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MUTUALLY INTERACTING SUPERPROCESSES WITH MIGRATION

LINA JI ,* *Shenzhen MSU-BIT University*
HUILI LIU ,** *Hebei Normal University*
JIE XIONG,*** *Southern University of Science and Technology*

Abstract

A system of mutually interacting superprocesses with migration is constructed as the limit of a sequence of branching particle systems arising from population models. The uniqueness in law of the superprocesses is established using the pathwise uniqueness of a system of stochastic partial differential equations, which is satisfied by the corresponding system of distribution function-valued processes.

Keywords: Distribution function-valued; Martingale problem; superprocess; stochastic partial differential equation; well-posedness

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1. Introduction

Superprocesses, describing the evolution of a large population undergoing random reproduction and spatial motion, were first constructed as high-density limits of branching particle systems by Watanabe [20]. The connection between superprocesses and stochastic evolution equations was investigated by Dawson [1]. Since then, ample systematic research results have been published; see e.g. [2], [7], and [13]. Those with immigration, a class of generalizations of superprocesses, have also attracted the interest of many researchers. We refer to [13], [14], [15], and the references therein for immigration structure and related properties. Let $M_F(\mathbb{R})$ be the collection of all finite Borel measures on $\mathbb{R}$. Set $M_F(\mathbb{R})^2 = M_F(\mathbb{R}) \times M_F(\mathbb{R})$. Let $C_b^k(\mathbb{R})$ (resp. $C_0^k(\mathbb{R})$) be the collection of all bounded (resp. compactly supported) continuous functions on $\mathbb{R}$ with bounded derivatives up to kth order. Let $(\mu_t)_{t\geq 0}$ be a continuous $M_F(\mathbb{R})$-valued process solving the following martingale problem (MP): for all $f \in C_b^2(\mathbb{R})$, the process

$$M_t^f = \langle \mu_t, f\rangle - \langle \mu_0, f\rangle - \int_0^t \left(\frac{1}{2}\langle \mu_s, f''\rangle + \langle \kappa, f\rangle\right) \mathrm{d}s \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\langle M^f\rangle_t = \gamma \int_0^t \langle \mu_s, f^2\rangle \,\mathrm{d}s, \tag{1.2}$$

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* Postal address: Faculty of Computational Mathematics and Cybernetics, Shenzhen MSU-BIT University, Shenzhen, Guangdong, 518172, China.
** Postal address: School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei, 050024, China. Email address: liuhuili@hebtu.edu.cn
*** Postal address: Department of Mathematics & National Center for Applied Mathematics (Shenzhen), Southern University of Science and Technology, Shenzhen, Guangdong, 518055, China.

where $\gamma > 0$ and $\kappa \in M_F(\mathbb{R})$. The corresponding model is super-Brownian motion (SBM) when $\kappa = 0$. The uniqueness in law of SBM can be obtained by its log-Laplace transform

$$\log \mathbb{E} \exp(-\langle \mu_t, f \rangle) = -\langle \mu_0, u_t \rangle$$

(see Watanabe [20]), where u_t is the unique solution to the following log-Laplace equation:

$$\begin{cases} \dfrac{\partial u_t(x)}{\partial t} = \dfrac{1}{2} \triangle u_t(x) - \dfrac{\gamma}{2} u_t(x)^2, \\ u_0(x) = f(x). \end{cases}$$

Xiong [22] studied the stochastic partial differential equation (SPDE) satisfied by the distribution function-valued process of SBM, and approached its uniqueness from a different point of view. Related work can also be found in [3] and [8].

For a finite measure κ, the corresponding model is the superprocess with immigration, which was constructed in [11] through the cumulant semigroup; see also [12] and [14]. In the case of κ being interactive, i.e. $\kappa = \kappa(\mu_s)$, the existence of a solution to the MP (1.1, 1.2) has been verified in Méléard [17], where the result is applicable to the situation with interactive immigration, branching rate, and spatial motion. By the approach of pathwise uniqueness for SPDEs satisfied by the distribution function-valued process, Mytnik and Xiong [19] established the well-posedness of MPs for superprocesses with interactive immigration. See also [23] for related work.

In fact there exist some populations distributed in different colonies, such as the mutually catalytic branching model; see [4], [5], [6], [16], and [18]. The evolution of this model can be illustrated by interacting superprocesses. A sudden event may induce mass migration and lead to an increment of population size in one colony and a decrement in the other. For instance, war causes large numbers of people to move into a neighbouring country, and radiation mutates normal cells, and so on. Therefore it is natural to study superprocesses with interactive migration between different colonies. In this paper we consider a continuous $M_F(\mathbb{R})^2$-valued process $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\geq 0}$, called a *mutually interacting superprocess with migration*, which solves the following MP: for all $f, g \in C_0^2(\mathbb{R})$, the processes

$$\begin{cases} M_t^f = \langle \mu_t^{\mathbb{1}}, f \rangle - \langle \mu_0^{\mathbb{1}}, f \rangle - \dfrac{1}{2} \displaystyle\int_0^t \langle \mu_s^{\mathbb{1}}, f'' \rangle \, \mathrm{d}s - b_{\mathbb{1}} \int_0^t \langle \mu_s^{\mathbb{1}}, f \rangle \, \mathrm{d}s \\ \qquad + \displaystyle\int_0^t \langle \mu_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}) f \rangle \, \mathrm{d}s, \\ \hat{M}_t^g = \langle \mu_t^{\mathbb{2}}, g \rangle - \langle \mu_0^{\mathbb{2}}, g \rangle - \dfrac{1}{2} \displaystyle\int_0^t \langle \mu_s^{\mathbb{2}}, g'' \rangle \, \mathrm{d}s - b_{\mathbb{2}} \int_0^t \langle \mu_s^{\mathbb{2}}, g \rangle \, \mathrm{d}s \\ \qquad - \langle \chi, g \rangle \displaystyle\int_0^t \langle \mu_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}) \rangle \, \mathrm{d}s \end{cases} \tag{1.3}$$

are two continuous martingales with quadratic variation and covariation processes

$$\begin{cases} \langle M^f \rangle_t = \gamma_{\mathbb{1}} \displaystyle\int_0^t \langle \mu_s^{\mathbb{1}}, f^2 \rangle \, \mathrm{d}s, \\ \langle \hat{M}^g \rangle_t = \gamma_{\mathbb{2}} \displaystyle\int_0^t \langle \mu_s^{\mathbb{2}}, g^2 \rangle \, \mathrm{d}s, \\ \langle M^f, \hat{M}^g \rangle_t = 0, \end{cases} \tag{1.4}$$

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where χ is a probability measure on $\mathbb{R}$, $\gamma_{\mathbb{1}}$ and $\gamma_{\mathbb{2}}$ are positive constants, and the migration intensity $\eta(\cdot, \cdot, \cdot)$ is a non-negative bounded continuous function on $\mathbb{R} \times M_F(\mathbb{R})^2$.

The *purpose* of this paper is to establish the well-posedness of the MP (1.3, 1.4), i.e. the existence and uniqueness in law of such mutually interacting superprocesses with migration. As far as we know, this is the first attempt to discuss the well-posedness of mutually interactive superprocesses with migration. The structure of interactive migration makes the model more complex and increases the challenge of constructing a solution to the MP. Simultaneously, the traditional method of moment duality fails to prove the uniqueness of such a process. We formulate the process as the limit empirical measure of a sequence of branching particle systems. The uniqueness in law of the superprocesses is demonstrated by the pathwise uniqueness of the solution to a system of mutually interacting SPDEs, which are satisfied by the corresponding distribution function-valued processes.

We introduce some notation. Let $D(\mathbb{R}_+, M_F(\mathbb{R})^2)$ (resp. $C(\mathbb{R}_+, M_F(\mathbb{R})^2)$) denote the space of càdlàg (resp. continuous) paths from $\mathbb{R}_+$ to $M_F(\mathbb{R})^2$ furnished with the Skorokhod topology. Let $D(\mathbb{R}_+, \mathbb{R}^2)$ be the collection of càdlàg paths from $\mathbb{R}_+$ to $\mathbb{R}^2$. Let $C_{b,m}(\mathbb{R})$ be the subset of $C_b(\mathbb{R})$ consisting of non-decreasing bounded continuous functions on $\mathbb{R}$. Write $\langle \mu, f\rangle$ as the integral of $f \in C_b^2(\mathbb{R})$ with respect to measure $\mu \in M_F(\mathbb{R})$. For any $f, g \in C_b^2(\mathbb{R})$, define $\langle f, g\rangle_1 = \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x$. Let $H_0 = L^2(\mathbb{R})$ be the Hilbert space consisting of all square-integrable functions with the norm $\|\cdot\|_0$ given by $\|f\|_0^2 = \int_{\mathbb{R}} f^2(x)\mathrm{e}^{-|x|}\mathrm{d}x$ for any $f \in H_0$. Set $\nu_i(x) = \nu_i((-\infty, x])$ as the distribution functions of $\nu_i \in M_F(\mathbb{R})$ for any $x \in \mathbb{R}$ and $i = 1, 2$. Define the distance ρ on $M_F(\mathbb{R})$ by

$$\rho(\nu_1, \nu_2) = \int_{\mathbb{R}} \mathrm{e}^{-|x|}|\nu_1(x) - \nu_2(x)|\mathrm{d}x.$$

It is easy to see that under metric distance ρ, $M_F(\mathbb{R})$ is a Polish space whose topology coincides with that given by weak convergence of measures. Let $\lfloor x\rfloor$ denote the integer part of x. Moreover, we always assume that all random variables in this paper are defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let $\mathbb{E}$ be the corresponding expectation.

The rest of this paper is organized as follows. In Section 2 a sequence of branching particle systems arising from population models is formulated. In Section 3 the existence of solution to the MP (1.3, 1.4) is established through the tightness of a sequence of measure-valued stochastic processes arising as the empirical measures of the branching particle systems. In Section 4 we verify the equivalence between the MP (1.3, 1.4) and SPDEs satisfied by the distribution function-valued processes and further prove the pathwise uniqueness of the SPDEs by an extended Yamada–Watanabe argument. Throughout the paper we use the letter K, with or without subscripts, to denote constants whose exact value is unimportant and may change from line to line.

2. A related branching model with migration

There exists a population living in two colonies with labels $\{\mathbb{1}, \mathbb{2}\}$. Initially, each colony has n particles, spatially distributed in $\mathbb{R}$. Write $k \sim t$ to denote the kth living particle at time t in each colony. For any $s \geq t$, let $X_{k\sim t}^{n,\mathbb{i}}(s)$ denote the corresponding particle's location at time s in colony $\mathbb{i}$ with $\mathbb{i} \in \{\mathbb{1}, \mathbb{2}\}$ if it is alive up to time s. The motions of the particles during their lifetimes are modelled by independent Brownian motions. For any $\mathbb{i} \in \{\mathbb{1}, \mathbb{2}\}$, $k \sim t$, accompanying the corresponding particle with a standard Brownian motion $\{B_{k\sim t}^{\mathbb{i}}(s) : s \geq 0\}$, we have

$$X_{k\sim t}^{n,\mathbb{i}}(s) \stackrel{d}{=} X_{k\sim t}^{n,\mathbb{i}}(t) + B_{k\sim t}^{\mathbb{i}}(s) - B_{k\sim t}^{\mathbb{i}}(t) \quad \text{for all } s \geq t.$$

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For $s < t$, $X^{n,\mathbb{i}}_{k\sim t}(s)$ represents the corresponding particle's location at time s if it is alive at s; otherwise the same notation represents its ancestor's location at s.

We start by introducing the branching particle systems. In colony $\mathbb{1}$ there exist independent branching and emigration, and there are also independent branching and immigration in colony $\mathbb{2}$. The branching mechanisms in the two colonies are also independent. However, the emigration and immigration are interactive. The particles in colony $\mathbb{1}$ can move to colony $\mathbb{2}$, but not reciprocally. During branching/emigration/immigration events, all the particles move according to independent Brownian motions.

- (Measure-valued process in colony $\mathbb{i}$ with $\mathbb{i} = \mathbb{1}, \mathbb{2}$.) Let $\mu^{n,\mathbb{i}}_t$ be the empirical distribution of particles living in colony $\mathbb{i}$, that is, for any $f \in C^2_b(\mathbb{R})$, we have

$$\langle \mu^{n,\mathbb{i}}_t, f\rangle = \frac{1}{n}\sum_{k\sim t} f\big(X^{n,\mathbb{i}}_{k\sim t}(t)\big),$$

  where the sum $k \sim t$ includes all those particles alive at t in each colony.

- (The behaviour of particles living in colony $\mathbb{1}$.) For a particle k alive at time t in colony $\mathbb{1}$, we consider (conditionally independent) random times $\tau^{\mathbb{1}}_{k\sim t}$ (corresponding to a reproduction event) and $\rho_{k\sim t}$ (corresponding to a migration event) such that

$$\mathbb{P}\big(\tau^{\mathbb{1}}_{k\sim t} > t+h \mid \mathcal{F}_t\big) = \mathrm{e}^{-\lambda_{n,\mathbb{1}} h}$$

  and

$$\begin{aligned}\mathbb{P}(\rho_{k\sim t} > t+h \mid \mathcal{F}_t) &= \mathbb{E}\bigg[\exp\bigg\{-\int_t^{t+h} \eta\big(X^{n,\mathbb{1}}_{k\sim t}(s),\, \mu^{n,\mathbb{1}}_s,\, \mu^{n,\mathbb{2}}_s\big)\,\mathrm{d}s\bigg\}\,\bigg|\, \mathcal{F}_t\bigg] \\ &\approx \mathrm{e}^{-\eta\left(X^{n,\mathbb{1}}_{k\sim t}(t),\mu^{n,\mathbb{1}}_t,\mu^{n,\mathbb{2}}_t\right)h}\end{aligned}$$

  when h is sufficiently small, where $\lambda_{n,\mathbb{1}}$ is the branching rate of those particles living in colony $\mathbb{1}$ with $\lambda_{n,\mathbb{1}}/n \to \lambda_{\mathbb{1}}$ as $n \to \infty$, and $\eta(\cdot,\cdot,\cdot) \in C^+_b(\mathbb{R}\times M_F(\mathbb{R})^2)$ is the migration intensity. If $\tau^{\mathbb{1}}_{k\sim t} < \rho_{k\sim t}$, then at time $\tau^{\mathbb{1}}_{k\sim t}$, particle k dies and gives birth to a random number $\xi^{n,\mathbb{1}}$ of offspring at the position of colony $\mathbb{1}$ where particle k died, with $\mathbb{E}[\xi^{n,\mathbb{1}}] = 1 + \beta_{n,\mathbb{1}}/n$, $\mathrm{Var}[\xi^{n,\mathbb{1}}] = \sigma^2_{n,\mathbb{1}}$ satisfying $\beta_{n,\mathbb{1}} \to \beta_{\mathbb{1}}$ and $\sigma_{n,\mathbb{1}} \to \sigma_{\mathbb{1}}$ as $n \to \infty$. If $\rho_{k\sim t} < \tau^{\mathbb{1}}_{k\sim t}$, then at time $\rho_{k\sim t}$, particle k migrates to colony $\mathbb{2}$ and settles down at a random position according to a probability measure χ.

- (The behaviour of particles living in colony $\mathbb{2}$.) For a particle k alive at time t in colony $\mathbb{2}$, we consider random time $\tau^{\mathbb{2}}_{k\sim t}$ (corresponding to a reproduction event) such that

$$\mathbb{P}\big(\tau^{\mathbb{2}}_{k\sim t} > t+h \mid \mathcal{F}_t\big) = \mathrm{e}^{-\lambda_{n,\mathbb{2}} h},$$

  where $\lambda_{n,\mathbb{2}}$ is the branching rate of those particles living in colony $\mathbb{2}$ with $\lambda_{n,\mathbb{2}}/n \to \lambda_{\mathbb{2}}$ as $n \to \infty$. Then, at time $\tau^{\mathbb{2}}_{k\sim t}$, particle k dies and gives birth to a random number $\xi^{n,\mathbb{2}}$ of offspring at the position of colony $\mathbb{2}$ where particle k died, with $\mathbb{E}[\xi^{n,\mathbb{2}}] = 1 + \beta_{n,\mathbb{2}}/n$, $\mathrm{Var}\,[\xi^{n,\mathbb{2}}] = \sigma^2_{n,\mathbb{2}}$ satisfying $\beta_{n,\mathbb{2}} \to \beta_{\mathbb{2}}$ and $\sigma_{n,\mathbb{2}} \to \sigma_{\mathbb{2}}$ as $n \to \infty$.

3. Existence of a solution to the martingale problem

In this section we study the convergence of a sequence of measure-valued processes arising as the empirical measures of the proposed branching particle systems in Section 2, and show

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that the limit is a weak solution to the MP (1.3, 1.4). We denote $\sum_{i=1}^{0} f_i = 0$ for any f_i. For any $t > 0$, let $h > 0$ be sufficiently small. It follows from the construction of the branching particle systems that

$$\langle \mu_{t+h}^{n,\mathbb{1}}, f\rangle = \frac{1}{n}\sum_{k\sim t}\big(f\big(X_{k\sim t}^{n,\mathbb{1}}(t+h)\big) + \Delta_{k\sim t}^{n,\mathbb{1},h}(f) - D_{k\sim t}^{n,\mathbb{1},h}(f)\big) + \frac{1}{n}\sum_{k\sim t}\Upsilon_{k\sim t}^{n,\mathbb{1},h}(f),$$

with

$$\Delta_{k\sim t}^{n,\mathbb{1},h}(f) = \left[\sum_{i=1}^{\xi_{k\sim t}^{n,\mathbb{1}}} f\big(X_{k\sim t}^{n,\mathbb{1},i}(t+h)\big) - f\big(X_{k\sim t}^{n,\mathbb{1}}(t+h)\big)\right]\mathbb{I}_{\{\tau_{k\sim t}^{\mathbb{1}} < t+h < \rho_{k\sim t}\}}, \tag{3.1}$$

$$D_{k\sim t}^{n,\mathbb{1},h}(f) = f\big(X_{k\sim t}^{n,\mathbb{1}}(t+h)\big)\mathbb{I}_{\{\rho_{k\sim t} < t+h < \tau_{k\sim t}^{\mathbb{1}}\}}, \tag{3.2}$$

where $\xi_{k\sim t}^{n,\mathbb{1}}$, $k = 1, 2, \ldots$ are independent and identically distributed (i.i.d.) copies of $\xi^{n,\mathbb{1}}$, $X_{k\sim t}^{n,\mathbb{1},i}(t+h)$, $i = 1, \ldots, \xi_{k\sim t}^{n,\mathbb{1}}$ are defined as i.i.d. copies of $X_{k\sim t}^{n,\mathbb{1}}(t+h)$, $\mathbb{I}_{\{\cdot\}}$ is the indicator function, and $\Upsilon_{k\sim t}^{n,\mathbb{1},h}(f)$ is the correction term due to particle k and its descendants having more than one branching or migration event in the time interval $(t, t+h]$. Since f is bounded, one can check that the correction term satisfies

$$\mathbb{E}\big[|\Upsilon_{k\sim t}^{n,\mathbb{1},h}(f)| \mid \mathcal{F}_t\big] \le \mathrm{O}(h^2),$$

where $\mathrm{O}(h^2)$ is an infinitesimal of h with order no less than 2. In fact it is hard to obtain the explicit expression for the correction term. However, the order indicates that it is very tiny and can make itself disappear in the limit form. The conditional independence of $\rho_{k\sim t}$ and $\tau_{k\sim t}^{\mathbb{1}}$ implies that

$$\begin{aligned}\mathbb{E}\big[\mathbb{I}_{\{\tau_{k\sim t}^{\mathbb{1}} < t+h < \rho_{k\sim t}\}} \mid \mathcal{F}_t\big] &= \mathbb{P}\big(\tau_{k\sim t}^{\mathbb{1}} < t+h \mid \mathcal{F}_t\big)\mathbb{P}\big(t+h < \rho_{k\sim t} \mid \mathcal{F}_t\big)\\ &\approx \big(1 - \mathrm{e}^{-\lambda_{n,\mathbb{1}}h}\big)\mathrm{e}^{-\eta(X_{k\sim t}^{n,\mathbb{1}}(t),\mu_t^{n,\mathbb{1}},\mu_t^{n,\mathbb{2}})h}\\ &\approx \lambda_{n,\mathbb{1}}h\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big[\mathbb{I}_{\{\rho_{k\sim t} < t+h < \tau_{k\sim t}^{\mathbb{1}}\}} \mid \mathcal{F}_t\big] &\approx \big(1 - \mathrm{e}^{-\eta(X_{k\sim t}^{n,\mathbb{1}}(t),\mu_t^{n,\mathbb{1}},\mu_t^{n,\mathbb{2}})h}\big)\mathrm{e}^{-\lambda_{n,\mathbb{1}}h}\\ &\approx \eta\big(X_{k\sim t}^{n,\mathbb{1}}(t), \mu_t^{n,\mathbb{1}}, \mu_t^{n,\mathbb{2}}\big)h.\end{aligned}$$

The corresponding conditional probability can be replaced by the approximate value when h is sufficiently small. Applying Itô's formula, we have

$$f\big(X_{k\sim t}^{n,\mathbb{1}}(t+h)\big) = f\big(X_{k\sim t}^{n,\mathbb{1}}(t)\big) + \int_t^{t+h} f'\big(X_{k\sim t}^{n,\mathbb{1}}(s)\big)\mathrm{d}B_{k\sim t}^{\mathbb{1}}(s) + \frac{1}{2}\int_t^{t+h} f''\big(X_{k\sim t}^{n,\mathbb{1}}(s)\big)\,\mathrm{d}s. \tag{3.3}$$

Consequently

$$\begin{aligned}\langle \mu^{n,\mathbb{1}}_{t+h}, f\rangle &= \langle \mu^{n,\mathbb{1}}_{t}, f\rangle + \frac{1}{n}\sum_{k\sim t}\int_t^{t+h} f'\big(X^{n,\mathbb{1}}_{k\sim t}(s)\big)\mathrm{d}B^{\mathbb{1}}_{k\sim t}(s)\\ &\quad + \frac{1}{n}\sum_{k\sim t}\frac{1}{2}\int_t^{t+h} f''\big(X^{n,\mathbb{1}}_{k\sim t}(s)\big)\,\mathrm{d}s\\ &\quad + \frac{1}{n}\sum_{k\sim t}\big(\Delta^{n,\mathbb{1},h}_{k\sim t}(f) - D^{n,\mathbb{1},h}_{k\sim t}(f) + \Upsilon^{n,\mathbb{1},h}_{k\sim t}(f)\big).\end{aligned}$$

Given any $t>0$, let us discretize the time interval $[0, t]$ by the step size h. Set $j=\lfloor t/h\rfloor$. Then

$$\begin{aligned}\langle \mu^{n,\mathbb{1}}_{t}, f\rangle &= \langle \mu^{n,\mathbb{1}}_{0}, f\rangle + \sum_{\ell=0}^{j}\big(\langle \mu^{n,\mathbb{1}}_{(\ell+1)h\wedge t}, f\rangle - \langle \mu^{n,\mathbb{1}}_{\ell h}, f\rangle\big)\\ &= \langle \mu^{n,\mathbb{1}}_{0}, f\rangle + \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\int_{\ell h}^{(\ell+1)h\wedge t} f'\big(X^{n,\mathbb{1}}_{k\sim \ell h}(s)\big)\mathrm{d}B^{\mathbb{1}}_{k\sim \ell h}(s)\\ &\quad + \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\frac{1}{2}\int_{\ell h}^{(\ell+1)h\wedge t} f''\big(X^{n,\mathbb{1}}_{k\sim \ell h}(s)\big)\,\mathrm{d}s\\ &\quad + \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\big(\Delta^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f) - D^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f) + \Upsilon^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f)\big).\end{aligned}$$

Let $M^{n,f}_i$, $i=1, 2, 3$ be martingales with

$$M^{n,f}_1(t) = \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\int_{\ell h}^{(\ell+1)h\wedge t} f'\big(X^{n,\mathbb{1}}_{k\sim \ell h}(s)\big)\mathrm{d}B^{\mathbb{1}}_{k\sim \ell h}(s), \tag{3.4}$$

$$M^{n,f}_2(t) = \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\big\{\Delta^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f) - \mathbb{E}\big[\Delta^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f)\mid \mathcal{F}_{\ell h}\big]\big\}, \tag{3.5}$$

$$M^{n,f}_3(t) = \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\big\{D^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f) - \mathbb{E}\big[D^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f)\mid \mathcal{F}_{\ell h}\big]\big\}. \tag{3.6}$$

Moreover, $A^{n,f}$ is defined as

$$\begin{aligned}A^{n,f}(t) &= \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\frac{1}{2}\int_{\ell h}^{(\ell+1)h\wedge t} f''\big(X^{n,\mathbb{1}}_{k\sim \ell h}(s)\big)\,\mathrm{d}s + \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\Upsilon^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f)\\ &\quad + \sum_{\ell=0}^{j}\frac{1}{n}\sum_{k\sim \ell h}\mathbb{E}\big[\Delta^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f) - D^{n,\mathbb{1},h\wedge(t-\ell h)}_{k\sim \ell h}(f)\mid \mathcal{F}_{\ell h}\big].\end{aligned}$$

It follows from (3.3) that

$$\mathbb{E}\big[f\big(X^{n,\mathbb{1}}_{k\sim \ell h}((\ell+1)h\wedge t)\big)\mid \mathcal{F}_{\ell h}\big] = f\big(X^{n,\mathbb{1}}_{k\sim \ell h}(\ell h)\big) + \mathrm{O}(h\wedge(t-\ell h)),$$

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where $\mathrm{O}(h \wedge (t-\ell h)) \leq \frac{1}{2} \sup_{x\in\mathbb{R}} |f''(x)| h \leq Kh$ holds for any given h and n since $f \in C_b^2(\mathbb{R})$. Combining with (3.1) and (3.2), we obtain the conditional expectations

$$\mathbb{E}\big[\Delta_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) \mid \mathcal{F}_{\ell h}\big] = \frac{\beta_{n,\mathbb{1}}\lambda_{n,\mathbb{1}}}{n} f\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big)(h \wedge (t-\ell h)) + \mathrm{O}((h\wedge(t-\ell h))^2)$$

and

$$\begin{aligned}\mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) \mid \mathcal{F}_{\ell h}\big] \\ = f\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big)\eta\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h),\, \mu_{\ell h}^{n,\mathbb{1}},\, \mu_{\ell h}^{n,\mathbb{2}}\big)(h\wedge(t-\ell h)) + \mathrm{O}((h\wedge(t-\ell h))^2),\end{aligned}$$

where $\mathrm{O}((h\wedge(t-\ell h))^2) \leq Kh^2$ holds for any given h and n. Consequently

$$\begin{aligned}A^{n,f}(t) = {} & \frac{1}{2}\int_0^t \big\langle \mu_s^{n,\mathbb{1}}, f''\big\rangle \,\mathrm{d}s + \sum_{\ell=0}^{j} \frac{\beta_{n,\mathbb{1}}\lambda_{n,\mathbb{1}}}{n} \langle \mu_{\ell h}^{n,\mathbb{1}}, f\rangle (h\wedge(t-\ell h)) \\ & - \sum_{\ell=0}^{j} \big\langle \mu_{\ell h}^{n,\mathbb{1}},\, \eta\big(\cdot,\, \mu_{\ell h}^{n,\mathbb{1}},\, \mu_{\ell h}^{n,\mathbb{2}}\big) f\big\rangle (h\wedge(t-\ell h)) \\ & + \sum_{\ell=0}^{j} \big\langle \mu_{\ell h}^{n,\mathbb{1}},\, 1\big\rangle\, \mathrm{O}((h\wedge(t-\ell h))^2) \\ & + \sum_{\ell=0}^{j} \frac{1}{n} \sum_{k\sim\ell h} \Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f),\end{aligned} \tag{3.7}$$

with the last term satisfying

$$\sum_{\ell=0}^{j} \frac{1}{n} \sum_{k\sim\ell h} \mathbb{E}\big[\big|\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\big| \mid \mathcal{F}_{\ell h}\big] \leq \sum_{\ell=0}^{j} \big\langle \mu_{\ell h}^{n,\mathbb{1}},\, 1\big\rangle\, \mathrm{O}((h\wedge(t-\ell h))^2). \tag{3.8}$$

One can see that

$$\big\langle \mu_t^{n,\mathbb{1}}, f\big\rangle = \big\langle \mu_0^{n,\mathbb{1}}, f\big\rangle + M_1^{n,f}(t) + M_2^{n,f}(t) - M_3^{n,f}(t) + A^{n,f}(t). \tag{3.9}$$

Carrying out steps similar to those above in colony $\mathbb{2}$, for any $g \in C_b^2(\mathbb{R})$, we have

$$\big\langle \mu_t^{n,\mathbb{2}}, g\big\rangle = \big\langle \mu_0^{n,\mathbb{2}}, g\big\rangle + \hat{M}_1^{n,g}(t) + \hat{M}_2^{n,g}(t) + \hat{M}_3^{n,g}(t) + \hat{A}^{n,g}(t), \tag{3.10}$$

where

$$\hat{M}_1^{n,g}(t) = \sum_{\ell=0}^{\lfloor t/h \rfloor} \frac{1}{n} \sum_{k\sim\ell h} \int_{\ell h}^{(\ell+1)h\wedge t} g'\big(X_{k\sim\ell h}^{n,\mathbb{2}}(s)\big)\,\mathrm{d}B_{k\sim\ell h}^{\mathbb{2}}(s) \tag{3.11}$$

and

$$\hat{M}_2^{n,g}(t) = \sum_{\ell=0}^{\lfloor t/h \rfloor} \frac{1}{n} \sum_{k\sim\ell h} \big\{\Delta_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g) - \mathbb{E}\big[\Delta_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g) \mid \mathcal{F}_{\ell h}\big]\big\} \tag{3.12}$$

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are martingales with

$$\Delta_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)=\left[\sum_{i=1}^{\xi_{k\sim\ell h}^{n,\mathbb{2}}} g\big(X_{k\sim\ell h}^{n,\mathbb{2},i}((\ell+1)h\wedge t)\big)-g\big(X_{k\sim\ell h}^{n,\mathbb{2}}((\ell+1)h\wedge t)\big)\right]\mathbb{I}_{\{\tau_{k\sim\ell h}^{\mathbb{2}}<(\ell+1)h\wedge t\}}$$

and $X_{k\sim\ell h}^{n,\mathbb{2},i}((\ell+1)h\wedge t)$ having the same distribution as $X_{k\sim\ell h}^{n,\mathbb{2}}((\ell+1)h\wedge t)$. Moreover,

$$\hat{M}_3^{n,g}(t)=\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Big\{D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)-\mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)\mid\mathcal{F}_{\ell h}\big]\Big\} \tag{3.13}$$

is also a martingale with

$$D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)=\langle\chi,g\rangle\,\mathbb{I}_{\{\rho_{k\sim\ell h}\le(\ell+1)h\wedge t\}}.$$

We emphasize that the sum '$k\sim\ell h$' in (3.13) involves those particles living in colony $\mathbb{1}$ at time ℓh. In addition,

$$\begin{aligned}\hat{A}^{n,g}(t)=&\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\frac{1}{2}\int_{\ell h}^{(\ell+1)h\wedge t} g''\big(X_{k\sim\ell h}^{n,\mathbb{2}}(s)\big)\,\mathrm{d}s\\ &+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\mathbb{E}\big[\Delta_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)\mid\mathcal{F}_{\ell h}\big]+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)\mid\mathcal{F}_{\ell h}\big]\\ &+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Upsilon_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Xi_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g),\end{aligned}$$

where the second and third terms are due to birth events in colony $\mathbb{2}$ and migration from colony $\mathbb{1}$, respectively, the fourth term is the correction term due to those particles living in colony $\mathbb{2}$ and their descendants having more than one branching, and the last term is the correction term due to those particles living in colony $\mathbb{1}$ that have not only migrated to colony $\mathbb{2}$ but also given birth to offspring after settlements. Therefore the sum '$k\sim\ell h$' in the second or fourth term involves those particles living in colony $\mathbb{2}$ at ℓh, and the sum '$k\sim\ell h$' in the third or last term involves those particles living in colony $\mathbb{1}$ at ℓh. Considering the possibility of more than one branching or migration event in the time interval $(\ell h,(\ell+1)h\wedge t]$ and the boundedness of g, one can see that

$$\mathbb{E}\big[|\Upsilon_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)|\mid\mathcal{F}_{\ell h}\big]+\mathbb{E}\big[|\Xi_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)|\mid\mathcal{F}_{\ell h}\big]\le \mathrm{O}((h\wedge(t-\ell h))^2).$$

Consequently

$$\begin{aligned}\hat{A}^{n,g}(t)=&\frac{1}{2}\int_0^t\big\langle\mu_s^{n,\mathbb{2}},g''\big\rangle\,\mathrm{d}s+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{\beta_{n,\mathbb{2}}\lambda_{n,\mathbb{2}}}{n}\big\langle\mu_{\ell h}^{n,\mathbb{2}},g\big\rangle(h\wedge(t-\ell h))\\ &+\sum_{\ell=0}^{\lfloor t/h\rfloor}\langle\chi,g\rangle\big\langle\mu_{\ell h}^{n,\mathbb{1}},\eta\big(\cdot,\mu_{\ell h}^{n,\mathbb{1}},\mu_{\ell h}^{n,\mathbb{2}}\big)\big\rangle(h\wedge(t-\ell h))\end{aligned}$$

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$$+\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}}+\mu_{\ell h}^{n,\mathbb{2}},1\big\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2)$$
$$+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Upsilon_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Xi_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g), \tag{3.14}$$

with the terms in the last line satisfying

$$\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\mathbb{E}\big[|\Upsilon_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)|\mid\mathcal{F}_{\ell h}\big]+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\mathbb{E}\big[|\Xi_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g)|\mid\mathcal{F}_{\ell h}\big]$$
$$\le\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}}+\mu_{\ell h}^{n,\mathbb{2}},1\big\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2).$$

The following propositions give the quadratic variation and covariation processes for those martingales $M_i^{n,f}(t)$ and $\hat{M}_i^{n,g}(t)$ with $i=1,2,3$.

Proposition 3.1. *The variation processes of $M_i^{n,f}(t)$ with $i=1,2,3$ are as follows:*

$$\big\langle M_1^{n,f}\big\rangle_t=\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n^2}\sum_{k\sim\ell h}\int_{\ell h}^{(\ell+1)h\wedge t}\big|f'\big(X_{k\sim\ell h}^{n,\mathbb{1}}(s)\big)\big|^2\mathrm{d}s, \tag{3.15}$$
$$\big\langle M_2^{n,f}\big\rangle_t=\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}},f^2\big\rangle\bigg(\sigma_{n,\mathbb{1}}^2+\frac{\beta_{n,\mathbb{1}}^2}{n^2}\bigg)\frac{\lambda_{n,\mathbb{1}}}{n}(h\wedge(t-\ell h))$$
$$+\frac{1}{n}\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}},1\big\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2), \tag{3.16}$$
$$\big\langle M_3^{n,f}\big\rangle_t=\frac{1}{n}\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}},f^2(\cdot)\eta\big(\cdot,\mu_{\ell h}^{n,\mathbb{1}},\mu_{\ell h}^{n,\mathbb{2}}\big)\big\rangle(h\wedge(t-\ell h))$$
$$+\frac{1}{n}\sum_{\ell=0}^{\lfloor t/h\rfloor}\big\langle\mu_{\ell h}^{n,\mathbb{1}},1\big\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2). \tag{3.17}$$

Proof. Notice that (3.15) is obtained directly by (3.4). Moreover, by (3.1) one can see that

$$\mathbb{E}\big[\big(\Delta_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\big)^2\mid\mathcal{F}_{\ell h}\big]=\bigg(\sigma_{n,\mathbb{1}}^2+\frac{\beta_{n,\mathbb{1}}^2}{n^2}\bigg)f^2\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big)\lambda_{n,\mathbb{1}}(h\wedge(t-\ell h))$$
$$+\mathrm{O}((h\wedge(t-\ell h))^2),$$

where $\mathrm{O}((h\wedge(t-\ell h))^2)\le Kh^2$ holds for any given h and n. Applying Lemma 8.12 of [21], the quadratic variations of $M_2^{n,f}$ are as follows:

$$\big\langle M_2^{n,f}\big\rangle_t=\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n^2}\mathbb{E}\Bigg\{\Bigg[\sum_{k\sim\ell h}\big(\Delta_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)-\mathbb{E}\big(\Delta_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\mid\mathcal{F}_{\ell h}\big)\Bigg]^2\,\Bigg|\,\mathcal{F}_{\ell h}\Bigg\}$$

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$$= \sum_{\ell=0}^{\lfloor t/h \rfloor} \frac{1}{n^2} \sum_{k\sim\ell h} \Big[f^2\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big) \Big(\sigma_{n,\mathbb{1}}^2 + \frac{\beta_{n,\mathbb{1}}^2}{n^2} \Big) \lambda_{n,\mathbb{1}}(h \wedge (t-\ell h)) + \mathrm{O}((h \wedge (t-\ell h))^2) \Big]$$
$$= \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, f^2 \rangle \Big(\sigma_{n,\mathbb{1}}^2 + \frac{\beta_{n,\mathbb{1}}^2}{n^2} \Big) \frac{\lambda_{n,\mathbb{1}}}{n} (h \wedge (t-\ell h)) + \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, 1 \rangle \, \mathrm{O}((h \wedge (t-\ell h))^2).$$

Further, by (3.2) we have

$$\mathbb{E}\big[\big(D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\big)^2 \mid \mathcal{F}_{\ell h}\big] = f^2\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big) \eta\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h), \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big)(h \wedge (t-\ell h)) + \mathrm{O}((h \wedge (t-\ell h))^2).$$

In the same way, the quadratic variation process of $M_3^{n,f}(t)$ is derived as follows:

$$\langle M_3^{n,f} \rangle_t = \sum_{\ell=0}^{\lfloor t/h \rfloor} \frac{1}{n^2} \mathbb{E}\left\{ \left[\sum_{k\sim\ell h} \big(D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) - \mathbb{E}\big(D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) \mid \mathcal{F}_{\ell h}\big)\big) \right]^2 \,\Big|\, \mathcal{F}_{\ell h} \right\}$$
$$= \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, f^2(\cdot) \eta\big(\cdot, \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big) \rangle (h \wedge (t-\ell h)) + \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, 1 \rangle \, \mathrm{O}((h \wedge (t-\ell h))^2).$$

This completes the proof. □

As above, the quadratic variation processes for $\hat{M}_i^{n,g}(t)$ with $i = 1, 2, 3$ are stated in the proposition below without proof.

Proposition 3.2. *The variation processes of $\hat{M}_i^{n,g}(t)$ with $i = 1, 2, 3$ are as follows:*

$$\langle \hat{M}_1^{n,g} \rangle_t = \sum_{\ell=0}^{\lfloor t/h \rfloor} \frac{1}{n^2} \sum_{k\sim\ell h} \int_{\ell h}^{(\ell+1)h\wedge t} \big| g'\big(X_{k\sim\ell h}^{n,\mathbb{2}}(s)\big) \big|^2 \mathrm{d}s, \tag{3.18}$$

$$\langle \hat{M}_2^{n,g} \rangle_t = \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{2}}, g^2 \rangle \Big(\sigma_{n,\mathbb{2}}^2 + \frac{\beta_{n,\mathbb{2}}^2}{n^2} \Big) \frac{\lambda_{n,\mathbb{2}}}{n} (h \wedge (t-\ell h)) + \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{2}}, 1 \rangle \, \mathrm{O}((h \wedge (t-\ell h))^2), \tag{3.19}$$

$$\langle \hat{M}_3^{n,g} \rangle_t = \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \chi, g \rangle^2 \langle \mu_{\ell h}^{n,\mathbb{1}}, \eta\big(\cdot, \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big) \rangle (h \wedge (t-\ell h)) + \frac{1}{n} \sum_{\ell=0}^{\lfloor t/h \rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, 1 \rangle \, \mathrm{O}((h \wedge (t-\ell h))^2). \tag{3.20}$$

From the construction of the model, one can check that the martingales in Proposition 3.1 and 3.2 are mutually uncorrelated except for $M_3^{n,f}$ and $\hat{M}_3^{n,g}$, whose covariation process is demonstrated in the following proposition.

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Proposition 3.3. *The covariation process of* $M_3^{n,f}$ *and* $\hat{M}_3^{n,g}$ *is*

$$\langle M_3^{n,f}, \hat{M}_3^{n,g}\rangle_t = \sum_{\ell=0}^{\lfloor t/h\rfloor} \frac{1}{n}\langle \chi, g\rangle\langle \mu_{\ell h}^{n,\mathbb{1}}, f\eta(\cdot, \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}})\rangle(h\wedge(t-\ell h)) + \sum_{\ell=0}^{\lfloor t/h\rfloor} \frac{1}{n}\langle \mu_{\ell h}^{n,\mathbb{1}}, 1\rangle^2\, \mathrm{O}((h\wedge(t-\ell h))^2).$$

Proof. For simplicity of notation, we set

$$I_1(\ell, k) = \frac{1}{n}\big\{D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) - \mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) \mid \mathcal{F}_{\ell h}\big]\big\},$$
$$I_2(\ell, k) = \frac{1}{n}\big\{D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g) - \mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{2},h\wedge(t-\ell h)}(g) \mid \mathcal{F}_{\ell h}\big]\big\}.$$

It follows from (3.6) and (3.13) that

$$M_3^{n,f}(t) = \sum_{\ell=0}^{\lfloor t/h\rfloor}\sum_{k\sim\ell h} I_1(\ell, k) \quad \text{and} \quad \hat{M}_3^{n,g}(t) = \sum_{\ell=0}^{\lfloor t/h\rfloor}\sum_{k\sim\ell h} I_2(\ell, k),$$

where $\sum_{k\sim\ell h}$ includes all alive particles in colony $\mathbb{1}$ at time ℓh. Again applying Lemma 8.12 of [21], one can check that

$$\langle M_3^{n,f}, \hat{M}_3^{n,g}\rangle_t = \sum_{\ell=0}^{\lfloor t/h\rfloor} \mathbb{E}\Bigg[\sum_{k_1\sim\ell h, k_2\sim\ell h} I_1(\ell, k) I_2(\ell, k) \,\Bigg|\, \mathcal{F}_{\ell h}\Bigg].$$

For $k_1 = k_2 = k$, the kth living particle in colony $\mathbb{1}$ emigrates to colony $\mathbb{2}$, and we have

$$\begin{aligned}
&\mathbb{E}[I_1(\ell, k) I_2(\ell, k) \mid \mathcal{F}_{\ell h}] \\
&= \frac{\langle \chi, g\rangle}{n^2}\big\{\mathbb{E}\big[D_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) \mid \mathcal{F}_{\ell h}\big]\big(1 - \mathbb{E}\big[\mathbb{I}_{\{\rho_{k\sim\ell h}\le(\ell+1)h\wedge t\}} \mid \mathcal{F}_{\ell h}\big]\big)\big\} \\
&= \frac{\langle \chi, g\rangle}{n^2} f\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big)\eta\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h), \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big)(h\wedge(t-\ell h)) \\
&\quad \times \big[1 - \eta\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h), \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big)(h\wedge(t-\ell h))\big] + \frac{\mathrm{O}((h\wedge(t-\ell h))^2)}{n^2} \\
&= \frac{\langle \chi, g\rangle}{n^2} f\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h)\big)\eta\big(X_{k\sim\ell h}^{n,\mathbb{1}}(\ell h), \mu_{\ell h}^{n,\mathbb{1}}, \mu_{\ell h}^{n,\mathbb{2}}\big)(h\wedge(t-\ell h)) + \frac{\mathrm{O}((h\wedge(t-\ell h))^2)}{n^2}.
\end{aligned}$$

Further, for the case $k_1 \ne k_2$ we obtain that

$$\mathbb{E}\big[D_{k_1\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f) D_{k_2\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(g) \mid \mathcal{F}_{\ell h}\big] = 0,$$

since the probability that two particles emigrate from colony $\mathbb{1}$ to $\mathbb{2}$ simultaneously is 0, which implies

$$\begin{aligned}&\mathbb{E}\big[I_1(\ell, k_1)I_2(\ell, k_2) \mid \mathcal{F}_{\ell h}\big]\\&\quad= -\frac{\langle \chi, g\rangle}{n^2}\mathbb{E}\big[D^{n,\mathbb{1},h\wedge(t-\ell h)}_{k_1\sim \ell h}(f) \mid \mathcal{F}_{\ell h}\big]\cdot \mathbb{E}\big[\mathbb{1}_{\{\rho_{k_2\sim\ell h}\le(\ell+1)h\wedge t\}} \mid \mathcal{F}_{\ell h}\big]\\&\quad= \frac{\mathrm{O}((h\wedge(t-\ell h))^2)}{n^2}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\big\langle M^{n,f}_3, \hat{M}^{n,g}_3\big\rangle_t\\&\quad= \sum_{\ell=0}^{\lfloor t/h\rfloor}\sum_{k\sim\ell h}\frac{\langle \chi, g\rangle}{n^2} f\big(X^{n,\mathbb{1}}_{k\sim\ell h}(\ell h)\big)\eta\big(X^{n,\mathbb{1}}_{k\sim\ell h}(\ell h), \mu^{n,\mathbb{1}}_{\ell h}, \mu^{n,\mathbb{2}}_{\ell h}\big)(h\wedge(t-\ell h))\\&\qquad+ \sum_{\ell=0}^{\lfloor t/h\rfloor}\sum_{k_1\sim\ell h, k_2\sim\ell h}\frac{1}{n^2}\mathrm{O}((h\wedge(t-\ell h))^2)\\&\quad= \sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{\langle \chi, g\rangle}{n}\big\langle \mu^{n,\mathbb{1}}_{\ell h}, f\eta\big(\cdot, \mu^{n,\mathbb{1}}_{\ell h}, \mu^{n,\mathbb{2}}_{\ell h}\big)\big\rangle(h\wedge(t-\ell h))\\&\qquad+ \sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\big\langle \mu^{n,\mathbb{1}}_{\ell h}, 1\big\rangle^2\mathrm{O}((h\wedge(t-\ell h))^2).\end{aligned}$$

The result follows. □

In the following we make some estimations (Lemmas 3.1–3.4) and then prove the tightness of the empirical measure for the branching particle systems.

Lemma 3.1. *Assume that*

$$\sup_n \mathbb{E}\big[\big\langle \mu^{n,\mathbb{1}}_0, 1\big\rangle^{2p} + \big\langle \mu^{n,\mathbb{2}}_0, 1\big\rangle^{2p}\big] < \infty \quad \textit{for some } p \ge 1.$$

For any $T > 0$, there exists a constant $K = K(p, T)$ such that

$$\sup_n \mathbb{E}\bigg[\sup_{t\le T}\big(\big\langle \mu^{n,\mathbb{1}}_t, 1\big\rangle^{2p} + \big\langle \mu^{n,\mathbb{2}}_t, 1\big\rangle^{2p}\big)\bigg] < K.$$

Proof. Replacing f and g with 1 in (3.9) and (3.10), by Doob's inequality one can check that

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\le T}\big\langle \mu^{n,\mathbb{1}}_t, 1\big\rangle^{2p}\bigg] &\le K_1 + K_1\sum_{i=1}^{3}\mathbb{E}\big[|M^{n,1}_i(T)|^{2p}\big] + K_1\mathbb{E}\bigg[\sup_{t\le T}|A^{n,1}(t)|^{2p}\bigg]\\&= K_1 + K_1\sum_{i=1}^{3}\mathbb{E}\big[\big\langle M^{n,1}_i\big\rangle^p_T\big] + K_1\mathbb{E}\bigg[\sup_{t\le T}|A^{n,1}(t)|^{2p}\bigg] \qquad (3.20)\end{aligned}$$

and similarly

$$\mathbb{E}\Big[\sup_{t\le T}\langle \mu_t^{n,\mathbb{2}},1\rangle^{2p}\Big]\le K_1+K_1\sum_{i=1}^{3}\mathbb{E}\big[\langle \hat{M}_i^{n,1}\rangle_T^p\big]+K_1\mathbb{E}\Big[\sup_{t\le T}|\hat{A}^{n,1}(t)|^{2p}\Big],$$

where K_1 is a constant depending on p. By 3.7, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\le T}|A^{n,1}(t)|^{2p}\Big]\\
&\le \mathbb{E}\Bigg[\sup_{t\le T}\Bigg|\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{\beta_{n,\mathbb{1}}\lambda_{n,\mathbb{1}}}{n}\langle\mu_{\ell h}^{n,\mathbb{1}},1\rangle(h\wedge(t-\ell h))-\sum_{\ell=0}^{\lfloor t/h\rfloor}\langle\mu_{\ell h}^{n,\mathbb{1}},\eta(\cdot,\mu_{\ell h}^{n,\mathbb{1}},\mu_{\ell h}^{n,\mathbb{2}})\rangle(h\wedge(t-\ell h))\\
&\quad+\sum_{\ell=0}^{\lfloor t/h\rfloor}\langle\mu_{\ell h}^{n,\mathbb{1}},1\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2)+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\Bigg|^{2p}\Bigg]\\
&\le \mathbb{E}\Bigg[\sup_{t\le T}\Bigg|\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{\beta_{n,\mathbb{1}}\lambda_{n,\mathbb{1}}}{n}\langle\mu_{\ell h}^{n,\mathbb{1}},1\rangle(h\wedge(t-\ell h))-\sum_{\ell=0}^{\lfloor t/h\rfloor}\langle\mu_{\ell h}^{n,\mathbb{1}},\eta(\cdot,\mu_{\ell h}^{n,\mathbb{1}},\mu_{\ell h}^{n,\mathbb{2}})\rangle(h\wedge(t-\ell h))\\
&\quad+\sum_{\ell=0}^{\lfloor t/h\rfloor}\langle\mu_{\ell h}^{n,\mathbb{1}},1\rangle\,\mathrm{O}((h\wedge(t-\ell h))^2)+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\mathbb{E}\big[\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\mid\mathcal{F}_{\ell h}\big]\\
&\quad+\sum_{\ell=0}^{\lfloor t/h\rfloor}\frac{1}{n}\sum_{k\sim\ell h}\big(\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)-\mathbb{E}\big[\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)\mid\mathcal{F}_{\ell h}\big]\big)\Bigg|^{2p}\Bigg],
\end{aligned}$$

where $\Upsilon_{k\sim\ell h}^{n,\mathbb{1},h\wedge(t-\ell h)}(f)$ is expressed as the sum of two parts. The upper bound of the first part is given by (3.8). The second part can be treated in the same way as other martingales. Recall that $\lambda_{n,\mathbb{1}}/n\to\lambda_{\mathbb{1}}$ and $\beta_{n,\mathbb{1}}\to\beta_{\mathbb{1}}$ as $n\to\infty$; η is bounded and the step size h is sufficiently small. Moreover,

$$\sum_{\ell=0}^{\lfloor T/h\rfloor}\langle\mu_{\ell h}^{n,\mathbb{1}},1\rangle(h\wedge(t-\ell h))$$

is bounded by

$$\int_0^T\sup_{t\le s}\langle\mu_t^{n,\mathbb{1}},1\rangle\,\mathrm{d}s.$$

By Hölder's inequality, there is a constant K_2 depending on p and T such that

$$\mathbb{E}\Big[\sup_{t\le T}|A^{n,1}(t)|^{2p}\Big]\le K_2\int_0^T\mathbb{E}\Big[\sup_{t\le s}\langle\mu_t^{n,\mathbb{1}},1\rangle^{2p}\Big]\,\mathrm{d}s.$$

The other terms of (3.20) can be treated in the same way. Then we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\le T}\langle\mu_t^{n,\mathbb{1}},1\rangle^{2p}\Big]&\le K_1+K_2\int_0^T\mathbb{E}\Big[\sup_{t\le s}\langle\mu_t^{n,\mathbb{1}},1\rangle^{p}\Big]\,\mathrm{d}s+K_2\int_0^T\mathbb{E}\Big[\sup_{t\le s}\langle\mu_t^{n,\mathbb{1}},1\rangle^{2p}\Big]\,\mathrm{d}s\\
&\le(K_1+K_2T)+K_2\int_0^T\mathbb{E}\Big[\sup_{t\le s}\langle\mu_t^{n,\mathbb{1}},1\rangle^{2p}\Big]\,\mathrm{d}s,
\end{aligned}$$

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where the last inequality follows from $x \le x^2 + 1$ for any $x \ge 0$. Let $K_3 = K_1 + K_2 T$, which depends on p and T. Similarly, by (3.14), (3.18)–(3.20), one can see that

$$\mathbb{E}\left[\sup_{t\le T}\langle \mu_t^{n,\mathbb{2}}, 1\rangle^{2p}\right] \le K_3 + K_2 \int_0^T \mathbb{E}\left[\sup_{t\le s}\left(\langle \mu_t^{n,\mathbb{2}}, 1\rangle^{2p} + \langle \mu_t^{n,\mathbb{1}}, 1\rangle^{2p}\right)\right] \mathrm{d}s.$$

As a consequence, we obtain

$$\mathbb{E}\left[\sup_{t\le T}\left(\langle \mu_t^{n,\mathbb{1}}, 1\rangle^{2p} + \langle \mu_t^{n,\mathbb{2}}, 1\rangle^{2p}\right)\right] \le 2K_3 + 2K_2 \int_0^T \mathbb{E}\left[\sup_{t\le s}\left(\langle \mu_t^{n,\mathbb{1}}, 1\rangle^{2p} + \langle \mu_t^{n,\mathbb{2}}, 1\rangle^{2p}\right)\right] \mathrm{d}s.$$

The result follows from Gronwall's inequality. □

Lemma 3.2. *Under the condition of Lemma 3.1, for any* $0 \le s \le t \le T$, $f, g \in C_b^2(\mathbb{R})$, *and* $i = 1, 2, 3$, *we have*

$$\mathbb{E}\left[\left|\langle M_i^{n,f}\rangle_t - \langle M_i^{n,f}\rangle_s\right|^p + \left|\langle \hat{M}_i^{n,g}\rangle_t - \langle \hat{M}_i^{n,g}\rangle_s\right|^p\right] \le K|t-s|^p.$$

Proof. We start with the case of $i = 2$. It follows from (3.16) and Hölder's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle M_2^{n,f}\rangle_t - \langle M_2^{n,f}\rangle_s\right|^p\right] \\
&= \mathbb{E}\Bigg|\langle \mu_s^{n,\mathbb{1}}, f^2\rangle\left(\sigma_{n,\mathbb{1}}^2 + \frac{\beta_{n,\mathbb{1}}^2}{n^2}\right)\lambda_{n,\mathbb{1}}((\lfloor s/h\rfloor + 1)h - s) \\
&\quad + \sum_{\ell=\lfloor s/h\rfloor+1}^{\lfloor t/h\rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, f^2\rangle\left(\sigma_{n,\mathbb{1}}^2 + \frac{\beta_{n,\mathbb{1}}^2}{n^2}\right)\frac{\lambda_{n,\mathbb{1}}}{n}(h \wedge (t-\ell h)) \\
&\quad + \frac{1}{n}\sum_{\ell=\lfloor s/h\rfloor+1}^{\lfloor t/h\rfloor} \langle \mu_{\ell h}^{n,\mathbb{1}}, 1\rangle\, \mathrm{O}((h\wedge(t-\ell h))^2) + \frac{1}{n}\langle \mu_s^{n,\mathbb{1}}, 1\rangle\, \mathrm{O}(((\lfloor s/h\rfloor + 1)h - s)^2)\Bigg|^p \\
&\le K\mathbb{E}\left[\int_s^t \sup_{s\le u\le r}\langle \mu_u^{n,\mathbb{1}}, 1\rangle \mathrm{d}r\right]^p \\
&\le K|t-s|^p \mathbb{E}\left[\left(\sup_{s\le r\le t}\langle \mu_r^{n,\mathbb{1}}, 1\rangle\right)^p\right] \\
&\le K|t-s|^p \left[\mathbb{E}\left[\left(\sup_{s\le r\le t}\langle \mu_r^{n,\mathbb{1}}, 1\rangle\right)^{2p}\right]\right]^{1/2} \\
&= K|t-s|^p \left[\mathbb{E}\left[\sup_{s\le r\le t}\langle \mu_r^{n,\mathbb{1}}, 1\rangle^{2p}\right]\right]^{1/2} \\
&\le K|t-s|^p,
\end{aligned}$$

where the last inequality follows from Lemma 3.1. As above, a similar estimation can be carried out for $\langle \hat{M}_2^{n,g}\rangle_t$, which implies the result for $i = 2$. For $i = 1, 3$ we can derive the results analogously. □

By the same approach as for Lemma 3.2, the following lemmas are presented without proof.

Lemma 3.3. *Under the conditions of Lemma 3.1, for any* $0 \le s \le t \le T$, $f, g \in C_b^2(\mathbb{R})$, *we have*

$$\mathbb{E}\left[\left|\langle M_3^{n,f}, \hat{M}_3^{n,g}\rangle_t - \langle M_3^{n,f}, \hat{M}_3^{n,g}\rangle_s\right|^p\right] \le K|t-s|^p.$$

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Lemma 3.4. *Under the conditions of Lemma 3.1, for any $0 \le s \le t \le T$, $f, g \in C_b^2(\mathbb{R})$, we have*

$$\mathbb{E}\big[|A^{n,f}(t) - A^{n,f}(s)|^{2p} + |\hat{A}^{n,g}(t) - \hat{A}^{n,g}(s)|^{2p}\big] \le K|t-s|^{2p}.$$

Theorem 3.1. *Assume that*

$$\sup_n \mathbb{E}\big[\langle \mu_0^{n,\mathbb{1}}, 1\rangle^{2p} + \langle \mu_0^{n,\mathbb{2}}, 1\rangle^{2p}\big] < \infty \quad \textit{for some } p \ge 1.$$

The sequence

$$\big\{\big(\mu_t^{n,\mathbb{1}}, \mu_t^{n,\mathbb{2}}\big)_{t\in[0,T]} : n \ge 1\big\}$$

is tight in $D([0, T], M_F(\mathbb{R})^2)$. Furthermore, the limit $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\ge 0}$ is a solution to the MP (1.3, 1.4) with $b_{\mathbb{1}} = \beta_{\mathbb{1}}\lambda_{\mathbb{1}}$, $b_{\mathbb{2}} = \beta_{\mathbb{2}}\lambda_{\mathbb{2}}$, $\gamma_{\mathbb{1}} = \sigma_{\mathbb{1}}^2\lambda_{\mathbb{1}}$, and $\gamma_{\mathbb{2}} = \sigma_{\mathbb{2}}^2\lambda_{\mathbb{2}}$.

Proof. Suppose that $\{h_n : n \ge 1\}$ is a sequence in $(0, +\infty)$ satisfying $\lim_{n\to+\infty} h_n = 0$. For simplicity, we use $M_i^{n,f}$ and $\hat{M}_i^{n,g}$ to denote the martingales defined by (3.4)–(3.6) and (3.11)–(3.13) with respect to $h = h_n$. In fact, by Jakubowski's criterion (see e.g. Dawson [2, Theorem 3.6.4]), the tightness of

$$\big\{\big(\mu_t^{n,\mathbb{1}}, \mu_t^{n,\mathbb{2}}\big)_{t\in[0,T]} : n \ge 1\big\}$$

in $D([0, T], M_F(\mathbb{R})^2)$ is obtained by the tightness of

$$\big\{\big(\langle\mu_t^{n,\mathbb{1}}, f\rangle, \langle\mu_t^{n,\mathbb{2}}, g\rangle\big)_{t\in[0,T]} : n \ge 1\big\}$$

in $D([0, T], \mathbb{R}^2)$ for any $f, g \in C_b^2(\mathbb{R})$.

Denote

$$M^{n,f}(t) = \sum_{i=1}^{3} M_i^{n,f}(t) \quad \text{and} \quad \hat{M}^{n,g}(t) = \sum_{i=1}^{3} \hat{M}_i^{n,g}(t).$$

Then

$$\langle M^{n,f}\rangle_t = \sum_{i=1}^{3}\langle M_i^{n,f}\rangle_t, \quad \langle \hat{M}^{n,g}\rangle_t = \sum_{i=1}^{3}\langle \hat{M}_i^{n,g}\rangle_t \quad \text{and} \quad \langle M^{n,f}, \hat{M}^{n,g}\rangle_t = \big\langle M_3^{n,f}, \hat{M}_3^{n,g}\big\rangle_t.$$

For any $0 \le s \le t \le T$ and $p \ge 1$, by Hölder's inequality and Lemmas 3.2 and 3.4, we have

$$\begin{aligned}
&\mathbb{E}\big[\big|\langle\mu_t^{n,\mathbb{1}}, f\rangle - \langle\mu_s^{n,\mathbb{1}}, f\rangle\big|^{2p}\big] \\
&\quad \le K\mathbb{E}\big[|A^{n,f}(t) - A^{n,f}(s)|^{2p}\big] + K\mathbb{E}\big[|\langle M^{n,f}\rangle_t - \langle M^{n,f}\rangle_s|^{p}\big] \\
&\quad \le K\mathbb{E}\big[|A^{n,f}(t) - A^{n,f}(s)|^{2p}\big] + K\mathbb{E}\big[|\langle {M_2}^{n,f}\rangle_t - \langle {M_2}^{n,f}\rangle_s|^{p}\big] \\
&\qquad + K\mathbb{E}\big[|\langle {M_1}^{n,f}\rangle_t - \langle {M_1}^{n,f}\rangle_s|^{p}\big] + K\mathbb{E}\big[|\langle {M_3}^{n,f}\rangle_t - \langle {M_3}^{n,f}\rangle_s|^{p}\big] \\
&\quad \le K|t-s|^p + K|t-s|^{2p}.
\end{aligned}$$

On the other hand, it follows that

$$\mathbb{E}\big[|\langle\mu_t^{n,\mathbb{2}}, g\rangle - \langle\mu_s^{n,\mathbb{2}}, g\rangle|^{2p}\big] \le K|t-s|^p + K|t-s|^{2p}.$$

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Combining with Lemma 3.3, the tightness of

$$\big(\langle\mu_t^{n,\mathbb{1}}, f\rangle\big)_{t\in[0,T]},\quad \big(\langle\mu_t^{n,\mathbb{2}}, g\rangle\big)_{t\in[0,T]},\quad (A^{n,f}(t))_{t\in[0,T]},\quad (\hat{A}^{n,g}(t))_{t\in[0,T]},$$
$$(\langle M^{n,f}\rangle_t)_{t\in[0,T]},\quad (\langle \hat{M}^{n,g}\rangle_t)_{t\in[0,T]}\quad \text{and}\quad (\langle M^{n,f}, \hat{M}^{n,g}\rangle_t)_{t\in[0,T]}$$

follows from [9, Theorem VI.4.1], which implies that $\big(\mu_t^{n,\mathbb{1}}, \mu_t^{n,\mathbb{2}}\big)_{t\in[0,T]}$ is tight. Thus there is a subsequence $\big(\mu_t^{n_k,\mathbb{1}}, \mu_t^{n_k,\mathbb{2}}\big)_{t\in[0,T]}$ converging in law as $k\to\infty$. Suppose that $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\in[0,T]}$ is the weak limit. For any $f, g\in C_b^2(\mathbb{R})$, we have

$$\begin{aligned}&\big(\langle\mu^{n_k,\mathbb{1}}, f\rangle, A^{n_k,f}, \langle M^{n_k,f}\rangle, \langle\mu^{n_k,\mathbb{2}}, g\rangle, \hat{A}^{n_k,g}, \langle\hat{M}^{n_k,g}\rangle, \langle M^{n_k,f}, \hat{M}^{n_k,g}\rangle\big)\\ &\quad\longrightarrow \big(\langle\mu^{\mathbb{1}}, f\rangle, A^f, \langle M^f\rangle, \langle\mu^{\mathbb{2}}, g\rangle, \hat{A}^g, \langle\hat{M}^g\rangle, \langle M^f, \hat{M}^g\rangle\big)\end{aligned}$$

weakly as $k\to\infty$, where

$$A^f(t) = \frac{1}{2}\int_0^t \langle\mu_s^{\mathbb{1}}, f''\rangle\,\mathrm{d}s + b_{\mathbb{1}}\int_0^t\langle\mu_s^{\mathbb{1}}, f\rangle\,\mathrm{d}s - \int_0^t\langle\mu_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})f\rangle\,\mathrm{d}s,$$
$$\hat{A}^g(t) = \frac{1}{2}\int_0^t \langle\mu_s^{\mathbb{2}}, g''\rangle\,\mathrm{d}s + b_{\mathbb{2}}\int_0^t\langle\mu_s^{\mathbb{2}}, g\rangle\,\mathrm{d}s + \int_0^t\langle\chi, g\rangle\langle\mu_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\rangle\,\mathrm{d}s$$

with $b_{\mathbb{1}} = \beta_{\mathbb{1}}\lambda_{\mathbb{1}}$, $b_{\mathbb{2}} = \beta_{\mathbb{2}}\lambda_{\mathbb{2}}$. For any $f, g\in C_b^2(\mathbb{R})$ and $t\in[0, T]$, we see that

$$\langle M^{n_k,f}, \hat{M}^{n_k,g}\rangle_t = \langle M_3^{n_k,f}, \hat{M}_3^{n_k,g}\rangle_t \to 0 \quad \text{as } k\to\infty$$

by Proposition 3.3. Moreover, $\langle M_i^{n_k,f}\rangle_t \to 0$ and $\langle\hat{M}_i^{n_k,g}\rangle_t\to 0$ with $i=1, 3$ as $k\to\infty$ by (3.15), (3.17), (3.18), and (3.20). We can pass to the limit to conclude that $M^f(t)$ and $\hat{M}^g(t)$ are martingales with quadratic variations

$$\langle M^f\rangle_t = \gamma_{\mathbb{1}}\int_0^t\langle\mu_s^{\mathbb{1}}, f^2\rangle\,\mathrm{d}s \quad \text{and}\quad \langle\hat{M}^g\rangle_t = \gamma_{\mathbb{2}}\int_0^t\langle\mu_s^{\mathbb{2}}, g^2\rangle\,\mathrm{d}s,$$

where $\gamma_{\mathbb{1}} = \sigma_{\mathbb{1}}^2\lambda_{\mathbb{1}}$ and $\gamma_{\mathbb{2}} = \sigma_{\mathbb{2}}^2\lambda_{\mathbb{2}}$. Letting $T\to\infty$, it implies that $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\geq 0}$ is a solution to the MP (1.3, 1.4). The result follows. □

4. Uniqueness of the solution to the martingale problem

In this section we first derive the SPDEs satisfied by the distribution function-valued processes of the mutually interacting superprocesses with migration, and then establish its equivalence with the MP (1.3, 1.4). Moreover, the pathwise uniqueness of the SPDEs is proved by an extended Yamada–Watanabe argument.

4.1. A related system of SPDEs

For any $y\in\mathbb{R}$, we write

$$u_t^{\mathbb{1}}(y) = \mu_t^{\mathbb{1}}((-\infty, y]) \quad\text{and}\quad u_t^{\mathbb{2}}(y) = \mu_t^{\mathbb{2}}((-\infty, y]) \tag{4.1}$$

as the distribution function-valued processes for the mutually interacting superprocesses with migration $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\geq 0}$. For any $x, y\in\mathbb{R}\cup\{\pm\infty\}$, $\nu_1, \nu_2\in M_F(\mathbb{R})$, and $\eta(\cdot,\cdot,\cdot)\in C_b^+(\mathbb{R}\times M_F(\mathbb{R})^2)$, let

$$\dot{\eta}(y, \nu_1, \nu_2) = \int_{-\infty}^y \eta(x, \nu_1, \nu_2)\nu_1(\mathrm{d}x).$$

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Let $W^{\mathbb{i}}(\mathrm{d}s\,\mathrm{d}a)$ be independent space–time white noise random measures on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\,\mathrm{d}a$ and $\mathbb{i} \in \{\mathbb{1}, \mathbb{2}\}$. We consider the following SPDEs: for any $t \geq 0$ and $y \in \mathbb{R}$,

$$\begin{cases} u_t^{\mathbb{1}}(y) = u_0^{\mathbb{1}}(y) + \sqrt{\gamma_{\mathbb{1}}} \displaystyle\int_0^t \int_0^{u_s^{\mathbb{1}}(y)} W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \left(\frac{\Delta}{2} u_s^{\mathbb{1}}(y) + b_{\mathbb{1}} u_s^{\mathbb{1}}(y) \right) \mathrm{d}s \\ \qquad - \displaystyle\int_0^t \dot{\eta}\big(y, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big)\,\mathrm{d}s, \\ u_t^{\mathbb{2}}(y) = u_0^{\mathbb{2}}(y) + \sqrt{\gamma_{\mathbb{2}}} \displaystyle\int_0^t \int_0^{u_s^{\mathbb{2}}(y)} W^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \left(\frac{\Delta}{2} u_s^{\mathbb{2}}(y) + b_{\mathbb{2}} u_s^{\mathbb{2}}(y) \right) \mathrm{d}s \\ \qquad + \dot{\chi}(y) \displaystyle\int_0^t \dot{\eta}\big(+\infty, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big)\,\mathrm{d}s, \end{cases} \tag{4.2}$$

where $\dot{\chi}$ is the distribution function of χ, i.e. $\dot{\chi}(y) = \chi((-\infty, y])$.

Definition 4.1. We say that the SPDEs (4.2) have a weak solution if there exists a $C_{b,m}(\mathbb{R})^2$-valued process $\big(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}\big)_{t\geq 0}$ on a stochastic basis such that for any $f,\ g \in C_0^2(\mathbb{R})$ and $t \geq 0$, the following holds:

$$\begin{cases} \langle u_t^{\mathbb{1}}, f\rangle_1 = \langle u_0^{\mathbb{1}}, f\rangle_1 + \sqrt{\gamma_{\mathbb{1}}} \displaystyle\int_0^t \int_0^\infty \int_{\mathbb{R}} f(y) \mathbb{I}_{\{a \leq u_s^{\mathbb{1}}(y)\}} \mathrm{d}y W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a) \\ \qquad + \displaystyle\int_0^t \left[\left\langle \frac{\Delta}{2} u_s^{\mathbb{1}}, f \right\rangle_1 + b_{\mathbb{1}} \langle u_s^{\mathbb{1}}, f\rangle_1 - \big\langle \dot{\eta}\big(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big), f\big\rangle_1 \right] \mathrm{d}s, \\ \langle u_t^{\mathbb{2}}, g\rangle_1 = \langle u_0^{\mathbb{2}}, g\rangle_1 + \sqrt{\gamma_{\mathbb{2}}} \displaystyle\int_0^t \int_0^\infty \int_{\mathbb{R}} g(y) \mathbb{I}_{\{a \leq u_s^{\mathbb{2}}(y)\}} \mathrm{d}y W^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a) \\ \qquad + \displaystyle\int_0^t \left[\left\langle \frac{\Delta}{2} u_s^{\mathbb{2}}, g \right\rangle_1 + b_{\mathbb{2}} \langle u_s^{\mathbb{2}}, g\rangle_1 + \langle \dot{\chi}, g\rangle_1 \dot{\eta}\big(+\infty, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big) \right] \mathrm{d}s. \end{cases}$$

Proposition 4.1. *Suppose that $\big(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}\big)_{t\geq 0}$ is a weak solution to the system of SPDEs (4.2). Then the corresponding measure-valued process $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\geq 0}$ is a solution to the MP (1.3, 1.4).*

Proof. For a non-decreasing continuous function h on $\mathbb{R}$, the inverse function is defined as $h^{-1}(a) = \inf\{x : h(x) > a\}$. Then, for any $f, g \in C_0^3(\mathbb{R})$, we have

$$\begin{aligned} \langle \mu_t^{\mathbb{1}}, f\rangle &= -\langle u_t^{\mathbb{1}}, f'\rangle_1 \\ &= -\langle u_0^{\mathbb{1}}, f'\rangle_1 - \sqrt{\gamma_{\mathbb{1}}} \int_0^t \int_0^\infty \int_{\mathbb{R}} f'(y) \mathbb{I}_{\{a \leq u_s^{\mathbb{1}}(y)\}} \mathrm{d}y W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a) \\ &\quad - \int_0^t \left(\left\langle \frac{\Delta}{2} u_s^{\mathbb{1}}, f' \right\rangle_1 + b_{\mathbb{1}} \langle u_s^{\mathbb{1}}, f'\rangle_1 - \big\langle \dot{\eta}\big(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big), f'\big\rangle_1 \right) \mathrm{d}s \\ &= \langle \mu_0^{\mathbb{1}}, f\rangle + \sqrt{\gamma_{\mathbb{1}}} \int_0^t \int_0^\infty f\big(\big(u_s^{\mathbb{1}}\big)^{-1}(a)\big) W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a) \\ &\quad + \int_0^t \left(\left\langle \mu_s^{\mathbb{1}}, \frac{1}{2} f'' \right\rangle + b_{\mathbb{1}} \langle \mu_s^{\mathbb{1}}, f\rangle - \big\langle \mu_s^{\mathbb{1}}, \eta\big(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big) f\big\rangle \right) \mathrm{d}s \end{aligned}$$

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and

$$\begin{aligned}
\langle \mu_t^{\mathbb{2}}, g \rangle &= -\langle u_t^{\mathbb{2}}, g' \rangle_1 \\
&= -\langle u_0^{\mathbb{2}}, g' \rangle_1 - \sqrt{\gamma_{\mathbb{2}}} \int_0^t \int_0^\infty \int_{\mathbb{R}} g'(y) \mathbb{1}_{\{a \le u_s^{\mathbb{2}}(y)\}} \mathrm{d}y W^{\mathbb{2}}(\mathrm{d}s\, \mathrm{d}a) \\
&\quad - \int_0^t \left(\left\langle \frac{\triangle}{2} u_s^{\mathbb{2}}, g' \right\rangle_1 + b_{\mathbb{2}} \langle u_s^{\mathbb{2}}, g' \rangle_1 + \langle \dot{\chi}, g' \rangle_1 \dot{\eta}\big(+\infty, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big) \right) \mathrm{d}s \\
&= \langle \mu_0^{\mathbb{2}}, g \rangle + \sqrt{\gamma_{\mathbb{2}}} \int_0^t \int_0^\infty g\big((u_s^{\mathbb{2}})^{-1}(a) \big) W^{\mathbb{2}}(\mathrm{d}s\, \mathrm{d}a) \\
&\quad + \int_0^t \left(\left\langle \mu_s^{\mathbb{2}}, \frac{1}{2} g'' \right\rangle + b_{\mathbb{2}} \langle \mu_s^{\mathbb{2}}, g \rangle + \langle \chi, g \rangle \dot{\eta}\big(+\infty, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big) \right) \mathrm{d}s.
\end{aligned}$$

Thus M_t^f and $\hat{M}_t^g$ are martingales with quadratic variation processes

$$\begin{aligned}
\langle M^f \rangle_t &= \gamma_{\mathbb{1}} \int_0^t \int_0^\infty f^2\big((u_s^{\mathbb{1}})^{-1}(a) \big)\, \mathrm{d}s\, \mathrm{d}a \\
&= \gamma_{\mathbb{1}} \int_0^t \int_{\mathbb{R}} f^2(y) \mathrm{d}s\, \mathrm{d}\big(u_s^{\mathbb{1}}(y)\big) \\
&= \gamma_{\mathbb{1}} \int_0^t \langle \mu_s^{\mathbb{1}}, f^2 \rangle\, \mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{M}^g \rangle_t &= \gamma_{\mathbb{2}} \int_0^t \int_0^\infty g^2\big((u_s^{\mathbb{2}})^{-1}(a) \big)\, \mathrm{d}s\, \mathrm{d}a \\
&= \gamma_{\mathbb{2}} \int_0^t \int_{\mathbb{R}} g^2(y) \mathrm{d}s\, \mathrm{d}\big(u_s^{\mathbb{2}}(y)\big) \\
&= \gamma_{\mathbb{2}} \int_0^t \langle \mu_s^{\mathbb{2}}, g^2 \rangle\, \mathrm{d}s.
\end{aligned}$$

The independence of $W^{\mathbb{1}}$ and $W^{\mathbb{2}}$ leads to $\langle M^f, \hat{M}^g \rangle_t = 0$. Therefore $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t \ge 0}$ is a solution to the MP (1.3, 1.4). This completes the proof. □

Proposition 4.2. *Suppose that* $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t \ge 0}$ *is a solution to the MP* (1.3, 1.4) *and* $\eta(\cdot, \nu_1, \nu_2) \in C_0^1(\mathbb{R})$ *for any* $\nu_1, \nu_2 \in M_F(\mathbb{R})$*. Then the random field* $\big(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}\big)_{t \ge 0}$ *defined by* (4.1) *is a weak solution to the SPDEs* (4.2)*.*

Proof. Let $f, g \in C_0^2(\mathbb{R})$ and set

$$\tilde{f}(y) = \int_y^\infty f(x) \mathrm{d}x, \quad \tilde{g}(y) = \int_y^\infty g(x) \mathrm{d}x.$$

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Then we have

$$\begin{aligned}\langle u_t^{\mathbb{1}}, f\rangle_1 &= \langle \mu_t^{\mathbb{1}}, \tilde{f}\rangle \\ &= \langle \mu_0^{\mathbb{1}}, \tilde{f}\rangle + \int_0^t \left\langle \mu_s^{\mathbb{1}}, \frac{1}{2}\tilde{f}''\right\rangle \mathrm{d}s + b_{\mathbb{1}} \int_0^t \langle \mu_s^{\mathbb{1}}, \tilde{f}\rangle \,\mathrm{d}s \\ &\quad - \int_0^t \langle \mu_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\tilde{f}\rangle \,\mathrm{d}s + M_t^{\tilde{f}} \\ &= \langle u_0^{\mathbb{1}}, f\rangle_1 + \int_0^t \left\langle u_s^{\mathbb{1}}, \frac{1}{2}f''\right\rangle_1 \mathrm{d}s + b_{\mathbb{1}} \int_0^t \langle u_s^{\mathbb{1}}, f\rangle_1 \mathrm{d}s \\ &\quad + \int_0^t \langle u_s^{\mathbb{1}}, (\eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\tilde{f})'\rangle_1 \mathrm{d}s + M_t^{\tilde{f}}.\end{aligned}$$

Note that

$$\begin{aligned}\langle u_s^{\mathbb{1}}, (\eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\tilde{f})'\rangle_1 &= \langle u_s^{\mathbb{1}}, \eta'(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\tilde{f} - \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})f\rangle_1 \\ &= \langle u_s^{\mathbb{1}}\eta'(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}), \tilde{f}\rangle_1 - \langle u_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})f\rangle_1 \\ &= \left\langle \int_{-\infty}^{\cdot} u_s^{\mathbb{1}}(x)\eta'(x, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\mathrm{d}x, f\right\rangle_1 - \langle u_s^{\mathbb{1}}, \eta(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})f\rangle_1 \\ &= -\left\langle \int_{-\infty}^{\cdot} \eta(x, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\mathrm{d}u_s^{\mathbb{1}}(x), f\right\rangle_1 \\ &= -\langle \dot{\eta}(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}), f\rangle_1.\end{aligned}$$

Therefore we continue to have

$$\begin{aligned}\langle u_t^{\mathbb{1}}, f\rangle_1 &= \langle u_0^{\mathbb{1}}, f\rangle_1 + \frac{1}{2}\int_0^t \langle u_s^{\mathbb{1}}, f''\rangle_1 \mathrm{d}s \\ &\quad + b_{\mathbb{1}} \int_0^t \langle u_s^{\mathbb{1}}, f\rangle_1 \mathrm{d}s - \int_0^t \langle \dot{\eta}(\cdot, \mu_s^{\mathbb{2}}, \mu_s^{\mathbb{1}}), f\rangle_1 \mathrm{d}s + M_t^{\tilde{f}}.\end{aligned} \tag{4.3}$$

Similarly, we will have

$$\begin{aligned}\langle u_t^{\mathbb{2}}, g\rangle_1 &= \langle u_0^{\mathbb{2}}, g\rangle_1 + \frac{1}{2}\int_0^t \langle u_s^{\mathbb{2}}, g''\rangle_1 \mathrm{d}s + b_{\mathbb{2}} \int_0^t \langle u_s^{\mathbb{2}}, g\rangle_1 \mathrm{d}s \\ &\quad + \langle \dot{\chi}, g\rangle_1 \int_0^t \dot{\eta}(+\infty, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}})\,\mathrm{d}s + \hat{M}_t^{\tilde{g}}.\end{aligned} \tag{4.4}$$

Let $S'(\mathbb{R})$ be the space of Schwarz distributions and define the $S'(\mathbb{R})$-valued processes N_t and $\hat{N}_t$ by $N_t(f) = M_t^{\tilde{f}}$ and $\hat{N}_t(g) = \hat{M}_t^{\tilde{g}}$ for any $f, g \in C_0^\infty(\mathbb{R})$. Then N_t and $\hat{N}_t$ are $S'(\mathbb{R})$-valued continuous square-integrable martingales with

$$\begin{aligned}\langle N(f)\rangle_t &= \langle M^{\tilde{f}}\rangle_t \\ &= \gamma_{\mathbb{1}} \int_0^t \int_{\mathbb{R}} \tilde{f}^2(y)\mu_s^{\mathbb{1}}(\mathrm{d}y)\,\mathrm{d}s \\ &= \int_0^t \int_0^\infty (\sqrt{\gamma_{\mathbb{1}}})^2 \tilde{f}^2((u_s^{\mathbb{1}})^{-1}(a))\mathrm{d}a\,\mathrm{d}s \\ &= \int_0^t \int_0^\infty \left(\sqrt{\gamma_{\mathbb{1}}}\int_{\mathbb{R}} \mathbb{1}_{\{a \le u_s^{\mathbb{1}}(y)\}} f(y)\mathrm{d}y\right)^2 \mathrm{d}a\,\mathrm{d}s\end{aligned}$$

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and

$$\begin{aligned}\langle\hat{N}(g)\rangle_t &= \langle\hat{M}^{\tilde{g}}\rangle_t \\ &= \gamma_{\mathbb{2}}\int_0^t\int_{\mathbb{R}}\tilde{g}^2(y)\mu_s^{\mathbb{2}}(\mathrm{d}y)\,\mathrm{d}s \\ &= \int_0^t\int_0^\infty\big(\sqrt{\gamma_{\mathbb{2}}}\big)^2\tilde{g}^2\big((u_s^{\mathbb{2}})^{-1}(a)\big)\mathrm{d}a\,\mathrm{d}s \\ &= \int_0^t\int_0^\infty\bigg(\sqrt{\gamma_{\mathbb{2}}}\int_{\mathbb{R}}\mathbb{I}_{\{a\le u_s^{\mathbb{2}}(y)\}}g(y)\mathrm{d}y\bigg)^2\mathrm{d}a\,\mathrm{d}s.\end{aligned}$$

Moreover, one can see that $\langle N(f), \hat{N}(g)\rangle_t = \langle M^{\tilde{f}}, \hat{M}^{\tilde{g}}\rangle_t = 0$. By Theorem III-7 and Corollary III-8 of [10], on some extension of the probability space, one can define two independent Gaussian white noises $W^{\mathbb{i}}(\mathrm{d}s\,\mathrm{d}a)$, $\mathbb{i}=\mathbb{1},\mathbb{2}$ on $\mathbb{R}_+\times\mathbb{R}$ with intensity $\mathrm{d}s\,\mathrm{d}a$ such that

$$N_t(f) = \int_0^t\int_0^\infty\int_{\mathbb{R}}\sqrt{\gamma_{\mathbb{1}}}\,\mathbb{I}_{\{a\le u_s^{\mathbb{1}}(x)\}}f(x)\mathrm{d}x W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a)$$

and

$$\hat{N}_t(g) = \int_0^t\int_0^\infty\int_{\mathbb{R}}\sqrt{\gamma_{\mathbb{2}}}\,\mathbb{I}_{\{a\le u_s^{\mathbb{2}}(x)\}}g(x)\mathrm{d}x W^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a).$$

Plugging back into (4.3) and (4.4), one can see that $\big(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}\big)_{t\ge 0}$ is a solution to (4.2). □

4.2. Uniqueness for SPDEs

This subsection is devoted to proving the pathwise uniqueness of the solution to the system of SPDEs (4.2). By Propositions 4.1 and 4.2, the uniqueness of the solution to the MP (1.3, 1.4) is then a direct consequence. We apply the approach of an extended Yamada–Watanabe argument to smooth functions. This is an adaptation to the argument of Proposition 3.1 of [19].

Before going deep into the uniqueness theorem, we introduce some notation. Let $\Phi\in C_0^\infty(\mathbb{R})^+$ such that $\mathrm{supp}(\Phi)\subseteq(-1,1)$ and the total integral is 1. Define $\Phi_m(x)=m\Phi(mx)$. Notice that $\Phi_m\to\delta_0$ weakly in the sense that for all $f\in C_b(\mathbb{R})$,

$$\lim_{m\to\infty}\int_{\mathbb{R}}\Phi_m(x)f(x)\mathrm{d}x = \int_{\mathbb{R}}\delta_0(x)f(x)\mathrm{d}x = f(0).$$

Let $\{a_k\}$ be a decreasing sequence defined recursively by $a_0=1$ and $\int_{a_k}^{a_{k-1}} z^{-1}\mathrm{d}z = k$ for $k\ge 1$. Let ψ_k be non-negative functions in $C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\psi_k)\subseteq(a_k, a_{k-1})$ and $\int_{a_k}^{a_{k-1}}\psi_k(z)\mathrm{d}z = 1$ and $\psi_k(z)\le 2(kz)^{-1}$ for all $z\in\mathbb{R}$. Let

$$\phi_k(z) = \int_0^{|z|}\mathrm{d}y\int_0^y\psi_k(x)\mathrm{d}x \quad \text{for all } z\in\mathbb{R}.$$

Then $\phi_k(z)\uparrow|z|$, $|\phi_k'(z)|\le 1$ and $|z|\phi_k''(z)\le 2k^{-1}$. Let

$$J(x) = \int_{\mathbb{R}}\mathrm{e}^{-|x|}\varrho(x-y)\mathrm{d}y,$$

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where ϱ is the mollifier given by $\varrho(x) = C\exp\{-1/(1-x^2)\}\mathbb{I}_{\{|x|<1\}}$, and C is a constant such that $\int_{\mathbb{R}} \varrho(x)\mathrm{d}x = 1$. Then, for any $m \in \mathbb{Z}_+$, there are positive constants c_m and C_m such that

$$c_m\, \mathrm{e}^{-|x|} \le |J^{(m)}(x)| \le C_m\, \mathrm{e}^{-|x|} \quad \text{for all } x \in \mathbb{R}. \tag{4.5}$$

Suppose that $\big(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}\big)_{t\ge 0}$ and $\big(\tilde{u}_t^{\mathbb{1}}, \tilde{u}_t^{\mathbb{2}}\big)_{t\ge 0}$ are two weak solutions to the system of SPDEs (4.2) with the same initial values; $\big(\mu_t^{\mathbb{1}}, \mu_t^{\mathbb{2}}\big)_{t\ge 0}$ and $\big(\tilde{\mu}_t^{\mathbb{1}}, \tilde{\mu}_t^{\mathbb{2}}\big)_{t\ge 0}$ stand for their corresponding measure-valued processes, namely $u_t^{\mathbb{i}}(y) = \mu_t^{\mathbb{i}}(-\infty, y]$ and $\tilde{u}_t^{\mathbb{i}}(y) = \tilde{\mu}_t^{\mathbb{i}}(-\infty, y]$ for $\mathbb{i} = \mathbb{1}, \mathbb{2}$. Let

$$v_t^{\mathbb{i}}(y) = u_t^{\mathbb{i}}(y) - \tilde{u}_t^{\mathbb{i}}(y) \quad \text{and} \quad \bar{G}_s^{\mathbb{i}}(a, y) = \mathbb{I}_{\{a\le u_s^{\mathbb{i}}(y)\}} - \mathbb{I}_{\{a\le \tilde{u}_s^{\mathbb{i}}(y)\}}.$$

Moreover, we denote

$$\begin{aligned} I_1^{m,k,\mathbb{i}} &= \frac{1}{2}\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\langle v_s^{\mathbb{i}}, \Delta_y\Phi_m(x-\cdot)\rangle_1 J(x)\mathrm{d}x\,\mathrm{d}s\bigg], \\ I_2^{m,k,\mathbb{i}} &= \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1 J(x)\mathrm{d}x\,\mathrm{d}s\bigg], \end{aligned} \tag{4.6}$$

$$I_3^{m,k,\mathbb{i}} = \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}^+} \phi_k''\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\bigg|\int_{\mathbb{R}} \bar{G}_s^{\mathbb{i}}(a, y)\Phi_m(x-y)\mathrm{d}y\bigg|^2 \mathrm{d}aJ(x)\mathrm{d}x\,\mathrm{d}s\bigg].$$

Proposition 4.3. *For* $\mathbb{i} = \mathbb{1}, \mathbb{2}$ *we have*

$$\mathbb{E}\bigg[\int_{\mathbb{R}} \phi_k\big(\langle v_t^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)J(x)\mathrm{d}x\bigg] = I_1^{m,k,\mathbb{i}} + b_{\mathbb{i}} I_2^{m,k,\mathbb{i}} + \frac{\gamma_{\mathbb{i}}}{2} I_3^{m,k,\mathbb{i}} + I_4^{m,k,\mathbb{i}},$$

where $I_1^{m,k,\mathbb{i}}$, $I_2^{m,k,\mathbb{i}}$, *and* $I_3^{m,k,\mathbb{i}}$ *are given by* (4.6),

$$I_4^{m,k,\mathbb{1}} = -\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\Phi_m(x-y)\bar{\xi}_s(y)\mathrm{d}yJ(x)\mathrm{d}x\,\mathrm{d}s\bigg] \tag{4.7}$$

and

$$I_4^{m,k,\mathbb{2}} = \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\langle \dot{\chi}, \Phi_m(x-\cdot)\rangle_1 \bar{\xi}_s(\infty)J(x)\mathrm{d}x\,\mathrm{d}s\bigg], \tag{4.8}$$

with $\bar{\xi}_s(\cdot) = \dot{\eta}\big(\cdot, \mu_s^{\mathbb{1}}, \mu_s^{\mathbb{2}}\big) - \dot{\eta}\big(\cdot, \tilde{\mu}_s^{\mathbb{1}}, \tilde{\mu}_s^{\mathbb{2}}\big)$.

Proof. It follows from (4.2) that

$$v_t^{\mathbb{1}}(y) = \sqrt{\gamma_{\mathbb{1}}}\int_0^t\int_0^\infty \bar{G}_s^{\mathbb{1}}(a, y)W^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \bigg(\frac{\Delta_y}{2}v_s^{\mathbb{1}}(y) + b_{\mathbb{1}}v_s^{\mathbb{1}}(y) - \bar{\xi}_s(y)\bigg)\,\mathrm{d}s$$

and

$$v_t^{\mathbb{2}}(y) = \sqrt{\gamma_{\mathbb{2}}}\int_0^t\int_0^\infty \bar{G}_s^{\mathbb{2}}(a, y)W^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \bigg(\frac{\Delta_y}{2}v_s^{\mathbb{2}}(y) + b_{\mathbb{2}}v_s^{\mathbb{2}}(y) + \dot{\chi}(y)\bar{\xi}_s(\infty)\bigg)\,\mathrm{d}s.$$

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Consequently we have

$$\begin{aligned}\langle v_t^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1 &= \sqrt{\gamma_{\mathbb{1}}}\int_0^t\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar{G}_s^{\mathbb{1}}(a,y)\Phi_m(x-y)\mathrm{d}yW^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a)\\ &\quad + b_{\mathbb{1}}\int_0^t \langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1 \mathrm{d}s - \int_0^t\int_{\mathbb{R}} \Phi_m(x-y)\bar{\xi}_s(y)\mathrm{d}y\,\mathrm{d}s\\ &\quad + \frac{1}{2}\int_0^t \langle v_s^{\mathbb{1}}, \Delta_y\Phi_m(x-\cdot)\rangle_1 \mathrm{d}s \end{aligned}\tag{4.9}$$

and

$$\begin{aligned}\langle v_t^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1 &= \sqrt{\gamma_{\mathbb{2}}}\int_0^t\int_{\mathbb{R}^+}\int_{\mathbb{R}} \bar{G}_s^{\mathbb{2}}(a,y)\Phi_m(x-y)\mathrm{d}yW^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a)\\ &\quad + \frac{1}{2}\int_0^t \langle v_s^{\mathbb{2}}, \Delta_y\Phi_m(x-\cdot)\rangle_1 \mathrm{d}s + b_{\mathbb{2}}\int_0^t \langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1 \mathrm{d}s\\ &\quad + \int_0^t \langle \dot{\chi}, \Phi_m(x-\cdot)\rangle_1 \bar{\xi}_s(\infty)\,\mathrm{d}s. \end{aligned}\tag{4.10}$$

Applying Itô's formula to (4.9) and (4.10), we can easily get

$$\begin{aligned}&\phi_k\big(\langle v_t^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\\ &= \sqrt{\gamma_{\mathbb{1}}}\int_0^t\int_{\mathbb{R}^+} \phi_k'\big(\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\int_{\mathbb{R}} \bar{G}_s^{\mathbb{1}}(a,y)\Phi_m(x-y)\mathrm{d}yW^{\mathbb{1}}(\mathrm{d}s\,\mathrm{d}a)\\ &\quad + \int_0^t \phi_k'\big(\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\Big[\frac{1}{2}\langle v_s^{\mathbb{1}}, \Delta_y\Phi_m(x-\cdot)\rangle_1 + b_{\mathbb{1}}\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\Big]\mathrm{d}s\\ &\quad + \frac{\gamma_{\mathbb{1}}}{2}\int_0^t\int_{\mathbb{R}^+} \phi_k''\big(\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\bigg|\int_{\mathbb{R}} \bar{G}_s^{\mathbb{1}}(a,y)\Phi_m(x-y)\mathrm{d}y\bigg|^2 \mathrm{d}a\,\mathrm{d}s\\ &\quad - \int_0^t\int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{1}}, \Phi_m(x-\cdot)\rangle_1\big)\Phi_m(x-y)\bar{\xi}_s(y)\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\phi_k\big(\langle v_t^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\\ &= \sqrt{\gamma_{\mathbb{2}}}\int_0^t\int_{\mathbb{R}^+} \phi_k'\big(\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\int_{\mathbb{R}} \bar{G}_s^{\mathbb{2}}(a,y)\Phi_m(x-y)\mathrm{d}yW^{\mathbb{2}}(\mathrm{d}s\,\mathrm{d}a)\\ &\quad + \int_0^t \phi_k'\big(\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\Big[\frac{1}{2}\langle v_s^{\mathbb{2}}, \Delta_y\Phi_m(x-\cdot)\rangle_1 + b_{\mathbb{2}}\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\Big]\mathrm{d}s\\ &\quad + \frac{\gamma_{\mathbb{2}}}{2}\int_0^t\int_{\mathbb{R}^+} \phi_k''\big(\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\bigg|\int_{\mathbb{R}} \bar{G}_s^{\mathbb{2}}(a,y)\Phi_m(x-y)\mathrm{d}y\bigg|^2 \mathrm{d}a\,\mathrm{d}s\\ &\quad + \int_0^t \phi_k'\big(\langle v_s^{\mathbb{2}}, \Phi_m(x-\cdot)\rangle_1\big)\langle \dot{\chi}, \Phi_m(x-\cdot)\rangle_1 \bar{\xi}_s(\infty)\,\mathrm{d}s.\end{aligned}$$

Taking the expectations of $\big\langle \phi_k\big(\langle v_t^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big), J(x)\big\rangle_1$ with $\mathbb{i}=\mathbb{1},\mathbb{2}$, we obtain the desired results. □

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Lemma 4.1. *For* $\mathbb{i} = \mathbb{1}, \mathbb{2}$ *we have*

$$2I_1^{m,k,\mathbb{i}} \le \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \langle |v_s^{\mathbb{i}}|, \Phi_m(x-\cdot)\rangle_1 |J''(x)| \mathrm{d}x\, \mathrm{d}s\Big].$$

Proof. Note that

$$2I_1^{m,k,\mathbb{i}} = \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\langle v_s^{\mathbb{i}}, \Delta_y \Phi_m(x-\cdot)\rangle_1 J(x)\mathrm{d}x\, \mathrm{d}s\Big].$$

Since $\Delta_y \Phi_m(x-y) = \Delta_x \Phi_m(x-y)$, we have

$$\begin{aligned}
2I_1^{m,k,\mathbb{i}} &= \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\Delta_x\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1 J(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&= -\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \Big(\frac{\partial}{\partial x}\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\Big)^2 \phi_k''\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) J(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&\quad - \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1 \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) J'(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&\le -\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1 \phi_k'\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) J'(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&= -\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}\big(\phi_k\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\big) J'(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&= \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \phi_k\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) J''(x)\mathrm{d}x\, \mathrm{d}s\Big],
\end{aligned}$$

where the inequality follows from the fact that $\phi_k''(z) = \psi_k(|z|) \ge 0$. Use $\phi_k(z) \le |z|$ to get

$$\phi_k\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) \le \big|\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big| \le \langle |v_s^{\mathbb{i}}|, \Phi_m(x-\cdot)\rangle_1.$$

This implies the result. □

Lemma 4.2. *For* $\mathbb{i} = \mathbb{1}, \mathbb{2}$ *we have*

$$I_3^{m,k,\mathbb{i}} \le 4\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \phi_k''\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big)\langle |v_s^{\mathbb{i}}|, \Phi_m(x-\cdot)\rangle_1 J(x)\mathrm{d}x\, \mathrm{d}s\Big].$$

Proof. It is easy to see that

$$\begin{aligned}
I_3^{m,k,\mathbb{i}} &\le \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^+} \phi_k''\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) \int_{\mathbb{R}} \big(\bar{G}_s^{\mathbb{i}}(a, y)\big)^2 \Phi_m(x-y)\mathrm{d}y\, \mathrm{d}a J(x)\mathrm{d}x\, \mathrm{d}s\Big] \\
&\le 4\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \phi_k''\big(\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1\big) \int_{\mathbb{R}} |v_s^{\mathbb{i}}(y)|\Phi_m(x-y)\mathrm{d}y J(x)\mathrm{d}x\, \mathrm{d}s\Big].
\end{aligned}$$

The result follows. □

Theorem 4.1. *Assume that there exists a constant* K *such that*

$$|\xi(x, v_1, v_2) - \xi(x, \tilde{v}_1, \tilde{v}_2)| \le K[\rho(v_1, \tilde{v}_1) + \rho(v_2, \tilde{v}_2)]$$

for any $x \in \mathbb{R} \cup \{\pm\infty\}$ and $\nu_i, \tilde{\nu}_i \in M_F(\mathbb{R})$ with $i = 1, 2$. Then pathwise uniqueness holds for the SPDEs (4.2), namely, if (4.2) has two weak solutions defined on the same stochastic basis with the same initial values, then the solutions coincide almost surely.

Proof. Suppose $(u_t^{\mathbb{1}}, u_t^{\mathbb{2}})_{t\geq 0}$ and $(\tilde{u}_t^{\mathbb{1}}, \tilde{u}_t^{\mathbb{2}})_{t\geq 0}$ are two weak solutions on the same stochastic basis with the same initial values. It is sufficient to show that $(u_t^{\mathbb{1}}, u_t^{\mathbb{2}}) = (\tilde{u}_t^{\mathbb{1}}, \tilde{u}_t^{\mathbb{2}})$ for all $t \geq 0$ almost surely. Recall that $v_t^{\mathbb{i}}(y) = u_t^{\mathbb{i}}(y) - \tilde{u}_t^{\mathbb{i}}(y)$. We subsequently estimate the values of $I_\ell^{m,k,\mathbb{i}}$ with $\ell = 1, 2, 3, 4$ and $\mathbb{i} = \mathbb{1}, \mathbb{2}$. Since

$$\lim_{m\to\infty}\langle v_s^{\mathbb{i}}, \Phi_m(x-\cdot)\rangle_1 = v_s^{\mathbb{i}}(x) \quad \text{and} \quad \lim_{m\to\infty}\langle |v_s^{\mathbb{i}}|, \Phi_m(x-\cdot)\rangle_1 = |v_s^{\mathbb{i}}(x)|$$

for Lebesgue-a.e. x and any $s \geq 0$ almost surely, by Lemma 4.1 and the dominated convergence theorem, we have

$$\limsup_{k,m\to\infty} 2I_1^{m,k,\mathbb{i}} \leq \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} |v_s^{\mathbb{i}}(x)| \cdot |J''(x)| \mathrm{d}x\, \mathrm{d}s\bigg].$$

By (4.5), there exists a constant K such that

$$\limsup_{k,m\to\infty} 2I_1^{m,k,\mathbb{i}} \leq K\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} |v_s^{\mathbb{i}}(x)| \cdot |J(x)| \mathrm{d}x\, \mathrm{d}s\bigg]. \tag{4.11}$$

Using $|\phi_k'(z)| \leq 1$ and the dominated convergence theorem, we can easily get

$$\limsup_{k,m\to\infty} |I_2^{m,k,\mathbb{i}}| \leq \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} |v_s^{\mathbb{i}}(x)| \cdot |J(x)| \mathrm{d}x\, \mathrm{d}s\bigg]. \tag{4.12}$$

Recall that $\phi_k''(z)|z| \leq 2k^{-1}$. By Lemma 4.2 one can prove that

$$\limsup_{m\to\infty} I_3^{m,k,\mathbb{i}} \leq K\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \phi_k''\big(v_s^{\mathbb{i}}(x)\big)|v_s^{\mathbb{i}}(x)| J(x) \mathrm{d}x\, \mathrm{d}s\bigg] = \mathrm{O}(k^{-1}). \tag{4.13}$$

Recall that χ is a finite measure on $\mathbb{R}$ and $|\phi_k'(z)| \leq 1$. By (4.7), (4.8) we have

$$\begin{aligned}\limsup_{k,m\to\infty} |I_4^{m,k,\mathbb{i}}| &\leq \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \big[|\bar{\xi}_s(x)| + |\bar{\xi}_s(\infty)|\big] \cdot |J(x)| \mathrm{d}x\, \mathrm{d}s\bigg] \\ &\leq K \int_0^t \mathbb{E}\big[\rho\big(\mu_s^{\mathbb{1}}, \tilde{\mu}_s^{\mathbb{1}}\big) + \rho\big(\mu_s^{\mathbb{2}}, \tilde{\mu}_s^{\mathbb{2}}\big)\big]\, \mathrm{d}s \\ &\leq K\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \big(|v_s^{\mathbb{1}}(x)| + |v_s^{\mathbb{2}}(x)|\big) J(x) \mathrm{d}x\, \mathrm{d}s\bigg]. \end{aligned} \tag{4.14}$$

By Proposition 4.3 and putting (4.11)–(4.14) together, one can see that

$$\mathbb{E}\bigg[\int_{\mathbb{R}} \big(|v_t^{\mathbb{1}}(x)| + |v_t^{\mathbb{2}}(x)|\big) J(x) \mathrm{d}x\bigg] \leq K\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \big(|v_s^{\mathbb{1}}(x)| + |v_s^{\mathbb{2}}(x)|\big) J(x) \mathrm{d}x\, \mathrm{d}s\bigg].$$

Then Gronwall's inequality implies that

$$\mathbb{E}\bigg[\int_{\mathbb{R}} \big(|v_t^{\mathbb{1}}(x)| + |v_t^{\mathbb{2}}(x)|\big) J(x) \mathrm{d}x\bigg] = 0$$

for any $t \geq 0$. Therefore $|v_t^{\mathbb{1}}(x)| = |v_t^{\mathbb{2}}(x)| = 0$ for any $t \geq 0$ and $x \in \mathbb{R}$ almost surely. Pathwise uniqueness follows. □

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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