GROUPS WITH A CYCLIC SYLOW SUBGROUP

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Dedicated to the memory of TADASI NAKAYAMA

§ 1. Introduction

By focussing attention on indecomposable modular representations J. G. Thompson [11] has recently simplified and generalized some classical results of R. Brauer [1] concerning groups which have a Sylow group of prime order. In this paper this approach will be used to prove some results which generalize theorems of R. Brauer [2] and H. F. Tuan [12].

We will say that a finite group $\mathfrak{G}$ is of *type* $L_2(p)$ if every composition factor is either a p-group or a p'-group or is isomorphic to $PSL_2(p)$. Thus in particular every p-solvable group is of type $L_2(p)$. It is well known that every subgroup of a group of type $L_2(p)$ is again of type $L_2(p)$.

THEOREM 1. *Let $\mathfrak{G}$ be a finite group with a cyclic S_p-subgroup $\mathfrak{P}$ for some prime p. Assume that $\mathfrak{G}$ is not of type $L_2(p)$. Suppose that $\mathfrak{G}$ has a faithful indecomposable representation $\mathfrak{L}$ of degree $d \leq p$ in a field of characteristic p. Then $p \neq 2$, $|\mathfrak{P}| = p$, $\mathfrak{L}|\mathfrak{P}$ is indecomposable and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P} \times \mathbf{Z}(\mathfrak{G})$. Furthermore $d \geq 2/3(p-1)$ and $d \geq \frac{7}{10}p - \frac{1}{2}$ in case $p \geq 13$.*

It is known [9] that the multiplier of $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathfrak{A}_7$, respectively has a non-trivial complex representation of degree 2, 3, 4 respectively. Hence this is the case in any algebraically closed field. The new simple group discovered by Z. Janko [8] has a 7-dimensional representation in the field of 11 elements. Thus for $p \leq 11$ the estimate in Theorem 1 is the best possible (since d is an integer). However it follows easily from the last statement that $d \geq 2/3(p-1)$ is never the best possible estimate for $p \geq 13$. By modifying the argument in section 4 slightly it can be shown that for $p \geq 13$ the estimate can be improved

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provided $|\mathbf{N}_{\mathfrak{G}}(\mathfrak{P}) : \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})|$ is sufficiently large. In particular it is easy to show that if $\mathfrak{G} = \mathfrak{G}'$, $|\mathbf{N}_{\mathfrak{G}}(\mathfrak{P}) : \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})| = p-1$ and $p \geq 13$ then $d \geq \frac{3(p-1)}{4}$. This is in sharp contrast to the case of Janko's group where $p=11$, $d=7$ and $|\mathbf{N}_{\mathfrak{G}}(\mathfrak{P}) : \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})| = 10$. It would be of interest to determine the best possible lower bound for d in case $p \geq 13$. Since the Symmetric group on p letters has a faithful representation of degree $p-2$ in the field of p elements one cannot do better than $p-3$. However this is probably much too large in general.

Theorem 1 is easily seen to imply some results of Brauer [2] and Tuan [12] concerning groups $\mathfrak{G}$ which have a faithful irreducible complex representations of "small" degree relative to the size of some prime dividing $|\mathfrak{G}|$. As another application of these methods the following can be proved.

THEOREM 2. *Suppose the S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is not normal in $\mathfrak{G}$ and $\mathbf{Z}(\mathfrak{G}) = \langle 1 \rangle$. Assume that $\mathfrak{G}$ has a complex irreducible representation of degree d with $\frac{p-1}{2} < d < p-1$. Let $|\mathbf{N}_{\mathfrak{G}}(\mathfrak{P}) : \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})| = e$. Then $\mathfrak{G}$ is simple and $e \equiv \frac{p-1}{e} \equiv 0 \pmod 2$. Thus in particular $p \equiv 1 \pmod 4$.*

The only known groups which satisfy the hypotheses of Theorem 2 are $PSL_2(p)$ with $p \equiv 1 \pmod 4$ and $d-1 = e = \frac{p-1}{2}$, and $PSL_2(p-1)$ where $p-1 = 2^a$ for some integer $a > 1$ with $e=2$ and $d=p-2$.

§ 2. Preliminaries

Let K be a field and $\mathfrak{G}$ a group. If M, N are $K\mathfrak{G}$-modules then $M+N$ denotes their direct sum and $aM = M + \cdots + M$ a times for any nonnegative integer a. The kernel of M is the kernel of the representation of $\mathfrak{G}$ corresponding to M. If $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ then $M_{|\mathfrak{H}}$ denotes the restriction of M to $\mathfrak{H}$ and for any $K\mathfrak{H}$-module L, $L^{\mathfrak{G}}$ is the $K\mathfrak{G}$-module induced by L. The contragradient module of M is denoted by M^*. The remainder of the notation and terminology is standard.

Basic properties of modules will be used continually. In particular the Mackey decomposition [3, (44.2)] and a fundamental result of D. G. Higman [3, (63.5)] are of importance. Also a theorem of Schanuel will be used [6, (1.6 e)] or [10, p. 270]. The following result is a simple consequence of the Mackey decomposition, the proof of [3, (51.2)] and Fitting's Lemma.

(2.1) *Suppose that K is a field of characteristic p. Let $\mathfrak{P}$ be a p-group and $\mathfrak{H}$*

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*a p'-group. A $K(\mathfrak{P}\times\mathfrak{H})$-module is indecomposable if and only if it is of the form $V\otimes W$ where V is an indecomposable $K\mathfrak{P}$-module and W is an irreducible $K\mathfrak{H}$-*module.

An exposition of the fundamentals of the theory of blocks can be found in [3, Chapter XII]. The following special cases of some results of R. Brauer [2] will be needed.

Suppose S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ has order p for some prime p. Assume further that $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})=\mathfrak{P}\times\mathbf{Z}(\mathfrak{G})$. Let $e=|\mathbf{N}_{\mathfrak{G}}(\mathfrak{P}):\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})|$.

(2.2) *If ζ is an irreducible complex character of $\mathfrak{G}$ with $1<\zeta(1)<p-1$ then $e<p-1$ and either $\zeta(1)=e$ or $\zeta(1)=p-e$. In the latter case ζ does not contain the principal Brauer character as a modular constituent. Furthermore if B is the p-block of $\mathfrak{G}$ containing ζ then B contains exactly $\frac{p-1}{e}$ irreducible complex character of degree $\zeta(1)$, any two of which are p-conjugate and hence coincide as Brauer characters.*

(2.3) *If $\mathbf{Z}(\mathfrak{G})=\langle 1\rangle$ and $e=2$ then the degree of any irreducible modular representation of $\mathfrak{G}$ is 1, $p-2$ or at least p.*

The following result of Tuan [12, Theorem C] is also useful.

(2.4) *Any modular irreducible representation of $\mathfrak{G}$ in the principal block can be written in the field of p elements.*

The proofs of (2.2), (2.3) and (2.4) can be simplified considerably using the methods of [11].

§ 3. Local Results

Throughout this section K is a field of characteristic p. $\mathfrak{E}\mathfrak{P}$ is a Frobenius group with Frobenius kernel $\mathfrak{P}$ where $|\mathfrak{P}|=p$ and $\mathfrak{E}\cap\mathfrak{P}=\langle 1\rangle$. The one dimensional K-representation α of $\mathfrak{E}\mathfrak{P}$ is defined by

(3.1) $$G^{-1}PG=P^{\alpha(G)} \text{ for } P\in\mathfrak{P},\ G\in\mathfrak{E}\mathfrak{P}.$$

The following result is a reformulation of [11, Lemma 2].

LEMMA 3.1. *Let λ be a one dimensional K-representation of $\mathfrak{P}\mathfrak{E}$ and let $1\leq s\leq p$. Then there exists an indecomposable $K\mathfrak{P}\mathfrak{E}$-module V_s^λ such that* $\dim_K V_s^\lambda=s$, *$V^\lambda_{s|\mathfrak{P}}$ is indecomposable and if U is the unique submodule of V_s^λ with* $\dim_K U$

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$=1$ *then* $uG=\lambda(G)u$ *for all* $u \in U$, $G \in \mathfrak{E}\mathfrak{P}$. *Furthermore every nonzero indecomposable* $K\mathfrak{P}\mathfrak{E}$*-module is isomorphic to some* V_s^λ ; $V_s^\lambda \approx V_t^\mu$ *if and only if* $s=t$, $\lambda=\mu$; V_s^λ *is projective if and only if* $s=p$.

Throughout this section V_s^λ will be defined as in Lemma 3.1 and for any λ, $V_0^\lambda=0$. In case $\mathfrak{E}=\langle 1\rangle$ we will write $V_s=V_s^\lambda$. If $E\in\mathfrak{E}$ then $\det_s^\lambda(E)$ denotes the determinant of E acting as a linear transformation on V_s^λ and φ_s^λ denotes the Brauer character of $\mathfrak{P}\mathfrak{E}$ corresponding to V_s^λ.

LEMMA 3.2. *Let* $0\le i\le s\le p$. *Then* V_s^λ *has a unique submodule* U_i *with* $\dim_K U_i = i$. *Furthermore* $U_i\approx V_i^\lambda$ *and* $V_s^\lambda/U_i\approx V_{s-i}^{\lambda\alpha^{-i}}$.

Proof. Since every irreducible $K\mathfrak{P}\mathfrak{E}$-module is 1-dimensional V_s^λ has an i-dimensional submodule U_i for $0\le i\le s$. As $V_{s|\mathfrak{P}}^\lambda$ is indecomposable each U_i is uniquely determined. By Lemma 3.1. $U_1\subseteq U_i$ and so $U_i\approx V_i^\lambda$.

If $i=0$ or $i=s$ the last statement is clear. Suppose that $i=1$ and $s\ge 2$. Since $|\mathfrak{E}|\,|(p-1)$ the $K\mathfrak{E}$-module $U_{2|\mathfrak{E}}$ is a direct sum of two $K\mathfrak{E}$-modules. Choose a K-basis x, y of U_2 such that $y\in U_1$ and $xE=\mu(E)x$ for all $E\in\mathfrak{E}$ and some 1-dimensional K-representation of $\mathfrak{E}$. Then for suitable $P\in\mathfrak{P}$, $xP=x+y$. Thus for $E\in\mathfrak{E}$

$$\begin{aligned}x+\alpha(E)y=xP^{\alpha(E)}=xE^{-1}PE=\mu(E^{-1})xPE&=\mu(E^{-1})xE+\mu(E^{-1})yE\\ =x+\mu(E^{-1})\lambda(E)y.&\end{aligned}$$

Hence $\mu(E)=\lambda(E)\alpha^{-1}(E)$ for all $E\in\mathfrak{E}$. If $\bar{x}$ denotes the image of x in V_s^λ/U_1 this implies that if $G=PE$, $P\in\mathfrak{P}$, $E\in\mathfrak{E}$ then

$$\bar{x}G=\bar{x}E=\lambda\alpha^{-1}(E)\bar{x}=\lambda\alpha^{-1}(G)\bar{x}$$

Thus $V_s^\lambda/U_i\approx V_{s-1}^{\lambda\alpha^{-1}}$. Since $V_s^\lambda/U_1\approx(V_s^\lambda/U_1)/(U_i/U_1)$ for $i\ge 1$ the result follows by induction on i.

LEMMA 3.3. $(V_s^\lambda)^*\approx V_s^{\lambda^{-1}\alpha^{(s-1)}}$. $\det_s^\lambda(E)=\lambda^s\alpha^{-s(s-1)/2}(E)$ *for* $E\in\mathfrak{E}$. *Let* $\mathfrak{E}=\langle E_0\rangle$. *Then* $\varphi_s^{\alpha^j}(E_0)=\varepsilon^j\left(\sum_{i=0}^{s-1}\varepsilon^{-i}\right)$ *for a suitable primitive* $|\mathfrak{E}|$*th root of unity* ε *and all* s *and* j.

Proof. This is an immediate consequence of Lemma 3.2.

LEMMA 3.4. $V_s^\lambda\otimes V_p^\mu\approx\sum_{i=0}^{s-1}V_p^{\lambda\mu\alpha^{-i}}$ *for* $0\le s\le p$.

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Proof. Let M_μ be the 1-dimensional $K\mathfrak{E}$-module corresponding to the representation $\mu|\mathfrak{E}$. It is easily seen (and well known) that $V_p^\mu \approx M_\mu^{\mathfrak{E}\mathfrak{P}}$. By Lemma 3.2 $V_s^\lambda|_{\mathfrak{E}} \approx \sum_{i=0}^{s-1} M_{\lambda\alpha^{-i}}$. Thus [3, p. 325].

$$V_s^\lambda \otimes V_p^\mu \approx (V_s^\lambda|_{\mathfrak{E}} \otimes M_\mu)^{\mathfrak{E}\mathfrak{P}} \approx \left(\sum_{i=0}^{s-1} M_{\lambda\mu\alpha^{-i}}\right)^{\mathfrak{E}\mathfrak{P}} \approx \sum_{i=0}^{s-1} V_p^{\lambda\mu\alpha^{-i}}$$

LEMMA 3.5. *If $0 \leq s \leq t$ and $s+t \leq p$ then*

$$V_s^\lambda \otimes V_t^\mu \approx \sum_{i=0}^{s-1} V_{s+t-1-2i}^{\lambda\mu\alpha^{-i}}.$$

Proof. It suffices to prove the result in case $|\mathfrak{E}| = p-1$. If $s=0$ or 1 it is immediate.

Suppose $s=2$. By [6, Theorem 3 (2.3 b)] $V_2 \otimes V_t \approx V_{t-1} + V_{t+1}$. Thus by Lemma 3.1 $V_2^\lambda \otimes V_t^\mu \approx V_{t-1}^\beta + V_{t+1}^\gamma$ for some β, γ. By Lemma 3.2 there exist K-bases $\{x_0, x_1\}$ of V_2^λ and $\{y_0, \dots, y_{t-1}\}$ of V_t^μ such that for $E \in \mathfrak{E}$ and all i

$$x_iE = \lambda\alpha^{-i}(E)x_i, \quad y_iE = \mu\alpha^{-i}(E)y_i.$$

Furthermore if U is the submodule of $V_2^\lambda \otimes V_t^\mu$ consisting of all u with $uP = u$ for all $P \in \mathfrak{P}$ then $\dim_K U = 2$. Let $P \in \mathfrak{P}$. Then there exist $a, b \in K$ with $ab \neq 0$ such that

$$x_0P = x_0, \quad x_1P = x_1 + ax_0$$
$$y_0P = y_0, \quad y_1P = y_1 + by_0$$

Define

$$v_0 = x_0 \otimes y_0, \quad v_1 = \frac{1}{a} x_1 \otimes y_0 - \frac{1}{b} x_0 \otimes y_1.$$

Then $v_iP = v_i$ for $i = 0, 1$, and so $\{v_0, v_1\}$ is a basis of U. If $E \in \mathfrak{E}$ then

$$v_0E = \lambda\mu(E)v_0, \quad v_1E = \lambda\mu\alpha^{-1}(E)v_1$$

As $|\mathfrak{E}| \neq 1$, $\lambda\mu \neq \lambda\mu\alpha^{-1}$. Therefore $v_0 \in V_{t-1}^\beta$ and $\beta = \lambda\mu$ or $v_0 \in V_{t+1}^\gamma$ and $\gamma = \lambda\mu$.

Let $\langle x_0 \rangle \approx V_1^\lambda$ be the submodule of V_2^λ generated by x_0. Let $W = \langle x_0 \rangle \otimes V_t^\mu \approx V_t^{\mu\lambda}$. Thus W is indecomposable and $v_0 \in W$. Since $\dim_K W = t$ it follows that $W \cap U_{t+1} \neq 0$, where U_{t+1} is a submodule of $V_2^\lambda \otimes V_t^\mu$ with $U_{t+1} \approx V_{t+1}^\gamma$. By Lemma 3.2 $v_0 \in W \cap U_{t+1}$. Hence $v_0 \in U \cap U_{t+1}$ and $\gamma = \lambda\mu$. Thus $\beta = \lambda\mu\alpha^{-1}$ and the result is proved in case $s = 2$.

We proceed by induction on s. Assume that $s \geq 3$ and the result has been proved for $s-1$ and $s-2$. Then

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$$V_{s-1}^{\lambda}\otimes V_2^1\otimes V_t^{\mu}\approx(V_{s-2}^{\lambda\alpha^{-1}}\otimes V_t^{\mu})+(V_s^{\lambda}\otimes V_t^{\mu}).$$

Thus by induction

$$\sum_{i=0}^{s-2}(V_{s+t-2-2i}^{\lambda\mu\alpha^{-i}}\otimes V_2^1)\approx\sum_{i=0}^{s-3}V_{s+t-3-2i}^{\lambda\mu\alpha^{-i-1}}+(V_s^{\lambda}\otimes V_t^{\mu}).$$

Applying the first part of the lemma once again yields that

$$V_{s+t-1}^{\lambda\mu}+2\Big(\sum_{i=1}^{s-2}V_{s+t-1-2i}^{\lambda\mu\alpha^{-i}}\Big)+V_{t-s+1}^{\lambda\mu\alpha^{-(s-1)}}\approx\sum_{i=0}^{s-3}V_{s+t-3-2i}^{\lambda\mu\alpha^{-i-1}}+(V_s^{\lambda}\otimes V_t^{\mu}).$$

The result now follows from the Krull-Schmidt Theorem.

The next result is proved in a similar manner to [6, (2.5 a)].

LEMMA 3.6. *Suppose that* $1\leq b,\ c\leq p-1$ *and* $V_b^{\beta}\otimes V_c^{\gamma}\approx\sum_{i=0}^{k}V_{e_i}^{\sigma_i}$ *with* $e_i>0$ *for* $i=0,\ldots,k$. *Then*

$$\sum_{i=0}^{k}V_p^{\sigma_i\alpha^{b-e_i}}+(V_{p-b}^{\beta}\otimes V_c^{\gamma})\approx\sum_{j=0}^{c-1}V_p^{\beta\gamma\alpha^{-j}}+\sum_{i=0}^{k}V_{p-e_i}^{\sigma_i\alpha^{b-e_i}}.$$

Proof. By Lemma 3.2

$$0\to V_{p-b}^{\beta\alpha^{p-b}}\to V_p^{\beta\alpha^{p-b}}\to V_b^{\beta}\to 0.$$

is exact. Tensoring with V_c^{γ} yields that

$$0\to V_{p-b}^{\beta\alpha^{p-b}}\otimes V_c^{\gamma}\to V_p^{\beta\alpha^{p-b}}\otimes V_c^{\gamma}\to V_b^{\beta}\otimes V_c^{\gamma}\to 0$$

is exact. Also

$$0\to\sum_{i=0}^{k}V_{p-e_i}^{\sigma_i\alpha^{p-e_i}}\to\sum_{i=0}^{k}V_p^{\sigma_i\alpha^{p-e_i}}\to\sum_{i=0}^{k}V_{e_i}^{\sigma_i}\to 0$$

is exact. Thus Schanuel's Theorem and Lemma 3.4 imply that

$$\sum_{i=0}^{k}V_p^{\sigma_i\alpha^{p-e_i}}+(V_{p-b}^{\beta\alpha^{p-b}}\otimes V_c^{\gamma})\approx\sum_{j=0}^{c-1}V_p^{\beta\gamma\alpha^{p-b-j}}+\sum_{i=0}^{k}V_{p-e_i}^{\sigma_i\alpha^{p-e_i}}.$$

The result follows by tensoring this equation with $V_1^{\alpha^{b-p}}$

LEMMA 3.7. *If* $1\leq s\leq\dfrac{p-1}{2}$ *then*

$$V_s^{\lambda}\otimes V_s^{\mu}\approx\sum_{i=0}^{s-1}V_{2i+1}^{\lambda\mu\alpha^{i+1-s}}$$

$$V_{p-s}^{\lambda}\otimes V_{p-s}^{\mu}\approx\sum_{i=0}^{s-1}V_{2i+1}^{\lambda\mu\alpha^{s+i}}+\sum_{i=2s}^{p-1}V_p^{\lambda\mu\alpha^{i}}$$

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Proof. The first statement is a special case of Lemma 3.5. Also Lemma 3.5 yields that

$$V_s^\lambda \otimes V_{p-s}^\mu \approx \sum_{i=0}^{s-1} V_{p-1-2i}^{\lambda\mu\alpha^{-i}}$$

Apply Lemma 3.6 with $\beta=\lambda$, $\gamma=\mu$, $b=s$ and $c=p-s$. Then

$$\sum_{i=0}^{s-1} V_p^{\lambda\mu\alpha^{s+i-p+1}} + (V_{p-s}^\lambda \otimes V_{p-s}^\mu) \approx \sum_{j=0}^{p-s-1} V_p^{\lambda\mu\alpha^{-j}} + \sum_{i=0}^{s-1} V_{2i+1}^{\lambda\mu\alpha^{s+i-p+1}}$$

Since $\alpha^{p-1}(G)=1$ for all $G\in\mathfrak{C}\mathfrak{P}$ the Krull Schmidt Theorem implies the result.

LEMMA 3.8. *If* $1\leq s\leq \frac{p-1}{2}$ *then*

$$V_s^\lambda \otimes (V_s^\lambda)^* \approx \sum_{i=0}^{s-1} V_{2i+1}^{\alpha^i}$$

$$V_{p-s}^\lambda \otimes (V_{p-s}^\lambda)^* \approx \sum_{i=0}^{s-1} V_{2i+1}^{\alpha^i} + \sum_{i=s}^{p-s-1} V_p^{\alpha^i}$$

Proof. This follows directly from Lemmas 3.3 and 3.7 and the fact that $\alpha^{p-1}(G)=1$ for all $G\in\mathfrak{C}\mathfrak{P}$.

§ 4. Proof of Theorem 1

Throughout this section $\mathfrak{G}$ is a group which satisfies the hypotheses of Theorem 1. $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$. Since $d\leq p$ in Theorem 1 $\mathfrak{P}$ has exponent p and so $|\mathfrak{P}|=p$ as $\mathfrak{P}$ is cyclic. $\mathfrak{N}=\mathbf{N}_{\mathfrak{G}}(\mathfrak{P})$ and $\mathfrak{C}=\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})=\mathfrak{P}\times\mathfrak{H}$. By assumption $\mathfrak{N}\neq\mathfrak{G}$ and by Burnside's transfer theorem $\mathfrak{N}\neq\mathfrak{C}$. K is a field of characteristic p.

$\mathscr{M}=\{M\,|\,M$ is an indecomposable $K\mathfrak{G}$-module with $\dim_K M\leq p$ and $\mathfrak{P}$ is not in the kernel of $M\}$.

By assumption $\mathscr{M}$ is nonempty. If $M\in\mathscr{M}$ then M is a direct summand of $(M|_{\mathfrak{N}})^{\mathfrak{G}}$ by D. G. Higman's Theorem [3. § 63]. Thus $M|_{\mathfrak{N}}$ is indecomposable by the Mackey decomposition and if $\dim_K M<p$ then M is uniquely determined by $M|_{\mathfrak{N}}$. The Mackey decomposition and (2.1) imply that $M|_{\mathfrak{C}}=\sum_{i=1}^{a} U_i\otimes W_i$ where for each i U_i is an indecomposable $K\mathfrak{P}$-module and W_i is an irreducible $K\mathfrak{H}$-module. Furthermore $U_i\otimes W_i$ is conjugate to $U_j\otimes W_j$ for all i, j under the action of $\mathfrak{N}/\mathfrak{C}$. Thus $\dim_K U_i=c$, $\dim_K W_i=b$ are both independent of i and in the notation of section 3 $U_i\approx V_c$ for all i. Therefore

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$$M|_{\mathfrak{C}} \approx V_c \otimes \left(\sum_{i=1}^{a} W_i\right), \ \dim_K W_i = b. \tag{4.1}$$

The triple $a = a(M)$, $b = b(M)$, $c = c(M)$ is a set of invariants attached to M and (4.1) implies that

$$\dim_K M = a(M)\, b(M)\, c(M). \tag{4.2}$$

LEMMA 4.1. *Suppose that* $p \geq 5$. *If* $M \in \mathscr{M}$ *then* $\dim_K M > 2$.

Proof. Suppose $\dim_K M \leq 2$ for some $M \in \mathscr{M}$. Let $\mathfrak{K}$ be the kernel of M. Then $\mathfrak{G}/\mathfrak{K}$ is isomorphic to a subgroup of $GL_2(K)$. All finite subgroups of $GL_2(K)$ are known and it is easily seen that $\mathfrak{G}/\mathfrak{K}$ and hence $\mathfrak{G}$, is of type $L_2(p)$ contrary to assumption.

LEMMA 4.2. *Suppose that* $p \geq 5$. *If* $M \in \mathscr{M}$ *with* $\mathfrak{H}$ *in the kernel of* M *then* $\dim_K M > 3$.

Proof. Let $M \in \mathscr{M}$ with $\mathfrak{H}$ in the kernel of M. Suppose that $\dim_K M \leq 3$. By Lemma 4.1 it may be assumed that $\dim_K M = 3$ and M is absolutely irreducible. We will reach a contradiction by showing that $\mathfrak{G}$ is of type $L_2(p)$. By changing notation it may be assumed that $\mathfrak{G}' = \mathfrak{G}$ and M is faithful. Thus $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P}$. Let $\mathfrak{N} = \mathfrak{P}\mathfrak{E}$ with $\mathfrak{P} \cap \mathfrak{E} = \langle 1 \rangle$. Let $\mathfrak{E} = \langle E \rangle$. Let α be defined as in (3.1). Then $M|_{\mathfrak{N}} \approx V_3^{\lambda}$ for some one dimensional K-representation λ by Lemma 3.1 and (4.1). Lemmas 3.1, 3.3 and 3.8 imply that $M \otimes M^* = L_0 + L_1 + L_2$ where $\dim_K L_i = 2i+1$ and $L_i|_{\mathfrak{N}}^* = L_i|_{\mathfrak{N}}$. Thus M may be chosen so that $M|_{\mathfrak{N}}^* \approx M|_{\mathfrak{N}}$.

Since $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P}$ there is only one block of defect 1 [3, (86.10)]. Hence M is in the principal block of $\mathfrak{G}$. Thus if K_0 is the field of p elements there exists a K_0-representation $\mathfrak{F}$ of $\mathfrak{G}$ corresponding to M by (2.4). Since $M|_{\mathfrak{N}} \approx M|_{\mathfrak{N}}^*$ it follows from Lemma 3.3 that $\mathfrak{F}$ is equivalent to $\mathfrak{F}^*$. An argument of R. Brauer [2, p. 438] now implies that $\mathfrak{G}$ is isomorphic to a subgroup of $O_3(p)$. Since $O_3(p)$ is of type $L_2(p)$ so is $\mathfrak{G}$ contrary to assumption.

LEMMA 4.3. *Suppose that* $p \geq 5$. *If* $M \in \mathscr{M}$ *then* $c(M) > \frac{p-1}{2}$.

Proof. Suppose $M \in \mathscr{M}$ with $c = c(M) \leq \frac{p-1}{2}$. By Lemma 3.8 and (4.1)

$$M \otimes M^*|_{\mathfrak{C}} \approx \left(\sum_{i=0}^{c-1} V_{2i+1}\right) \otimes \left(\sum_{j=1}^{a}\sum_{k=1}^{a} W_i \otimes W_j^*\right).$$

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Thus no direct summand of $M\otimes M^*|_{\mathfrak{C}}$ is projective. Let W_0 be the trivial 1-dimensional $K\mathfrak{G}$-module. Then

$$M\otimes M^*|_{\mathfrak{C}}\approx\sum_{i=0}^{c-1}(V_{2i+1}\otimes W_0)+U$$

where U is a direct sum of indecomposable modules none of which are projective. Since $M\in\mathscr{M}$, $c>1$. Thus $V_3\otimes W_0$ is isomorphic to a direct summand of $M\otimes M^*|_{\mathfrak{C}}$. Let L be a direct summand of $M\otimes M^*$ such that $V_3\otimes W_0$ is isomorphic to a direct summand of $L|_{\mathfrak{C}}$. Since no direct summand of $L|_{\mathfrak{N}}$ is projective, $L|_{\mathfrak{N}}$ is indecomposable. As $\mathfrak{H}$ is in the kernel of $V_3\otimes W_0$ $\mathfrak{H}$ is also in the kernel of $L|_{\mathfrak{N}}$. Thus $L|_{\mathfrak{C}}$ is indecomposable by Lemma 3.1. Hence $\dim_K L=3$ contrary to Lemma 4.2.

LEMMA 4.4. *If $M\in\mathscr{M}$, $M|_{\mathfrak{P}}$ is indecomposable and $\mathfrak{H}=\mathbf{Z}(\mathfrak{G})$.*

Proof. If $\dim_K M=p$ then M is projective and so $M|_{\mathfrak{P}}$ is projective and hence indecomposable. Suppose that $\dim_K M\leq p-1$. If $p=3$ then $M|_{\mathfrak{P}}$ is indecomposable since $\mathfrak{P}$ is not in the kernel of M. If $p\geq 5$ then (4.2) and Lemma 4.3 imply that $a(M)=b(M)=1$. Thus by (4.1) $M|_{\mathfrak{P}}$ is indecomposable in any case. If $\mathfrak{F}$ is the K-representation of $\mathfrak{G}$ corresponding to M this implies that any p'-element in the commuting ring of $\mathfrak{F}|_{\mathfrak{P}}$ is a scalar. Thus $\mathfrak{H}=\mathbf{Z}(\mathfrak{G})$ as required.

The proof of Theorem 1 can now be given. If $p=2$ then $\mathfrak{G}$ is 2-solvable since $|\mathfrak{P}|=2$ contrary to assumption. Thus $p\neq 2$. In view of Lemma 4.4 it only remains to prove the inequalities. If $p=3$ the result is trivial and if $p=5$ it follows from Lemma 4.1. Hence it may be assumed that $p\geq 7$. It may further be assumed that $\mathfrak{G}=\mathfrak{G}'$ and K is algebraically closed without loss of generality.

Choose $L\in\mathscr{M}$ with $\dim_K L$ minimal. Let $d=p-s=\dim_K L$. It may be assumed that L is faithful. By Lemma 4.3 and (4.1)

(4.3) $$L|_{\mathfrak{C}}\approx V_{p-s}\otimes W,\ \dim_K W=1,\ s\leq\frac{p-1}{2}.$$

Since $\mathfrak{N}/\mathfrak{C}$ is cyclic and $\mathfrak{H}\subseteq\mathbf{Z}(\mathfrak{N})$ it follows that $\mathfrak{N}/\mathfrak{P}$ is abelian. Thus there exists a $K\mathfrak{N}$-module W_1 whose kernel contains $\mathfrak{P}$ such that $W_1|_{\mathfrak{H}}=W$. Then

(4.4) $$L|_{\mathfrak{N}}\otimes L^*|_{\mathfrak{N}}\approx(L|_{\mathfrak{N}}\otimes W_1^*)\otimes(L|_{\mathfrak{N}}\otimes W_1^*)^*.$$

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Furthermore

$$(L|_{\mathfrak{N}} \otimes W_1^*)|_{\mathfrak{C}} \approx V_{p-s} \otimes W_0$$

where W_0 denotes the trivial 1-dimensional $K\mathfrak{H}$-module. Let $\bar{\mathfrak{N}} = \mathfrak{N}/\mathfrak{H}$. Thus $L|_{\mathfrak{N}} \otimes W_1^*$ is a $K\bar{\mathfrak{N}}$-module. Hence (4.3), (4.4) and Lemma 3.8 imply that in the notation of section 3

$$L|_{\mathfrak{N}} \otimes L^*|_{\mathfrak{N}} \approx \sum_{i=0}^{s-1} V_{2i+1}^{\alpha^i} + \sum_{i=s}^{p-s-1} V_p^{\alpha^i}, \tag{4.5}$$

where each V_j^{λ} is a $K\bar{\mathfrak{N}}$-module.

Higman's Theorem and (4.5) imply that

$$L \otimes L^* \approx \sum_{i=0}^{s-1} L_i + A$$

where each L_i is indecomposable, A is projective and $L_i|_{\mathfrak{N}}$ has $V_{2i+1}^{\alpha^i}$ has a direct summand. Let

$$L_i|_{\mathfrak{N}} = V_{2i+1}^{\alpha^i} + \sum_{j=1}^{m_i} V_p^{\mu_{ij}} \tag{4.6}$$

Thus L_0 is the 1-dimensional trivial $K\mathfrak{G}$-module. By (4.5)

$$\{\mu_{ij} \mid j = 1, \ldots, m_i;\ i = 0, \ldots, s-1\} \subseteq \{\alpha^i \mid s \le i \le p-s-1\} \tag{4.7}$$

Suppose that $p - s < 2/3(p-1)$. Then $p < 3s - 1$. By (4.7)

$$\sum_{i=1}^{s-1} m_i \le (p-s-1) - s + 1 = p - 2s < s - 1.$$

Hence at least $(s-1) - (p-2s)$ of the m_i are zero. Thus $m_k = 0$ for some k with

$$1 \le k \le (s-1) - \{(s-1) - (p-2s)\} = p - 2s.$$

Thus by (4.6)

$$\dim_K L_k = 2k + 1 \le 2p - 4s + 1 = (p-s) + (p+1-3s) < p - s = d.$$

Hence $L_k \in \mathscr{M}$ contrary to the minimality of d. Therefore in proving Theorem 1 it may be assumed that $p \ge 13$ and $d = p - s \ge 2/3(p-1)$ or equivalently

$$s \le \frac{p+2}{3}, \quad p \ge 13. \tag{4.8}$$

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Choose $E \in \mathfrak{G}$ so that $\mathfrak{N} = \langle E, \mathfrak{G} \rangle$. Since $\mathfrak{G} = \mathfrak{G}'$ E must have determinant 1 when considered as a linear transformation on the K-space L_i for $i = 0, \ldots, s-1$. Thus by (4.6) and Lemma 3.3.

$$(4.9) \qquad \left(\prod_{j=1}^{m_i} \mu_{ij}\right)\alpha^{m_i(p-1)/2}(E) = \left(\prod_{j=1}^{m_i} \mu_{ij^p}\right)\alpha^{-m_ip(p-1)/2}(E) = 1.$$

Hence if $m_i = 1$ then $\mu_{i1}(E) = \alpha^{(p-1)/2}(E) = \pm 1$. Since $\mathfrak{G}$ is not of type $L_2(p)$, $E \neq 1$. Thus for any k either $\alpha^k(E) \neq \alpha^{(p-1)/2}(E)$ or $\alpha^{k+1}(E) \neq \alpha^{(p-1)/2}(E)$. Consequently (4.5) and (4.6) imply that at most $\frac{p+1-2s}{2}$ of m_i's are equal to 1.

Suppose first that $2s-1 < p-s = d$. The minimality of d and (4.6) yield that $m_i \neq 0$ for $i = 1, \ldots, s-1$. Thus by (4.6)

$$s-1 \leq \frac{p+1-2s}{2} + \frac{1}{2}\left\{p-2s-\frac{(p+1-2s)}{2}\right\} = \frac{1}{4}(3p-6s+1).$$

Hence $s \leq \frac{3p+5}{10}$ and so $d = p-s \geq \frac{7p}{10} - \frac{1}{2}$ as required.

Assume now that $2s-1 \geq p-s$. Thus $s \geq 5$. The minimality of d yields that $m_i \neq 0$ for $i = 1, \ldots, s-2$.
Thus by (4.6)

$$s-2 \leq \frac{p+1-2s}{2} + \frac{1}{2}\{p-2s-(p+1-2s)\} = \frac{1}{4}(3p-6s+1)$$

Therefore

$$10s \leq 3p+9 = 9s+6$$

Hence $s \leq 6$ and $p \neq 13$ so $p \leq 3s-1 \leq 17$. Thus $s = 6$ and $p = 17$. Furthermore $\dim_K L_5 = 11 = d$. Since $\mathfrak{H}$ is in the kernel of L_5 it may be assumed L was chosen initially such that $\mathfrak{H}$ is in the kernel of L. Hence since L is faithful it may be assumed that $\mathfrak{H} = \langle 1 \rangle$. Thus L is in the principal p-block. The minimality of d implies that L is an irreducible $K\mathfrak{G}$-module. Therefore $|\langle E \rangle| = |\mathfrak{N} : \mathfrak{P}| > 2$ by (2.3). Thus for any k either $\alpha^k(E) \neq \alpha^{(p-1)/2}(E)$ or $\alpha^{k+1}(E) \neq \alpha^{(p-1)/2}(E)$ or $\alpha^{k+2}(E) \neq \alpha^{(p-1)/2}(E)$. Thus by (4.9) at most $\frac{p+2-2s}{3} < 3$ of the m_i's are equal to 1 and so by (4.6).

$$4 = s-2 \leq 2 + \frac{1}{2}(5-2) < 4.$$

This contradiction establishes Theorem 1 in all cases.

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§ 5. Proof of Theorem 2

Throughout this section $\mathfrak{G}$ is a group which satisfies the hypotheses but not the conclusion of Theorem 2. $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{N} = \mathbf{N}_{\mathfrak{G}}(\mathfrak{P})$. ζ is an irreducible faithful complex character of degree d.

LEMMA 5.1. *$\mathfrak{G}$ is simple. $|\mathfrak{P}| = p$ and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P}$*

Proof. Let $\mathfrak{G}_0$ be the subgroup of $\mathfrak{G}$ generated by all p-elements in $\mathfrak{G}$. Thus $\mathfrak{G}_0 \triangleleft \mathfrak{G}$. Let $\zeta_{|\mathfrak{G}_0} = \sum_{i=1}^{n} \omega_i$ where each ω_i is an irreducible character of $\mathfrak{G}_0$. Since the ω_i are conjugate under the action of $\mathfrak{G}$ they all have the same degree. Thus if $n > 1$, $\omega_i(1) < \frac{p-1}{2}$ for each i and so by [5] $\mathfrak{P} \triangleleft \mathfrak{G}$ contrary to assumption. Hence $\zeta_{|\mathfrak{G}_0} = \omega$ is irreducible. Thus $\mathbf{Z}(\mathfrak{G}_0) = \mathbf{Z}(\mathfrak{G}) = \langle 1 \rangle$.

Suppose that $|\mathfrak{P}| \neq p$. Then there exists $\mathfrak{P}_1 \triangleleft \mathfrak{G}$ with $|\mathfrak{P} : \mathfrak{P}_1| = p$ [4]. Hence $\mathfrak{P} \subseteq \mathbf{C}_{\mathfrak{G}}(\mathfrak{P}_1) \triangleleft \mathfrak{G}$ and so $\mathfrak{G}_0 \subseteq \mathbf{C}_{\mathfrak{G}}(\mathfrak{P}_1)$. Thus $\mathfrak{P}_1 \subseteq \mathbf{Z}(\mathfrak{G}_0) = \langle 1 \rangle$ and so $|\mathfrak{P}| = p$.

Suppose that $\mathfrak{A} \triangleleft \mathfrak{G}_0$, $\mathfrak{A} \neq \mathfrak{G}_0$ Then $\mathfrak{A}$ is a p'-group. Hence $\mathfrak{A} \triangleleft \mathfrak{A}\mathfrak{P}$ and $\mathfrak{A}\mathfrak{P}$ is p-solvable. Since $\mathfrak{A}\mathfrak{P}$ has a faithful complex representation of degree $d < p-1$ it follows that $\mathfrak{A}\mathfrak{P}$ has a K-representation whose kernel is in $\mathfrak{P}$ for a suitable field K of characteristic p. Thus by Theorem B of Hall and Higman [7] (see also [11] for a simplification of part of the proof.) $\mathfrak{P} \subseteq \mathbf{C}_{\mathfrak{G}_0}(\mathfrak{A}) \triangleleft \mathfrak{G}_0$. Thus $\mathfrak{A} \subseteq \mathbf{Z}(\mathfrak{G}_0) = \langle 1 \rangle$. Therefore $\mathfrak{G}_0$ is simple.

By (2.2)

$$e = |\mathfrak{N} : \mathbf{C}(\mathfrak{P})| = p - \zeta(1) = p - \omega(1) = |\mathbf{N}_{\mathfrak{G}_0}(\mathfrak{P}) : \mathbf{C}_{\mathfrak{G}_0}(\mathfrak{P})|.$$

Since $\mathfrak{G} = \mathfrak{G}_0\mathfrak{N}$ this yields that $\mathfrak{G} = \mathfrak{G}_0\mathbf{C}_{\mathfrak{G}}(\mathfrak{P})$. If $\mathfrak{G}$ is not of type $L_2(p)$ then Theorem 1 implies that $\mathfrak{G} = \mathfrak{G}_0$ and $\mathfrak{P} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{P})$ completing the proof of the Lemma. Suppose that $\mathfrak{G}$ is of type $L_2(p)$. Thus $\mathfrak{G}_0 \approx PSL_2(p)$. Since $PSL_2(p)$ admits no outer automorphism which leaves all the elements in a S_p-subgroup fixed it follows that $\mathfrak{G} = \mathfrak{G}_0 \approx PSL_2(p)$. Thus $\mathfrak{G}$ is simple since $p > 3$ and $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P}$ as required.

Let F be a finite extension field of the field of p-adic numbers which is a splitting field for $\mathfrak{G}$ and all its subgroups and contains all the $|\mathfrak{G}|$th roots of unity. Let R be the ring of local integers in F, let $\mathfrak{p}$ be the maximal ideal in R and let $K = R/\mathfrak{p}$. It is well known that there exists an $R\mathfrak{G}$-module Z which affords the character ζ. Let $\bar{Z} = Z/\mathfrak{p}Z$.

LEMMA 5.2. *$\bar{Z}$ is absolutely irreducible.*

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Proof. Since F contains all $|\mathfrak{G}|$th roots of unity K is a splitting field of $\mathfrak{G}$. Thus it suffices to show that $\bar{Z}$ is irreducible. By (2.2) and Lemma 5.1 every modular irreducible constituent of $\bar{Z}$ is faithful. Hence if $\bar{Z}$ is reducible then $\mathfrak{G}$ has a faithful K-representation of degree at most $d/2 < \frac{p-1}{2}$. Hence by Theorem 1 $\mathfrak{G}$ is of type $L_2(p)$ and so $\mathfrak{G}$ is isomorphic to $PSL_2(p)$ by Lemma 5.1. In this case it is well known that $e = \frac{p-1}{2}$ and $p \equiv 1 \pmod 4$ contrary to assumption.

Let $\mathfrak{N} = \mathfrak{P}\mathfrak{E}$ with $\mathfrak{P} \cap \mathfrak{E} = \langle 1 \rangle$ and let $\mathfrak{E} = \langle E \rangle$. Let α be defined as in (3.1). Let ε be a primitive e^{th} root of unity in R such that the image of ε in $R/\mathfrak{p}$ is $\alpha(E)$.

LEMMA 5.3. $\bar{Z}_{|\mathfrak{N}} \not\approx V^1_{p-e}$

Proof. Suppose that $\bar{Z}_{|\mathfrak{N}} \approx V^1_{p-e}$. Let $\left\{\zeta_i \mid i = 1, \ldots, \frac{p-1}{e}\right\}$ be all the irreducible complex characters of $\mathfrak{G}$ which are algebraically conjugate to ζ. Then by (2.2) the ζ_i are all equal as Brauer characters. Thus if U is an $R\mathfrak{G}$-module affording the character Φ such that $\bar{U} = U/\mathfrak{p}U$ is the projective indecomposable $K\mathfrak{G}$-module corresponding to $\bar{Z}$ then $\Phi = \sum_{i=1}^{p-1/e} \zeta_i + \theta$ for some character θ. Thus [11, Theorem 1] there exists an $R\mathfrak{G}$-module M which affords the character $\sum_{i=1}^{p-1/e} \zeta_i$ such that $\bar{M} = M/\mathfrak{p}M$ is indecomposable. Since $\dim_K \bar{M} = \left(\frac{p-1}{e} - 1\right)p + 1$ Higman's theorem and Lemma 3.1 imply that

$$\bar{M}_{|\mathfrak{N}} \approx V_1^{\alpha^k} + \sum_{j=1}^{(p-1)/e-1} V_p^{\alpha^{a(j)}}$$

for suitable k and $a(j)$. Let ψ be the Brauer character afforded by $\bar{M}$. Then Lemma 3.3 implies that

$$\psi(E) = \varepsilon^k + \sum_{j=1}^{(p-1)/e-1} \varepsilon^{a(j)} \left(\sum_{t=0}^{p-1} \varepsilon^{-t} \right) = \varepsilon^k + \sum_{j=1}^{(p-1)/e-1} \varepsilon^{a(j)}$$

$$\zeta_i(E) = \sum_{t=0}^{p-e-1} \varepsilon^{-t} = 1$$

Since $\psi(E) = \sum_{i=1}^{(p-1)/e} \zeta_i(E)$ this yields that $k = 1$ and $a(j) = 1$ for all j. Hence $\bar{M}_{|\mathfrak{N}} \approx V_1^1 + A$ for some projective $K\mathfrak{N}$-module A. Let L_0 be the trivial 1-dimensional $K\mathfrak{G}$-module. Then $L_{0|\mathfrak{N}} \approx V_1^1 + B$ for some projective $K\mathfrak{N}$-module B. Hence by Higman's Theorem $\bar{M}$ and L_0 are both direct summands of $(V_1^1)^{\mathfrak{G}}$ contrary to the Mackey decomposition. This contradiction establishes the lemma.

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LEMMA 5.4. *$e \equiv 0 \pmod 2$, $\bar{Z}|_{\mathfrak{N}} \approx V_{p-1}^{\alpha^{e/2}}$ and $\frac{p-1}{e} \equiv 0 \pmod 2$*

Proof. Let $\bar{Z}|_{\mathfrak{N}} \approx V_{p-e}^{\alpha^k}$. By Lemma 3.3 $\zeta(E) = \varepsilon^k$. Since $\mathbf{C}_{\mathfrak{G}}(\mathfrak{P}) = \mathfrak{P}$ (2.2) implies that $\zeta(E)$ is rational. Hence $\varepsilon^k = \pm 1$. If $\varepsilon^k = 1$ then $e|k$ and so $\bar{Z}|_{\mathfrak{N}} \approx V_{p-e}^1$ contrary to Lemma 5.3. Hence $\varepsilon^k = -1$. Therefore $e \equiv 0 \pmod 2$ and $\bar{Z}|_{\mathfrak{N}} \approx V_{p-e}^{\alpha^{e/2}}$.

Since $\mathfrak{G}$ is simple $\det_{p-s}^{\alpha^{e/2}}(E) = 1$. Thus by Lemma 3.3

$$1 = \alpha^{e/2(p-e)} \alpha^{-(p-e)(p-e-1)/2}(E) = -\alpha^{-(p-e)(p-e-1)/2}(E) = -\alpha^{-(p-e-1)/2}(E).$$

Thus $\frac{p-e-1}{2} \equiv e/2 \pmod e$ and so $\frac{p-1}{2} \equiv 0 \pmod e$. Hence $\frac{p-1}{e} \equiv 0 \pmod 2$ as required.

Theorem 2 now follows from Lemmas 5.1 and 5.4.

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