J. Aust. Math. Soc. **112** (2022), 367–390
doi:10.1017/S1446788720000208

SECOND ORDER SUBEXPONENTIALITY AND INFINITE DIVISIBILITY

TOSHIRO WATANABE

(Received 2 January 2020; accepted 11 May 2020; first published online 22 June 2020)

Communicated by N. Ross

Abstract

We characterize the second order subexponentiality of an infinitely divisible distribution on the real line under an exponential moment assumption. We investigate the asymptotic behaviour of the difference between the tails of an infinitely divisible distribution and its Lévy measure. Moreover, we study the second order asymptotic behaviour of the tail of the tth convolution power of an infinitely divisible distribution. The density version for a self-decomposable distribution on the real line without an exponential moment assumption is also given. Finally, the regularly varying case for a self-decomposable distribution on the half line is discussed.

2010 *Mathematics subject classification*: primary 60E07; secondary 60G50, 60G51.

Keywords and phrases: second order subexponentiality, local subexponentiality, infinite divisibility.

1. Introduction and results

The subexponentiality of infinitely divisible distributions on the half line was characterized by Embrechts *et al.* [6] and on the real line by Pakes [16]. The subexponentiality of an infinitely divisible distribution implies the asymptotic equivalence between the tails of the distribution and its Lévy measure. In this paper, we characterize the second order subexponentiality of an infinitely divisible distribution on the real line in terms of its Lévy measure under an exponential moment assumption. The second order subexponentiality yields a higher asymptotic relation than the usual subexponentiality between the tails of an infinitely divisible distribution and its Lévy measure.

In what follows, we denote by $\mathbb{R}$ the real line and by $\mathbb{R}_+$ the half line $[0, \infty)$. Denote by $\mathbb{N}$ the totality of positive integers. The symbol $\delta_a(dx)$ stands for the delta measure at $a \in \mathbb{R}$. Let η and ρ be probability distributions on $\mathbb{R}$. We denote by $\eta * \rho$ the convolution of η and ρ and by ρ^{n*} the nth convolution power of ρ with the understanding that

$\rho^{0*}(dx) = \delta_0(dx)$. The characteristic function of ρ is denoted by $\widehat{\rho}(z)$, namely, for $z \in \mathbb{R}$,

$$\widehat{\rho}(z) := \int_{-\infty}^{\infty} e^{izx}\rho(dx).$$

For a measure ξ on $\mathbb{R}$, we denote by $\bar{\xi}(x)$ the tail $\xi((x,\infty))$ for $x > 0$. For positive functions $f(x)$ and $g(x)$ on $[a,\infty)$ for some $a \in \mathbb{R}$, we define the relation $f(x) \sim g(x)$ by $\lim_{x\to\infty} f(x)/g(x) = 1$. We say that $f(x,A) \sim cf(x)$ as $x \to \infty$ and then $A \to \infty$ if

$$\lim_{A\to\infty}\lim_{x\to\infty} f(x,A)/f(x) = c > 0.$$

We say that $f(x,A) = o(f(x))$ as $x \to \infty$ and then $A \to \infty$ if

$$\limsup_{A\to\infty}\limsup_{x\to\infty} |f(x,A)|/f(x) = 0.$$

Definition 1.1.

(i) A nonnegative measurable function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathbf{L}$ if $g(x+a) \sim g(x)$ for every $a \in \mathbb{R}$.

(ii) Let $\Delta := (0,c]$ with $c > 0$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{L}_\Delta$ if $\rho((x,x+c]) \in \mathbf{L}$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{L}_{\mathrm{loc}}$ if $\rho \in \mathcal{L}_\Delta$ for each $\Delta := (0,c]$ with $c > 0$.

(iii) Let $\Delta := (0,c]$ with $c > 0$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}_\Delta$ if $\rho \in \mathcal{L}_\Delta$ and $\rho^{2*}((x,x+c]) \sim 2\rho((x,x+c])$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}_{\mathrm{loc}}$ if $\rho \in \mathcal{S}_\Delta$ for each $\Delta := (0,c]$ with $c > 0$.

If a distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{L}_{\mathrm{loc}}$, then, for $c > 0$,

$$\rho((x,x+c]) \sim c\rho((x,x+1]),$$

and, for every $\delta > 0$, $e^{\delta x}\rho((x,x+1]) \to \infty$ as $x \to \infty$. See (2.6) in the proof of Theorem 2.1 of Watanabe and Yamamuro [23] and Lemma 2.17 of Foss *et al.* [7]. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}$ if $\bar{\rho}(x) \in \mathbf{L}$ and $\overline{\rho^{2*}}(x) \sim 2\bar{\rho}(x)$. Distributions in the classes $\mathcal{S}$ and $\mathcal{S}_{\mathrm{loc}}$ are called *subexponential* and *locally subexponential*, respectively.

Definition 1.2. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}^2_{\mathrm{loc}}$ if the following three conditions hold.

(1) $\rho \in \mathcal{S}_{\mathrm{loc}}$.

(2) $\int_{-\infty}^{\infty} |x|\rho(dx) < \infty$.

(3) We have

$$\overline{\rho^{2*}}(x) = 2\bar{\rho}(x) + 2m(\rho)\rho((x,x+1]) + o(\rho((x,x+1])) \tag{1-1}$$

as $x \to \infty$. Here we denote by $m(\rho)$ the mean of ρ, namely,

$$m(\rho) := \int_{-\infty}^{\infty} x\rho(dx).$$

It is different from the absolute mean of ρ in the two-sided case.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

The subclasses $\mathcal{S}_\Delta$, $\mathcal{S}_{\mathrm{loc}}$, and $\mathcal{S}^2_{\mathrm{loc}}$ of the class $\mathcal{S}$ were respectively introduced by Asmussen *et al.* [1], Watanabe and Yamamuro [23], and Lin [13]. Lin [13] treated the one-sided case and used the symbol $\mathcal{S}_2$ for the class $\mathcal{S}^2_{\mathrm{loc}}$. Distributions in the class $\mathcal{S}^2_{\mathrm{loc}}$ are called *second order subexponential*. Infinitely divisible distributions on $\mathbb{R}$ in the classes $\mathcal{S}_\Delta$ and $\mathcal{S}_{\mathrm{loc}}$ are found in Watanabe and Yamamuro [22, 23] and Shimura and Watanabe [18]. Lin [13] gave some sufficient conditions in order that a distribution on $\mathbb{R}_+$ belongs to the class $\mathcal{S}^2_{\mathrm{loc}}$. See [13, Proposition 2.4 and Corollary 2.1]. He showed that the lognormal distribution, Weibull distribution with parameter $\beta \in (0, 1)$, and Pareto distribution with parameter $\alpha > 1$ belong to the class $\mathcal{S}^2_{\mathrm{loc}}$. Geluk and Pakes [9] and Geluk [8] treated another second order subexponentiality. Let μ be an infinitely divisible distribution on $\mathbb{R}$. Then its characteristic function $\widehat{\mu}(z)$ is represented as

$$\widehat{\mu}(z) = \exp\Big(\int_{-\infty}^{\infty}\Big(e^{izx} - 1 - \frac{izx}{1+x^2}\Big)\nu(dx) + i\gamma z - \frac{1}{2}az^2\Big),$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, and ν is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\nu(dx) < \infty.$$

The measure ν is called the Lévy measure of μ. See Sato [17]. Throughout the paper, we assume that the tail $\bar{\nu}(c)$ is positive for all $c > 0$. For $c > 0$, define a normalized distribution $\nu_{(c)}$ as

$$\nu_{(c)}(dx) := 1_{(c,\infty)}(x)\frac{\nu(dx)}{\bar{\nu}(c)}.$$

Here the symbol $1_{(c,\infty)}(x)$ stands for the indicator function of the set (c, ∞). Denote by μ^{t*} the tth convolution power of μ for $t > 0$. Note that μ^{t*} is the distribution of X_t for a certain Lévy process $\{X_t\}$.

THEOREM 1.3. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν. Assume that there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty} \exp(-\epsilon x)\mu(dx) < \infty$. Then we have the following results.*

(i) *$\mu \in \mathcal{S}^2_{\mathrm{loc}}$ if and only if $\nu_{(1)} \in \mathcal{S}^2_{\mathrm{loc}}$.*

(ii) *If $\mu \in \mathcal{S}^2_{\mathrm{loc}}$, then*

$$\bar{\nu}(x) = \bar{\mu}(x) - m(\mu)\mu((x, x+1]) + o(\mu((x, x+1])) \tag{1-2}$$

as $x \to \infty$, equivalently,

$$\bar{\mu}(x) = \bar{\nu}(x) + m(\mu)\nu((x, x+1]) + o(\nu((x, x+1])) \tag{1-3}$$

as $x \to \infty$.

(iii) *Conversely, if (1-2) with finite $m(\mu)$, $\mu \in \mathcal{S}_{\mathrm{loc}}$, and*

$$(\bar{\mu}(x))^2 = o(\mu((x, x+1]))$$

as $x \to \infty$ hold, then $\mu \in \mathcal{S}^2_{\mathrm{loc}}$.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Remark 1.4. An exponential moment assumption in the above theorem is necessary for the restriction of the class $\mathcal{S}_{\rm loc}$ in the two-sided case. See Jiang *et al.* [10] for a detailed account. The one-sided compound Poisson case in the above theorem is already known by Lin [13, Theorem 2.1]. Omey and Willekens [15] studied an infinitely divisible distribution on $\mathbb{R}_+$ with the density of the normalized Lévy measure in the class $\mathcal{S}_d^2$. For the definition of the class $\mathcal{S}_d^2$, see Section 4. It is the density version of the class $\mathcal{S}_{\rm loc}^2$. However, they could not characterize the density of an infinitely divisible distribution on $\mathbb{R}_+$ with its density in the class $\mathcal{S}_d^2$ because they did not know Lemmas 2.1 and 4.3 below.

Corollary 1.5. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν. Assume that there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty} \exp(-\epsilon x)\mu(dx) < \infty$. Then we have the following results.*

(i) *$\mu \in \mathcal{S}_{\rm loc}^2$ if and only if $\mu^{t*} \in \mathcal{S}_{\rm loc}^2$ for some $t > 0$, equivalently, for all $t > 0$.*

(ii) *If $\mu \in \mathcal{S}_{\rm loc}^2$, then, for all $t > 0$,*

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x) + (t^2 - t)m(\mu)\mu((x, x+1]) + o(\mu((x, x+1])) \qquad (1\text{-}4)$$

as $x \to \infty$.

Remark 1.6. Let μ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure ν. If $\mu \in \mathcal{S}_{\rm loc}$, $m(\mu) < \infty$, and μ satisfies (1-4) for $t = t_0, t_0 + 1$ with some $t_0 > 0$, then $\mu \in \mathcal{S}_{\rm loc}^2$.

An infinitely divisible distribution μ on $\mathbb{R}$ is called *self-decomposable* if, for every $b \in (0, 1)$, there is a distribution ρ_b on $\mathbb{R}$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho_b}(z).$$

An infinitely divisible distribution μ on $\mathbb{R}$ is self-decomposable if and only if $\nu(dx) = k(x)/|x|\,dx$ with $k(x)$ being nonnegative and increasing on $(-\infty, 0)$ and nonnegative and decreasing on $(0, \infty)$. An infinitely divisible distribution μ on $\mathbb{R}$ is called nondegenerate if it is not a delta measure. Every nondegenerate self-decomposable distribution μ on $\mathbb{R}$ is absolutely continuous and unimodal. Many important statistical distributions are known to be self-decomposable. However, their Lévy measures and the tth convolution powers are often not explicitly known. See Sato [17]. In Section 4, we shall prove the following result (the classes $\mathcal{S}_d$ and $\mathcal{S}_d^2$ are defined in Definitions 4.1 and 4.2 below).

Theorem 1.7. *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}$ with Lévy measure $\nu(dx) = k(x)/|x|\,dx$. Assume that $\int_{-\infty}^{0-} |x|\mu(dx) < \infty$. Then the following hold.*

(i) *$p(x) \in \mathcal{S}_d^2$ if and only if $\nu_{(1)} \in \mathcal{S}_{\rm loc}^2$, equivalently,*

$$\frac{1}{\bar{\nu}(1)} 1_{(1,\infty)}(x)k(x)/x \in \mathcal{S}_d^2.$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

(ii) *If $p(x) \in \mathcal{S}_d^2$, then*

$$\bar{\nu}(x) = \bar{\mu}(x) - m(\mu)p(x) + o(p(x)) \tag{1-5}$$

as $x \to \infty$, equivalently,

$$\bar{\mu}(x) = \bar{\nu}(x) + m(\mu)k(x)/x + o(k(x)/x) \tag{1-6}$$

as $x \to \infty$.

(iii) *Conversely, if (1-5) with finite $m(\mu)$, $p(x) \in \mathcal{S}_d$, and $(\bar{\mu}(x))^2 = o(p(x))$ as $x \to \infty$ hold, then $p(x) \in \mathcal{S}_d^2$.*

Corollary 1.8. *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}$ with Lévy measure ν. Let $p^t(x)$ be the density of $\mu^{t*}(dx)$ for $t > 0$. Then the following hold.*

(i) *$p(x) \in \mathcal{S}_d^2$ if and only if $p^t(x) \in \mathcal{S}_d^2$ for some $t > 0$, equivalently, for all $t > 0$.*

(ii) *If $p(x) \in \mathcal{S}_d^2$, then, for all $t > 0$,*

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x) + (t^2 - t)m(\mu)p(x) + o(p(x))$$

as $x \to \infty$.

In Section 6, we shall discuss self-decomposable distributions on $\mathbb{R}_+$ with regularly varying densities as follows.

Proposition 1.9. *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}_+$ with Lévy measure ν. Assume that $p(x) \sim x^{-\alpha-1}l(x)$ for $0 \le \alpha \le 1$ with $l(x)$ being slowly varying as $x \to \infty$. Define slowly varying functions $l^*(x)$ and $l_*(x)$ as $l^*(x) = \int_1^x l(u)/u\,du$ and $l_*(x) = \int_x^\infty l(u)/u\,du$ for $x > 1$. Then we have the following results.*

(i) *Let $0 < \alpha < 1$ and define $K(\alpha)$ as*

$$K(\alpha) := \frac{(2\alpha - 1)(\Gamma(1-\alpha))^2}{2\alpha\Gamma(2-2\alpha)}.$$

Then

$$\bar{\nu}(x) = \bar{\mu}(x)(1 - K(\alpha)x^{-\alpha}l(x) + o(x^{-\alpha}l(x))) \tag{1-7}$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)(1 + (t-1)K(\alpha)x^{-\alpha}l(x) + o(x^{-\alpha}l(x))) \tag{1-8}$$

as $x \to \infty$.

(ii) *Let $\alpha = 1$. Assume that $l^*(\infty) = \infty$. Then*

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 - \frac{l^*(x)}{x} + o\Big(\frac{l^*(x)}{x}\Big)\Big) \tag{1-9}$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 + (t-1)\frac{l^*(x)}{x} + o\Big(\frac{l^*(x)}{x}\Big)\Big) \tag{1-10}$$

as $x \to \infty$.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

(iii) *Let $\alpha = 1$. Assume that $l^*(\infty) < \infty$. Then*

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 - \frac{m(\mu)}{x} + o\Big(\frac{1}{x}\Big)\Big) \tag{1-11}$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 + (t-1)\frac{m(\mu)}{x} + o\Big(\frac{1}{x}\Big)\Big) \tag{1-12}$$

as $x \to \infty$.

(iv) *Let $\alpha = 0$. Then*

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 + \frac{l_*(x)}{2} + o(l_*(x))\Big) \tag{1-13}$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 - (t-1)\frac{l_*(x)}{2} + o(l_*(x))\Big). \tag{1-14}$$

The organization of this paper is as follows. In Section 2, we give preliminaries for the proof of Theorem 1.3 and its corollary. In Section 3, we prove Theorem 1.3 and its corollary. In Section 4, we treat the self-decomposable case and prove Theorem 1.7 and its corollary. In Section 5, three examples of the results are given. In Section 6, we give some remarks on the regularly varying case and prove Proposition 1.9.

2. Preliminaries

Watanabe and Yamamuro [23] used the main results of Watanabe [20] on the convolution equivalence of infinitely divisible distributions on $\mathbb{R}$ to prove the following two lemmas.

LEMMA 2.1 [23, Corollary 2.1]. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν. Assume that there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty} \exp(-\epsilon x)\mu(dx) < \infty$. Then the following are equivalent.*

(1) $\mu \in \mathcal{S}_{\mathrm{loc}}$.

(2) $\nu_{(1)} \in \mathcal{S}_{\mathrm{loc}}$.

(3) $\nu_{(1)} \in \mathcal{L}_{\mathrm{loc}}$ *and* $\mu((x, x+c]) \sim \nu((x, x+c])$ *for all* $c > 0$.

REMARK 2.2. Since $\mathcal{S}_{\mathrm{loc}} \subset \mathcal{S}$, we see that if condition (1) holds in the above lemma, then

$$\bar{\mu}(x) \sim \bar{\nu}(x) \in \mathbf{L}.$$

LEMMA 2.3 [23, Corollary 3.1]. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν. Assume that there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty} \exp(-\epsilon x)\mu(dx) < \infty$. If $\mu^{t*} \in \mathcal{S}_{\mathrm{loc}}$ for some $t > 0$, then $\mu^{t*} \in \mathcal{S}_{\mathrm{loc}}$ for all $t > 0$ and*

$$\mu^{t*}((x, x+c]) \sim t\mu((x, x+c])$$

for all $t > 0$ and for all $c > 0$.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Lin [13] proved the following three lemmas.

LEMMA 2.4 [13, Theorem 2.1]. *Let ρ be a distribution on $\mathbb{R}_+$. Let $\{p_n\}_{n=0}^{\infty}$ be a nonnegative sequence with $p_n > 0$ for some $n \geq 2$ and $\sum_{n=0}^{\infty} p_n = 1$ satisfying $\sum_{n=0}^{\infty} p_n(1+\epsilon_1)^n < \infty$ for some $\epsilon_1 > 0$. Define a distribution η on $\mathbb{R}_+$ as*

$$\eta(dx) := \sum_{n=0}^{\infty} p_n \rho^{n*}(dx).$$

Then we have the following results.

(i) *If $\rho \in \mathcal{S}^2_{\mathrm{loc}}$, then we have $\eta \in \mathcal{S}^2_{\mathrm{loc}}$ and*

$$\overline{\eta}(x) = \Big(\sum_{n=1}^{\infty} np_n\Big)\bar{\rho}(x) + \Big(\sum_{n=2}^{\infty} n(n-1)p_n\Big)m(\rho)\rho((x,x+1]) + o(\rho((x,x+1])) \tag{2-1}$$

as $x \to \infty$.

(ii) *Conversely, if (2-1) with finite $m(\rho)$, $\rho \in \mathcal{S}_{\mathrm{loc}}$, and*

$$(\bar{\rho}(x))^2 = o(\rho((x,x+1]))$$

as $x \to \infty$ hold, then $\rho \in \mathcal{S}^2_{\mathrm{loc}}$.

REMARK 2.5. We can see from the proof of Theorem 2.1 of [13] that even in the case of $p_n < 0$ for some $n \geq 0$, assertion (i) of the above lemma is still true if $\sum_{n=0}^{\infty} |p_n|(1+\epsilon_1)^n < \infty$ for some $\epsilon_1 > 0$.

LEMMA 2.6 [13, Proposition 2.3]. *Let ρ and η be distributions on $\mathbb{R}_+$. If $\rho \in \mathcal{S}^2_{\mathrm{loc}}$ and there are $K > 0$ and $c \in \mathbb{R}$ such that*

$$\lim_{x\to\infty} \frac{\bar{\eta}(x) - K\bar{\rho}(x)}{\rho((x,x+1])} = c,$$

then $\eta \in \mathcal{S}^2_{\mathrm{loc}}$.

LEMMA 2.7 [13, Lemma 3.4]. *Let ρ be a distribution on $\mathbb{R}_+$. Assume that $m(\rho) < \infty$, $\rho \in \mathcal{L}_{\mathrm{loc}}$, and $(\bar{\rho}(x))^2 = o(\rho(x,x+1])$ as $x \to \infty$. Then the relation (1-1) implies that $\rho \in \mathcal{S}_{\mathrm{loc}}$.*

Let $\delta := \bar{\nu}(c)$ for $c > 0$. Define a compound Poisson distribution μ_1 and a distribution σ on $\mathbb{R}_+$ as

$$\mu_1 := e^{-\delta} \sum_{n=0}^{\infty} \frac{\delta^n}{n!}(\nu_{(c)})^{n*} \tag{2-2}$$

and

$$\sigma := \frac{e^{-\delta}}{1-e^{-\delta}} \sum_{n=1}^{\infty} \frac{\delta^n}{n!}(\nu_{(c)})^{n*}. \tag{2-3}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Lemma 2.8. *We can choose a sufficiently large $c > 0$ such that $0 < e^{\delta} - 1 < 1$ and*

$$\nu_{(c)} = -\frac{1}{\delta}\sum_{n=1}^{\infty}\frac{(1-e^{\delta})^n}{n}\sigma^{n*}. \tag{2-4}$$

Proof. Under the assumption that $0 < e^{\delta} - 1 < 1$, we define a signed measure η as

$$\eta := -\frac{1}{\delta}\sum_{n=1}^{\infty}\frac{(1-e^{\delta})^n}{n}\sigma^{n*}.$$

Let ρ be a signed measure on $\mathbb{R}_+$. Denote by $L_\rho(t)$ for $t \geq 0$ the Laplace transform of ρ, that is, $L_\rho(t) := \int_{0-}^{\infty} e^{-tx}\rho(dx)$. We have

$$\begin{aligned} L_\eta(t) &= -\frac{1}{\delta}\sum_{n=1}^{\infty}\frac{(1-e^{\delta})^n}{n}(L_\sigma(t))^n \\ &= \frac{1}{\delta}\log(1-(1-e^{\delta})L_\sigma(t)). \end{aligned}$$

We see from (2-3) that

$$L_\sigma(t) = (e^{\delta}-1)^{-1}(\exp(\delta L_{\nu_{(c)}}(t)) - 1).$$

Thus,

$$\begin{aligned} L_\eta(t) &= \frac{1}{\delta}\log(\exp(\delta L_{\nu_{(c)}}(t))) \\ &= L_{\nu_{(c)}}(t) \end{aligned}$$

and hence we have $\eta = \nu_{(c)}$, that is, (2-4). □

3. Proof of Theorem 1.3 and its corollary

Proof of Theorem 1.3. Let μ be an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν. We define an infinitely divisible distribution μ_2 by $\mu = \mu_1 * \mu_2$. As in Lemma 2.8, we choose a sufficiently large $c > 0$ such that $0 < e^{\delta} - 1 < 1$ and $\int_{0-}^{\infty} y\mu_2(dy) > 0$. Assume that there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty}\exp(-\epsilon x)\mu(dx) < \infty$. Note that the Lévy measure ν_2 of μ_2 is $1_{(-\infty,c]}(x)\nu(dx)$ and hence $\int_{-\infty}^{-1}\exp(-\epsilon x)\nu_2(dx) < \infty$ and, for every $b > 0$, $\int_1^{\infty}\exp(bx)\nu_2(dx) < \infty$. Then we see from Theorem 25.17 of Sato [17] that $\int_{-\infty}^{\infty}\exp(-\epsilon x)\mu_2(dx) < \infty$ and, for every $b > 0$, $\int_{-\infty}^{\infty}\exp(bx)\mu_2(dx) < \infty$. Hence, for every $b > 0$, $\overline{\mu_2}(x) = o(e^{-bx})$ as $x \to \infty$. We find from Lemma 2.1 that $\mu \in \mathcal{S}_{\rm loc}$ if and only if $\mu_1 \in \mathcal{S}_{\rm loc}$. Since $\int_{-\infty}^{\infty}\exp(-\epsilon x)\mu(dx) < \infty$, we have $\int_{-\infty}^{\infty}|x|\mu(dx) < \infty$ if and only if $\int_{-\infty}^{\infty}|x|\mu_1(dx) < \infty$. Suppose that $\mu \in \mathcal{S}_{\rm loc}$ and $\int_{-\infty}^{\infty}|x|\mu(dx) < \infty$, that is, $\mu_1 \in \mathcal{S}_{\rm loc}$ and $\int_{-\infty}^{\infty}|x|\mu_1(dx) < \infty$. We have

$$\begin{aligned} \bar{\mu}(x) - \bar{\mu}_1(x) &= \overline{\mu_1 * \mu_2}(x) - \bar{\mu}_1(x) \\ &= \int_{0-}^{\infty}\mu_1((x-y,x])\mu_2(dy) - \int_{-\infty}^{0-}\mu_1((x,x-y])\mu_2(dy) \\ &= I_1 - I_2. \end{aligned} \tag{3-1}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Note that if $\int_{-\infty}^{0-} |y|\mu_2(dy) = 0$, then $I_2 = 0$. We find that

$$I_1 := I_{11} + I_{12} + I_{13},$$

where, for $A > 0$,

$$I_{11} := \int_{0-}^{A+} \mu_1((x-y, x])\mu_2(dy),$$

$$I_{12} := \int_{A+}^{x/2+} \mu_1((x-y, x])\mu_2(dy),$$

and

$$I_{13} := \int_{x/2+}^{\infty} \mu_1((x-y, x])\mu_2(dy).$$

We have by $\mu_1 \in \mathcal{S}_{\mathrm{loc}} \subset \mathcal{L}_{\mathrm{loc}}$ that

$$\begin{aligned} I_{11} &\sim \mu_1((x, x+1]) \int_{0-}^{A+} y\mu_2(dy) \\ &\sim \mu_1((x, x+1]) \int_{0-}^{\infty} y\mu_2(dy) \end{aligned}$$

as $x \to \infty$ and then $A \to \infty$. For any $\epsilon_1 \in (0, 1)$, there is $C_1 = C_1(\epsilon_1) > 0$ such that, for $0 \le y \le x/2$ and for sufficiently large $x > 0$,

$$\frac{\mu_1((x-y, x])}{\mu_1((x, x+1])} \le C_1 e^{\epsilon_1 y}.$$

Thus,

$$\begin{aligned} I_{12} &\le \mu_1((x, x+1])C_1 \int_{A+}^{x/2+} e^{\epsilon_1 y}\mu_2(dy) \\ &= o(\mu_1((x, x+1])) \end{aligned}$$

as $x \to \infty$ and then $A \to \infty$. We have

$$I_{13} \le \bar{\mu}_2(x/2) = o(e^{-x}) = o(\mu_1((x, x+1]))$$

as $x \to \infty$. Thus,

$$I_1 \sim \mu_1((x, x+1]) \int_{0-}^{\infty} y\mu_2(dy). \tag{3-2}$$

Since $\mu_1 \in \mathcal{S}_{\mathrm{loc}}$, for any $\epsilon_2 \in (0, \epsilon)$, there is $C_2 = C_2(\epsilon_2) > 0$ such that, for $y < 0$ and for sufficiently large $x > 0$,

$$\frac{\mu_1((x, x-y])}{\mu_1((x, x+1])} \le C_2 e^{\epsilon_2 |y|}.$$

Thus, by the dominated convergence theorem, we see that $I_2 = 0$ or

$$I_2 \sim \mu_1((x, x+1]) \int_{-\infty}^{0-} |y|\mu_2(dy). \tag{3-3}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Hence, we find from (3-1)–(3-3) that

$$\bar{\mu}(x) - \bar{\mu}_1(x) = m(\mu_2)\mu_1((x, x+1]) + o(\mu_1((x, x+1])) \tag{3-4}$$

as $x \to \infty$. By an argument analogous to the above equation,

$$\overline{\mu^{2*}}(x) - \overline{\mu_1^{2*}}(x) = m(\mu_2^{2*})\mu_1^{2*}((x, x+1]) + o(\mu_1^{2*}((x, x+1]))$$

as $x \to \infty$. Since $\mu_1^{2*}((x, x+1]) \sim 2\mu_1((x, x+1])$ and $m(\mu_2^{2*}) = 2m(\mu_2)$,

$$\overline{\mu^{2*}}(x) - \overline{\mu_1^{2*}}(x) = 4m(\mu_2)\mu_1((x, x+1]) + o(\mu_1((x, x+1]))$$

as $x \to \infty$. Thus, we see from (3-4) that

$$\begin{aligned} &\overline{\mu^{2*}}(x) - 2\bar{\mu}(x) \\ &\quad = \overline{\mu_1^{2*}}(x) - 2\overline{\mu_1}(x) + 2m(\mu_2)\mu_1((x, x+1]) + o(\mu_1((x, x+1])) \end{aligned} \tag{3-5}$$

as $x \to \infty$. Since $m(\mu) = m(\mu_1) + m(\mu_2)$ and we find from Lemma 2.1 that

$$\mu((x, x+1]) \sim \nu((x, x+1]) \sim \mu_1((x, x+1]), \tag{3-6}$$

we have by (3-5) that

$$\overline{\mu^{2*}}(x) = 2\bar{\mu}(x) + 2m(\mu)\mu((x, x+1]) + o(\mu((x, x+1]))$$

as $x \to \infty$ if and only if

$$\overline{\mu_1^{2*}}(x) = 2\bar{\mu}_1(x) + 2m(\mu_1)\mu_1((x, x+1]) + o(\mu_1((x, x+1]))$$

as $x \to \infty$. Hence, $\mu \in \mathcal{S}^2_{\text{loc}}$ if and only if $\mu_1 \in \mathcal{S}^2_{\text{loc}}$. Since, for $x > 0$,

$$\frac{\overline{\mu_1}(x)}{1 - e^{-\delta}} = \bar{\sigma}(x),$$

we see from Lemma 2.6 that $\sigma \in \mathcal{S}^2_{\text{loc}}$ if and only if $\mu_1 \in \mathcal{S}^2_{\text{loc}}$. We find from (2-3), Lemma 2.8, and Remark 2.5 that if $\sigma \in \mathcal{S}^2_{\text{loc}}$, then $\nu_{(c)} \in \mathcal{S}^2_{\text{loc}}$ for sufficiently large $c > 0$. We see from (2-2) and Lemma 2.4 that if $\nu_{(c)} \in \mathcal{S}^2_{\text{loc}}$, then $\mu_1 \in \mathcal{S}^2_{\text{loc}}$. Thus, for sufficiently large $c > 0$, $\mu \in \mathcal{S}^2_{\text{loc}}$ if and only if $\nu_{(c)} \in \mathcal{S}^2_{\text{loc}}$. Since, for sufficiently large $x > 0$,

$$\overline{\nu_{(c)}}(x) = \frac{\bar{\nu}(1)}{\bar{\nu}(c)}\overline{\nu_{(1)}}(x),$$

we obtain from Lemma 2.6 that $\nu_{(1)} \in \mathcal{S}^2_{\text{loc}}$ if and only if $\nu_{(c)} \in \mathcal{S}^2_{\text{loc}}$ for sufficiently large $c > 0$. Thus, we have $\mu \in \mathcal{S}^2_{\text{loc}}$ if and only if $\nu_{(1)} \in \mathcal{S}^2_{\text{loc}}$. We have proved assertion (i).

Next we prove assertion (ii). Assume that $\mu \in \mathcal{S}^2_{\text{loc}}$, equivalently, $\nu_{(c)} \in \mathcal{S}^2_{\text{loc}}$ for $c > 0$. Note that $m(\mu_1) = \delta m(\nu_{(c)})$. We see from Lemma 2.4 that

$$\begin{aligned} \overline{\mu_1}(x) &= e^{-\delta}\sum_{n=1}^{\infty}\frac{\delta^n}{(n-1)!}\overline{\nu_{(c)}}(x) \\ &\quad + e^{-\delta}\sum_{n=2}^{\infty}\frac{\delta^n}{(n-2)!}m(\nu_{(c)})\nu_{(c)}((x, x+1]) + o(\nu_{(c)}((x, x+1])) \\ &= \bar{\nu}(x) + m(\mu_1)\nu((x, x+1]) + o(\nu((x, x+1])) \end{aligned} \tag{3-7}$$

as $x \to \infty$. Thus, we obtain (1-2) and (1-3) from (3-4) and (3-6).

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Next we prove assertion (iii). We see from (3-4) that the assumption that (1-2) with finite $m(\mu)$, $\mu \in \mathcal{S}_{\rm loc}$, and $(\bar{\mu}(x))^2 = o(\mu((x, x+1]))$ as $x \to \infty$ is equivalent to that (3-7) with finite $m(\nu_{(c)})$, $\nu_{(c)} \in \mathcal{S}_{\rm loc}$, and $(\overline{\nu_{(c)}}(x))^2 = o(\nu_{(c)}((x, x+1]))$ as $x \to \infty$. This implies from Lemma 2.4 that $\nu_{(c)} \in \mathcal{S}^2_{\rm loc}$, equivalently, $\mu \in \mathcal{S}^2_{\rm loc}$.

Proof of Corollary 1.5. We see from Theorem 1.3 that $\mu^{t*} \in \mathcal{S}^2_{\rm loc}$ for some $t > 0$, equivalently, for all $t > 0$ if and only if $\nu_{(1)} \in \mathcal{S}^2_{\rm loc}$. Hence, assertion (i) is true.

Next we prove assertion (ii). Suppose that $\mu \in \mathcal{S}^2_{\rm loc}$. Then we find from (i) that $\mu^{t*} \in \mathcal{S}^2_{\rm loc}$ for all $t > 0$. We see from (1-2) that

$$\overline{\mu^{t*}}(x) = t\bar{\nu}(x) + m(\mu^{t*})\mu^{t*}((x, x+1]) + o(\mu^{t*}((x, x+1]))$$

as $x \to \infty$. Note that $m(\mu^{t*}) = tm(\mu)$ and from Lemma 2.3 that

$$\mu^{t*}((x, x+1]) \sim t\mu((x, x+1]).$$

Thus, we have by (1-2) that

$$\begin{aligned}\overline{\mu^{t*}}(x) &= t\bar{\nu}(x) + t^2 m(\mu)\mu((x, x+1]) + o(\mu((x, x+1]))\\ &= t\bar{\mu}(x) + (t^2 - t)m(\mu)\mu((x, x+1]) + o(\mu((x, x+1]))\end{aligned}$$

as $x \to \infty$. We have proved (1-4). □

Proof of Remark 1.6. Assume that $\mu \in \mathcal{S}_{\rm loc}$, $m(\mu) < \infty$, and μ satisfies (1-4) for $t = t_0, t_0 + 1$ with some $t_0 > 0$. Then

$$\begin{aligned}&\overline{\mu^{(t_0+1)*}}(x) - (t_0+1)\bar{\mu}(x) - t_0(\overline{\mu^{2*}}(x) - 2\bar{\mu}(x))\\ &\quad = \int_{0-}^{x+} \overline{\mu^{t_0*}}(x-y)\mu(dy) + \bar{\mu}(x) - (t_0+1)\bar{\mu}(x)\\ &\qquad - t_0 \int_{0-}^{x+} \bar{\mu}(x-y)\mu(dy) - t_0\bar{\mu}(x) + 2t_0\bar{\mu}(x)\\ &\quad = \int_{0-}^{x+} (\overline{\mu^{t_0*}}(x-y) - t_0\bar{\mu}(x-y))\mu(dy)\\ &\quad = I_1 + I_2 + I_3,\end{aligned} \tag{3-8}$$

where, for $0 < 2A < x$,

$$\begin{aligned}I_1 &:= \int_{0-}^{A+} (\overline{\mu^{t_0*}}(x-y) - t_0\bar{\mu}(x-y))\mu(dy),\\ I_2 &:= \int_{A+}^{(x-A)+} (\overline{\mu^{t_0*}}(x-y) - t_0\bar{\mu}(x-y))\mu(dy),\end{aligned}$$

and

$$I_3 := \int_{(x-A)+}^{x+} (\overline{\mu^{t_0*}}(x-y) - t_0\bar{\mu}(x-y))\mu(dy).$$

We divide the proof into three cases: $t_0 > 1$; $t_0 = 1$; and $0 < t_0 < 1$. Let $t_0 > 1$. By the assumption,

$$\begin{aligned} I_1 &\sim t_0(t_0-1)m(\mu)\int_{0-}^{A+}\mu((x-y,x-y+1])\mu(dy) \\ &\sim t_0(t_0-1)m(\mu)\mu((x,x+1]) \end{aligned} \tag{3-9}$$

as $x \to \infty$ and then $A \to \infty$. We find from $\mu \in \mathcal{S}_{loc}$ that there is $\epsilon > 0$ such that

$$\begin{aligned} |I_2| &\le (1+\epsilon)t_0(t_0-1)m(\mu)\int_{A+}^{(x-A)+}\mu((x-y,x-y+1])\mu(dy) \\ &= o(\mu((x,x+1])) \end{aligned} \tag{3-10}$$

as $x \to \infty$ and then $A \to \infty$. By using integration by parts,

$$\begin{aligned} I_3 &= \int_{0-}^{A+}(\bar{\mu}(x-y)-\bar{\mu}(x))\mu^{t_0*}(dy) \\ &\quad - t_0\int_{0-}^{A+}(\bar{\mu}(x-y)-\bar{\mu}(x))\mu(dy) \\ &\quad + (\overline{\mu^{t_0*}}(A)-t_0\bar{\mu}(A))(\bar{\mu}(x-A)-\bar{\mu}(x)) \\ &= K_1 - K_2 + K_3. \end{aligned}$$

As $x \to \infty$ and then $A \to \infty$,

$$K_1 \sim m(\mu^{t_0*})\mu((x,x+1]) = t_0 m(\mu)\mu((x,x+1])$$

and

$$K_2 \sim t_0 m(\mu)\mu((x,x+1]).$$

Note from $m(\mu) < \infty$ that $\overline{\mu^{t_0*}}(A)A \to 0$ and $\overline{\mu}(A)A \to 0$ as $A \to \infty$. Thus,

$$\limsup_{A\to\infty}\limsup_{x\to\infty}\frac{|K_3|}{\mu((x,x+1])} \le \limsup_{A\to\infty}(\overline{\mu^{t_0*}}(A)+t_0\bar{\mu}(A))A = 0.$$

Thus,

$$I_3 = o(\mu((x,x+1])) \tag{3-11}$$

as $x \to \infty$ and then $A \to \infty$. Thus, from (3-8)–(3-11) and the assumption, we obtain that

$$\begin{aligned} &(t_0+1)t_0 m(\mu)\mu((x,x+1]) - t_0(\overline{\mu^{2*}}(x)-2\overline{\mu}(x)) \\ &\qquad = t_0(t_0-1)m(\mu)\mu((x,x+1]) + o(\mu((x,x+1])) \end{aligned} \tag{3-12}$$

as $x \to \infty$. Hence,

$$\overline{\mu^{2*}}(x) = 2\overline{\mu}(x) + 2m(\mu)\mu((x,x+1]) + o(\mu((x,x+1])) \tag{3-13}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

as $x \to \infty$. That is, $\mu \in \mathcal{S}^2_{loc}$. Next let $t_0 = 1$. Then we have (3-13) and hence $\mu \in \mathcal{S}^2_{loc}$. Finally, let $0 < t_0 < 1$. In the same way, we see that, as $x \to \infty$ and then $A \to \infty$,

$$-I_1 \sim t_0(1-t_0)m(\mu)\mu((x, x+1]),$$
$$I_2 = o(\mu((x, x+1])),$$

and

$$I_3 = o(\mu((x, x+1])).$$

Thus, we have (3-12) and (3-13) and hence $\mu \in \mathcal{S}^2_{loc}$. □

4. Self-decomposable case

Let $f(x)$ and $g(x)$ be probability density functions on $\mathbb{R}$. We denote by $f \otimes g(x)$ the convolution of $f(x)$ and $g(x)$ and by $f^{n\otimes}(x)$ the nth convolution power of $f(x)$ for $n \in \mathbb{N}$.

Definition 4.1.

(i) A probability density function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathcal{L}_d$ if $g(x) \in \mathbf{L}$.
(ii) A probability density function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathcal{S}_d$ if $g(x) \in \mathcal{L}_d$ and $g^{2\otimes}(x) \sim 2g(x)$.

Definition 4.2. A probability density function $g(x)$ on $\mathbb{R}$ belongs to the class $\mathcal{S}^2_d$ if the following three conditions hold.

(1) $g(x) \in \mathcal{S}_d$.
(2) $\int_{-\infty}^{\infty} |x| g(x)\, dx < \infty$.
(3) For $\rho(dx) := g(x)\, dx$,

$$\overline{\rho^{2*}}(x) = 2\bar{\rho}(x) + 2m(\rho)g(x) + o(g(x))$$

as $x \to \infty$.

The classes $\mathcal{S}_d$ and $\mathcal{S}^2_d$ were introduced by Chover *et al.* [5] and Omey and Willekens [15], respectively. Densities in the classes $\mathcal{S}_d$ and $\mathcal{S}^2_d$ are called *subexponential* and *second order subexponential*, respectively. See also Foss *et al.* [7] and Klüppelberg [12] for the class $\mathcal{S}_d$. An infinitely divisible distribution on $\mathbb{R}_+$ with its density in the class $\mathcal{S}_d$ is found in Watanabe [21].

Let $\mu(dx) = p(x)\, dx$ be a nondegenerate self-decomposable distribution on $\mathbb{R}$. We assume that $k(x)$ is positive for all $x > 0$. We define self-decomposable distributions $\xi_1(dx) = p_1(x)\, dx$ and $\xi_2(dx) = p_2(x)\, dx$ as $\mu = \xi_1 * \xi_2$ and

$$\widehat{\xi_1}(z) := \exp\Big(\int_0^{\infty} (e^{izx} - 1)\frac{k(x \vee d)}{x}\, dx\Big)$$

for $d > 0$. We choose sufficiently large $d > 0$ such that $\int_0^{\infty} y\xi_2(dy) > 0$. Watanabe and Yamamuro [23] proved the following two lemmas.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Lemma 4.3 (Theorem 1.3 of [23] and its proof). *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}$ with $\nu(dx) = k(x)/|x|\,dx$. The following are equivalent.*

(1) $\mu \in \mathcal{S}_{\mathrm{loc}}$.
(2) $p(x) \in \mathcal{S}_d$.
(3) $p_1(x) \in \mathcal{S}_d$.
(4) $(1/\bar{\nu}(1))1_{(1,\infty)}(x)k(x)/x \in \mathcal{S}_d$.
(5) $k(x) \in \mathbf{L}$ *and* $p(x) \sim p_1(x) \sim k(x)/x$.

Remark 4.4. Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}$ with Lévy measure $\nu(dx) = k(x)/|x|\,dx$. We see from Lemma 4.3 that $\mu \in \mathcal{S}^2_{\mathrm{loc}}$ if and only if $p(x) \in \mathcal{S}^2_d$ and that $\nu_{(1)} \in \mathcal{S}^2_{\mathrm{loc}}$ if and only if $(1/\bar{\nu}(1))1_{(1,\infty)}(x)k(x)/x \in \mathcal{S}^2_d$.

Lemma 4.5 [23, Theorem 1.4]. *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}$. Let $p^t(x)$ be the density of $\mu^{t*}(dx)$ for $t > 0$. If $p^t(x) \in \mathcal{S}_d$ for some $t > 0$, then $p^t(x) \in \mathcal{S}_d$ for all $t > 0$ and*

$$p^t(x) \sim tp(x)$$

for all $t > 0$.

Proposition 4.6. *Let $\mu(dx) = p(x)\,dx$ be a self-decomposable distribution on $\mathbb{R}_+$. If $(\bar{\mu}(x))^2 = o(\mu((x, x+1]))$ as $x \to \infty$, $m(\mu) < \infty$, and*

$$\overline{\mu^{2*}}(x) = 2\bar{\mu}(x) + 2m(\mu)\mu((x, x+1]) + o(\mu((x, x+1]))$$

as $x \to \infty$, then $p(x) \in \mathcal{S}^2_d$.

Proof. Assume that $(\bar{\mu}(x))^2 = o(\mu((x, x+1]))$ as $x \to \infty$, $m(\mu) < \infty$, and

$$\overline{\mu^{2*}}(x) = 2\bar{\mu}(x) + 2m(\mu)\mu((x, x+1]) + o(\mu((x, x+1]))$$

as $x \to \infty$. Note that

$$\overline{\mu^{2*}}(x) - 2\bar{\mu}(x) + (\bar{\mu}(x))^2 = \int_{0-}^{x+} (\bar{\mu}(x-y) - \bar{\mu}(x))\mu(dy).$$

Thus, by the assumption,

$$\int_{0-}^{x+} (\bar{\mu}(x-y) - \bar{\mu}(x))\mu(dy) \sim 2m(\mu)\mu((x, x+1]).$$

We shall prove that, for every $m \in \mathbb{N}$,

$$\lim_{x\to\infty} \frac{\mu((x-m, x-m+1])}{\mu((x, x+1])} = 1. \tag{4-1}$$

Since μ is unimodal, we see that, for every $m \in \mathbb{N}$,

$$\liminf_{x\to\infty} \frac{\mu((x-m, x-m+1])}{\mu((x, x+1])} \geq 1.$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Suppose that there are some $c > 1$, $m_0 \in \mathbb{N}$, and a increasing sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = \infty$ such that

$$\lim_{n\to\infty} \frac{\mu((x_n - m_0, x_n - m_0 + 1])}{\mu((x_n, x_n + 1])} = c.$$

We have

$$\int_{0-}^{x_n+} (\bar{\mu}(x_n - y) - \bar{\mu}(x_n))\mu(dy) = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{0-}^{m+} (\bar{\mu}(x_n - y) - \bar{\mu}(x_n))\mu(dy),$$
$$I_2 := \int_{m+}^{(x_n-m)+} (\bar{\mu}(x_n - y) - \bar{\mu}(x_n))\mu(dy),$$

and

$$I_3 := \int_{(x_n-m)+}^{x_n+} (\bar{\mu}(x_n - y) - \bar{\mu}(x_n))\mu(dy).$$

By unimodality, we have for $0 \le y \le m_0$

$$\liminf_{n\to\infty} \frac{\mu((x_n - y, x_n])}{\mu((x_n, x_n + 1])} \ge y$$

and for $y \ge m_0$

$$\liminf_{n\to\infty} \frac{\mu((x_n - y, x_n])}{\mu((x_n, x_n + 1])} \ge c(y - m_0) + m_0.$$

Thus, we have, by Fatou's lemma,

$$\begin{aligned}
&\liminf_{m\to\infty} \liminf_{n\to\infty} \frac{I_1}{\mu((x_n, x_n + 1])} \\
&\quad \ge \int_{0-}^{m_0+} \liminf_{n\to\infty} \frac{\mu((x_n - y, x_n])}{\mu((x_n, x_n + 1])}\mu(dy) \\
&\qquad + \liminf_{m\to\infty} \int_{m_0+}^{m+} \liminf_{n\to\infty} \frac{\mu((x_n - y, x_n])}{\mu((x_n, x_n + 1])}\mu(dy) \\
&\quad \ge \int_{0-}^{m_0+} y\mu(dy) + \liminf_{m\to\infty} \int_{m_0+}^{m+} (c(y - m_0) + m_0)\mu(dy) \\
&\quad = \int_{0-}^{m_0+} y\mu(dy) + \int_{m_0+}^{\infty} (c(y - m_0) + m_0)\mu(dy) > m(\mu).
\end{aligned} \tag{4-2}$$

Clearly,

$$\liminf_{n\to\infty} \frac{I_2}{\mu((x_n, x_n + 1])} \ge 0. \tag{4-3}$$

By using integration by parts, we see that, for sufficiently large n,

$$I_3 = I_1 + (\bar{\mu}(x_n - m) - \bar{\mu}(x_n))(\bar{\mu}(m) - \bar{\mu}(x_n)) \ge I_1.$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Thus, we obtain from (4-2) that

$$\liminf_{m\to\infty}\liminf_{n\to\infty}\frac{I_3}{\mu((x_n,x_n+1])}>m(\mu). \tag{4-4}$$

Hence, we have by (4-2), (4-3), and (4-4) that

$$\liminf_{n\to\infty}\frac{1}{\mu((x_n,x_n+1])}\int_{0-}^{x_n+}(\bar{\mu}(x_n-y)-\bar{\mu}(x_n))\mu(dy)>2m(\mu).$$

This is a contradiction. Thus, we have proved (4-1). By unimodality, it implies that $p(x)\in\mathcal{L}_d$. Thus, by Lemma 2.7, we have proved that $\mu\in\mathcal{S}^2_{\rm loc}$ and hence $p(x)\in\mathcal{S}^2_d$. □

Proof of Theorem 1.7. Since the support of the Lévy measure of ξ_2 has an upper bound, we find from Sato [17, Theorem 25.17] that, for every $b>0$, $\int_0^\infty e^{bx}\xi_2(dx)<\infty$ and hence $\overline{\xi_2}(x)=o(e^{-bx})$ as $x\to\infty$. We have

$$\begin{aligned}\bar{\mu}(x)-\bar{\xi}_1(x)&=\overline{\xi_1*\xi_2}(x)-\bar{\xi}_1(x)\\&=\int_0^\infty\xi_1((x-y,x])\xi_2(dy)-\int_{-\infty}^0\xi_1((x,x-y])\xi_2(dy)\\&=I_1-I_2.\end{aligned}$$

Note that if $\int_{-\infty}^0|y|\xi_2(dy)=0$, then $I_2=0$. Suppose that $p(x)\in\mathcal{S}_d$, equivalently by Lemma 4.3, $p_1(x)\in\mathcal{S}_d$. Recall that $p_1(x)\in\mathbf{L}$ and ξ_1 is unimodal. Thus, there are $C>0$ and $\epsilon>0$ such that, for $0<y<x/2$ and for sufficiently large $x>0$,

$$\xi_1((x-y,x])\le Ce^{\epsilon y}p_1(x).$$

Note that $\int_0^\infty e^{\epsilon y}\xi_2(dy)<\infty$ and

$$\int_{x/2}^\infty\xi_1((x-y,x])\xi_2(dy)\le\bar{\xi}_2(x/2)=o(e^{-x})=o(p_1(x))$$

as $x\to\infty$. Thus, by the dominated convergence theorem,

$$I_1\sim p_1(x)\int_0^\infty y\xi_2(dy).$$

Since ξ_1 is unimodal, we have, for $y<0$ and for sufficiently large $x>0$,

$$\xi_1((x,x-y])\le p_1(x)|y|.$$

Since $\int_{-\infty}^0|y|\mu(dy)<\infty$, we see from Sato [17, Theorem 25.3] that $\int_{-\infty}^\infty|y|\xi_2(dy)<\infty$. Thus, by the dominated convergence theorem, $I_2=0$ or

$$I_2\sim p_1(x)\int_{-\infty}^0|y|\xi_2(dy).$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Note from Lemma 4.3 that $p_1(x) \sim p(x)$. Hence,

$$\begin{aligned}\bar{\mu}(x) &= \bar{\xi}_1(x) + m(\xi_2)p_1(x) + o(p_1(x)) \\ &= \bar{\xi}_1(x) + m(\xi_2)p(x) + o(p(x))\end{aligned} \tag{4-5}$$

as $x \to \infty$. Note that $p^{2\otimes}(x) \sim 2p(x)$ and $m(\xi_2^{2*}) = 2m(\xi_2)$. In the same way,

$$\begin{aligned}\overline{\mu^{2*}}(x) &= \overline{\xi_1^{2*}}(x) + m(\xi_2^{2*})p^{2\otimes}(x) + o(p^{2\otimes}(x)) \\ &= \overline{\xi_1^{2*}}(x) + 4m(\xi_2)p(x) + o(p(x))\end{aligned}$$

as $x \to \infty$. Hence, we obtain from (4-5) that

$$\begin{aligned}&\overline{\mu^{2*}}(x) - 2\overline{\mu}(x) \\ &\quad = \overline{\xi_1^{2*}}(x) - 2\overline{\xi_1}(x) + 2m(\xi_2)p(x) + o(p(x))\end{aligned}$$

as $x \to \infty$. Since $\int_{-\infty}^{\infty} |y|\xi_2(dy) < \infty$, we see from Sato [17, Theorem 25.3] that $\int_{-\infty}^{\infty} |x|\mu(dx) < \infty$ if and only if $0 < m(\xi_1) < \infty$. Thus,

$$\overline{\mu^{2*}}(x) = 2\overline{\mu}(x) + 2m(\mu)p(x) + o(p(x))$$

as $x \to \infty$ if and only if

$$\overline{\xi_1^{2*}}(x) = 2\overline{\xi_1}(x) + 2m(\xi_1)p_1(x) + o(p_1(x))$$

as $x \to \infty$. Thus, under the assumption of $\int_{-\infty}^{0} |y|\mu(dy) < \infty$, we have $p(x) \in \mathcal{S}_d^2$ if and only if $p_1(x) \in \mathcal{S}_d^2$, equivalently, $\xi_1 \in \mathcal{S}_{\text{loc}}^2$. We find from Theorem 1.3 that $\xi_1 \in \mathcal{S}_{\text{loc}}^2$ if and only if $\nu_{(1)} \in \mathcal{S}_{\text{loc}}^2$. That is, $p(x) \in \mathcal{S}_d^2$ if and only if $\nu_{(1)} \in \mathcal{S}_{\text{loc}}^2$, equivalently, $(1/\bar{\nu}(1))1_{(1,\infty)}(x)k(x)/x \in \mathcal{S}_d^2$.

Next we prove assertion (ii). If $p(x) \in \mathcal{S}_d^2$, then $\nu_{(1)} \in \mathcal{S}_{\text{loc}}^2$ and hence, by Theorem 1.3,

$$\begin{aligned}\bar{\xi}_1(x) &= \bar{\nu}(x) + m(\xi_1)\nu((x, x+1]) + o(\nu((x, x+1])) \\ &= \bar{\nu}(x) + m(\xi_1)p(x) + o(p(x)) \\ &= \bar{\nu}(x) + m(\xi_1)p_1(x) + o(p_1(x))\end{aligned} \tag{4-6}$$

as $x \to \infty$. Thus, it follows from (4-5) that (1-5) and (1-6) hold.

Next we prove assertion (iii). The assumptions that (1-5) with finite $m(\mu)$, $p(x) \in \mathcal{S}_d$, and $(\bar{\mu}(x))^2 = o(p(x))$ as $x \to \infty$ imply that (4-6) with finite $m(\xi_1)$, $\xi_1 \in \mathcal{S}_{\text{loc}}$, and $(\bar{\xi}_1(x))^2 = o(\xi_1((x, x+1]))$ as $x \to \infty$ hold. Thus, we see from (iii) of Theorem 1.3 that $\xi_1 \in \mathcal{S}_{\text{loc}}^2$, that is, $p_1(x) \in \mathcal{S}_d^2$. It follows from the proof of (i) that $p(x) \in \mathcal{S}_d^2$. □

Proof of Corollary 1.8. By an argument analogous to the proof of Corollary 1.5, we can easily prove the corollary from Theorem 1.7 and Lemma 4.5. □

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

5. Examples

By using a method of Klüppelberg [11] and Baltrunas [2], Lin [13] proved that the standard lognormal distribution, Weibull distribution with parameter $\beta \in (0,1)$, and Pareto distribution with parameter $\alpha > 1$ belong to the class $\mathcal{S}^2_{\mathrm{loc}}$. Those distributions are all self-decomposable, so their densities also belong to the class $\mathcal{S}^2_d$. See Sato [17] and Steutel and van Harn [19] for their self-decomposability. The following examples are direct consequences of Theorem 1.3 and Corollary 1.5 and hence their proofs are omitted.

Example 5.1. Let μ be the standard lognormal distribution with Lévy measure $\nu(dx) = k(x)/x\,dx$. Then we have the density

$$p(x) := \frac{1}{\sqrt{2\pi}x}\exp\Big(-\frac{(\log x)^2}{2}\Big)$$

for $x > 0$. Embrechts *et al.* [6] showed that μ is subexponential and that

$$\bar{\nu}(x) \sim \bar{\mu}(x) \sim \frac{x}{\log x}p(x)$$

and

$$\overline{\mu^{t*}}(x) \sim t\bar{\mu}(x).$$

Watanabe and Yamamuro [23] proved a conjecture of Bondesson [4]. That is,

$$k(x) \sim xp(x).$$

We have

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 - \sqrt{e}\frac{\log x}{x} + o\Big(\frac{\log x}{x}\Big)\Big)$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 + (t-1)\sqrt{e}\frac{\log x}{x} + o\Big(\frac{\log x}{x}\Big)\Big)$$

as $x \to \infty$.

Example 5.2. Let μ be a Weibull distribution with Lévy measure ν and parameter $\beta \in (0,1)$. Then

$$\bar{\mu}(x) := \exp(-x^{\beta})$$

for $x \in \mathbb{R}_+$,

$$\bar{\nu}(x) = \bar{\mu}(x)(1 - \Gamma(\beta^{-1})x^{\beta-1} + o(x^{\beta-1}))$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)(1 + (t-1)\Gamma(\beta^{-1})x^{\beta-1} + o(x^{\beta-1}))$$

as $x \to \infty$.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

EXAMPLE 5.3. Let μ be a Pareto distribution with Lévy measure ν and parameter $\alpha > 1$. Then

$$\bar{\mu}(x) := (1+x)^{-\alpha}$$

for $x \in \mathbb{R}_+$,

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 - \frac{\alpha}{\alpha-1}x^{-1} + o(x^{-1})\Big)$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 + (t-1)\frac{\alpha}{\alpha-1}x^{-1} + o(x^{-1})\Big)$$

as $x \to \infty$.

6. Remarks on the regularly varying case

We cannot find from our results the relations of Example 5.3 for a Pareto distribution with parameter $0 < \alpha \le 1$ because it does not belong to the class $\mathcal{S}^2_{\text{loc}}$. However, we can get the analogous relations by using the following lemma of Omey and Willekens [14]. Theorem 4.3 of [14] is a direct consequence from [14, Theorem 2.3] for a compound Poisson distribution on $\mathbb{R}_+$, but there is a mistake in the case of finite mean for an infinitely divisible distribution on $\mathbb{R}_+$. So, we restore and prove it for an infinitely divisible distribution on $\mathbb{R}_+$.

LEMMA 6.1 [14, Theorem 4.3]. *Let μ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure ν. Assume that $\nu(dx)$ has a density $q(x)$ on $(1,\infty)$ such that $q(x) \sim x^{-\alpha-1}l(x)$ for $0 \le \alpha \le 1$ with $l(x)$ being slowly varying as $x \to \infty$. Define a constant $C(\alpha)$ for $0 < \alpha < 1$ as*

$$C(\alpha) := \frac{(1-\alpha)(2\alpha-1)(\Gamma(1-\alpha))^2}{2\alpha\Gamma(2-2\alpha)}.$$

(i) *We have for $0 < \alpha < 1$,*

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{q(x)\int_1^x \bar{\nu}(u)\,du} = C(\alpha). \tag{6-1}$$

(ii) *For $\alpha = 1$, if $\int_1^\infty \bar{\nu}(u)\,du = \infty$, then*

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{q(x)\int_1^x \bar{\nu}(u)\,du} = 1. \tag{6-2}$$

(iii) *For $\alpha = 1$, if $\int_1^\infty \bar{\nu}(u)\,du < \infty$, then*

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{q(x)m(\mu)} = 1. \tag{6-3}$$

(iv) *For $\alpha = 0$,*

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{(\bar{\nu}(x))^2} = -\frac{1}{2}. \tag{6-4}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Proof. Let μ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure ν. Assume that $\nu(dx)$ has a density $q(x)$ on $(1,\infty)$ such that $q(x) \sim x^{-\alpha-1}l(x)$ for $0 \le \alpha \le 1$ with $l(x)$ being slowly varying as $x \to \infty$. Define a compound Poisson distribution μ_1 on $\mathbb{R}_+$ as (2-2) for $c = 1$. Define an infinitely divisible distribution μ_2 on $\mathbb{R}_+$ as $\mu = \mu_1 * \mu_2$. Then we have by [14, Theorem 2.3], for $0 \le \alpha \le 1$, that the lemma is true by substituting μ_1 for μ. Thus, we can assume that $\mu_2(dx) \neq \delta_0(dx)$. We see from Sato [17, Theorem 25.17] that, for every $b > 0$, $\int_{0-}^{\infty} \exp(bx)\mu_2(dx) < \infty$ and hence $\overline{\mu_2}(x) = o(e^{-bx})$ as $x \to \infty$. We have

$$\begin{aligned}\bar{\mu}(x) - \bar{\mu}_1(x) &= \overline{\mu_1 * \mu_2}(x) - \bar{\mu}_1(x)\\ &= \int_{0-}^{\infty} \mu_1((x-y,x])\mu_2(dy)\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \int_{0-}^{A+} \mu_1((x-y,x])\mu_2(dy),\\ I_2 &:= \int_{A+}^{x/2+} \mu_1((x-y,x])\mu_2(dy),\end{aligned}$$

and

$$I_3 := \int_{x/2+}^{\infty} \mu_1((x-y,x])\mu_2(dy).$$

Since $q(x) \sim x^{-\alpha-1}l(x)$, $\nu_{(1)} \in \mathcal{S}_{\mathrm{loc}}$ and hence, by Lemma 2.1, $\mu_1 \in \mathcal{S}_{\mathrm{loc}}$. Thus,

$$\begin{aligned}I_1 &\sim \mu_1((x,x+1]) \int_{0-}^{A+} y\mu_2(dy)\\ &\sim \mu_1((x,x+1]) \int_{0-}^{\infty} y\mu_2(dy)\end{aligned}$$

as $x \to \infty$ and then $A \to \infty$. Since $\mu_1 \in \mathcal{S}_{\mathrm{loc}}$, there are $C > 0$ and $\epsilon > 0$ such that, for $0 \le y \le x/2$ and for sufficiently large $x > 0$,

$$\mu_1((x-y,x]) \le Ce^{\epsilon y}\mu_1((x,x+1]).$$

Thus,

$$\begin{aligned}I_2 &\le \mu_1((x,x+1]) \int_{A+}^{x/2+} Ce^{\epsilon y}\mu_2(dy)\\ &= o(\mu_1((x,x+1]))\end{aligned}$$

as $x \to \infty$ and then $A \to \infty$. We have

$$I_3 \le \bar{\mu}_2(x/2) = o(e^{-x}) = o(\mu_1((x,x+1]))$$

as $x \to \infty$. Thus,

$$\bar{\mu}(x) - \bar{\mu}_1(x) \sim m(\mu_2)\mu_1((x,x+1]). \tag{6-5}$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Note from Lemma 4.3 that, for $0 < \alpha < 1$ or $\alpha = 1$ with $\int_1^\infty \bar{\nu}(u)\,du = \infty$,

$$\mu_1((x, x+1]) \sim q(x) = o\Big(q(x)\int_1^x \bar{\nu}(u)\,du\Big)$$

as $x \to \infty$. For $\alpha = 0$, we have by Lemma 4.3 that

$$\mu_1((x, x+1]) \sim q(x) = o((\bar{\nu}(x))^2)$$

as $x \to \infty$. Thus, except for the case of $\alpha = 1$ with finite $m(\mu_1)$, the lemma is true. In the case of $\alpha = 1$ with finite $m(\mu_1)$, we see from (6-3) with substituting μ_1 for μ and (6-5) that the lemma is true. □

Proof of Proposition 1.9. Assume that $p(x) \sim x^{-\alpha-1}l(x)$ for $0 \le \alpha \le 1$ with $l(x)$ being slowly varying as $x \to \infty$. First we prove (i). Let $0 < \alpha < 1$. Since $p(x) \in \mathcal{S}_d$, we have by Lemma 4.3,

$$q(x) \sim x^{-\alpha-1}l(x).$$

By Karamata's theorem (see [3, Theorem 1.5.11]),

$$\bar{\nu}(x) \sim \bar{\mu}(x) \sim \frac{x^{-\alpha}l(x)}{\alpha}$$

and

$$\int_1^x \bar{\nu}(u)\,du \sim \frac{x^{1-\alpha}l(x)}{\alpha(1-\alpha)}.$$

Thus, we see from (6-1) of Lemma 6.1 that

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{x^{-2\alpha}(l(x))^2} = \frac{K(\alpha)}{\alpha}.$$

Thus, we have (1-7). In the same way,

$$\lim_{x\to\infty} \frac{\overline{\mu^{t*}}(x) - t\bar{\nu}(x)}{x^{-2\alpha}(l(x))^2} = t^2\frac{K(\alpha)}{\alpha}.$$

Hence, we get (1-8) by (1-7). Next we prove (ii). Assume that $p(x) \sim x^{-2}l(x)$. Then, by Karamata's theorem, we have $\bar{\mu}(x) \sim x^{-1}l(x)$. We have by Lemma 4.3,

$$q(x) \sim x^{-2}l(x).$$

We see from Karamata's theorem that $\bar{\nu}(x) \sim x^{-1}l(x)$ and

$$\int_1^x \bar{\nu}(u)\,du \sim l^*(x)$$

and that $\int_1^\infty \bar{\nu}(u)\,du = \infty$ from $l^*(\infty) = \infty$. Thus, we see from (6-2) of Lemma 6.1 that

$$\lim_{x\to\infty} \frac{\bar{\mu}(x) - \bar{\nu}(x)}{x^{-2}l(x)l^*(x)} = 1.$$

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

Thus, we have (1-9). In the same way,

$$\lim_{x\to\infty}\frac{\overline{\mu^{t*}}(x)-t\bar{\nu}(x)}{x^{-2}l(x)l^*(x)}=t^2.$$

Hence, we get (1-10) by (1-9). Next we prove (iii). As in (ii), we have $q(x)\sim p(x)\sim x^{-2}l(x)$, $\bar{\nu}(x)\sim\bar{\mu}(x)\sim x^{-1}l(x)$, and

$$\int_1^x \bar{\nu}(u)\,du\sim l^*(x).$$

We see that $\int_1^\infty \bar{\nu}(u)\,du<\infty$ from $l^*(\infty)<\infty$. Thus, we find from (6-3) of Lemma 6.1 that

$$\lim_{x\to\infty}\frac{\bar{\mu}(x)-\bar{\nu}(x)}{x^{-2}l(x)m(\mu)}=1.$$

Thus, we have (1-11). In the same way,

$$\lim_{x\to\infty}\frac{\overline{\mu^{t*}}(x)-t\bar{\nu}(x)}{x^{-2}l(x)m(\mu)}=t^2.$$

Hence, we get (1-12) by (1-11). Next we prove (iv). Assume that $p(x)\sim x^{-1}l(x)$. Then we see from Lemma 4.3 that $q(x)\sim x^{-1}l(x)$. Thus,

$$\bar{\mu}(x)\sim\bar{\nu}(x)\sim l_*(x).$$

We find from (6-4) of Lemma 6.1 that

$$\lim_{x\to\infty}\frac{\bar{\mu}(x)-\bar{\nu}(x)}{(l_*(x))^2}=-\frac{1}{2}.$$

Thus, we have (1-13). In the same way,

$$\lim_{x\to\infty}\frac{\overline{\mu^{t*}}(x)-t\bar{\nu}(x)}{(l_*(x))^2}=-\frac{t^2}{2}.$$

Hence, we get (1-14) by (1-13). □

Finally, we give the relations for a Pareto distribution with parameter $0<\alpha\le 1$ as an example of Proposition 1.9. They are different from the relations of Example 5.3.

Example 6.2. Let μ be a Pareto distribution with Lévy measure ν and parameter $0<\alpha\le 1$. Then

$$\bar{\mu}(x):=(1+x)^{-\alpha}$$

for $x\in\mathbb{R}_+$.

(i) Let $0<\alpha<1$. Then

$$\bar{\nu}(x)=\bar{\mu}(x)(1-\alpha K(\alpha)x^{-\alpha}+o(x^{-\alpha}))$$

as $x\to\infty$, and, for $t>0$,

$$\overline{\mu^{t*}}(x)=t\bar{\mu}(x)(1+(t-1)\alpha K(\alpha)x^{-\alpha}+o(x^{-\alpha}))$$

as $x\to\infty$.

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press

(ii) Let $\alpha = 1$. Then

$$\bar{\nu}(x) = \bar{\mu}(x)\Big(1 - \frac{\log x}{x} + o\Big(\frac{\log x}{x}\Big)\Big)$$

as $x \to \infty$, and, for $t > 0$,

$$\overline{\mu^{t*}}(x) = t\bar{\mu}(x)\Big(1 + (t-1)\frac{\log x}{x} + o\Big(\frac{\log x}{x}\Big)\Big)$$

as $x \to \infty$.

References

[1] S. Asmussen, S. Foss and D. Korshunov, 'Asymptotics for sums of random variables with local subexponential behaviour', *J. Theoret. Probab.* **16** (2003), 489–518.

[2] A. Baltrunas, 'Second order behaviour of ruin probabilities', *Scand. Actuar. J.* **2** (1999), 120–133.

[3] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation* (Cambridge University Press, Cambridge, 1984).

[4] L. Bondesson, 'On the Lévy measure of the lognormal and the logcauchy distributions', *Methodol. Comput. Appl. Probab.* **4** (2002), 243–256.

[5] J. Chover, P. Ney and S. Wainger, 'Functions of probability measures', *J. Anal. Math.* **26** (1973), 255–302.

[6] P. Embrechts, C. M. Goldie and N. Veraverbeke, 'Subexponentiality and infinite divisibility', *Z. Wahrscheinlichkeitstheorie verw. Geb.* **49** (1979), 335–347.

[7] S. Foss, D. Korshunov and S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions*, 2nd edn, Springer Series in Operations Research and Financial Engineering (Springer, New York, 2013).

[8] J. L. Geluk, 'Second order tail behaviour of a subordinated probability distribution', *Stochastic Process. Appl.* **40** (1992), 325–337.

[9] J. L. Geluk and A. G. Pakes, 'Second order subexponential distributions', *J. Aust. Math. Soc.* **51** (1991), 73–87.

[10] T. Jiang, Y. Wang, Z. Cui and Y. Chen, 'On the almost decrease of a subexponential density', *Statist. Probab. Lett.* **153** (2019), 71–79.

[11] C. Klüppelberg, 'Subexponential distributions and integrated tails', *J. Appl. Probab.* **25** (1988), 132–141.

[12] C. Klüppelberg, 'Subexponential distributions and characterizations of related classes', *Probab. Theory Related Fields* **82** (1989), 259–269.

[13] J. Lin, 'Second order subexponential distributions with finite mean and their applications to subordinated distributions', *J. Theoret. Probab.* **25** (2012), 834–853.

[14] E. Omey and E. Willekens, 'Second order behaviour of the tail of a subordinated probability distribution', *Stochastic Process. Appl.* **21** (1986), 339–353.

[15] E. Omey and E. Willekens, 'Second-order behaviour of distributions subordinate to a distribution with finite mean', *Comm. Statist. Stochastic Models* **3** (1987), 311–342.

[16] A. G. Pakes, 'Convolution equivalence and infinite divisibility', *J. Appl. Probab.* **41** (2004), 407–424.

[17] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, 68 (Cambridge University Press, Cambridge, 2013).

[18] T. Shimura and T. Watanabe, 'Subexponential densities of compound Poisson sums and the supremum of a random walk', Preprint, 2020, arXiv:2001.11362.

[19] F. W. Steutel and K. van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, Pure and Applied Mathematics: A Series of Monographs and Textbooks, 259 (Marcel Dekker, New York–Basel, 2004).

[20] T. Watanabe, 'Convolution equivalence and distributions of random sums', *Probab. Theory Related Fields* **142** (2008), 367–397.

[21] T. Watanabe, 'Subexponential densities of infinitely divisible distributions on the half line', *Lith. Math. J.* (2020), to appear.
[22] T. Watanabe and K. Yamamuro, 'Local subexponentiality of infinitely divisible distributions', *J. Math-for-Ind.* **1** (2009), 81–90.
[23] T. Watanabe and K. Yamamuro, 'Local subexponentiality and self-decomposability', *J. Theoret. Probab.* **23** (2010), 1039–1067.

TOSHIRO WATANABE, Center for Mathematical Sciences,
The University of Aizu, Aizu-Wakamatsu,
Fukushima 965-8580, Japan
e-mail: markov2000t@yahoo.co.jp

https://doi.org/10.1017/S1446788720000208 Published online by Cambridge University Press