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APPROXIMATE WEAK AMENABILITY OF $I_0(\mathrm{SL}(2, \mathbb{R}))$

POURUS BHARUCHA

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Abstract

We demonstrate that $I_0(\mathrm{SL}(2, \mathbb{R}))$ fails to be approximately weakly amenable.

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NOTATION. Given a Banach algebra A and a Banach A-bimodule X, a *derivation* $D : A \to X$ is a continuous linear map which satisfies

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A).$$

One defines the derivation generated by $x \in X$ by

$$\mathrm{ad}_x(a) = a \cdot x - x \cdot a \quad (a \in A),$$

and any derivation of this form is said to be *inner*. Denoting the collection of derivations from A to X by $Z^1(A, X)$, and the collection of inner derivations from A to X by $N^1(A, X)$, it is standard to write $H^1(A, X) = Z^1(A, X)/N^1(A, X)$, the *first continuous cohomology group* of A with coefficients in X. Given a Banach algebra A, the Banach A-bimodule which we will use most often is A^* under the natural actions [1, Example 2.6.2 v].

We are now in a position to recall the definitions of weak amenability and approximate weak amenability for a Banach algebra, and refer the reader to [5] and [3] for other standard definitions and results.

DEFINITION 1. A Banach algebra A is *weakly amenable* if each derivation $D : A \to A^*$ is inner. A Banach algebra A is *approximately weakly amenable* if every derivation $D : A \to A^*$ is *approximately inner*, that is, there is a net $(a^*_\alpha) \subset A^*$ such that

$$D(a) = \lim_\alpha (a \cdot a^*_\alpha - a^*_\alpha \cdot a) \quad (a \in A).$$

It is standard that for any locally compact group G, the group algebra $L^1(G)$ is weakly amenable [5, Theorem 4.2.3] and, in the case where A is commutative, A is weakly amenable if and only if its unitization $A^\sharp$ is weakly amenable [1, Corollary 2.8.70]. For all Banach algebras, if A is weakly amenable, then $A^\sharp$ is weakly amenable [2, Proposition 1.4], and the converse to this was an open problem for many years [2, p. 25]. Finally, Johnson and White, in the now well-known preprint [4], showed that there exists a Banach algebra A which fails to be weakly amenable, but such that its unitization is weakly amenable. For their example, we recall the definition of an augmentation ideal.

Definition 2. For a locally compact group G with left Haar measure h, the *augmentation ideal* of $L^1(G)$ is

$$I_0(G) = \left\{ f \in L^1(G) : \int_G f\, dh = 0 \right\}.$$

Corollary 5.6 of [4] states that the augmentation ideal $I_0(\mathrm{SL}(2, \mathbb{R}))$ fails to be weakly amenable whilst its unitization satisfies weak amenability. The purpose of this note is to demonstrate that $I_0(\mathrm{SL}(2, \mathbb{R}))$ in fact fails to be even approximately weakly amenable, thereby establishing that the negation of approximate weak amenability is not preserved by taking unitizations.

The results in [4] are established using cohomological arguments, which do not reveal any information about the specific form of the noninner derivations. However [4, Theorem 5.2] does prove that there is effectively only one noninner derivation into the dual module, that is,

$$\frac{Z^1(I_0(\mathrm{SL}(2, \mathbb{R})), I_0(\mathrm{SL}(2, \mathbb{R}))^*)}{N^1(I_0(\mathrm{SL}(2, \mathbb{R})), I_0(\mathrm{SL}(2, \mathbb{R}))^*)}$$

is one-dimensional. It turns out that this is enough information for our purposes, and in particular the group structure of $\mathrm{SL}(2, \mathbb{R})$ turns out to be superfluous to our needs. We need a technical lemma; it states that the space of inner derivations cannot be complemented in the space of approximately inner derivations. On the surface this makes intuitive sense because one would expect that the inner derivations approximating an approximately inner derivation would always somehow 'contribute' to the projection onto the inner derivations.

Lemma 3. *Let A be a Banach algebra, X a Banach A-bimodule. Suppose that there exists an approximately inner, noninner derivation in $Z^1(A, X)$, and further that the space $N^1(A, X)$ of inner derivations is norm-closed in $Z^1(A, X)$. Then $N^1(A, X)$ is not complemented in $Z^1(A, X)$.*

Proof. Suppose that $N^1(A, X)$ is complemented in $Z^1(A, X)$, that is, there exists a continuous projection P of $Z^1(A, X)$ onto $N^1(A, X)$. If we could show, for each nonzero approximately inner derivation $D : A \to X$, that the derivation $P(D)$ is nonzero, this would suffice, because subsequently, for every approximately

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inner noninner derivation D, $P(D) \neq 0$, and so we would have $D - P(D)$ being approximately inner and noninner, but $P(D - P(D)) \neq 0$, giving a contradiction.

So let $D : A \to X$ be an approximately inner noninner derivation. Then there exists $(x_\alpha) \subset X$ such that for $a \in A$,

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) = \lim_\alpha \mathrm{ad}_{x_\alpha}(a).$$

To obtain the desired contradiction, assume that $P(D) = 0$. Write $d_\alpha = D - \mathrm{ad}_{x_\alpha}$; d_α is approximately inner. We have that $P(d_\alpha) = -P(\mathrm{ad}_{x_\alpha}) = -\mathrm{ad}_{x_\alpha}$.

Since $D \neq 0$ there must exist $a \in A$ with $D(a) \neq 0$. By the definition of approximately inner, for $\delta > 0$ we must be able to find an α such that

$$\|d_\alpha(a)\| < \delta.$$

Let us specifically choose $\delta = \|D(a)\|/(3\|P\|)$. Then via the projections,

$$\|\mathrm{ad}_{x_\alpha}(a)\| = \|P(d_\alpha)(a)\| < \|D(a)\|/3.$$

But also via the triangle inequality,

$$\|\mathrm{ad}_{x_\alpha}(a)\| = \|D(a) - d_\alpha(a)\| \geq \|D(a)\| - \|D(a)\|/3 = 2\|D(a)\|/3$$

since $\|P\| \geq 1$; this is the desired contradiction. □

It is worth mentioning that one can establish a stronger result which is of independent interest (below) but for our purposes the above proof, while being technical, is illustrative in terms of the derivations involved.

PROPOSITION 4. *Let Z be a norm-closed complemented subspace of a strong operator (so)-closed subspace W of $B(X, Y)$. Then Z is itself so-closed. So if Z is proper and so-dense, it cannot be complemented.*

PROOF. Let $P : W \to Z$ be a continuous projection. Take $(T_\alpha) \subset Z$, $T_\alpha \overset{\mathrm{so}}{\to} T$. So for any $x \in X$, $T_\alpha x \to Tx$. Then by continuity of P, we have $PT_\alpha x \to PTx$. Since P is a projection onto Z, we have $PT_\alpha = T_\alpha$. It follows that $PTx = Tx$. Thus $PT = T$, so that $T \in Z$. □

We now make use of the structure we have determined for the derivations.

COROLLARY 5. *Let A be a Banach algebra, X a Banach A-bimodule. Suppose that $H^1(A, X)$ has finite dimension. Then every derivation $D : A \mapsto X$ which fails to be inner also fails to be approximately inner.*

PROOF. By hypothesis, $N^1(A, X)$ has finite codimension in $Z^1(A, X)$. We note that $N^1(A, X)$ is the range of the continuous linear map from X into $Z^1(A, X)$ given by $x \mapsto \mathrm{ad}_x$ and so $N^1(A, X)$ must be closed by [6, Lemma 3.3]. Thus $N^1(A, X)$ is a closed complemented subspace of $Z^1(A, X)$ and so, by Lemma 3, any derivation $D : A \to X$ which is not inner also fails to be approximately inner. □

This is exactly what we need for the example of Johnson and White.

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COROLLARY 6. $I_0(\mathrm{SL}(2, \mathbb{R}))$ *is not approximately weakly amenable.*

PROOF. By [4, Theorem 5.2], writing $A = I_0(\mathrm{SL}(2, \mathbb{R}))$, we have that $H^1(A, A^*)$ is one-dimensional. One just applies Corollary 5, with $X = A^*$. □

REMARK 7. Lemma 3 tells us that in an approximately amenable nonamenable Banach algebra, such as $c_0(A_n)$ where the A_n are each unital and amenable with amenability constants increasing to infinity [3, Example 6.1], there are indeed infinitely many genuinely different derivations which are not inner, but may be approximated by inner ones in the strong topology. Less formally, when there are approximately inner noninner derivations into a bimodule, there are a lot of them.

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POURUS BHARUCHA, Mathematical Sciences Institute,
Australian National University, Canberra, ACT 0200, Australia
e-mail: pourus@gmail.com