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Global existence for the magnetohydrodynamic system in critical spaces

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In this article, we show that the magnetohydrodynamic system in $\mathbb{R}^N$ with variable density, variable viscosity and variable conductivity has a local weak solution in the Besov space $\dot{B}^{N/p_1}_{p_1,1}(\mathbb{R}^N) \times \dot{B}^{(N/p_2)-1}_{p_2,1}(\mathbb{R}^N) \times \dot{B}^{(N/p_2)-1}_{p_2,1}(\mathbb{R}^N)$ for all $1 < p_2 < +\infty$ and some $1 < p_1 \leqslant 2N/3$ if the initial density approaches a positive constant. Moreover, this solution is unique if we impose the restrictive condition $1 < p_2 \leqslant 2N$. We also prove that the constructed solution exists globally in time if the initial data are small. In particular, this allows us to work in the framework of Besov space with negative regularity indices and this fact is particularly important when the initial data are strongly oscillating.

1. Introduction

In this paper we study the existence and uniqueness of solutions for the magnetohydrodynamic (MHD) system with variable viscosity and variable density, which describes the coupling between the inhomogeneous Navier–Stokes system and the Maxwell equation:

$$\left.\begin{aligned} \partial_t\rho + \operatorname{div}(\rho u) &= 0,\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - 2\operatorname{div}(\mu(\rho)\mathcal{M}) + \nabla\left(\Pi + \frac{B^2}{2}\right) &= \rho f + \operatorname{div}(B\otimes B),\\ \partial_t B - \operatorname{div}\left(\frac{\nabla B}{\sigma(\rho)}\right) &= B\cdot\nabla u - u\cdot\nabla B,\\ \operatorname{div} u = \operatorname{div} B &= 0,\\ (\rho,u,B)|_{t=0} &= (\rho_0,u_0,B_0), \end{aligned}\right\} \tag{MHD}$$

where $\mathcal{M} = \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the symmetrical part of the gradient, the external force f is given, $\mu(\cdot) > 0$ is the viscosity of the fluid, $\sigma(\cdot) > 0$ is the conductivity and $\Pi(t,x)$ is the pressure in the fluid. Moreover, we suppose that σ and μ are C^∞ functions and that

$$0 < \underline{\sigma} \leqslant \frac{1}{\sigma} \leqslant \bar{\sigma} < \infty \quad\text{and}\quad 0 < \underline{\mu} \leqslant \mu. \tag{1.1}$$

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The homogeneous case ($\rho = \text{const.}$) of the system (MHD) was studied by Duvaut and Lions [12]. They established the local existence and uniqueness of a solution in the classical Sobolev spaces $H^s(\mathbb{R}^N)$, $s \geqslant N$. They also proved the global existence of the solution for small initial data.

The inhomogeneous case has been studied by many authors: in particular we mention Gerbeau and Le Bris [15] and Desjardins and Le Bris [11], who studied the global existence of weak solutions of finite energy in $\mathbb{R}^3$ and in the torus $\mathcal{T}^3$. On the other hand, the local existence of strong solutions was recently considered by Abidi and Hmidi [2]. They also proved the global existence of strong solutions when the initial data are small in some Sobolev spaces.

The principal aim of this paper is to study the strong solutions in some Sobolev–Besov critical spaces of negative regularity index. Working with initial data in Besov spaces of negative regularity allows us to choose the initial velocity and the initial magnetic field to be very irregular (even discontinuous) functions. On the other hand, working in spaces of negative regularity allows us to prove that the system (MHD) is globally well posed for strongly oscillating initial data.

In the following, we suppose that the initial density verifies $\inf_x \rho_0(x) > 0$ and, thus, by the maximum principle for the transport equation, we have $\inf_x \rho(t,x) > 0$. We also suppose that the density of the fluid is a small perturbation of a constant density which we choose to be equal to 1. This implies that we can use the transform $a = (1/\rho) - 1$, which allows us to work with the following system:

$$\left.\begin{aligned} \partial_t a + u\cdot\nabla a = 0,\qquad\qquad\qquad\qquad\qquad \\ \partial_t u + u\cdot\nabla u + (1+a)\left\{\nabla \Pi + \nabla\left(\frac{B^2}{2}\right) - 2\operatorname{div}(\tilde{\mu}(a)\mathcal{M})\right\} \\ = f + (1+a)B\cdot\nabla B, \\ \partial_t B - \operatorname{div}(\tilde{\sigma}(a)\nabla B) = B\cdot\nabla u - u\cdot\nabla B, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (a,u,B)|_{t=0} = (a_0,u_0,B_0), \end{aligned}\right\}\qquad \widetilde{\text{MHD}}$$

where

$$\tilde{\mu}(a) = \mu\left(\frac{1}{1+a}\right) \quad\text{and}\quad \tilde{\sigma}(a) = \frac{1}{\sigma(1/(1+a))}$$

are regular functions.

Let us recall the theorem proved in [2]. We denote by $\mathcal{P}$ the Leray projector on the divergence-free vector fields and by $\mathcal{Q} = I - \mathcal{P}$ the projector on the gradient-type vector fields. The Besov spaces are defined in the next section.

Theorem 1.1 (Abidi and Hmidi [2]). *Let $1 < p < 6$. There exists a constant c depending on p and on the functions μ and σ such that, for $u_0, B_0 \in \dot{B}_{p1}^{(3/p)-1}(\mathbb{R}^3)$ with*

$$\operatorname{div} u_0 = \operatorname{div} B_0 = 0, \quad f \in L^1(\mathbb{R}_+; \dot{B}_{p1}^{(3/p)-1}(\mathbb{R}^3)),$$

with $\mathcal{Q}f$ belonging to $L^2_{\text{loc}}(\mathbb{R}_+; \dot{B}_{p1}^{(3/p)-2}(\mathbb{R}^3))$ and $a_0 \in \dot{B}_{p1}^{3/p}(\mathbb{R}^3)$, where

$$\|a_0\|_{\dot{B}_{p1}^{3/p}} \leqslant c.$$

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there exists a $T \in (0, +\infty]$ *such that the system* $(\widetilde{\mathrm{MHD}})$ *has a solution* $(a, u, B, \nabla \Pi)$*, where*

$$a \in C_b([0,T); \dot{B}^{3/p}_{p1}) \cap \tilde{L}^{\infty}([0,T); \dot{B}^{3/p}_{p1}),$$
$$u, B \in C_b([0,T); \dot{B}^{(3/p)-1}_{p1}) \cap L^1(0,T; \dot{B}^{(3/p)+1}_{p1}).$$

Moreover, there is a sufficiently small constant $c_1 > 0$ *such that, if*

$$\|u_0\|_{\dot{B}^{(3/p)-1}_{p1}} + \|B_0\|_{\dot{B}^{(3/p)-1}_{p1}} + \|f\|_{L^1(\mathbb{R}_+; \dot{B}^{(3/p)-1}_{p1})} \leqslant c_1 \inf(\mu^1, \sigma^1),$$

with $\mu^1 = \mu(1)$ *and* $\sigma^1 = \tilde{\sigma}(1)$*, then* $T = +\infty$*. If* $1 < p \leqslant 3$*, then this solution is unique.*

This result can easily be generalized to the case of fluid evolving in the whole space $\mathbb{R}^N$. However, the result does not provide uniqueness for $N < p \leqslant 2N$, which would allow one to conclude that the system $(\widetilde{\mathrm{MHD}})$ is globally well posed for strongly oscillating initial data. Addressing the issue of uniqueness is the principal motivation of our work.

In order to have a clearer idea of uniqueness, let us note that the system $(\widetilde{\mathrm{MHD}})$ can be written as a coupled system of a transport equation for the density and a Navier–Stokes-type equation for the couple (u, B). Let us note also that the stabilizing effect of strongly oscillating initial data is well known for the classical homogeneous Navier–Stokes equation. Indeed, for the Navier–Stokes system in the homogeneous case ($\rho, B = \text{const.}, \mu > 0$), i.e.

$$\left.\begin{aligned} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi &= 0, \\ \operatorname{div} u &= 0, \\ u|_{t=0} &= u_0, \end{aligned}\right\} \qquad (\mathrm{NS}_\mu)$$

the classical approach is to obtain global existence and uniqueness of solutions for small initial data in the Besov space $\dot{B}^{-1+(N/p)}_{p,1}(\mathbb{R}^N)$ for all $1 < p < \infty$ (see [4]). The Cannone–Meyer–Planchon result generalizes the classical theorem by Fujita and Kato [14], which gives the existence and uniqueness of solutions in the framework of classical Sobolev spaces $\dot{H}^{(N-1)/2}(\mathbb{R}^N)$, to Besov spaces of negative regularity index. The interest in such a result comes from the fact that initial data which are large in $\dot{H}^{(N/2)-1}(\mathbb{R}^N)$ become small in the presence of oscillations in the norm of the space $\dot{B}^{-1+(N/p)}_{p,1}$ when $N < p < +\infty$. In particular, we find that the very fast oscillations of the initial data stabilize the Navier–Stokes system in the sense that the solution exists globally in time.

THEOREM 1.2 (Cannone *et al.* [4]). *Let* $1 < p < +\infty$ *and let* $u_0 \in \dot{B}^{(N/p)-1}_{p,1}(\mathbb{R}^N)$ *be a divergence-free vector field. There then exists a time* $T > 0$ *such that system* (NS_μ) *has a unique solution,*

$$u \in C_b([0,T); \dot{B}^{(N/p)-1}_{p,1}) \cap L^1(0,T; \dot{B}^{(N/p)+1}_{p,1}).$$

Moreover, there is a constant $c > 0$ *small enough such that if*

$$\|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}} \leqslant c\mu,$$

then $T = \infty$*.*

In this paper we will show the existence and uniqueness of global solution for system $(\widetilde{\mathrm{MHD}})$ for strongly oscillating initial data. For this it will be necessary to work in spaces with negative index of regularity. Let us note that the result of [2] does not make it possible to construct a unique global solution for the data in spaces of negative index, since we have uniqueness of the solution only in the case when $1 < p \leqslant N$. Also let us note that we have existence of a global weak solution when $N < p < 2N$ for small data. We prove, in fact, that the $(\widetilde{\mathrm{MHD}})$ system is globally well posed for oscillating initial data, when $(1/\rho_0) - 1 \in \dot{B}^{N/p_1}_{p_1,1}$ and $u_0, B_0 \in \dot{B}^{(N/p_2)-1}_{p_2,1}$ with $p_1 \leqslant p_2$ and

$$\frac{1}{p_1} \leqslant \frac{1}{p_2} + \frac{1}{N} \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} \geqslant \frac{2}{N}.$$

Note in particular that we obtain the Abidi and Hmidi results as a particular case of our theorem by taking $p_1 = p_2$. The improvement obtained in our results is due directly to the fact that we work with the density, velocity field and magnetic field belonging to Besov spaces with different index of integrability. The method of the proof is based on the regularizing effect for the heat equation (for more precise details, see [6]). To be more precise, we point out a result of harmonic analysis due to Danchin [9], which is a Poincaré-type inequality for functions localized in frequencies. This enables us to gain two derivatives on the solution from the heat equation starting from the Laplacian. Thus, for initial data in $\dot{B}^{-1+(N/p_2)}_{p_2,1}(\mathbb{R}^N)$, we find that the solution belongs to the space $L^1([0,T]; \dot{B}^{1+(N/p_2)}_{p_2,1})$, which is a subspace of $L^1(\mathrm{Lip}(\mathbb{R}^N))$. This is the principal reason why one cannot work with the initial data $u_0 \in \dot{B}^{-1+(N/p_2)}_{p_2,r}$ for $r > 1$.

We prove an existence result in critical Besov spaces (for the definition see the next section). Our principal result is as follows.

THEOREM 1.3. *Let $1 < p_1 \leqslant p_2 < +\infty$ be such that*

$$\frac{1}{p_1} \leqslant \frac{1}{p_2} + \frac{1}{N} \quad \text{and} \quad \frac{1}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

There exists a positive constant c depending on p_1, p_2 and on functions μ, σ such that, for

$$u_0, B_0 \in \dot{B}^{(N/p_2)-1}_{p_2,1}(\mathbb{R}^N) \quad \text{with } \operatorname{div} u_0 = \operatorname{div} B_0 = 0,$$

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}^{(N/p_2)-1}_{p_2,1}(\mathbb{R}^N)) \text{ with } \mathcal{Q}f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}^{(N/p_2)-2}_{p_2,1}(\mathbb{R}^N)) \text{ and}$$

$$a_0 \in \dot{B}^{N/p_1}_{p_1,1}(\mathbb{R}^N), \quad \text{where } \|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}} \leqslant c,$$

there exists $T(u_0, B_0, f) > 0$ such that system $(\widetilde{\mathrm{MHD}})$ has a solution $(a, u, B, \nabla \Pi)$ with

$$a \in C_b([0,T]; \dot{B}^{N/p_1}_{p_1,1}) \cap \tilde{L}^\infty([0,T]; \dot{B}^{N/p_1}_{p_1,1}),$$
$$u, B \in C_b([0,T]; \dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1(0,T; \dot{B}^{(N/p_2)+1}_{p_2,1})$$

and $\nabla \Pi \in L^{2/(2-\eta)}_T(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$ with

$$0 \leqslant \eta < \inf\left(1, \frac{2N}{p_2}\right) \quad \text{and} \quad \frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

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Moreover, there exists a positive constant c_1 *such that if*

$$\|u_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}} + \|B_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^{(N/p_2)-1}_{p_2,1})} \leqslant c_1 \inf(\mu^1, \sigma^1),$$

with $\mu^1 = \tilde{\mu}(1)$*,* $\sigma^1 = \tilde{\sigma}(1)$*, then* $T = +\infty$*.*

If, in addition, we have that $1 < p_2 \leqslant 2N$*, and*

$$\frac{1}{p_1} + \frac{1}{p_2} \geqslant \frac{2}{N},$$

then such a solution is unique.

Proof. The proof of theorem 1.3 is carried out in two stages. Firstly, we show the uniqueness result that is based on a logarithmic estimate combined with Osgood lemma. Secondly, for the existence part we proceed as follows: we regularize both the initial data and the system $(\widetilde{\mathrm{MHD}})$, for which we establish the existence of solutions. After this step, we show that we can bound the existence time from below. Finally, we prove that the sequence of the approximate solutions converges to a solution satisfying our initial problem.

Remark 1.4. In the case of variable viscosity and variable conductivity, we need the more restrictive condition $p_1 \leqslant p_2$. This condition does not appear in the case where the viscosity is constant (see [3]).

Remark 1.5. This theorem allows us to construct a solution (local in time in general, respectively global in time when the initial data is small compared with viscosity), for $u_0, B_0 \in \dot{B}^{-1+(N/p_2)}_{p_2,1}(\mathbb{R}^N)$ and all $1 < p_2 < +\infty$. In fact, it is sufficient, for example, to consider the density such that $a_0 = \rho_0^{-1} - 1 \in \dot{B}^1_{N1}(\mathbb{R}^N)$ when $N \leqslant p_2 < +\infty$. In the case when $1 < p_2 < N$ we take, for example, $p_1 = p_2$ (other choices are possible; it suffices, for example, to take p_1, which verifies $\sup(1, Np_2/(N+p_2)) < p_1 \leqslant p_2$).

On the other hand, we obtain a unique solution for all $u_0, B_0 \in \dot{B}^{-1+(N/p_2)}_{p_2,1}(\mathbb{R}^N)$ and for all $1 < p_2 \leqslant 2N$. In order to obtain this, it suffices to consider, for example, $a_0 = \rho_0^{-1} - 1 \in \dot{B}^{N/p_1}_{p_1,1}(\mathbb{R}^N)$ with $p_1 = 2N/3$ when $N \leqslant p_2 \leqslant 2N$ and it suffices to take $\sup(1, Np_2/(N+p_2)) < p_1 \leqslant p_2$ when $1 < p_2 < N$.

Remark 1.6. In particular, theorem 1.3 implies the existence of a unique global solution for system $(\widetilde{\mathrm{MHD}})$, when the initial data (ρ_0, u_0, B_0) have the particular form

$$\begin{aligned}
a_0 &= \rho_0^{-1} - 1 \in \mathcal{S}(\mathbb{R}^3),\\
u_0 &= \varepsilon^{-\alpha} \sin\left(\frac{x_3}{\varepsilon}\right)(-\partial_2\phi^1, \partial_1\phi^1, 0),\\
B_0 &= \varepsilon^{-\beta} \sin\left(\frac{x_3}{\varepsilon}\right)(-\partial_2\phi^2, \partial_1\phi^2, 0),
\end{aligned}$$

with $\alpha, \beta \in [0,1)$, $\inf_{x\in\mathbb{R}^3} \rho_0 > 0$ and $\phi^i \in \mathcal{S}(\mathbb{R}^3)$, with a_0 small and $\varepsilon > 0$ small enough. Indeed, it is easy to verify the following assertion. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$, $k \in \mathbb{R}^3$,

$|k| \neq 0$ and $(\sigma, p, r) \in \mathbb{R}_+^* \times [1, \infty)^2$. Then, the function $\phi_\varepsilon(x) = \phi(x)\mathrm{e}^{\mathrm{i}x\cdot k/\varepsilon}$ is small in the space $\dot{B}_{p,r}^{-\sigma}$. More precisely, we have

$$\|\phi_\varepsilon\|_{\dot{B}_{p,r}^{-\sigma}} \leqslant C(\phi)\varepsilon^\sigma,$$

where $C(\phi) = \|\phi\|_{\dot{B}_{p,r}^{\sigma}}$.

2. Preliminaries

2.1. Notation

Let X be a Banach space and let $p \in [1, \infty]$. We denote by $L^p(0, T; X)$ the set of measurable functions $f : (0, T) \to X$, such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(0, T)$, and we denote by $C([0, T); X)$ the space of continuous functions on $[0, T)$ with values in X, $C_b([0, T); X) := C([0, T); X) \cap L^\infty(0, T; X)$. Let $\mu^1 = \mu(1)$, $\tilde{\mu}(a) = \mu(1/(1 + a))$, $\tilde{\sigma}(a) = 1/(\sigma(1/(1 + a)))$, $\sigma^1 = \tilde{\sigma}(1)$ and, for $1 \leqslant p \leqslant \infty$, we denote by p' the conjugate exponent of p given by $1/p + 1/p' = 1$.

2.2. Littlewood–Paley theory

In this section, we briefly recall the Littlewood–Paley theory and we define the functional spaces in which we will work. To this order, we use a unit dyadic (see, for example, [5]). Let $\mathcal{C} \subset \mathbb{R}^N$ be the annulus centred in 0, with the small radius $\frac{3}{4}$, and the big radius $\frac{8}{3}$. There exist two positive radially symmetric functions χ and φ belonging respectively to $C_0^\infty(B(0, \frac{4}{3}))$ and to $C_0^\infty(\mathcal{C})$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all } \xi \neq 0 \quad \text{and} \quad \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^N.$$

We define the following operators:

$$\Delta_q u = \varphi(2^{-q}D)u \quad \text{for all } q \in \mathbb{Z} \quad \text{and} \quad S_q u = \sum_{p\leqslant q-1} \Delta_p u \quad \text{for all } q \in \mathbb{Z}.$$

Moreover, we have

$$u = \sum_{q\in\mathbb{Z}} \Delta_q u \quad \text{for all } u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}[\mathbb{R}^N],$$

where $\mathcal{P}[\mathbb{R}^N]$ is the set of polynomials (see, for example, [17]). Moreover, the Littlewood–Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if } |k - q| \geqslant 2 \quad \text{and} \quad \Delta_k(S_{q-1}u\Delta_q u) \equiv 0 \quad \text{if } |k - q| \geqslant 5. \tag{2.1}$$

Definition 2.1. For $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^N)$, we define

$$\|u\|_{\dot{B}_{pr}^s} = \left(\sum_{q\in\mathbb{Z}} 2^{rqs}\|\Delta_q u\|_{L^p}^r\right)^{1/r}$$

with the usual change for the case when $r = +\infty$. Then for $s < N/p$ and $s \leqslant N/p$, $r = 1$, we define

$$\dot{B}_{pr}^s = \{u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{\dot{B}_{pr}^s} < \infty\};$$

otherwise, we define $\dot{B}^s_{pr}$ as the adherence in $\mathcal{S}'$ of functions belonging to the Schwartz space, for the norm $\|\cdot\|_{\dot{B}^s_{pr}}$.

Let us now recall the Bernstein inequality (see, for example, [5]) which allows us to obtain some embedding of spaces.

LEMMA 2.2 (Bernstein). *Let (r_1, r_2) be a couple of non-negative real numbers such that $r_1 < r_2$. There then exists a non-negative constant C such that for any integer k, any couple (a, b) such that $1 \leqslant a \leqslant b \leqslant \infty$ and every function u in $L^a(\mathbb{R}^N)$, we have*

$$\operatorname{supp} \mathcal{F}u \in B(0, \lambda r_1) \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leqslant C^k \lambda^{k+N(1/a-1/b)} \|u\|_{L^a},$$

$$\operatorname{supp} \mathcal{F}u \in C(0, \lambda r_1, \lambda r_2) \implies C^{-k}\lambda^k \|u\|_{L^a} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leqslant C^k \lambda^k \|u\|_{L^a}.$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use the spaces $\tilde{L}^\rho_T(\dot{B}^s_{pr})$ introduced in [7].

DEFINITION 2.3. Let $s \in \mathbb{R}$, $(r, \rho, p) \in [1, +\infty]^3$ and $T \in]0, +\infty]$. We then say that $f \in \tilde{L}^\rho_T(\dot{B}^s_{pr})$, if

$$\|f\|_{\tilde{L}^\rho_T(\dot{B}^s_{pr})} := \left(\sum_{q\in\mathbb{Z}} 2^{qrs} \left(\int_0^T \|\Delta_q f(t)\|^\rho_{L^p}\, \mathrm{d}t\right)^{r/\rho}\right)^{1/r} < \infty,$$

with the usual change if $r = \infty$.

For $\theta \in [0, 1]$, we have

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{pr})} \leqslant \|u\|^\theta_{\tilde{L}^{\rho_1}_T(\dot{B}^{s_1}_{pr})} \|u\|^{1-\theta}_{\tilde{L}^{\rho_2}_T(\dot{B}^{s_2}_{pr})} \tag{2.2}$$

with

$$\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2} \quad \text{and} \quad s = \theta s_1 + (1-\theta)s_2.$$

Note that the Minkowski inequality implies that

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{pr})} \leqslant \|u\|_{L^\rho_T(\dot{B}^s_{pr})} \quad \text{if } \rho \leqslant r \quad \text{and} \quad \|u\|_{L^\rho_T(\dot{B}^s_{pr})} \leqslant \|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{pr})} \quad \text{if } r \leqslant \rho.$$

We now give the product laws in Besov spaces based on different Lebesgue spaces. These product laws are studied in detail in [3].

PROPOSITION 2.4. *Let $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1, \infty]^6$ such that*

$$\frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{p_2}, \quad p_1 \leqslant \lambda_2, \quad p_2 \leqslant \lambda_1, \qquad \frac{1}{p} \leqslant \frac{1}{p_1} + \frac{1}{\lambda_1} \leqslant 1 \quad \text{and} \quad \frac{1}{p} \leqslant \frac{1}{p_2} + \frac{1}{\lambda_2} \leqslant 1.$$

Then, we have the following inequality.

If

$$s_1 + s_2 + N \inf\left(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\right) > 0, \quad s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1} \quad \text{and} \quad s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2},$$

then

$$\|uv\|_{\dot{B}^{s_1+s_2-N(1/p_1+1/p_2-1/p)}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,r}}\|v\|_{\dot{B}^{s_2}_{p_2,\infty}}. \tag{2.3}$$

When

$$s_1+\frac{N}{\lambda_2}=\frac{N}{p_1} \quad \textit{and} \quad s_2+\frac{N}{\lambda_1}=\frac{N}{p_2},$$

we replace

$$\|u\|_{\dot{B}^{s_1}_{p_1,r}}\|v\|_{\dot{B}^{s_2}_{p_2,\infty}} \quad \textit{and} \quad \|v\|_{\dot{B}^{s_2}_{p_2,\infty}}$$

by

$$\|u\|_{\dot{B}^{s_1}_{p_1,1}}\|v\|_{\dot{B}^{s_2}_{p_2,r}} \quad \textit{and} \quad \|v\|_{\dot{B}^{s_2}_{p_2,\infty}\cap L^\infty},$$

respectively. If

$$s_1+\frac{N}{\lambda_2}=\frac{N}{p_1} \quad \textit{and} \quad s_2+\frac{N}{\lambda_1}=\frac{N}{p_2},$$

we take $r=1$.

If

$$s_1+s_2=0, \quad s_1\in\left(\frac{N}{\lambda_1}-\frac{N}{p_2},\frac{N}{p_1}-\frac{N}{\lambda_2}\right] \quad \textit{and} \quad \frac{1}{p_1}+\frac{1}{p_2}\leqslant 1,$$

then

$$\|uv\|_{\dot{B}^{-N(1/p_1+1/p_2-1/p)}_{p,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{p_1,1}}\|v\|_{\dot{B}^{s_2}_{p_2,\infty}}. \tag{2.4}$$

If $|s|<N/p$ *for* $p\geqslant 2$ *and* $-N/p'<s<N/p$ *otherwise, we have*

$$\|uv\|_{\dot{B}^{s}_{p,r}} \lesssim \|u\|_{\dot{B}^{s}_{p,r}}\|v\|_{\dot{B}^{N/p}_{p,\infty}\cap L^\infty}. \tag{2.5}$$

Remark 2.5. In the following, p will be equal to p_1 or to p_2 and

$$\frac{1}{\lambda}=\begin{cases}\dfrac{1}{p_1}-\dfrac{1}{p_2} & \text{if } p_1\leqslant p_2,\\[2mm] \dfrac{1}{p_2}-\dfrac{1}{p_1} & \text{if } p_2\leqslant p_1.\end{cases}$$

Remark 2.6. Note that, for $p_1=p_2$, we obtain the classical product laws. On the other hand, if $s_i<N/p_i$, $s_1+s_2>0$ and $p_1\leqslant p_2$, we obtain that $uv\in\dot{B}^{s_1+s_2-(N/p_1)}_{p_2,1}$; otherwise, if $s_i<N/p_2$, we obtain $uv\in\dot{B}^{s_1+s_2-(N/p_2)}_{p_1,1}$. The interpretation of this is that in a product law we can cancel a smaller number of derivatives than usual, if we measure these derivatives with an L^p Lebesgue space with small $p\geqslant 1$.

Remark 2.7. Proposition 2.4 is also satisfied in $\tilde{L}^\rho_t(\dot{B}^s_{p,r})$. For example, the inequality (2.5) becomes

$$\|uv\|_{\dot{B}^{s}_{p,r}} \lesssim \|u\|_{\dot{B}^{s}_{p,r}}\|v\|_{\dot{B}^{N/p}_{p,\infty}\cap L^\infty}$$

whenever $|s|<N/p$ for $p\geqslant 2$ and $-N/p'<s<N/p$, $1\leqslant\rho,\rho_1,\rho_2\leqslant\infty$ and $1/\rho=1/\rho_1+1/\rho_1$.

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3. Estimates for the transport and Stokes equations

We note that the MHD system with variable density consists of a transport equation for the density and a Stokes equation for the velocity vector field. We begin by giving the necessary estimates for the transport and the non-stationary Stokes equations (for the proofs, see [3]).

PROPOSITION 3.1. *Let* $(p_1, p_2) \in [1, +\infty]^2$ *and* $-1 - N\inf(1/p_2, 1/p_1') < s < 1 + N\inf(1/p_1, 1/p_2)$, *where* p_1' *is the conjugate exponent of* p_1 (*respectively,* $-1 - N\inf(1/p_2, 1/p_1') < s \leqslant 1 + N\inf(1/p_1, 1/p_2)$) *and* $r \in [1, +\infty]$ (*respectively,* $r = 1$). *Let* u *a free-divergence vector field such that*

$$\nabla u \in L^1(0, T; \dot{B}^{N/p_2}_{p_2,r} \cap L^\infty)$$

(*respectively,* $u \in L^1(0, T; \dot{B}^{(N/p_2)+1}_{p_2,1})$). *We suppose that*

$$\rho_0 \in \dot{B}^s_{p_1,r}, \quad f \in L^1(0, T; \dot{B}^s_{p_1,r}).$$

Let $\rho \in L^\infty(0, T; \dot{B}^s_{p_1,r}) \cap C([0, T]; \mathcal{S}')$ *be a solution of the following system:*

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= f, \\ \rho|_{t=0} &= \rho_0. \end{aligned}$$

There then exists a non-negative constant C *depending on* N *and* s *such that*

$$\|\rho\|_{\tilde{L}^\infty_T(\dot{B}^s_{p_1,r})} \leqslant \mathrm{e}^{CU(t)} \biggl(\|\rho_0\|_{\dot{B}^s_{p_1,r}} + \int_0^t \|f(\tau)\|_{\dot{B}^s_{p_1,r}} \,\mathrm{d}\tau \biggr), \tag{3.1}$$

where

$$U(t) = \int_0^t \|\nabla u(\tau)\|_{\dot{B}^{N/p_2}_{p_2,r} \cap L^\infty} \,\mathrm{d}\tau \quad \textit{and} \quad U(t) = \int_0^t \|u(\tau)\|_{\dot{B}^{(N/p_2)+1}_{p_2,1}} \,\mathrm{d}\tau,$$

respectively.

PROPOSITION 3.2. *Let* $p \in]1, \infty[$ *and* $-1 - N\inf(1/p, 1/p') < s < N/p$, *where* p *is the conjugate exponent of* p. *Let* u_0 *be a divergence-free vector field with the components in* $\dot{B}^s_{p,r}$ *and* g *a vector field with the components in* $\tilde{L}^1_T(\dot{B}^s_{p,r})$. *Let* u *and* v *be two divergence-free vector fields such that* ∇v *has the coefficients in* $L^1(0, T; \dot{B}^{N/p}_{p,r} \cap L^\infty)$ (*respectively,* $L^1_T(\dot{B}^{N/p}_{p,1})$) *and* $u \in C([0, T); \dot{B}^s_{p,r}) \cap \tilde{L}^1_T(\dot{B}^{s+2}_{p,r})$. *Let* u *be a solution of the non-stationary Stokes system*

$$\left.\begin{aligned} \partial_t u + v \cdot \nabla u - \nu \Delta u + \nabla \Pi &= g, \\ \operatorname{div} u &= 0, \\ u|_{t=0} &= u_0. \end{aligned}\right\} \tag{L}$$

There then exists $C > 0$ *depending on* N *and* s *such that* u *verifies the following estimate:*

$$\begin{aligned} \|u\|_{\tilde{L}^\infty_T(\dot{B}^s_{p,r})} &+ \nu \|u\|_{\tilde{L}^1_T(\dot{B}^{s+2}_{p,r})} + \|\nabla \Pi\|_{\tilde{L}^1_T(\dot{B}^s_{p,r})} \\ &\leqslant \exp\bigl(C\|\nabla v\|_{L^1_T(\dot{B}^{N/p}_{p,r} \cap L^\infty)}\bigr) \bigl\{ \|u_0\|_{\dot{B}^s_{p,r}} + C\|g\|_{\tilde{L}^1_T(\dot{B}^s_{p,r})} \bigr\}. \end{aligned} \tag{3.2}$$

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Moreover, if $2 \leqslant p$ and $s = -1 - N/p$, then we have the following estimate:

$$\|u\|_{L^\infty_T(\dot{B}^s_{p,\infty})} + \nu\|u\|_{\tilde{L}^1_T(\dot{B}^{2+s}_{p,\infty})} + \|\nabla \Pi\|_{\tilde{L}^1_T(\dot{B}^s_{p,\infty})} \leqslant \exp\bigl(C\|\nabla v\|_{L^1_T(\dot{B}^{N/p}_{p,1})}\bigr)\bigl\{\|u_0\|_{\dot{B}^s_{p,\infty}} + C\|g\|_{\tilde{L}^1_T(\dot{B}^s_{p,\infty})}\bigr\}. \quad (3.3)$$

Let us now recall the Osgood lemma [13], which allows us to infer uniqueness of the solution in the critical case (see § 4.3).

LEMMA 3.3 (Osgood). *Let $\rho \geqslant 0$ be a measurable function, let γ be a locally integrable function and let μ be a positive, continuous and non-decreasing function which verifies the following condition:*

$$\int_0^1 \frac{\mathrm{d}r}{\mu(r)} = +\infty.$$

In addition, let a be a positive real number and let ρ satisfy the inequality

$$\rho(t) \leqslant a + \int_0^t \gamma(s)\mu(\rho(s))\,\mathrm{d}s.$$

Then, if $a = 0$, the function ρ vanishes.
If $a \neq 0$, then we have

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leqslant \int_0^t \gamma(s)\,\mathrm{d}s \quad \textit{with } \mathcal{M}(x) = \int_x^1 \frac{\mathrm{d}r}{\mu(r)}.$$

Finally, we recall the following result of logarithmic interpolation (see [10, proposition 2.8]).

LEMMA 3.4. *Let $(p, \lambda) \in [1, +\infty]^2$, $s \in \mathbb{R}$, $t \in \mathbb{R}_+$, $\varepsilon \in (0, 1]$ and $u \in \tilde{L}^\lambda_t(\dot{B}^{s-\varepsilon}_{p,\infty}) \cap \tilde{L}^\lambda_t(\dot{B}^s_{p,1}) \cap \tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p,\infty})$. Then*

$$\|u\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,1})} \lesssim \frac{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,\infty})}}{\varepsilon} \log\left(\mathrm{e} + \frac{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^{s-\varepsilon}_{p,\infty})} + \|u\|_{\tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p,\infty})}}\right).$$

4. Proof of theorem 1.3

We will proceed in two steps. First we prove the uniqueness of the solution which is principally based on a logarithmic estimate and on the Osgood lemma, which is useful in the case of logarithmic estimates. The second part is devoted to the proof of existence of the solution.

4.1. Uniqueness

Let $1 \leqslant p_2 \leqslant 2N$ and $1 < p_1 \leqslant p_2$ be such that $1/p_1 + 1/p_2 \geqslant 2/N$ and $1/p_1 \leqslant 1/p_2 + 1/N$. We denote by $(a^i, u^i, \nabla \Pi^i)$ for $1 \leqslant i \leqslant 2$ two solutions of the (MHD) system. We define

$$(\mathcal{M}^i, \delta\mathcal{M}) = (\tfrac{1}{2}(\nabla u^i + {}^t\nabla u^i), \mathcal{M}^2 - \mathcal{M}^1)$$

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and

$$(\delta a, \delta u, \nabla \delta \Pi, \delta B) = (a^2 - a^1, u^2 - u^1, \nabla \Pi^2 - \nabla \Pi^1, B^2 - B^1).$$

We can easily check that

$$\begin{aligned}
\partial_t \delta a + u^2 \cdot \nabla \delta a &= -\delta u \cdot \nabla a^1, \\
\partial_t \delta u + u^2 \cdot \nabla \delta u - \mu^1 \Delta \delta u + \nabla \delta \Pi &= H(a^i, u^i, \nabla \Pi^i, B^i), \\
\partial_t \delta B + u^2 \cdot \nabla \delta B - \sigma^1 \Delta \delta B &= G(a^i, u^i, B^i), \\
\operatorname{div} \delta u = \operatorname{div} \delta B &= 0,
\end{aligned}$$

where

$$\begin{aligned}
&H(a^i, u^i, \nabla \Pi^i, B^i) \\
&\quad = -\delta u \cdot \nabla u^1 + a^1(\mu^1 \Delta \delta u - \nabla \delta \Pi) + \delta a(\mu^1 \Delta u^2 - \nabla \Pi^2) \\
&\qquad + 2 \operatorname{div}[(\tilde{\mu}(a^2) - \mu^1)\delta \mathcal{M}] \\
&\qquad + 2\delta a \operatorname{div}[(\tilde{\mu}(a^2) - \mu^1)\mathcal{M}^2] + 2a^1 \operatorname{div}[(\tilde{\mu}(a^1) - \mu^1)\delta \mathcal{M}] \\
&\qquad + 2 \operatorname{div}[(\tilde{\mu}(a^2) - \tilde{\mu}(a^1))\mathcal{M}^1] + 2a^1 \operatorname{div}[(\tilde{\mu}(a^2) - \tilde{\mu}(a^1))\mathcal{M}^2] - \tfrac{1}{2}\delta a \nabla (B^2)^2 \\
&\qquad - \tfrac{1}{2}(1 + a^1)\nabla((B^2)^2 - (B^1)^2) + (1 + a^1)(B^2 \cdot \nabla \delta B + \delta B \cdot \nabla B^1) + \delta a B^2 \cdot \nabla B^2
\end{aligned}$$

and

$$\begin{aligned}
G(a^i, u^i, B^i) = B^2 \cdot \nabla \delta u + \delta B \cdot \nabla u^1 - \delta u \cdot \nabla B^1 \\
+ \operatorname{div}\{(\tilde{\sigma}(a^2) - \tilde{\sigma}(a^1))\nabla B^2\} + \operatorname{div}\{(\tilde{\sigma}(a^1) - \sigma^1)\nabla \delta B\}.
\end{aligned}$$

In our discussion we will distinguish between two cases: the first case deals with the situation where $1/p_1 + 1/p_2 > 2/N$ and the second case concerns $1/p_1 + 1/p_2 = 2/N$. The distinction between the two cases appears on the level of the product laws that we use.

4.2. The case where $N \geqslant 3$, $1 \leqslant p_2 < 2N$ and $1/p_1 + 1/p_2 > 2/N$

We have established the following result.

PROPOSITION 4.1. *Let $(a^i, u^i, \nabla \Pi^i, B^i)$, with $i \in \{1, 2\}$, be two solutions of system $(\widetilde{\text{MHD}})$, corresponding to the same initial data*

$$a_0 \in \dot{B}^{N/p_1}_{p_1 1} \quad u_0, B_0 \in \dot{B}^{(N/p_2)-1}_{p_2 1}, \quad \operatorname{div} u_0 = \operatorname{div} B_0 = 0$$

and the external forcing term f belonging to $L^1_{\mathrm{loc}}([0, T^*); \dot{B}^{(N/p_2)-1}_{p_2,1})$ such that $\mathcal{Q}f$ belongs to $L^1_{\mathrm{loc}}([0, T^\star); \dot{B}^{(N/p_2)-2}_{p_2,1})$. Assume that for $i = 1, 2$ we have

$$\begin{aligned}
a^i &\in C([0, T^\star); \dot{B}^{N/p_1}_{p_1,1}(\mathbb{R}^N)), \\
u^i &\in C([0, T^\star); \dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{\mathrm{loc}}([0, T^\star); \dot{B}^{(N/p_2)+1}_{p_2,1}), \\
B^i &\in C([0, T^\star); \dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{\mathrm{loc}}([0, T^\star); \dot{B}^{(N/p)+1}_{p_2,1}), \\
\nabla \Pi^i &\in L^1_{\mathrm{loc}}([0, T^\star); \dot{B}^{(N/p_2)-1}_{p_2,1}).
\end{aligned}$$

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There exists a positive constant c such that if we have

$$\|a^1\|_{L^\infty_{T^\star}(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)} \leqslant c,$$

then $(a^2, u^2, \nabla\Pi^2, B^2) = (a^1, u^1, \nabla\Pi^1, B^1)$.

Proof. The first step of the proof consists in proving that $(\delta a, \delta u, \nabla\delta\Pi, \delta B) \in F^p_T$, where

$$F^p_T := C([0,T]; \dot B^{(N/p_1)-1}_{p_1,1}) \times (L^1_T(\dot B^{N/p_2}_{p_2,1}) \cap C([0,T]; \dot B^{(N/p_2)-2}_{p_2,1})) \times (L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})) \times L^1_T(\dot B^{N/p_2}_{p_2,1}) \cap C([0,T]; \dot B^{(N/p_2)-2}_{p_2,1}).$$

We define, for all $t \leqslant T$, the quantity

$$\begin{aligned}\gamma(t) &:= \|(\delta a, \delta u, \nabla\delta\Pi, \delta B)\|_{F^p_t} \\ &= \|\delta a\|_{L^\infty_t(\dot B^{(N/p_1)-1}_{p_1,1})} + \|\delta u\|_{L^\infty_t(\dot B^{(N/p_2)-2}_{p_2,1})} + \mu^1\|\delta u\|_{L^1_t(\dot B^{N/p_2}_{p_2,1})} \\ &\quad + \|\nabla\delta\Pi\|_{L^1_t(\dot B^{(N/p_2)-2}_{p_2,1})} + \|\delta B\|_{L^\infty_t(\dot B^{(N/p_2)-2}_{p_2,1})} + \sigma^1\|\delta B\|_{L^1_t(\dot B^{N/p_2}_{p_2,1})}.\end{aligned}$$

In order to prove that the solution belongs to the space F^p_T, it suffices to have $(a^i - a_0, \bar u^i, \nabla\bar\Pi^i, \bar B^i) \in F^p_T$, where we have defined $(\bar u^i, \nabla\bar\Pi^i, \bar B^i)$ by $u^i = u_L + \bar u^i$, $\nabla\Pi^i = \nabla\Pi^i_L + \nabla\bar\Pi^i$ et $B^i = B_L + \bar B^i$. The quantities $u_L, \nabla\Pi_L$ and B_L are defined by the system given below:

$$\begin{aligned}\partial_t u_L - \mu^1\Delta u_L + \nabla\Pi_L &= f, \\ \partial_t B_L - \sigma^1\Delta B_L &= 0, \\ \operatorname{div} u_L = \operatorname{div} B_L &= 0, \\ (u_L, B_L)|_{t=0} &= (u_0, B_0).\end{aligned}$$

Indeed, by [6, proposition 2.1] we see that u_L and B_L have their components in the space

$$C([0,T]; \dot B^{(N/p_2)-1}_{p_2,1}) \cap L^1(0,T; \dot B^{(N/p_2)+1}_{p_2,1})$$

and $\nabla\Pi_L \in L^1(0,T; \dot B^{(N/p_2)-1}_{p_2,1})$. The quantities $(\bar u^i, \nabla\bar\Pi^i, \bar B^i)$ verify

$$\left.\begin{aligned}\partial_t\bar u^i - \mu^1\Delta\bar u^i + \nabla\bar\Pi^i &= K(a^i, u^i, \nabla\Pi^i, B^i), \\ \partial_t\bar B^i - \sigma^1\Delta\bar B^i &= L(u^i, B^i), \\ \operatorname{div}\bar u^i = \operatorname{div}\bar B^i &= 0, \\ (\bar u^i, \bar B^i)|_{t=0} &= (0,0),\end{aligned}\right\} \qquad (\mathrm{MHD}_{\mathrm{mod}})$$

where

$$\begin{aligned}K(a^i, u^i, \nabla\Pi^i, B^i) = -u^i\cdot\nabla u^i + a^i(\mu^1\Delta u^i - \nabla\Pi^i) + (1+a^i)\operatorname{div}[(\tilde\mu(a^i) - \mu^1)\mathcal{M}^i] \\ - \tfrac12(1+a^i)\nabla {B^i}^2 + (1+a^i)B^i\cdot\nabla B^i\end{aligned}$$

and

$$L(u^i, B^i) = B^i\cdot\nabla u^i - u^i\cdot\nabla B^i + \operatorname{div}\{(\tilde\sigma(a^i) - \sigma^1)\nabla B^i\}.$$

We apply the operator $\mathcal{P}$ to the first equation of the system ($\mathrm{MHD}_{\mathrm{mod}}$) and we obtain

$$\partial_t \bar{u}^i - \mu^1 \Delta \bar{u}^i = \mathcal{P}(K(a^i, u^i, \nabla \Pi^i, B^i)). \tag{4.1}$$

In the same manner, the divergence operator applied to the same equation gives

$$\begin{aligned}\operatorname{div}((1 + a^i)\nabla \Pi^i) = \operatorname{div} \mathcal{Q} f & \\ & - \operatorname{div}(u^i \cdot \nabla u^i + \tfrac{1}{2}(1 + a^i)\nabla {B^i}^2 - (1 + a^i) B^i \cdot \nabla B^i) \\ & + \operatorname{div}(\mu^1(a^i \Delta u^i) + (1 + a^i) \operatorname{div}[(\tilde{\mu}(a^i) - \mu^1)\mathcal{M}^i]), \end{aligned} \tag{4.2}$$

Combining the inequality (2.2) together with the hypothesis concerning the solutions stated at the beginning, we find $u^i, B^i \in L^2_T(\dot{B}^{N/p_2}_{p_2,1})$. On the other hand inequality (2.5) gives

$$u^i \otimes u^i, \quad B^i \otimes B^i \quad \text{and} \quad {B^i}^2 \in L^2_T(\dot{B}^{(N/p_2)-1}_{p_2,1})$$

for $p_2 < 2N$, $N \geqslant 3$ and $1/p_1 + 1/p_2 > 2/N$. Inequality (2.3) then implies that

$$u^i \cdot \nabla u^i, a^i \Delta u^i \in L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1}), \quad (1+a^i) B^i \cdot \nabla B^i, (1+a^i)\nabla {B^i}^2 \in L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1}).$$

Now inequality (2.3) and Taylor's formula with a remainder in the integral form imply

$$\begin{aligned}\|(1 + a^i) \operatorname{div}[(\tilde{\mu}(a^i) - \mu^1)\mathcal{M}^i]\|_{L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1})} &\lesssim \big(1 + \|a^i\|_{L^\infty_T(\dot{B}^{N/p_1}_{p_1,\infty} \cap L^\infty)}\big) \\ &\quad \times \|(\tilde{\mu}(a^i) - \mu^1)\mathcal{M}^i\|_{L^2_T(\dot{B}^{(N/p_2)-1}_{p_2,1})} \\ &\lesssim \|a^i\|_{L^\infty_T(\dot{B}^{N/p_1}_{p_1,\infty} \cap L^\infty)} \|u^i\|_{L^2_T(\dot{B}^{N/p_2}_{p_2,1})}.\end{aligned}$$

We conclude also that the left-hand term of equality (4.2) belongs to $L^2_T(\dot{B}^{(N/p_2)-3}_{p_2,1})$. On the other hand, inequality (2.3) gives

$$\|a^i \nabla \Pi^i\|_{L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1})} \leqslant \|a^i\|_{L^\infty_T(\dot{B}^{N/p_1}_{p_1,\infty} \cap L^\infty)} \|\nabla \Pi^i\|_{L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1})}.$$

Consequently, the smallness condition on a^i together with (4.2) give that

$$\nabla \Pi^i \in L^2_T(\dot{B}^{(N/p_2)-2}_{p_2,1}).$$

This allows us to obtain, using the hypothesis concerning a^i and the inequality (2.3), that $a^i \nabla \Pi^i \in L^1_T(\dot{B}^{(N/p_2)-2}_{p_2,1})$. So we conclude that $K(a^i, u^i, \nabla \Pi^i, B^i)$ belongs to $L^1_T(\dot{B}^{(N/p_2)-2}_{p_2,1})$. In the similar manner we have $L(u^i, B^i) \in L^1_T(\dot{B}^{(N/p_2)-2}_{p_2,1})$. Since the operator $\mathcal{P}$ is continuous on the spaces $\dot{B}^s_{p,r}$, the terms on the left-hand side of equality (4.1) belong to $L^1_T(\dot{B}^{(N/p_2)-2}_{p_2,1})$. Consequently, applying [6, proposition 2.1], we obtain that

$$\bar{u}^i, \bar{B}^i \in L^1_T(\dot{B}^{N/p_2}_{p_2,1}) \cap C([0,T]; \dot{B}^{(N/p_2)-2}_{p_2,1}) \quad \text{and} \quad \nabla \bar{\Pi}^i \in L^1_T(\dot{B}^{(N/p_2)-2}_{p_2,1}).$$

For a^i, we write $\partial_t a^i = -u^i \cdot \nabla a^i$. Since

$$\frac{1}{p_1} \leqslant \frac{1}{N} + \frac{1}{p_2},$$

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therefore, the product laws (2.3) allow us to see that $\partial_t a^i$ belongs to $L^2_T(\dot B^{(N/p_1)-1}_{p_1,1})$, which, by the Cauchy–Schwarz inequality, gives

$$(a^i - a_0) \in C([0,T]; \dot B^{(N/p_1)-1}_{p_1,1}).$$

Finally, we have $(\delta a, \delta u, \nabla\delta\Pi, \delta B) \in F^p_T$.

Using propositions 3.1 and 3.2 we prove successively that, for all $t \leqslant T$,

$$\|\delta a\|_{L^\infty_t(\dot B^{(N/p_1)-1}_{p_1,1})} \lesssim \exp\bigl(C\|u^2\|_{L^1_t(\dot B^{(N/p_2)+1}_{p_2,1})}\bigr)\|\delta u\|_{L^1_t(\dot B^{N/p_2}_{p_2,1})}\|\nabla a^1\|_{L^\infty_t(\dot B^{(N/p_1)-1}_{p_1,1})},$$

$$\begin{aligned}\|\delta u\|_{L^\infty_t(\dot B^{(N/p_2)-2}_{p_2,1})} + \mu^1\|\delta u\|_{L^1_t(\dot B^{N/p_2}_{p_2,1})} + \|\nabla\delta\Pi\|_{L^1_t(\dot B^{(N/p_2)-2}_{p_2,1})}&\\ \lesssim \exp\bigl(C\|u^2\|_{L^1_t(\dot B^{(N/p_2)+1}_{p_2,1})}\bigr)\|H(a^i,u^i,\nabla\Pi^i,B^i)\|_{L^1_t(\dot B^{(N/p_2)-2}_{p_2,1})}&\end{aligned}$$

and

$$\begin{aligned}\|\delta B\|_{L^\infty_t(\dot B^{(N/p_2)-2}_{p_2,1})} + \sigma^1\|\delta B\|_{L^1_t(\dot B^{N/p_2}_{p_2,1})}&\\ \lesssim \exp\bigl(C\|u^2\|_{L^1_t(\dot B^{(N/p_2)+1}_{p_2,1})}\bigr)\|G(a^i,u^i,B^i)\|_{L^1_t(\dot B^{(N/p_2)-2}_{p_2,1})}.&\end{aligned}$$

We will estimate next the term $H(a^i,u^i,\nabla\Pi^i,B^i)$. Inequalities (2.5) and (2.3) give

$$\begin{aligned}&\|-\delta u\cdot\nabla u^1 + a^1(\mu^1\Delta\delta u - \nabla\delta\Pi) + \delta a(\mu^1\Delta u^2 - \nabla\Pi^2)\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\\ &\quad\lesssim \|\delta u\|_{L^2_T(\dot B^{(N/p_2)-1}_{p_2,1})}\|u^1\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\\ &\qquad + \|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\bigl(\|\Delta\delta u\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})} + \|\nabla\delta\Pi\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\bigr)\\ &\qquad + \|\delta a\|_{L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty})}\bigl(\|\Delta u^2\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})} + \|\nabla\Pi^2\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})}\bigr).\end{aligned}$$

Owing to (2.3) and Taylor's formula with a remainder in the integral form, we find that

$$\begin{aligned}\|\operatorname{div}[(\tilde\mu(a^1)-\mu^1)\delta\mathcal{M}] + a^1\operatorname{div}[(\tilde\mu(a^1)-\mu^1)\delta\mathcal{M}]\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}&\\ \lesssim \|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\|\delta u\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}&\end{aligned} \tag{4.3}$$

for $p_1 \leqslant p_2$. Using once more the inequality (2.3), Taylor's formula, inequality (2.5), and the fact that the space of Besov is stable by the action of a C^∞-function (see, for example, [16]), one obtains

$$\begin{aligned}\|\operatorname{div}[(\tilde\mu(a^2)-\tilde\mu(a^1))\mathcal{M}^2]\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}&\\ \lesssim \int_0^T \|\tilde\mu(a^2)-\tilde\mu(a^1)\|_{\dot B^{(N/p_1)-1}_{p_1,1}}\|\nabla u^2\|_{\dot B^{N/p_2}_{p_2,1}}\,\mathrm{d}t&\\ \lesssim \int_0^T \|\delta a\|_{\dot B^{(N/p_1)-1}_{p_1,1}}\|u^2\|_{\dot B^{(N/p_2)+1}_{p_2,1}}\,\mathrm{d}t.&\end{aligned}$$

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Combining inequality (2.3) with an interpolation result in the temporal variable, we prove that

$$
\begin{aligned}
&\|\delta a \nabla (B^2)^2\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\\
&\qquad \lesssim \|\delta a\|_{L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty})} \|(B^2)^2\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}\\
&\qquad \lesssim \|\delta a\|_{L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty})} \|B^2\|^2_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\\
&\qquad \lesssim \|\delta a\|_{L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty})} \|B^2\|_{L^\infty_T(\dot B^{(N/p_2)-1}_{p_2,1})} \|B^2\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})}.
\end{aligned}
$$

In the same manner we find

$$
\begin{aligned}
&\|(1+a^1)\nabla((B^2)^2-(B^1)^2)\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\\
&\qquad \lesssim \big(\|B^1\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})} + \|B^2\|_{L^2_T(\dot B^{(N/p_2)+1}_{p_2,1})}\big)\|\delta B\|_{L^2_T(\dot B^{(N/p_2)-1}_{p_2,1})}\\
&\qquad \lesssim \sum_{i=1}^{2} \|B^i\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\big(\|\delta B\|_{L^\infty_T(\dot B^{(N/p_2)-2}_{p_2,1})} + \|\delta B\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}\big).
\end{aligned}
$$

We have

$$
\frac{1}{p_1}+\frac{1}{p_2} > \frac{2}{N}, \quad p_2 < 2N \quad \text{and} \quad \left|\frac{1}{p_1}-\frac{1}{p_2}\right| \leqslant \frac{1}{N},
$$

so the inequalities (2.3) and (2.5) imply

$$
\begin{aligned}
&\|\delta a B^2 \cdot \nabla B^2\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\\
&\qquad \lesssim \|\delta a\|_{L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty})} \|B^2\|_{L^\infty_T(\dot B^{(N/p_2)-1}_{p_2,1})} \|B^2\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})}.
\end{aligned}
$$

Since one has

$$
\left|\frac{1}{p_1}-\frac{1}{p_2}\right| < \frac{1}{N}, \quad \frac{1}{p_1}+\frac{1}{p_2} > \frac{2}{N}, \quad p_1 \leqslant p_2,
$$

lemmas 3.1 and 3.2 of [1] remain valid. Thus, combining the preceding inequalities with these lemmas, we find

$$
\begin{aligned}
&\|H(a^i,u^i,\nabla \Pi^i, B^i)\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\\
&\quad \lesssim \gamma(t)\big\{\|(u^1,u^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})} + \|\nabla \Pi^2\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})} + \|(B^1,B^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})}\\
&\qquad\qquad + \|(B^1,B^2)\|_{L^2_T(\dot B^{(N/p_2)+1}_{p_2,1})} + \|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\big\}\\
&\qquad + \int_0^T \|\delta a(t)\|_{\dot B^{(N/p_1)-1}_{p_1,1}} \|u^2\|_{\dot B^{(N/p_2)+1}_{p_2,1}}\,\mathrm{d}t.
\end{aligned}
$$

We need now to estimate $G(a^i,u^i,B^i)$. Since $\operatorname{div} B^2 = 0$, using the inequalities of Bernstein and (2.5) together with an interpolation argument, we then obtain

$$
\begin{aligned}
\|B^2\cdot\nabla\delta u\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})} &\lesssim \|B^2\otimes\delta u\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})}\\
&\lesssim \|B^2\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})} \|\delta u\|_{L^2_T(\dot B^{(N/p_2)-1}_{p_2,1})}\\
&\lesssim \|B^2\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\big(\|\delta u\|_{L^\infty_T(\dot B^{(N/p_2)-2}_{p_2,1})} + \|\delta u\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}\big).
\end{aligned}
$$

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In the same manner, we have

$$\begin{aligned}\|\delta B\cdot\nabla u^1-\delta u\cdot\nabla B^1\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}&\\ \lesssim \|u^1\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\big(\|\delta B\|_{L^\infty_T(\dot B^{(N/p_2)-2}_{p_2,1})}+\|\delta B\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}\big)&\\ +\|B^1\|_{L^2_T(\dot B^{N/p_2}_{p_2,1})}\big(\|\delta u\|_{L^\infty_T(\dot B^{(N/p_2)-2}_{p_2,1})}+\|\delta u\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}\big).&\end{aligned}$$

Arguing similarly to the case of inequality (4.3), one finds that

$$\|\operatorname{div}\{(\tilde\sigma(a^1)-\sigma^1)\nabla\delta B\}\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}\lesssim\|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\|\delta B\|_{L^1_T(\dot B^{N/p_2}_{p_2,1})}.$$

Using the above estimates and [1, lemmas 3.1, 3.2], and arguing in the same manner as for the H term, we obtain finally that

$$\begin{aligned}\|G(a^i,u^i,\nabla\Pi^i,B^i)\|_{L^1_T(\dot B^{(N/p_2)-2}_{p_2,1})}&\\ \lesssim\gamma(t)\big(\|(u^1,u^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}+\|\nabla\Pi^2\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})}&\\ +\|(B^1,B^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}+\|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\big)&\\ +\int_0^T\|\delta a(t)\|_{\dot B^{(N/p_1)-1}_{p_1,1}}\|B^2\|_{\dot B^{(N/p_2)+1}_{p_2,1}}\,\mathrm{d}t.&\end{aligned}$$

Thus, one finds for $t\leqslant T$ that

$$\begin{aligned}\gamma(t)\lesssim\gamma(t)\big(\|(u^1,u^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}+\|\nabla\Pi^2\|_{L^1_T(\dot B^{(N/p_2)-1}_{p_2,1})}&\\ +\|a^1\|_{L^\infty_T(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}+\|(B^1,B^2)\|_{L^1_T(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}\big)&\\ +\int_0^T\gamma(t)\|(u^2,B^2)\|_{\dot B^{(N/p_2)+1}_{p_2,1}}\,\mathrm{d}t.&\end{aligned}$$

We choose a small time $T_1\leqslant T$ such that, for a constant $c>0$ small enough, we have the following inequality:

$$\|(u^1,u^2)\|_{L^1_{T_1}(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}+\|\nabla\Pi^2\|_{L^1_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})}\leqslant c$$

and

$$\|(B^1,B^2)\|_{L^1_{T_1}(\dot B^{(N/p_2)+1}_{p_2,1})\cap L^2_T(\dot B^{N/p_2}_{p_2,1})}\leqslant c.$$

Using the assumption that $\|a^1\|_{L^\infty_{T_1}(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)}\leqslant c$, for all $t\leqslant T_1$, we have

$$\gamma(t)\leqslant C\int_0^t\gamma(t)\|(u^2,B^2)\|_{\dot B^{(N/p_2)+1}_{p_2,1}}\,\mathrm{d}t.$$

Since the function

$$t\mapsto\|u^2\|_{\dot B^{(N/p_2)+1}_{p_2,1}}+\|B^2\|_{\dot B^{(N/p_2)+1}_{p_2,1}}$$

is locally integrable, we deduce by lemma 3.3 that $\gamma(t)=0$ for all $t\in[0,T_1]$. It is easy to see that this property is conserved on the whole time interval and we obtain finally that $\gamma(t) = 0$ for all $t \in [0, T]$. Thus, the proof is complete in the case when $1 < p_2 < 2N$. The above calculations are available for $p \neq 1$ (since they are based on proposition 3.2). The case $p = 1$ is deduced by injection. □

4.3. The case $1/p_1 + 1/p_2 = 2/N$ or $N = 2$ or $p_2 = 2N$

In this case the condition

$$\|a^1\|_{L^\infty_{T^\star}(\dot B^{N/p_1}_{p_1,\infty}\cap L^\infty)} \leqslant c$$

is not sufficient. To show uniqueness, one needs to suppose that $\|a^1\|_{L^\infty_{T^\star}(\dot B^{N/p_1}_{p_1,1})} \leqslant c$. More precisely, we have the following proposition.

Proposition 4.2. *Let* $(a^1, u^1, \nabla\Pi, B^1)$ *and* $(a^2, u^2, \nabla\Pi^2, B^2)$ *be two solutions of* $(\widetilde{\text{MHD}})$ *corresponding to the initial data* $a_0 \in \dot B^{N/p_1}_{p_1,1}$, $u_0, B_0 \in \dot B^{(N/p_2)-1}_{p_2,1}$, *where* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ *and* f *is such that its components are in* $L^1_{\text{loc}}([0, T^\star); \dot B^{(N/p_2)-1}_{p_2,1})$ *and* $\mathcal{Q}f$ *belongs to* $L^1_{\text{loc}}([0, T^\star); \dot B^{(N/p_2)-2}_{p_2,1})$. *We assume that, for* $i = 1, 2$, *we have*

$$\begin{aligned}
a^i &\in C([0, T^\star); \mathcal{S}') \cap L^\infty_{\text{loc}}([0, T^\star); \dot B^{N/p_1}_{p_1,1}),\\
u^i &\in C([0, T^\star); \dot B^{(N/p_2)-1}_{p_2,1}) \cap L^1_{\text{loc}}([0, T^\star); \dot B^{(N/p_2)+1}_{p_2,1}),\\
B^i &\in C([0, T^\star); \dot B^{(N/p_2)-1}_{p_2,1}) \cap L^1_{\text{loc}}([0, T^\star); \dot B^{(N/p_2)+1}_{p_2,1}),\\
\nabla\Pi^i &\in L^1_{\text{loc}}([0, T^\star); \dot B^{(N/p_2)-1}_{p_2,1}).
\end{aligned}$$

There then exists a positive constant c *which does not depend on these solutions such that the inequality*

$$\|a^1\|_{\tilde L^\infty_{T^\star}(\dot B^{N/p_1}_{p_1,1})} \leqslant c$$

implies $(a^2, u^2, \nabla\Pi^2, B^2) = (a^1, u^1, \nabla\Pi^1, B^1)$.

Proof. We need to prove first that $(\delta a, \delta u, \nabla\delta\Pi, \delta B) \in G_T$, where

$$\begin{aligned}
G_T := L^\infty_T(\dot B^{(N/p_1)-1}_{p_1,\infty}) \times \tilde L^1_T(\dot B^{N/p_2}_{p_2,\infty}) \cap L^\infty_T(\dot B^{-2+(N/p_2)}_{p_2,\infty}) \times \tilde L^1_T(\dot B^{-2+(N/p_2)}_{p_2,\infty})\\
\times \tilde L^1_T(\dot B^{N/p_2}_{p_2,\infty}) \cap L^\infty_T(\dot B^{-2+(N/p_2)}_{p_2,\infty}).
\end{aligned}$$

The estimates in this space allow us to obtain the uniqueness of the solution by the application of the Osgood lemma. We define

$$\begin{aligned}
\gamma(t) = \|\delta u\|_{L^\infty_T(\dot B^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta u\|_{\tilde L^1_T(\dot B^{N/p_2}_{p_2,\infty})}\\
+ \|\nabla\delta\Pi\|_{\tilde L^1_T(\dot B^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{L^\infty_T(\dot B^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{\tilde L^1_T(\dot B^{N/p_2}_{p_2,\infty})}.
\end{aligned}$$

The term G_T is dealt with in the same way as in the first case. The only difference to be noted appears in the treatment of the products of the type $a^i\nabla\Pi^i$. Here inequality (2.4) should be used to ensure that the left-hand side term of equality (4.1) belongs to $L^2_T(\dot B^{-1}_{3,\infty})$. Thus, [6, proposition 2.1] implies that $(\delta a, \delta u, \nabla\delta\Pi, \delta B) \in G_T$.

In this case it is sufficient to study the case $2/N = 1/p_1 + 1/p_2$, since one can deduce the other cases from this one. Indeed, if $p_2 = 2N$, then $p_1 = 2N/3$, since

$$\frac{1}{p_1} \leqslant \frac{1}{N} + \frac{1}{p_2} \quad \text{and} \quad \frac{2}{N} \leqslant \frac{1}{p_1} + \frac{1}{p_2}.$$

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Therefore, it is a particular case of $2/N = 1/p_1 + 1/p_2$. For $N = 2$, one starts with $p_2 = 4$ and $p_1 = \frac{4}{3}$. Afterwards, by injection, one will have uniqueness for $1 \leqslant p_1 \leqslant \frac{4}{3}$ and $1 \leqslant p_2 \leqslant 4$; by the same argument we obtain the uniqueness for $1 \leqslant p_1 \leqslant 4$ and $1 \leqslant p_2 \leqslant \frac{4}{3}$. Hence one can suppose that $2/N = 1/p_1 + 1/p_2$. Moreover, one can suppose that $p_2 \geqslant 2$ since inequality (3.3) is valid for $p \geqslant 2$. The case $p_2 \leqslant 2$ follows by injection.

Using propositions 3.1 and 3.2, we have

$$\|\delta a\|_{L_t^\infty(\dot B_{p_1,\infty}^{(N/p_1)-1})} \leqslant \exp\big(C\|\nabla u^2\|_{L_t^1(\dot B_{p_2,1}^{N/p_2})}\big)\|\delta u\cdot\nabla a^1\|_{L_t^1(\dot B_{p_1,\infty}^{(N/p_1)-1})}, \tag{4.4}$$

$$\begin{aligned}\|\delta u\|_{L_t^\infty(\dot B_{p_2,\infty}^{-2+(N/p_2)})} &+ \mu^1\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})} + \|\nabla\delta\Pi\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{-2+(N/p_2)})}\\ &\leqslant C\exp\big(C\|\nabla u^2\|_{L_t^1(\dot B_{p_2,1}^{N/p_2})}\big)\|H(a^i,u^i,\nabla\Pi^i,B^i)\|_{\tilde L_T^1(\dot B_{p_2,\infty}^{-2+(N/p_2)})}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\|\delta B\|_{L_t^\infty(\dot B_{p_2,\infty}^{-2+(N/p_2)})} &+ \sigma^1\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}\\ &\leqslant C\exp\big(C\|\nabla u^2\|_{L_t^1(\dot B_{p_2,1}^{N/p_2})}\big)\|G(a^i,u^i,B^i)\|_{\tilde L_T^1(\dot B_{p_2,\infty}^{-2+(N/p_2)})}.\end{aligned} \tag{4.6}$$

Combining the estimates of δa, inequality (2.3), the Bernstein and Minkowski inequalities, we obtain

$$\|\delta u\cdot\nabla a^1\|_{L_t^1(\dot B_{p_1,\infty}^{(N/p_1)-1})} \lesssim \|\delta u\|_{L_t^1(\dot B_{p_2,1}^{N/p_2})}\|a^1\|_{\tilde L_t^\infty(\dot B_{p_1,1}^{N/p_1})}. \tag{4.7}$$

By lemma 3.4 one has

$$\begin{aligned}&\|\delta u\|_{L_t^1(\dot B_{p_2,1}^{N/p_2})}\\ &\quad\lesssim \|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}\log\left(\mathrm{e}+\frac{\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{(N/p_2)-1})}+\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{1+(N/p_2)})}}{\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}}\right)\\ &\quad\lesssim \|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}\log\left(\mathrm{e}+\frac{t\sum_{i=1}^2\|u^i\|_{L_t^\infty(\dot B_{p_2,\infty}^{(N/p_2)-1})}+\sum_{i=1}^2\|u^i\|_{L_t^1(\dot B_{p_2,1}^{1+(N/p_2)})}}{\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}}\right).\end{aligned} \tag{4.8}$$

We will now estimate the term $H(a^i,u^i,\nabla\Pi^i,B^i)$. Since $\operatorname{div}\delta u = 0$ the inequalities of Bernstein, (2.5) (for $p_2 < 2N$) and (2.4) for $(p_2 = 2N)$ imply

$$\begin{aligned}\|\delta u\cdot\nabla u^1\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{-2+(N/p_2)})} &\lesssim \|\delta u\otimes u^1\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{-1+(N/p_2)})}\\ &\lesssim \|u^1\|_{\tilde L_t^2(\dot B_{p_2,1}^{N/p_2})}\|\delta u\|_{\tilde L_t^2(\dot B_{p_2,\infty}^{-1+(N/p_2)})}\\ &\lesssim \|u^1\|_{\tilde L_t^2(\dot B_{p_2,1}^{N/p_2})}\big(\|\delta u\|_{\tilde L_t^1(\dot B_{p_2,\infty}^{N/p_2})}+\|\delta u\|_{L_t^\infty(\dot B_{p_2,\infty}^{-2+(N/p_2)})}\big).\end{aligned}$$

Since $p_1 \leqslant p_2$, $2/N \leqslant 1/p_1 + 1/p_2 \leqslant 1$, owing to inequalities (2.4) and using the Bernstein inequality, we have

$$\begin{aligned}\|a^1(\mu^1\Delta\delta u - \nabla\delta\Pi) + \delta a(\mu^1\Delta u^2 - \nabla\Pi^2)\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\big(\|\delta u\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})} + \|\nabla\delta\Pi\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})}\big)\\ &\quad + \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\big(\|u^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} + \|\nabla\Pi^2\|_{\dot{B}^{-1+(N/p_2)}_{p_2,1}}\big)\,\mathrm{d}\tau.\end{aligned}$$

The Minkowski inequality, (2.4), (2.3) and the Taylor formula imply

$$\begin{aligned}\|\delta a\,\mathrm{div}[(\tilde{\mu}(a^2) - \mu^1)\mathcal{M}^2]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\|(\tilde{\mu}(a^2) - \mu^1)\mathcal{M}^2\|_{\dot{B}^{N/p_2}_{p_2,1}}\,\mathrm{d}\tau\\ &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\|\tilde{\mu}(a^2) - \mu^1\|_{\dot{B}^{N/p_1}_{p_1,1}}\|\nabla u^2\|_{\dot{B}^{N/p_2}_{p_2,1}}\,\mathrm{d}\tau\\ &\lesssim \|a^2\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\|u^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}}\,\mathrm{d}\tau.\end{aligned}$$

Using the Bernstein inequality and (2.3), we find

$$\begin{aligned}\|\,\mathrm{div}[(\tilde{\mu}(a^1) - \mu^1)\delta\mathcal{M}]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \|\tilde{\mu}(a^1) - \mu^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\|\nabla\delta u\|_{\tilde{L}^1_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})}\\ &\lesssim \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\|\delta u\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})}. \end{aligned} \tag{4.9}$$

This and the Minkowski inequality, (2.4), the Bernstein inequality, (2.3), Taylor's formula and (2.5) give

$$\begin{aligned}\|a^1\,\mathrm{div}[(\tilde{\mu}(a^2) - \tilde{\mu}(a^1))\mathcal{M}^2]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \int_0^t \|a^1\|_{\dot{B}^{N/p_1}_{p_1,1}}\|(\tilde{\mu}(a^2) - \tilde{\mu}(a^1))\mathcal{M}^2\|_{\dot{B}^{-1+(N/p_2)}_{p_2,\infty}}\,\mathrm{d}\tau\\ &\lesssim \int_0^t \|\tilde{\mu}(a^2) - \tilde{\mu}(a^1)\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\|\nabla u^2\|_{\dot{B}^{N/p_2}_{p_2,\infty}\cap L^\infty}\,\mathrm{d}\tau\\ &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\sum_{i=1}^2\|a^i\|_{\dot{B}^{N/p_1}_{p_1,1}}\|u^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}}\,\mathrm{d}\tau\\ &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}}\|u^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}}\,\mathrm{d}\tau. \end{aligned} \tag{4.10}$$

In the same manner we obtain the following estimates

$$\begin{aligned}\|a^1\,\mathrm{div}[(\tilde{\mu}(a^1) - \mu^1)\delta\mathcal{M}]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\|(\tilde{\mu}(a^1) - \mu^1)\delta\mathcal{M}\|_{\tilde{L}^1_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})}\\ &\lesssim \|a^1\|^2_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\|\delta u\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})}\end{aligned}$$

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and

$$\|\operatorname{div}[(\tilde{\mu}(a^2)-\tilde{\mu}(a^1))\mathcal{M}^2]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|u^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} \,\mathrm{d}\tau. \tag{4.11}$$

Using the Minkowski inequality, (2.3), the fact that $\dot{B}^{N/p_2}_{p_2,1}$ is an algebra and interpolation, we obtain

$$\begin{aligned}\|\delta a \nabla (B^2)^2\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|B^2\|^2_{\dot{B}^{N/p_2}_{p_2,1}} \,\mathrm{d}\tau \\ &\lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|B^2\|_{\dot{B}^{-1+(N/p_2)}_{p_2,1}} \|B^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} \,\mathrm{d}\tau.\end{aligned}$$

Since $\operatorname{div} B^2 = 0$, in an analogous manner, we obtain

$$\|\delta a B^2\cdot\nabla B^2\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|B^2\|_{\dot{B}^{-1+(N/p_2)}_{p_2,1}} \|B^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} \,\mathrm{d}\tau.$$

Owing to inequalities (2.4), (2.5) and a classical interpolation argument, we can write

$$\begin{aligned}&\|(1+a^1)\nabla((B^2)^2-(B^1)^2)\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \\ &\qquad\lesssim \big(1+\|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\big)\|(B^2)^2-(B^1)^2\|_{\tilde{L}^1_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})} \\ &\qquad\lesssim \sum_{i=1}^2 \|B^i\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \|\delta B\|_{\tilde{L}^2_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})} \\ &\qquad\lesssim \sum_{i=1}^2 \|B^i\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \big(\|\delta B\|_{L^\infty_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})}\big).\end{aligned}$$

Since $\operatorname{div}\delta B = \operatorname{div} B^2 = 0$, in the same way we will have

$$\begin{aligned}&\|(1+a^1)(\delta B\cdot\nabla B^1 + B^2\cdot\nabla\delta B)\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \\ &\qquad\lesssim \sum_{i=1}^2 \|B^i\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \big(\|\delta B\|_{L^\infty_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})}\big).\end{aligned}$$

Combining all these estimates, we are able to establish

$$\begin{aligned}&\|H(a^i,u^i,\nabla\Pi^i,B^i)\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \\ &\quad\lesssim \gamma(t)\big(\|(u^1,u^2)\|_{L^1_t(\dot{B}^{1+(N/p_2)}_{p_2,1})\cap\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \\ &\qquad\quad + \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})} + \|(B^1,B^2)\|_{L^1_t(\dot{B}^{1+(N/p_2)}_{p_2,1})\cap\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})}\big) \\ &\qquad + \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} g(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{4.12}$$

where g is a locally integrable function.

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We now give the estimates for G. Using the Bernstein inequality and (2.5) for $p_2 < 2N$, (2.4) for $p_2 = 2N$, by interpolation we obtain

$$\begin{aligned}
&\|B^2 \cdot \nabla \delta u + \delta B \cdot \nabla u^1 - \delta u \cdot \nabla B^1\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \\
&\quad \lesssim \|B^2 \otimes \delta u + \delta B \otimes u^1 - \delta u \otimes B^1\|_{\tilde{L}^1_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})} \\
&\quad \lesssim \|(B^1, B^2)\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \|\delta u\|_{\tilde{L}^2_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{\tilde{L}^2_t(\dot{B}^{-1+(N/p_2)}_{p_2,\infty})} \|u^1\|_{\tilde{L}^2_t(\dot{B}^{N/p_1}_{p_2,1})} \\
&\quad \lesssim \|(B^1, B^2, u^1)\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} \big(\|\delta B\|_{L^\infty_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta B\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})} \\
&\qquad\qquad + \|\delta u\|_{L^\infty_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} + \|\delta u\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})} \big).
\end{aligned}$$

We obtain identically to (4.11) and (4.9) that

$$\|\operatorname{div}[(\tilde{\sigma}(a^2) - \tilde{\sigma}(a^1))\nabla B^2]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \lesssim \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|B^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} \,\mathrm{d}\tau$$

and

$$\|\operatorname{div}[(\tilde{\sigma}(a^1) - \sigma^1)\nabla \delta B]\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \lesssim \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})} \|\delta B\|_{\tilde{L}^1_t(\dot{B}^{N/p_2}_{p_2,\infty})}.$$

We deduce from these estimates that

$$\begin{aligned}
\|G(a^i, u^i, B^i)\|_{\tilde{L}^1_t(\dot{B}^{-2+(N/p_2)}_{p_2,\infty})} \lesssim \gamma(t) \big(&\|(u^1, B^1, B^2)\|_{\tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} + \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})} \big) \\
&+ \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,\infty}} \|B^2\|_{\dot{B}^{1+(N/p_2)}_{p_2,1}} \,\mathrm{d}\tau.
\end{aligned}$$

Using the above estimate together with those given by (4.12), we have

$$\begin{aligned}
\gamma(t) \lesssim \gamma(t) \big(&\|(u^1, u^2, B^1, B^2)\|_{L^1_t(\dot{B}^{1+(N/p_2)}_{p_2,1}) \cap \tilde{L}^2_t(\dot{B}^{N/p_2}_{p_2,1})} + \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})} \big) \\
&+ \int_0^t \|\delta a\|_{\dot{B}^{-1+(N/p_1)}_{p_1,1}} g(\tau) \,\mathrm{d}\tau.
\end{aligned}$$

Using the above estimate, we may choose a sufficiently small time T_1 so that, using inequalities (4.4), (4.7), (4.8) and the smallness of a^1, for all $t \in [0, T_1]$, we obtain

$$\gamma(t) \lesssim \int_0^t \log\left(\mathrm{e} + \frac{\alpha(T)}{\|\delta u\|_{\tilde{L}^1_\tau(\dot{B}^{N/p_2}_{p_2,\infty})}} \right) \|\delta u\|_{\tilde{L}^1_\tau(\dot{B}^{N/p_2}_{p_2,\infty})} g(\tau) \,\mathrm{d}\tau,$$

with

$$\alpha(T) = \sum_{i=1}^{2} T \|u^i\|_{L^\infty_T(\dot{B}^{-1+(N/p_2)}_{p_2,1})} + \|u^i\|_{L^1_T(\dot{B}^{1+(N/p_2)}_{p_2,1})}.$$

Owing to the fact that $x \mapsto x \log(\mathrm{e} + (\alpha(T)/x))$ is an increasing function on $\mathbb{R}_+$, for all $t \in [0, T_1]$, we have

$$\gamma(t) \lesssim \int_0^t \gamma(\tau) \log\left(\mathrm{e} + \frac{\alpha(T)}{\gamma(\tau)} \right) g(\tau) \,\mathrm{d}\tau.$$

So, by lemma 3.3, we deduce that $\gamma(t) = 0$, for all $t \in [0, T_1]$. By inequality (4.4), this gives $\delta a = 0$. Standard arguments now yield the required conclusion. We note that the method used in this section (the logarithmic interpolation argument and the application of the Osgood lemma) is inspired by the proofs of the uniqueness given by Danchin [10] and was used by the authors in [3]. □

4.4. Existence

Throughout this section we assume that $p_1 \leqslant p_2$, $1/p_1 + 1/p_2 > 1/N$ and $1/p_1 \leqslant 1/N + 1/p_2$.

The proof of existence of a solution is performed in a standard manner. We begin by solving an approximate problem and we prove that the solutions are uniformly bounded. The last step consists in studying the convergence to a solution of the initial equation.

Step 1 (construction of a regular approximate solution). Let us recall first the following result (see [1, lemma 4.2]).

Lemma 4.3. *Assume that* $s_i \in \mathbb{R}$ *and* $(p_i, r_i) \in [1, \infty[^2$ *for* $i = 1, 2$. *Let* $G \in \dot{B}^{s_1}_{p_1 r_1}(\mathbb{R}^N)$. *There then exists* $G^n \in H^\infty(\mathbb{R}^N)$, *such that for all* $\varepsilon > 0$ *there is an* n_0 *such that*

$$\|G^n - G\|_{\dot{B}^{s_1}_{p_1 r_1}} \leqslant \varepsilon \quad \textit{for all } n \geqslant n_0.$$

If we have $\operatorname{div} G = 0$ *and* $\mathcal{Q}G \in \dot{B}^{s_2}_{p_2 r_2}$, *then we can choose* G^n *such that* $\operatorname{div} G^n = 0$ *and* $\mathcal{Q}G^n$ *is uniformly bounded with respect to* n *in the space* $\dot{B}^{s_2}_{p_2 r_2}$.

Owing to the above lemma there exist

$$a_0^n, u_0^n, B_0^n \in H^\infty(\mathbb{R}^N) \quad \text{and} \quad f^n \in L^1_{\mathrm{loc}}(\mathbb{R}_+; H^\infty(\mathbb{R}^N))$$

such that we have

$$\begin{aligned} \|a_0^n\|_{L^\infty} &\lesssim \|a_0\|_{L^\infty}, \\ \operatorname{div} u_0^n &= \operatorname{div} B_0^n = 0, \\ \|\mathcal{Q}f^n\|_{L^1_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}^{(N/p_2)-2}_{p_2,1})} &\lesssim \|\mathcal{Q}f\|_{L^1_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}^{(N/p_2)-2}_{p_2,1})}. \end{aligned}$$

Now, owing to [2, theorem 1.1], we deduce that system $(\widetilde{\mathrm{MHD}})$ with the initial data $(a_0^n, u_0^n, B_0^n, f^n)$ admits a unique local-in-time solution $(a^n, u^n, \nabla \Pi^n, B^n)$ verifying

$$\begin{aligned} a^n &\in C([0, T^n); H^{s+1}(\mathbb{R}^N)), \\ u^n, B^n &\in C([0, T^n); H^s(\mathbb{R}^N)) \cap \tilde{L}^1_{T^n}(H^{s+2}), \\ \nabla \Pi^n &\in L^1([0, T^n); H^s(\mathbb{R}^N)) \quad \text{with } s > \tfrac{1}{2}N - 1. \end{aligned}$$

Step 2 (estimates of the regularized solution).

Let $T \in [0, +\infty]$ be defined as $\inf_{n \in \mathbb{N}} T^n$. Our first goal is to prove that $T > 0$ such that $(a^n, u^n, \nabla \Pi^n, B^n)$ belongs to and is uniformly bounded in the space

$$E_T = (\tilde{L}^\infty_T(\dot{B}^{N/p_1}_{p_1,1}))(L^1_T(\dot{B}^{(N/p_2)+1}_{p_2,1}) \cap \tilde{L}^\infty_T(\dot{B}^{(N/p_2)-1}_{p_2,1}))L^1_T(\dot{B}^{(N/p_2)-1}_{p_2,1}) \\ \times (L^1_T(\dot{B}^{(N/p_2)+1}_{p_2,1}) \cap \tilde{L}^\infty_T(\dot{B}^{(N/p_2)-1}_{p_2,1})).$$

Let (u_L^n, Π_L^n) be a solution of the following non-stationary Stokes system

$$\left.\begin{aligned} \partial_t u_L^n - \mu^1 \Delta u_L^n + \nabla \Pi_L^n = f^n,\\ \partial_t B_L^n - \sigma^1 \Delta B_L^n = 0,\\ \operatorname{div} u_L^n = \operatorname{div} B_L^n = 0,\\ (u_L^n, B_L^n)|_{t=0} = (u_0^n, B_0^n). \end{aligned}\right\} \tag{L}$$

By construction, $u_0^n, B_0^n \in \dot{B}_{p_2,1}^{(N/p_2)-1} \cap H^s$ and $f^n \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \dot{B}_{p_2,1}^{(N/p_2)-1} \cap H^s)$. So, following [9, proposition 2.3], we have

$$(u_L^n, \nabla \Pi_L^n, B_L^n) \in L_t^\infty(\dot{B}_{p_2,1}^{(N/p_2)-1} \cap H^s) \times L_t^1(\dot{B}_{p_2,1}^{(N/p_2)-1} \cap H^s) \times L_t^\infty(\dot{B}_{p_2,1}^{(N/p_2)-1} \cap H^s)$$

and, moreover, $u_L^n, B_L^n \in L_t^1(\dot{B}_{p_2,1}^{(N/p_2)+1})$ for all $t > 0$.

Let $u^n = u_L^n + \bar{u}^n$, $\nabla \Pi^n = \nabla \Pi_L^n + \nabla \bar{\Pi}^n$ and $B^n = B_L^n + \bar{B}^n$. Then

$$\begin{aligned} (a^n, \bar{u}^n, \nabla \bar{\Pi}^n, \bar{B}^n) \in C([0, T^n); H^{s+1}(\mathbb{R}^N)) \times (C[0, T^n); H^s(\mathbb{R}^N))\\ \times L^1_{T^n}(H^s(\mathbb{R}^N)) \times C([0, T^n); H^s(\mathbb{R}^N)) \end{aligned}$$

and verifies

$$\left.\begin{aligned} \partial_t a^n + u^n \cdot \nabla a^n = 0,\\ \partial_t \bar{u}^n + u^n \cdot \nabla \bar{u}^n - \mu^1 \Delta \bar{u}^n + \nabla \bar{\Pi}^n = H(a^n, u^n, \nabla \Pi^n, B^n),\\ \partial_t \bar{B}^n + u^n \cdot \nabla \bar{B}^n - \sigma^1 \Delta \bar{B}^n\\ = -\operatorname{div}[(\tilde{\sigma}(a^n) - \sigma^1)\nabla B^n] + B^n \cdot \nabla u^n - u^n \cdot \nabla B_L^n,\\ \operatorname{div} \bar{u}^n = \operatorname{div} \bar{B}^n = 0,\\ (a^n, \bar{u}^n, \bar{B}^n)|_{t=0} = (a_0^n, 0, 0), \end{aligned}\right\} \tag{NL}$$

where

$$\begin{aligned} H(a^n, u^n, \nabla \Pi^n, B^n) = -u^n \cdot \nabla u_L^n + a^n(\mu^1 \Delta u^n - \nabla \Pi^n)\\ + 2(1 + a^n) \operatorname{div}\{(\tilde{\mu}(a^n) - \mu^1)\mathcal{M}^n\}\\ + (1 + a^n)(B^n \cdot \nabla B^n - \tfrac{1}{2} \nabla B^{n2}) \end{aligned}$$

with $\mathcal{M}^n = \frac{1}{2}(\nabla u^n +^t \nabla u^n)$. We find that $(a^n, \bar{u}^n, \nabla \bar{\Pi}^n, \bar{B}^n)$ belongs to E_{T^n} by following the arguments in [1].

Now we are in a position to prove that $T > 0$ such that $(a^n, u^n, \nabla \Pi^n, B^n)$ is bounded in E_T.

In what follows, we will use the following notation:

$$U^n(t) := \|\bar{u}^n\|_{\tilde{L}_t^\infty(\dot{B}_{p_2,1}^{(N/p_2)-1})} + \|\bar{u}^n\|_{L_t^1(\dot{B}_{p_2,1}^{(N/p_2)+1})} + \|\nabla \bar{\Pi}^n\|_{L_t^1(\dot{B}_{p_2,1}^{(N/p_2)-1})}$$

and

$$B^n(t) := \|\bar{B}^n\|_{\tilde{L}_t^\infty(\dot{B}_{p_2,1}^{(N/p_2)-1})} + \|\bar{B}^n\|_{L_t^1(\dot{B}_{p_2,1}^{(N/p_2)+1})}.$$

Since $1/p_1 \leqslant 1/N + 1/p_2$, according to proposition 3.1, we have

$$\begin{aligned}\|a^n\|_{\tilde{L}^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})} &\leqslant \exp\big(C\|u^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)\|a^n_0\|_{\dot{B}^{N/p_1}_{p_1,1}} \\ &\lesssim \exp\big(C\|u^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}.\end{aligned}$$

Moreover, proposition 3.2 implies that

$$U^n(T^n) \leqslant C\exp\big(C\|\nabla u^n\|_{L^1_{T^n}(\dot{B}^{N/p_2}_{p_2,1})}\big)\|H(a^n, u^n, \nabla \Pi^n, B^n)\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})}.$$

Since $1/p_1 + 1/p_2 > 1/N$, the inequality (2.3) implies that

$$\begin{aligned}&\|a^n(\mu^1\Delta u^n - \nabla \Pi^n)\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})} \\ &\qquad \lesssim \|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big(\|u^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})} + \|\nabla \Pi^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})}\big).\end{aligned} \tag{4.13}$$

From the Bernstein inequality, (2.3) and a classical interpolation argument, we may infer that

$$\begin{aligned}\|u^n\cdot\nabla u^n_L\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})} &\lesssim \|u^n\otimes u^n_L\|_{L^1_{T^n}(\dot{B}^{N/p_2}_{p_2,1})} \\ &\lesssim \|u^n\|_{L^2_{T^n}(\dot{B}^{N/p_2}_{p_2,1})}\|u^n_L\|_{L^2_{T^n}(\dot{B}^{N/p_2}_{p_2,1})}.\end{aligned} \tag{4.14}$$

Since $p_1 \leqslant p_2$ and $1/p_1 + 1/p_2 > 1/N$, the Bernstein inequality, estimate (2.3), and Taylor's formula imply that

$$\begin{aligned}&\|(1+a^n)\operatorname{div}\{(\tilde{\mu}(a^n)-\mu^1)\mathcal{M}^n\}\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})} \\ &\qquad\lesssim \big(1+\|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big)\|(\tilde{\mu}(a^n)-\mu^1)\mathcal{M}^n\|_{L^1_{T^n}(\dot{B}^{N/p_2}_{p_2,1})} \\ &\qquad\lesssim \big(1+\|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big)\|\tilde{\mu}(a^n)-\mu^1\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\|u^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})} \\ &\qquad\lesssim \big(1+\|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big)\|a^n\|_{\tilde{L}^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\|u^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})}\end{aligned}$$

and

$$\begin{aligned}&\|(1+a^n)(B^n\cdot\nabla B^n - \tfrac{1}{2}\nabla B^{n2})\|_{L^1_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})} \\ &\qquad\lesssim \big(1+\|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big)\big(\|B^n\otimes B^n\|_{L^1_{T^n}(\dot{B}^{N/p_2}_{p_2,1})} + \|B^{n2}\|_{L^1_{T^n}(\dot{B}^{N/p_2}_{p_2,1})}\big) \\ &\qquad\lesssim \big(1+\|a^n\|_{L^\infty_{T^n}(\dot{B}^{N/p_1}_{p_1,1})}\big)\|B^n\|_{L^\infty_{T^n}(\dot{B}^{(N/p_2)-1}_{p_2,1})}\|B^n\|_{L^1_{T^n}(\dot{B}^{(N/p_2)+1}_{p_2,1})}.\end{aligned}$$

For $\bar{B}^n$, we have

$$\partial_t\bar{B}^n + u^n\cdot\nabla\bar{B}^n - \sigma^1\Delta\bar{B}^n = -\operatorname{div}[(\tilde{\sigma}(a^n)-\sigma^1)\nabla B^n] + \bar{B}^n\cdot\nabla u^n + B^n_L\cdot\nabla u^n - u^n\cdot\nabla B^n_L,$$

By proposition 3.2 and inequalities (2.3) and (2.5) we deduce that, for $t\in[0,T^n]$,

$$\begin{aligned}B^n(t) &\leqslant C\exp\big(C\|u^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big) \\ &\quad\times\big\{\|B^n_L\|_{L^2_t(\dot{B}^{N/p_2}_{p_2,1})}\|u^n\|_{L^2_t(\dot{B}^{N/p_2}_{p_2,1})} + \|a^n\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\|B^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})} \\ &\qquad + \|u^n\|_{L^\infty_t(\dot{B}^{(N/p_2)-1}_{p_2,1})}\|B^n_L\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big\}.\end{aligned}$$

So, by interpolation, we have

$$\|v\|_{L_t^2(\dot B^{N/p_2}_{p_2,1})} \leqslant \|v\|^{1/2}_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})}\|v\|^{1/2}_{L_t^\infty(\dot B^{-1+(N/p_2)}_{p_2,1})}$$

$$\text{for all } v \in L_t^1(\dot B^{(N/p_2)+1}_{p_2,1}) \cap L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1});$$

thus,

$$\begin{aligned} B^n(t) \lesssim \exp\big(C\|u^n\|_{L_t^1(\dot B^{(N/p_1)+1}_{p_2,1})}\big) \\ \times \big\{\|B_L^n\|^{1/2}_{L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1})}\|B_L^n\|^{1/2}_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} \\ \times \|u^n\|^{1/2}_{L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1})}\|u^n\|^{1/2}_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} \\ + \|a^n\|_{\tilde L_t^\infty(\dot B^{N/p_1}_{p_1,1})}\|B^n\|_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} \\ + \|u^n\|_{L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1})}\|B_L^n\|_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})}\big\}. \end{aligned}$$

In the same manner, we have

$$\begin{aligned} U^n(t) \lesssim \exp\big(C\|u^n\|_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})}\big) \\ \times \big[\|u_L^n\|^{1/2}_{L^\infty_{t(\dot B^{(N/p_2)-1}_{p_2,1})}}\|u_L^n\|^{1/2}_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} \\ \times \|u^n\|^{1/2}_{L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1})}\|u^n\|^{1/2}_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} \\ + \|a^n\|_{\tilde L_t^\infty(\dot B^{N/p_1}_{p_1,1})}\big(1+\|a^n\|_{L_t^\infty(\dot B^{N/p_1}_{p_1,1})}\big) \\ \times \big(\|u^n\|_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})} + \|\nabla\Pi^n\|_{L_t^1(\dot B^{(N/p_2)-1}_{p_2,1})} \\ + \|B^n\|_{L_t^\infty(\dot B^{(N/p_2)-1}_{p_2,1})}\|B^n\|_{L_t^1(\dot B^{(N/p_2)+1}_{p_2,1})}\big)\big]. \end{aligned} \tag{4.15}$$

Let ζ be a small positive real number. There then exists $T_1 > 0$ such that

$$\|(u_L, B_L)\|_{L^1_{T_1}(\dot B^{(N/p_2)+1}_{p_2,1})} + \|\nabla\Pi_L\|_{L^1_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} \leqslant \zeta \tag{4.16}$$

and (see [9, proposition 2.3])

$$\|u_L\|_{\tilde L^\infty_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} \leqslant \|u_0\|_{\dot B^{(N/p_2)-1}_{p_2,1}} + \|\mathcal{P}f\|_{L^1_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} := U_0.$$

Consequently, we have

$$\|u_L^n\|_{L^1_{T_1}(\dot B^{(N/p_2)+1}_{p_2,1})} + \|\nabla\Pi_L^n\|_{L^1_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} \leqslant C\zeta \quad\text{and}\quad \|u_L^n\|_{\tilde L^\infty_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} \leqslant CU_0 \tag{4.17}$$

and

$$\|B_L^n\|_{L^1_{T_1}(\dot B^{(N/p_2)+1}_{p_2,1})} \leqslant C\zeta \quad\text{and}\quad \|B_L^n\|_{\tilde L^\infty_{T_1}(\dot B^{(N/p_2)-1}_{p_2,1})} \leqslant C\|B_0\|_{\dot B^{(N/p_2)-1}_{p_2,1}}. \tag{4.18}$$

In the following we can suppose that $T^n \leqslant T_1$; otherwise, we take a smaller T^n. Let $t \leqslant T^n$. Then

$$B^n(t) \leqslant C \exp\big(C(\zeta + \|\bar{u}^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})})\big) \times \big\{\zeta^{1/2}\big(U_0 + \|\bar{u}^n\|_{L^\infty_t(\dot{B}^{(N/p_2)-1}_{p_2,1})}\big)^{1/2}\big(\zeta + \|\bar{u}^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)^{1/2} + \|a^n\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})}\big(\zeta + \|\bar{B}^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big) + \zeta\big(U_0 + \|\bar{u}^n\|_{L^\infty_t(\dot{B}^{(N/p_2)-1}_{p_2,1})}\big)\big\}$$

and

$$\|a^n\|_{\tilde{L}^\infty_t(\dot{B}^{N/p_1}_{p_1,1})} \leqslant C \exp\big(C\big(\zeta + \|\bar{u}^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)\big)\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}. \tag{4.19}$$

Let $T_2 \leqslant T^n$ such that

$$\exp\big(C\big(\zeta + \|\bar{u}^n\|_{L^1_{T_2}(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)\big) < 2. \tag{4.20}$$

Therefore, if

$$16C^2\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}} \leqslant 1,$$

then

$$\|a^n\|_{\tilde{L}^\infty_{T_1}(\dot{B}^{N/p_1}_{p_1,1})} \leqslant 2C\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}} \tag{4.21}$$

and

$$\mathcal{B}^n(T_2) \leqslant 4C\big\{\zeta^{1/2}\|B_0\|^{1/2}_{\dot{B}^{(N/p_2)-1}_{p_2,1}}\big(U_0 + \|\bar{u}^n\|_{L^\infty_t(\dot{B}^{(N/p_2)-1}_{p_2,1})}\big)^{1/2} \times \big(\zeta + \|\bar{u}^n\|_{L^1_t(\dot{B}^{(N/p_2)+1}_{p_2,1})}\big)^{1/2} + 2C\zeta\|a_0\|_{\dot{B}^{N/p}_{p,1}} + \zeta\big(U_0 + \|\bar{u}^n\|_{L^\infty_t(\dot{B}^{(N/p_2)-1}_{p_2,1})}\big)\big\}. \tag{4.22}$$

Using inequalities (4.15) and (4.22) satisfied by $B^n = B^n_L + \bar{B}^n$, we obtain that

$$U^n(T_2) \leqslant C\big\{\zeta(U^n(T_2) + U_0) + 2C\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}\big(1 + 2C\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}\big)(\zeta + U^n(T_2)) + \zeta\|B_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}}\big(1 + 2C\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}\big)(U_0^2 + \zeta^2 + U^n(T_2)^2)\big\}.$$

Using (4.21) and the smallness of a_0, for ζ small enough, we obtain

$$U^n(T_2) \leqslant \zeta C\big(U_0, \|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}}, \|B_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}}\big). \tag{4.23}$$

Taking ζ small enough, we observe that inequality (4.20) is satisfied. Consequently, a standard argument then yields that $T_2 = T^n$. The same type of reasoning allows us to show that $T^n = T^1$, with uniform control.

In what follows, we give a precise estimate of the pressure term. Namely, we prove the following lemma.

Lemma 4.4. *Let $0 < \eta < \inf(1, 2N/p_2)$ be such that $1/N + \eta/N < 1/p_1 + 1/p_2$. Then $\nabla\Pi^n$ is uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$.*

Proof. Applying the divergence operator to the equation containing the pressure term, we obtain

$$\operatorname{div}((1+a^n)\nabla \Pi^n) = \operatorname{div}\{(1+a^n)(\operatorname{div}\{\tilde{\mu}(a^n)\mathcal{M}^n\} + B^n\cdot\nabla B^n - \tfrac{1}{2}\nabla B^{n2}) + \mathcal{Q}f^n - u^n\cdot\nabla u^n\}.$$

By construction of f^n and by interpolation, we have the result that $\mathcal{Q}f^n$ is uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$. Then, by interpolation, we have the result that u^n is uniformly bounded in $L^{2/(1-\eta)}_{T_1}(\dot{B}^{(N/p_2)-\eta}_{p_2,1})$. Since $\eta < 2N/p_2$, inequality (2.5) implies the estimate

$$\begin{aligned}\|u^n\cdot\nabla u^n\|_{L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})} &\lesssim \|u^n\otimes u^n\|_{L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-\eta}_{p_2,1})} \\ &\lesssim \|u^n\|_{L^{2/(1-\eta)}_{T_1}(\dot{B}^{(N/p_2)-\eta}_{p_2,1})}\|u^n\|_{L^2_{T_1}(\dot{B}^{N/p_2}_{p_2,1})},\end{aligned}$$

which shows that $u^n\cdot\nabla u^n$ is uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$. In the same way $(1+a^n)\operatorname{div}\{\tilde{\mu}(a^n)\mathcal{M}^n\}$, for $p_1 \leqslant p_2$, and $1/p_1 + 1/p_2 > 1/N$, the Bernstein inequality and (2.3) imply that $\operatorname{div}\{\tilde{\mu}(a^n)\mathcal{M}^n\}$ is uniformly bounded in

$$L^1_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^2_{T_1}(\dot{B}^{(N/p_2)-2}_{p_2,1}).$$

Therefore, by an interpolation argument, we find that $\operatorname{div}\{\tilde{\mu}(a^n)\mathcal{M}^n\}$ is uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$. Since

$$\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2},$$

inequality (2.3) implies that $(1+a^n)\operatorname{div}\{\tilde{\mu}(a^n)\mathcal{M}^n\}$ is uniformly bounded in

$$L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$$

in the same way as for $(1+a^n)(B^n\cdot\nabla B^n - \frac{1}{2}\nabla B^n)$, so $\nabla\Pi^n$ is also uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$. Indeed, using the fact that

$$\|a^n\|_{\tilde{L}^\infty_{T_1}(\dot{B}^{N/p_1}_{p_1,1})} \leqslant 2C\|a_0\|_{\dot{B}^{N/p_1}_{p_1,1}} \ll 1,$$

we deduced that the Π^n verify an elliptic equation which is a small perturbation of the Laplace equation. □

By the construction of the time of existence, we conclude that $T_1 = \infty$, provided that

$$\|u_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}} + \|B_0\|_{\dot{B}^{(N/p_2)-1}_{p_2,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^{(N/p_2)-1}_{p_2,1})} \leqslant c' \inf(\mu^1, \sigma^1).$$

4.4.1. *Passage to the limit*

Let us note first that by construction of (u_0^n, f^n), the sequence $(u_L^n, \nabla\Pi_L^n, B_L^n)$ converges strongly to the solution $(u_L, \nabla\Pi_L, B_L)$ of the system (L). However, to show that the weak limit of $(a^n, \bar{u}^n, \nabla\bar{\Pi}^n, \bar{B}^n)$ is a solution to the system (NL), we need to use some compactness arguments.

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We have already established that $(a^n, \bar{u}^n, \nabla\bar{\Pi}^n, \bar{B}^n)$ is uniformly bounded in

$$\tilde{L}^\infty_{T_1}(\dot{B}^{N/p_1}_{p_1,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}) \times L^1_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}).$$

Moreover, $\nabla\Pi^n$ is uniformly bounded in $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$.

So, in order to use the Ascoli theorem, it suffices to estimate the time derivative of a^n, $\bar{u}^n$ and $\bar{B}^n$ (see, for example, [8]). Following the proof of [2, lemma 4.8], the following lemma is shown to hold.

Lemma 4.5.

(i) *The sequence* $(\partial_t a^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $L^2_{T_1}(\dot{B}^{(N/p_1)-1}_{p_1,1})$.

(ii) *The sequence* $(\partial_t \bar{u}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$ *for*

$$0 < \eta < \inf\left(1, \frac{2N}{p_2}\right) \quad \textit{and} \quad \frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

(iii) *The sequence* $(\partial_t \bar{B}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $L^{2/(2-\eta)}_{T_1}(\dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$ *for*

$$0 < \eta < \inf\left(1, \frac{2N}{p_2}\right) \quad \textit{and} \quad \frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

From the above lemma, the Cauchy–Schwarz inequality and Hölder's inequality, we deduce the following corollary.

Corollary 4.6.

(i) *The sequence* $(a^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C^{1/2}([0,T_1]; \dot{B}^{(N/p_1)-1}_{p_1,1})$.

(ii) *The sequence* $(\bar{u}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C^{\eta/2}([0,T_1]; \dot{B}^{(N/p_2)-1-\eta}_{p_2,1})$ *for all* η *belonging to* $]0, \inf(1, 2N/p_2)[$ *and*

$$\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

(iii) *The sequence* $(\bar{B}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C^{\eta/2}([0,T_1]; \dot{B}^{(N/p_2)-2}_{p_2,1})$ *for all* η *belonging to* $]0, \inf(1, 2N/p_2)[$ *and*

$$\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}.$$

We recall that the injection of $\dot{B}^{s+\varepsilon}_{pq,\mathrm{loc}}$ in $B^s_{pq,\mathrm{loc}}$ (the inhomogeneous Besov space) is compact for all $\varepsilon > 0$ (see, for example, [18]).

Therefore, there exists a subsequence (still denoted by $(a^n, \bar{u}^n, \nabla\bar{\Pi}^n, \bar{B}^n)$) which converges to $(a, \bar{u}, \nabla\bar{\Pi}, \bar{B})$. Consequently, $(a, u, \nabla\Pi, B)$ is a solution of the $(\widetilde{\mathrm{MHD}})$ system belonging to

$$\tilde{L}^\infty_{T_1}(\dot{B}^{N/p_1}_{p_1,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}) \times L^1_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}).$$

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Concerning the continuity of u, we have used the fact that

$$\left.\begin{aligned} \partial_t u - \mu^1 \Delta u &= H(a, u, \nabla \Pi, B), \\ u|_{t=0} &= u_0, \end{aligned}\right\} \tag{H}$$

where

$$H(a, u, \nabla \Pi, B) = f - u \cdot \nabla u - (1 + a)(\nabla \Pi + \tfrac{1}{2}\nabla B^2 - B \cdot \nabla B) + 2(1 + a) \operatorname{div}\{(\tilde{\mu}(a) - \mu^1)\mathcal{M}\} + \mu^1 a \Delta u.$$

Since

$$(a, u, \nabla \Pi, B) \in \tilde{L}^\infty_{T_1}(\dot{B}^{N/p_1}_{p_1,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}) \times L^1_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \times \tilde{L}^\infty_{T_1}(\dot{B}^{(N/p_2)-1}_{p_2,1}) \cap L^1_{T_1}(\dot{B}^{(N/p_2)+1}_{p_2,1}),$$

proposition 2.4, implies that $H(a, u, \nabla \Pi, B) \in L^1_{T_1}(\dot{B}^{-1+(N/p_2)}_{p_2,1})$. And consequently, [6, proposition 2.1] ensured the continuity-in-time of u, in the same way as for B. To prove that a is continuous and that the L^∞-norm is conserved, we use the fact that $a = a_0 \circ \Psi^{-1}$, where Ψ is the flow of u. This completes the proof of theorem 1.3. □

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