathfrak{b})}\right) + \sum_{l=6,15} \kappa\left(\frac{3}{7}, -\frac{3l}{7} f_1^{(\mathfrak{b})} - \frac{8l}{7} f_2^{(\mathfrak{b})}\right) \\
&\quad + \sum_{l=7,14} \kappa\left(0, -\frac{3l}{7} f_1^{(\mathfrak{b})} - \frac{8l}{7} f_2^{(\mathfrak{b})}\right) + 12(\kappa\left(0, 2f_2^{(\mathfrak{b})}\right) + 12\kappa\left(0, f_2^{(\mathfrak{b})}\right)\Bigg).
\end{aligned}$$

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Following Theorem 8.1 carefully, one can compute and obtain that (the l's lie in the index set of the summation associated to $\kappa(m,\varphi_-)$ given in the expansion above)

$$\kappa\left(1, L_{p,-}^{(\mathfrak{a})}\right) = -\log 7,$$
$$\kappa\left(\frac{5}{7}, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = -\log 5,$$
$$\kappa\left(\frac{3}{7}, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = -\frac{1}{2}\log 3,$$
$$\kappa\left(\frac{3}{7}, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = -\frac{1}{2}\log 3,$$
$$\kappa\left(\frac{5}{7}, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = 0,$$
$$\kappa\left(\frac{6}{7}, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = -\log 3,$$
$$\kappa\left(0, -\frac{l}{7}f_1^{(\mathfrak{a})} - \frac{4l}{7}f_2^{(\mathfrak{a})}\right) = \kappa\left(0, 2f_2^{(\mathfrak{a})}\right) = \kappa\left(0, f_2^{(\mathfrak{a})}\right) = -\frac{1}{2}\log 3,$$

and

$$\kappa\left(1, L_{p,-}^{(\mathfrak{b})}\right) = 0,\ \kappa\left(\frac{3}{7}, -\frac{3l}{7}f_1^{(\mathfrak{b})} - \frac{8l}{7}f_2^{(\mathfrak{b})}\right) = -\frac{1}{2}\log 3,$$
$$\kappa\left(\frac{5}{7}, -\frac{3l}{7}f_1^{(\mathfrak{b})} - \frac{8l}{7}f_2^{(\mathfrak{b})}\right) = -\frac{1}{2}\log 5,$$
$$\kappa\left(\frac{6}{7}, -\frac{3l}{7}f_1^{(\mathfrak{b})} - \frac{8l}{7}f_2^{(\mathfrak{b})}\right) = -2\log 3,$$
$$\kappa\left(\frac{3}{7}, -\frac{3l}{7}f_1^{(\mathfrak{b})} - \frac{8l}{7}f_2^{(\mathfrak{b})}\right) = -\log 3,$$
$$\kappa\left(0, -\frac{3l}{7}f_1^{(\mathfrak{b})} - \frac{8l}{7}f_2^{(\mathfrak{b})}\right) = \kappa\left(0, 2f_2^{(\mathfrak{b})}\right) = \kappa\left(0, f_2^{(\mathfrak{b})}\right) = -\frac{1}{2}\log 3.$$

Plugging all of the above into the expansion for $\log\left|\operatorname{disc}\left(H_{\mathbb{Q}(\sqrt{-7}),3}(x)\right)\right|$, one obtains

$$\begin{aligned}
&\log\left|\operatorname{disc}\left(H_{\mathbb{Q}(\sqrt{-7}),3}(x)\right)\right| \\
&= -2\Bigg(-\log 7 + 2\times(-\log 5) + 2\times\left(-\frac{1}{2}\log 3\right) + 4\times\left(-\frac{1}{2}\log 3\right) \\
&\quad + 2\times 0 + 2\times(-\log 3) + 2\times\left(-\frac{1}{2}\log 3\right) + 12\times 2\times\left(-\frac{1}{2}\log 3\right)\Bigg) \\
&\quad - \Bigg(0 + 4\times\left(-\frac{1}{2}\log 3\right) + 4\times\left(-\frac{1}{2}\log 5\right) + 2\times(-2\log 3) + 2\times(-\log 3) \\
&\quad + 2\left(-\frac{1}{2}\log 3\right) + 12\times 2\times\left(-\frac{1}{2}\log 3\right)\Bigg) \\
&= \log(3^{57}5^{6}7^{2}).
\end{aligned}$$

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On the other hand, numerically, one can find that

$$H_{\mathbb{Q}(\sqrt{-7}),3}(x) = x^4 + 4131x^3 + 196830x^2 + 19131876x + 387420489,$$

whose discriminant is $-3^{57}5^{6}7^{2}$ given as in [5, Appendix 4, p. 306].

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References

[1] R. E. Borcherds, *Automorphic forms on* $O_{s+2,2}(\mathbb{R})$ *and infinite products*. Invent. Math. **120**(1995), 161–213.
[2] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*. Invent. Math. **132**(1998), 491–562.
[3] J. H. Bruinier, S. Kudla, and T. Yang, *Special values of Green functions at big CM points*. Int. Math. Res. Not. **9**(2012), 1917–1967.
[4] J. H. Bruinier and T. Yang, *Faltings heights of CM cycles and derivatives of L-functions*. Invent. Math. **177**(2009), 631–681.
[5] I. Chen and N. Yui, *Singular values of Thompson series*. In: K. T. Arasu, J. F. Dillon, K. Harada, S. Sehgal, and R. Solomon (eds.), Groups, difference sets, and the monster (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., 4, de Gruyter, Berlin, Germany, 1996, pp. 255–326.
[6] J. Conway and S. Norton, *Monstrous moonshine*. Bull. Lond. Math. Soc. **11**(1979), 308–339.
[7] D. Cox, *Primes of the form* x^2+ny^2. Wiley, Hoboken, NJ, 1989.
[8] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*. Math. Annalen. **278**(1987), 497–562.
[9] B. Gross and D. Zagier, *On singular moduli*. J. Reine Angew. Math. **355**(1985), 191–220.
[10] B. Gross and D. Zagier, *Heegner points and derivative of L-series*. Invent. Math. **85**(1986), 225–320.
[11] H. Iwaniec, *Spectral methods of automorphic forms*. Graduate Studies in Mathematics, 53, American Mathematical Society, Providence, RI, 2002.
[12] S. S. Kudla, *Integrals of Borcherds forms*. Compos. Math. **137**(2003), 293–349.
[13] S. S. Kudla and T. Yang, *Eisenstein series for* $\mathrm{SL}(2)$. Sci. China Math. **53**(2010), 2275–2316.
[14] J. Schofer, *Borcherds forms and generalizations of singular moduli*. J. Reine Angew. Math. **629**(2009), 1–36.
[15] G. Shimura, *Arithmetic of quadratic forms*. Springer, New York, NY, 2010.
[16] T. Yang and H. Yin, *Difference of modular functions and their CM value factorization*. Trans. Amer. Math. Soc. **371**(2019), 3451–3482.
[17] T. Yang, H. Yin, and P. Yu, *The lambda invariants at CM points*. Int. Math. Res. Not. **rnz230**(2019). https://doi.org/10.1093/imrn/rnz230
[18] D. Ye, *On the generating function of a canonical basis for* $M_0^{!,\infty}(\Gamma)$. Results Math. **74**(2019), no. 2.
[19] D. Ye, *Revisiting the Gross–Zagier discriminant formula*. Math. Nachr. **293**(2020), 1801–1826.
[20] D. Ye, *On a zeta function associated to a quadratic order*. Results Math. **75**(2020), no. 27.
[21] D. Ye, *Difference of a Hauptmodul for* $\Gamma_0(N)$ *and certain Gross–Zagier type CM value formulas*. Preprint, 2020. arxiv:1708.08783

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