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PRICING TIMER OPTIONS: SECOND-ORDER MULTISCALE STOCHASTIC VOLATILITY ASYMPTOTICS

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Abstract

We study the pricing of timer options in a class of stochastic volatility models, where the volatility is driven by two diffusions—one fast mean-reverting and the other slowly varying. Employing singular and regular perturbation techniques, full second-order asymptotics of the option price are established. In addition, we investigate an implied volatility in terms of effective maturity for the timer options, and derive its second-order expansion based on our pricing asymptotics. A numerical experiment shows that the price approximation formula has a high level of accuracy, and the implied volatility in terms of its effective maturity is illustrated.

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1. Introduction

In April 2007, Société Générale Corporate and Investment Banking (SG CIB) issued the timer option [18], an exotic option whose expiration date depends on the realized variance of the underlying asset. It is different from the vanilla option in that it has a random instead of a fixed maturity. The link to the realized variance of the underlying asset makes the timer option financially attractive. When the implied volatility is higher than the actual realized volatility, this product protects investors from overpaying for an option. The price of the vanilla option depends on the implied volatility, while the price of the timer option is connected to the actual realized volatility. This connection also makes the timer option a suitable financial instrument for volatility trading and hedging-like variance or volatility swaps [1]. Moreover, the

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timer option is interesting not only financially but also from a mathematical point of view.

In spite of the relatively simple payoff structure compared with other exotic options, pricing of timer options is challenging owing to the volatility-linked maturity structure. In the past few years, some research has been devoted to these exotic options. Li [15], Cui et al. [4], Liang et al. [16], and Zheng and Zeng [22] obtained some closed-form pricing formulas for timer options in specific types of stochastic volatility models. Zhang et al. [21] studied perpetual timer options under the Hull–White stochastic volatility model. They transformed the model into the Bessel process by time-change techniques and obtained an explicit analytical solution using a probabilistic method.

When more general models are under consideration, closed-form solutions are usually not available. In such a situation, one may appeal to a perturbed approximation of option prices. In this direction, Saunders [17] studied the pricing of the timer call option in a class of stochastic volatility models, where the volatility of the underlying asset follows an ergodic diffusion process fluctuating on a fast time scale. Fouque et al. [10] introduced a class of multiscale stochastic volatility models, motivated by several empirical studies [3, 11], in which there are two factors, a fast mean-reverting diffusion and a slowly varying diffusion, affecting the volatility of the underlying asset price. The pricing problems of various options have been studied under this class of multiscale stochastic volatility models (details can be found in [2, 6–8, 12] and the references therein).

In the present paper, we study the pricing of perpetual timer options, extending the model of Saunders [17] to a multiscale stochastic volatility model for the underlying asset and establishing full second-order asymptotics. The extension from the first-order asymptotics under the single fast-varying factor model to the second-order multiscale asymptotics is nontrivial owing to the increasing complexity resulting from the far greater numbers of parameters and equations involved in the derivation. We note that the asymptotic analysis employed in this paper is readily applicable to the timer put option. In addition, it will be interesting and may turn out to be useful to explore "implied volatility" for the timer options in some sense, though there is no fixed time for a maturity. Therefore, we consider implied volatility with expected expiration inferred from the pricing model, called effective implied volatility in the sequel. The pricing expansion is then translated to a second-order effective implied volatility approximation. Our numerical experiments demonstrate that the pricing formula is accurate, and the effective implied volatility is discussed.

The remainder of the paper is organized as follows. In Section 2, we first describe the class of multiscale stochastic volatility models employed in this work and then present our main result. Section 3 is devoted to establishing this formula in detail, and then a second-order expansion of the effective implied volatility is derived. A numerical example showing the use of the approximate formulas and assessing the accuracy is given in Section 4. Section 5 concludes our discussion.

2. Pricing model and main result

2.1. Pricing model In this paper, we price the timer option under a multiscale stochastic volatility model for the stock price. This class of models has been studied previously [12], and it has been developed as an effective framework in which the principal components of derivative prices can be efficiently captured. In our pricing model, most of the notation adopted in this paper conventionally follows that defined in [8, 12] and extensively used in the literature whenever this class of models is under consideration. Choosing a risk-neutral measure $\mathbb{P}^\star$ and a standard three-dimensional $\mathbb{P}^\star$-Brownian motion $W_t^\star$ [5], the evolution of the underlying asset price S satisfies the following stochastic differential equations (SDEs):

$$\begin{cases} dS_t = rS_t\,dt + f(Y_t, Z_t)S_t\,dW_t^{(0)\star} \\ dY_t = \Big(\dfrac{1}{\epsilon}\alpha(Y_t) - \dfrac{1}{\sqrt{\epsilon}}\Lambda(Y_t, Z_t)\beta(Y_t)\Big)\,dt + \dfrac{1}{\sqrt{\epsilon}}\beta(Y_t)\,dW_t^{(1)\star} \\ dZ_t = (\delta c(Z_t) - \sqrt{\delta}\Gamma(Y_t, Z_t)g(Z_t))\,dt + \sqrt{\delta}g(Z_t)\,dW_t^{(2)\star}, \end{cases} \tag{2.1}$$

where $W_t^\star = (W_t^{(0)\star}, W_t^{(1)\star}, W_t^{(2)\star})$ is a standard three-dimensional Brownian motion with correlation structure

$$\begin{pmatrix} W_t^{(0)\star} \\ W_t^{(1)\star} \\ W_t^{(2)\star} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \tilde{\rho}_{12} & \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2} \end{pmatrix} W_t.$$

Here W_t is a standard three-dimensional Brownian motion, and the coefficients ρ_1, ρ_2 and $\tilde{\rho}_{12}$ satisfy

$$|\rho_1| < 1, \quad \rho_2^2 + \tilde{\rho}_{12}^2 < 1 \quad \text{and} \quad \rho_{12} = \rho_1\rho_2 + \tilde{\rho}_{12}\sqrt{1-\rho_1^2}.$$

The risk-free interest rate r is a positive constant. The combined market prices of volatility risk are given by

$$\Lambda(y, z) = \sqrt{1-\rho_1^2}\,\frac{(\mu - r)}{f(y, z)} + \gamma(y, z)\rho_1,$$
$$\Gamma(y, z) = \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2}\,\frac{(\mu - r)}{f(y, z)} + \gamma(y, z)\tilde{\rho}_{12} + \xi(y, z)\rho_2,$$

where $\gamma(y, z)$ and $\xi(y, z)$ are bounded smooth functions of y and z. The volatility $f(y, z)$ is a positive function; ϵ and δ represent the time scales and allow us to study the problem in the fast and slow regimes at the same time. The fast factor Y is an ergodic process on J with a unique invariant distribution Π. We take $J = \mathbb{R}$, although in extensions to other cases, $\alpha(y)$, $\beta(y)$ and $c(z)$, $g(z)$ describe the dynamics of the process Y and Z, respectively, under the real-world measure $\mathbb{P}$.

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We employ this for the pricing of perpetual timer options. In the subsequent context, a timer option is understood as the perpetual timer option when there is no confusion. The timer call option has payoff $\max(S_\tau - K, 0)$ at random maturity τ, where K is the strike price. Given the predetermined variance budget B, $\tau = \inf\{t > 0, I_t = B\}$. Define

$$I_t = \int_0^t f^2(Y_s, Z_s)\, ds$$

as the cumulative realized variance. Thus, the dynamics are now described by

$$\begin{cases} dS_t = rS_t\, dt + f(Y_t, Z_t)S_t\, dW_t^{(0)\star} \\ dY_t = \Big(\dfrac{1}{\epsilon}\alpha(Y_t) - \dfrac{1}{\sqrt{\epsilon}}\Lambda(Y_t, Z_t)\beta(Y_t)\Big)\, dt + \dfrac{1}{\sqrt{\epsilon}}\beta(Y_t)\, dW_t^{(1)\star} \\ dZ_t = (\delta c(Z_t) - \sqrt{\delta}\Gamma(Y_t, Z_t)g(Z_t))\, dt + \sqrt{\delta}g(Z_t)\, dW_t^{(2)\star} \\ dI_t = f^2(Y_t, Z_t)\, dt. \end{cases}$$

It is clear that (S, Y, Z, I) is a four-dimensional Markov process. Denoting by $\mathbb{E}^\star$ the expectation with respect to $\mathbb{P}^\star$, the price of a timer call option with payoff function $\max(S_\tau - K, 0)$ is

$$P^{\epsilon,\delta}(t, S_t, Y_t, Z_t, I_t) = \mathbb{E}^\star[e^{-r(\tau-t)} \max(S_\tau - K, 0) \mid \mathcal{F}_t]. \tag{2.2}$$

Since the stochastic volatility setting under consideration is very general and complex, an analytic solution of (2.2) or (2.4) is difficult, if not impossible, to obtain. Thus, our goal is to derive an approximate price of the timer option. This approximation is of the form

$$P^{\epsilon,\delta} \approx \tilde{P}^{\epsilon,\delta} = P_{0,0} + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1} + \epsilon P_{2,0} + \delta P_{0,2} + \sqrt{\epsilon\delta}P_{1,1}, \tag{2.3}$$

where $P_{0,0}$ is the leading term of the price; $P_{1,0}$ is the first-order fast scale correction; $P_{0,1}$ is the first-order slow scale correction; and $P_{2,0}, P_{0,2}, P_{1,1}$ are the second-order fast scale term, second-order slow scale term and first fast–slow term, respectively. In the notation $P_{i,j}$, the subindex i corresponds to the power of $\sqrt{\epsilon}$, and the subindex j corresponds to the power of $\sqrt{\delta}$. The derivation of the expressions for these price approximation terms will be presented in the next section and summarized in Theorem 2.6.

2.2. Main result In this section, we briefly introduce the asymptotic analysis of the timer option pricing and give the main result of this paper. First, we make the following assumptions, which are necessary throughout the paper.

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ASSUMPTION 2.1. For all values of parameters $0 < \epsilon, \delta < 1$, the system of SDEs (2.1) has unique strong solutions (S_t, Y_t, Z_t) for all initial values (s, y, z).

ASSUMPTION 2.2. The market prices of volatility risk Λ and Γ appearing subsequently are bounded.

ASSUMPTION 2.3. The diffusion process $Y^{(1)}$ is defined by its infinitesimal generator $\mathcal{L}_0 = \beta^2(y)\partial^2_{yy}/2 + \alpha(y)\partial_y$ (so that, in distribution, $Y_t = Y^{(1)}_{t/\epsilon}$ under $\mathbb{P}$). Moreover, assume that $Y^{(1)}$ is an ergodic process with a unique invariant distribution Π, which has probability density π and does not explode in finite time. In particular, these conditions are satisfied by the Ornstein–Uhlenbeck (OU) process, which will be used in our numerical example in Section 4. Let $Z^{(1)}$ be a diffusion process with infinitesimal generator $\mathcal{M}_2 = g(z)\partial^2_{zz}/2 + c(z)\partial_z$ (so that, in distribution, $Z_t = Z^{(1)}_{\delta t}$).

ASSUMPTION 2.4. The volatility function f is measurable, bounded, and bounded away from zero. Furthermore, for all $y \in \mathbb{R}$, it is assumed that $f(y, z)$ is smooth with bounded derivatives, so that the *averaged effective variance* with respect to invariant density π defined by $\bar{\sigma}^2(z) = \int f^2(y, z)\pi(y)\, dy$ is finite and twice differentiable.

ASSUMPTION 2.5. Consider Poisson equations of the form $\mathcal{L}_0\phi(y, z) + \mathcal{X}(y, z) = 0$, where the solvability condition $\langle \mathcal{X}(y, z)\rangle = \int \mathcal{X}(y, z)\pi(y)\, dy = 0$ is satisfied, and $\mathcal{X}(y, z)$ is at most polynomially growing in y and z. We assume that the same polynomial growth condition holds for ϕ. In particular, this assumption is imposed on the solutions $\phi_i, i = 1, 2, 3, 4$ of the Poisson equations (3.17), (3.18), (3.25), (3.26) appearing in the subsequent analysis.

The analysis is based on two small parameters ϵ and δ, which govern two time scales for the two volatility driving processes. The driver Y runs on a fast time scale and rapidly converges to stationarity, which leads to a singular perturbation analysis. On the contrary, the factor Z fluctuates on a slow time scale, and hence the corresponding expansion is a regular perturbation problem. Applying a combination of singular and regular perturbations, we derive a second-order approximate formula for the timer option.

By application of the Feynman–Kac formula [12, Ch. 1.9.3], the pricing function $P^{\epsilon,\delta}$ in (2.2) is the solution of the following partial differential equation (PDE) with final condition

$$\begin{cases} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} = 0 \\ P^{\epsilon,\delta}(B, s, y, z) = \max(s - K, 0), \end{cases}$$

where the differential operator $\mathcal{L}^{\epsilon,\delta}$ is the sum of components

$$\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3,$$

and the operators $\mathcal{L}_i$ $(i = 0, 1, 2)$ and $\mathcal{M}_i$ $(i = 1, 2, 3)$ are defined by

$$\begin{cases} \mathcal{L}_0 = \alpha(y)\dfrac{\partial}{\partial y} + \dfrac{1}{2}\beta^2(y)\dfrac{\partial^2}{\partial y^2} \\ \mathcal{L}_1 = \beta(y)\Big(\rho_1 f(y,z)s\dfrac{\partial^2}{\partial s\partial y} - \Lambda(y,z)\dfrac{\partial}{\partial y}\Big) \\ \mathcal{L}_2 = \dfrac{\partial}{\partial t} + rs\dfrac{\partial}{\partial s} + f^2(y,z)\dfrac{\partial}{\partial x} + \dfrac{1}{2}f^2(y,z)s^2\dfrac{\partial^2}{\partial s^2} - r \\ \mathcal{M}_1 = -g(z)\Gamma(y,z)\dfrac{\partial}{\partial z} + \rho_2 f(y,z)g(z)s\dfrac{\partial^2}{\partial s\partial z} \\ \mathcal{M}_2 = c(z)\dfrac{\partial}{\partial z} + \dfrac{1}{2}g^2(z)\dfrac{\partial^2}{\partial z^2} \\ \mathcal{M}_3 = \rho_{12}\beta(y)g(z)\dfrac{\partial^2}{\partial y\partial z}. \end{cases} \tag{2.4}$$

Observe that $\mathcal{L}_2$ contains the derivative with t. In fact, given the value of the realized variance, the pricing problem for the perpetual timer option is independent of t [15, 17]. Therefore, $\mathcal{L}_2$ can be rewritten as

$$\mathcal{L}_2 = f^2\frac{\partial}{\partial x} + rs\frac{\partial}{\partial s} + \frac{s^2 f^2}{2}\frac{\partial^2}{\partial s^2} - r,$$

which can be regarded as the Black–Scholes operator with x instead of the time variable t [17]. Note that $\langle\mathcal{L}_2\rangle$ plays an important part in the derivation of the approximate price, where the bracket notation means the integration with respect to the invariant distribution Π of the process Y.

We expand $P^{\epsilon,\delta}$ as

$$P^{\epsilon,\delta} = \sum_{j\geq 0}\sqrt{\delta}^j P_j^\epsilon \quad \text{where } P_j^\epsilon = \sum_{i\geq 0}\sqrt{\epsilon}^i P_{i,j}.$$

Inserting this expression into (2.4), and equating both sides of the above equation with respect to the corresponding powers of $\sqrt{\delta}$, we have a system of pricing equations. Then, carrying out a singular perturbation analysis with respect to $\sqrt{\epsilon}$ for these equations, we will obtain the terms $P_{0,0}$, $P_{1,0}$, $P_{0,1}$, $P_{1,1}$, $P_{0,2}$ and $P_{2,0}$ explicitly. Note that there are many Poisson equations of asymptotic expansion terms for the timer option price. By Assumption 2.5, the source term $\mathcal{X}(y,z)$ must satisfy the solvability condition $\langle\mathcal{X}(y,z)\rangle = \int \mathcal{X}(y,z)\pi(y)\,dy = 0$, where the bracket notation denotes integration with respect to the invariant probability density π of the process Y. Thus, we derive equations about the operator $\langle\mathcal{L}_2\rangle$ for the terms. Matching boundary conditions, we have the following PDEs for the terms $P_{0,0}$, $P_{1,0}$, $P_{0,1}$, $P_{1,1}$, $P_{2,0}$ and $P_{0,2}$:

$$
\begin{aligned}
O(1) &: \langle \mathcal{L}_2 \rangle P_{0,0} = 0, \quad P_{0,0}(B,s,z) = \max(s-K,0), \\
O(\sqrt{\epsilon}) &: \langle \mathcal{L}_2 \rangle P_{1,0} = -\langle \mathcal{L}_1 P_{2,0} \rangle, \quad P_{1,0}(B,s,z) = 0, \\
O(\sqrt{\delta}) &: \langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 \rangle P_{0,0}, \quad P_{0,1}(B,s,z) = 0, \\
O(\sqrt{\epsilon\delta}) &: \langle \mathcal{L}_2 \rangle P_{1,1} = -\langle \mathcal{L}_1 P_{2,1} \rangle - \langle \mathcal{M}_3 P_{2,0} \rangle - \langle \mathcal{M}_1 \rangle P_{1,0}, \quad P_{1,1}(B,s,z) = 0, \\
O(\epsilon) &: P_{2,0}(x,s,y,z) = -\phi(y,z)(\partial_x + \tfrac{1}{2}\mathcal{D}_2)P_{0,0} + F_{2,0}(x,y,z), \\
\langle \mathcal{L}_2 \rangle F_{2,0} &= -\langle \mathcal{L}_1 P_{3,0} \rangle + \langle \phi f^2 \rangle (\partial_{xx} + \mathcal{D}_2 \partial_x + \tfrac{1}{4}\mathcal{D}_2^2) P_{0,0}, \quad F_{2,0}(B,s,z) = 0, \\
O(\delta) &: \langle \mathcal{L}_2 \rangle P_{0,2} = -\langle \mathcal{M}_1 \rangle P_{0,1} - \mathcal{M}_2 P_{0,0}, \quad P_{0,2}(B,s,z) = 0.
\end{aligned} \tag{2.5}
$$

Here, $\mathcal{D}_k = s^k \partial^k / \partial s^k, k = 1, 2, \ldots$, and $\phi(y,z)$ is the solution of $\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle$, with $\langle \phi \rangle = \int \phi(y,z)\pi(y)\,dy = 0$. The equations are from (3.10), (3.11), (3.22), (3.23), (3.16) and (3.29), respectively. The formal derivation of the price approximation is described in Section 3 in detail.

As mentioned in Section 2.1, $\langle \mathcal{L}_2 \rangle$ is regarded as the Black–Scholes operator. Obviously, for the leading term, $P_{0,0}(x,s,z) = P_{BS}(s, K, (B-x)/\bar{\sigma}^2(z), r, \bar{\sigma}(z))$, that is, the Black–Scholes formula with volatility σ set to $\bar{\sigma}(z)$ and the time for expiration set to $(B-x)/\bar{\sigma}^2(z)$, is given by

$$
P_{BS}\Big(s, K, \frac{B-x}{\bar{\sigma}^2(z)}, r, \bar{\sigma}(z)\Big) = sN(d_1) - Ke^{-r(B-x)/\bar{\sigma}^2(z)} N(d_2), \tag{2.6}
$$

where

$$
d_1 = \frac{\log(s/k) + \{r + \bar{\sigma}^2(z)/2\}(B-x)/\bar{\sigma}^2(z)}{\sqrt{B-x}}, \quad d_2 = d_1 - \sqrt{B-x},
$$

$$
N(z) = \int_{-\infty}^{z} e^{-y^2/2} \frac{1}{\sqrt{2\pi}}\, dy.
$$

We denote $\bar{\sigma}^2(z)$ as $\bar{\sigma}^2$, for short, in the following context. The other terms $P_{i,j} (0 < i + j \le 2)$ turn out to be related to the leading term $P_{0,0}$ in the sense that they are combinations of various Greeks of P_{BS} and some basis functions. The following theorem is the main result of this paper.

THEOREM 2.6. *Let $P_{i,j}$, $0 \le i + j \le 2$, in (2.3) be the unique classical solutions of the linear PDEs with terminal conditions given in (2.5). Then we have the following*

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expressions for $P_{i,j}$ in terms of $P_{BS}(s, K, (B-x)/\bar{\sigma}^2, r, \bar{\sigma})$ in (2.6):

$$
\begin{cases}
P_{0,0}(x,s,z) = P_{BS}(s, K, (B-x)/\bar{\sigma}^2, r, \bar{\sigma}) \\
P_{1,0}(x,s,z) = -\dfrac{B-x}{\bar{\sigma}^2}\Big[C_1(z)\Big(s\dfrac{\partial^2 P_{0,0}}{\partial s\partial x} + s^2\dfrac{\partial^2 P_{0,0}}{\partial s^2} + \dfrac{s^3}{2}\dfrac{\partial^3 P_{0,0}}{\partial s^3}\Big) \\
\qquad\qquad - C_2(z)\Big(\dfrac{\partial P_{0,0}}{\partial x} + \dfrac{s^2}{2}\dfrac{\partial^2 P_{0,0}}{\partial s^2}\Big)\Big] \\
P_{0,1}(x,s,z) = \dfrac{B-x}{\bar{\sigma}^2}(C_3(z)s\partial^2_{s\bar{\sigma}} + C_4(z)\partial_{\bar{\sigma}})P_{0,0} \\
P_{2,0}(x,s,y,z) = -\phi(y,z)\Big(\dfrac{\partial P_{0,0}}{\partial x} + \dfrac{s^2}{2}\dfrac{\partial^2 P_{0,0}}{\partial s^2}\Big) + F_{2,0}(x,s,z),
\end{cases}
$$

where $F_{2,0}(x,s,z)$ is defined in (3.19), $P_{0,2}$ in (3.30), $P_{1,1}$ in (3.27), $C_1(z)$ and $C_2(z)$ in (3.15), and $C_3(z)$ and $C_4(z)$ in (3.24).

Buyers of vanilla option often overpay for their options, because the implied volatility in the market is usually higher than the realized volatility. This was shown by an empirical analysis by SG CIB, which found that 80% of 3-month calls that had matured in the money were overpriced [18]. The principal pricing term $P_{0,0}$ provides a theoretical justification of the argument that the investor of the timer option only pays the real cost and does not suffer from high implied volatility. As ϵ and δ are small enough, $\bar{\sigma}$ is a natural estimation of the realized volatility I_t/t. In fact, we have

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \frac{1}{t}\int_0^t f^2(Y_s, Z_s)\,ds = \bar{\sigma}^2(Z_0),$$

where $\bar{\sigma}^2(z) = \int f^2(y,z)\pi(y)\,dy$. It is not difficult to obtain this result by the time scales of fluctuation in the volatility process [12, Ch. 3]. The volatility function $f(Y_t, Z_t)$ in the model is driven by Y_t and Z_t, which are both ergodic processes, running on a fast and a slow time scale, respectively. Thus, the principal pricing term $P_{0,0}$ is dependent on the realized volatility $\bar{\sigma}$. It shows that when the implied volatility in the market is higher than the realized volatility, the leading term $P_{0,0}$ in the timer option price is lower than the price of the corresponding vanilla option whose strike price is K and whose maturity is $(B-x)/\bar{\sigma}^2$. This theoretical justification was also provided with Heston's model (a review of Heston's model can be found in [13]) under the assumption that the risk-free rate r was 0%; for $r > 0\%$, a numerical example showed that a timer call option with variance budget $B = \sigma_0^2 T_0$ was less expensive than the European call option with maturity T_0 (using the notation of Li [15]; see further details therein).

Furthermore, by inverting the timer option approximation formula, we arrive at a second-order effective implied volatility expansion.

THEOREM 2.7. *With the terms $P_{i,j}, 0 \le i+j \le 2$, in Theorem 2.6, the approximation effective volatility formula can be obtained as*

$$I^{\epsilon,\delta} \approx \tilde{I}^{\epsilon,\delta} = I_{0,0} + \sqrt{\epsilon}I_{1,0} + \sqrt{\delta}I_{0,1} + \epsilon I_{2,0} + \delta I_{0,2} + \sqrt{\epsilon\delta}I_{1,1},$$

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where $I_{0,0}$ is the leading term of the effective volatility; $I_{1,0}, I_{0,1}$ and $I_{1,1}$ are the first-order fast term, first-order slow term and first-order fast–slow term, respectively; $I_{2,0}$ and $I_{0,2}$ the are second-order fast term and second-order slow term, respectively; and $I_{i,j}, 0 \le i+j \le 2$ satisfy (3.31).

A proof of this theorem via a detailed asymptotic analysis will be given in Section 3.

3. Second-order asymptotics for pricing and effective implied volatility

In this section, we give the derivation of the second-order two-factor timer option pricing asymptotics and effective implied volatility asymptotic expansion. This is similar to that carried out for European options [8].

3.1. Related lemmas To obtain expressions for the higher-order terms $P_{i,j}, 1 \le i + j \le 2$, it is desirable to use the following two lemmas.

LEMMA 3.1. *The Black–Scholes pricing function $P_{BS}(s, K, (B-x)/\bar{\sigma}^2(z), r, \bar{\sigma}(z))$ of the timer call option, given by (2.6), satisfies the following relationship between its vega (that is, $\partial P_{BS}/\partial_{\bar{\sigma}}$) and $\mathcal{D}_1 P_{BS}$:*

$$\frac{\partial P_{BS}}{\partial_{\bar{\sigma}}} = \frac{2r}{\bar{\sigma}} \frac{B-x}{\bar{\sigma}^2}(P_{BS} - \mathcal{D}_1 P_{BS}), Z,$$

where $P_{BS} = sN(d_1) - Ke^{-r(B-x)/\bar{\sigma}^2} N(d_2)$.

PROOF. A routine computation gives rise to the formula. □

LEMMA 3.2. *The Black–Scholes pricing function $P_{BS}(s, K, (B-x)/\bar{\sigma}^2, r, \bar{\sigma})$ of the timer call option, given by (2.6), satisfies, for nonnegative integers k and n,*

$$\langle \mathcal{L}_2 \rangle \frac{((B-x)/\bar{\sigma}^2)^{n+1}}{n+1} P(\mathcal{D}_k) P_{BS} = -\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^n P(\mathcal{D}_k) P_{BS},$$
$$\langle \mathcal{L}_2 \rangle \frac{((B-x)/\bar{\sigma}^2)^{n+1}}{n+2} P(\mathcal{D}_k) \partial_{\bar{\sigma}} P_{BS} = -\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^n P(\mathcal{D}_k) \partial_{\bar{\sigma}} P_{BS},$$
$$\langle \mathcal{L}_2 \rangle \frac{((B-x)/\bar{\sigma}^2)^{n+1}}{n+3} P(\mathcal{D}_k) \Big(\partial^2_{\bar{\sigma}\bar{\sigma}} - \frac{3}{\bar{\sigma}(n+2)} \partial_{\bar{\sigma}}\Big) P_{BS} = -\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^n P(\mathcal{D}_k) \partial^2_{\bar{\sigma}\bar{\sigma}} P_{BS},$$

where $P(\mathcal{D}_k)$ is some polynomial of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_k$.

PROOF. Using the fact $\mathcal{D}_k \mathcal{D}_m = \mathcal{D}_m \mathcal{D}_k$, for positive integers k and m, and $\langle \mathcal{L}_2 \rangle \mathcal{D}_k = \mathcal{D}_k \langle \mathcal{L}_2 \rangle$, we can derive the results via simple computation. □

By virtue of the two lemmas above, we are able to derive explicit expressions for $P_{i,j}, 1 \le i + j \le 2$. We carry out singular and regular perturbation asymptotics and establish the second-order approximation formula for the timer option price in the subsequent derivation.

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3.2. Second-order asymptotics for pricing

3.2.1. Zero-order term $P_{0,0}$ and first-order fast term $P_{1,0}$. In this section, we deal with $P_{0,0}$ and $P_{1,0}$, which are defined in (2.3). We first expand $P^{\epsilon,\delta}$ in the powers of $\sqrt{\delta}$ as

$$P^{\epsilon,\delta} = \sum_{j\geq 0} \sqrt{\delta}^j P_j^\epsilon, \tag{3.1}$$

where

$$P_j^\epsilon = \sum_{i\geq 0} \sqrt{\epsilon}^i P_{i,j}. \tag{3.2}$$

We substitute this expansion (3.1) into the PDE (2.4). With the decomposition (2.4) and collecting the terms based on increasing powers of $\sqrt{\delta}$, we have

$$\begin{aligned}&\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_0^\epsilon + \sqrt{\delta}\Big\{\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_1^\epsilon + \Big(\frac{\mathcal{M}_3}{\sqrt{\epsilon}} + \mathcal{M}_1\Big)P_0^\epsilon\Big\}\\ &\quad + \delta\Big\{\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_2^\epsilon + \Big(\frac{\mathcal{M}_3}{\sqrt{\epsilon}} + \mathcal{M}_1\Big)P_1^\epsilon + \mathcal{M}_2 P_0^\epsilon\Big\} + \cdots = 0. \end{aligned}\tag{3.3}$$

Equating the corresponding powers of $\sqrt{\delta}$ on both sides of equation (3.3), we have the following expressions:

$$\mathcal{O}(1) : 0 = \Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_0^\epsilon, \tag{3.4}$$

$$\mathcal{O}(\sqrt{\delta}) : 0 = \Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_1^\epsilon + \Big(\frac{\mathcal{M}_3}{\sqrt{\epsilon}} + \mathcal{M}_1\Big)P_0^\epsilon, \tag{3.5}$$

$$\mathcal{O}(\delta) : 0 = \Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\Big)P_2^\epsilon + \Big(\frac{\mathcal{M}_3}{\sqrt{\epsilon}} + \mathcal{M}_1\Big)P_1^\epsilon + \mathcal{M}_2 P_0^\epsilon. \tag{3.6}$$

Substituting expansions (3.2) into (3.4), and matching terms in powers of $\sqrt{\epsilon}$, we find that the terms $P_{0,0}$ and $P_{1,0}$ satisfy the equations

$$\mathcal{O}(1/\epsilon) : 0 = \mathcal{L}_0 P_{0,0},$$
$$\mathcal{O}(1/\sqrt{\epsilon}) : 0 = \mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_{0,0}.$$

Note that the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ defined in (2.4) take derivatives with respect to y, and we see that they are ordinary equations in y. We choose $P_{0,0}$ and $P_{1,0}$ to be independent of y so that $P_{0,0} = P_{0,0}(x, s, z)$ and $P_{1,0} = P_{1,0}(x, s, z)$ satisfy the above equations; thus, we seek $P_{0,0}$ and $P_{1,0}$ which are independent of y. Continuing the asymptotic analysis, we have:

$$\mathcal{O}(1) : 0 = \mathcal{L}_0 P_{2,0} + \cancel{\mathcal{L}_1 P_{1,0}} + \mathcal{L}_2 P_{0,0}, \tag{3.7}$$

$$\mathcal{O}(\sqrt{\epsilon}) : 0 = \mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}, \tag{3.8}$$

$$\mathcal{O}(\epsilon) : 0 = \mathcal{L}_0 P_{4,0} + \mathcal{L}_1 P_{3,0} + \mathcal{L}_2 P_{2,0}. \tag{3.9}$$

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Here, equations 3.7–3.9 are Poisson equations of the form $\mathcal{L}_0 P + \chi = 0$. In virtue of Assumption 2.5, the Poisson equation has a solution P if the following solvability condition holds [12, Ch. 3]: $\langle \chi \rangle = \int \chi(y)\pi(y)\,dy = 0$, where $\pi(y)$ is the invariant probability density of Y. Imposing the solvable condition on equations (3.7)–(3.9), we have:

$$\mathcal{O}(1) : 0 = \langle \mathcal{L}_2 \rangle P_{0,0}, \tag{3.10}$$

$$\mathcal{O}(\sqrt{\epsilon}) : 0 = \langle \mathcal{L}_1 P_{2,0} \rangle + \langle \mathcal{L}_2 \rangle P_{1,0}, \tag{3.11}$$

$$\mathcal{O}(\epsilon) : 0 = \langle \mathcal{L}_1 P_{3,0} \rangle + \langle \mathcal{L}_2 P_{2,0} \rangle. \tag{3.12}$$

We expand the terminal condition in (2.4) and have $P_{0,0}(B, s, z) = \max(s - K, 0)$, and $P_{1,0}(B, s, z) = 0$. We note that $P_{0,0}(x, s, z) = P_{BS}(s, K, (B - x)/\bar{\sigma}^2, r, \bar{\sigma})$, given by (2.6).

Before getting $P_{1,0}$ from equation (3.11), we first compute $\langle \mathcal{L}_1 P_{2,0} \rangle$. Using equation (3.10), equation (3.7) can be rewritten as

$$\mathcal{L}_0 P_{2,0} = -\mathcal{L}_2 P_{0,0} = -(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{0,0} = (\bar{\sigma}^2(z) - f^2(y, z))(\partial_x + \mathcal{D}_2/2) P_{0,0}.$$

Let $\phi(y, z)$ be a solution to the Poisson equation

$$\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle, \tag{3.13}$$

and we derive the expression for $P_{2,0}$ as

$$P_{2,0}(x, s, y, z) = -\phi(y, z)(\partial_x + \mathcal{D}_2/2) P_{0,0} + F_{2,0}(x, s, z). \tag{3.14}$$

Thus $P_{1,0}$ satisfies the PDE

$$\langle \mathcal{L}_2 \rangle P_{1,0} = C_1(z)\Big(s \frac{\partial^2 P_{0,0}}{\partial s \partial x} + s^2 \frac{\partial^2 P_{0,0}}{\partial s^2} + \frac{s^3}{2} \frac{\partial^3 P_{0,0}}{\partial s^3}\Big) - C_2(z)\Big(\frac{\partial P_{0,0}}{\partial x} + \frac{s^2}{2} \frac{\partial^2 P_{0,0}}{\partial s^2}\Big)$$

with terminal condition $P_{1,0}(B, s, z) = 0$, where

$$C_1(z) = \langle \rho_1 \beta(\cdot) f(\cdot, z) \partial_y \phi(\cdot, z) \rangle, \quad C_2(z) = \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \phi(\cdot, z) \rangle. \tag{3.15}$$

Using Lemma 3.2, we obtain the following solution $P_{1,0}$:

$$P_{1,0} = -\frac{B - x}{\bar{\sigma}^2}\Big[C_1(z)\Big(s \frac{\partial^2 P_{0,0}}{\partial s \partial x} + s^2 \frac{\partial^2 P_{0,0}}{\partial s^2} + \frac{s^3}{2} \frac{\partial^3 P_{0,0}}{\partial s^3}\Big) - C_2(z)\Big(\frac{\partial P_{0,0}}{\partial x} + \frac{s^2}{2} \frac{\partial^2 P_{0,0}}{\partial s^2}\Big)\Big].$$

3.2.2. Second-order fast term $P_{2,0}$. In this section, we deal with the term $P_{2,0}$, which is defined in (2.3). From equation (3.14), it turns out that the natural terminal condition $P_{2,0}(B, s, y, z) = 0$ is not possible. However, we can obtain the averaged terminal condition via the ergodicity, $\langle P_{2,0}(B, s, y, z) \rangle = 0$. In addition, the solution to the Poisson equation (3.13) is chosen here by imposing the condition $\langle \phi(\cdot, z) \rangle = 0$. From (3.14), using the fact that $\mathcal{D}_2$ and $\mathcal{L}_2$ can commute and $\langle \phi(\cdot, z) \rangle = 0$, we have:

$$\langle \mathcal{L}_2 P_{2,0} \rangle = -\langle \phi f^2 \rangle (\partial_{xx} + \mathcal{D}_2 \partial_x + \mathcal{D}_2^2/4) P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0}. \tag{3.16}$$

Let $\phi_1(y,z)$ and $\phi_2(y,z)$ be the solutions of the Poisson equations

$$\mathcal{L}_0\phi_1 = \beta(y)f(y,z)\phi_y'(y,z) - \langle\beta(\cdot)f(\cdot,z)\phi_y'(\cdot,z)\rangle, \tag{3.17}$$

$$\mathcal{L}_0\phi_2 = \beta(y)\Lambda(y,z)\phi_y'(y,z) - \langle\beta(\cdot)\Lambda(\cdot,z)\phi_y'(\cdot,z)\rangle, \tag{3.18}$$

respectively. According to equations (3.8), (3.11), (3.13), and (3.14), we compute $\mathcal{L}_1 P_{2,0}$, $\mathcal{L}_2 P_{1,0}$, and have the solution

$$P_{3,0} = (\rho_1\phi_1\mathcal{D}_1 - \phi_2)(\partial_x + \mathcal{D}_2/2)P_{0,0} - \phi(\partial_x + \mathcal{D}_2/2)P_{1,0} + F_{3,0},$$

for some $F_{3,0}$ independent of y. With equations (3.16) and (3.12), we have the following PDE for $F_{2,0}(x,s,z)$ and the terminal condition $F_{2,0}(B,s,z) = 0$:

$$\langle\mathcal{L}_2\rangle F_{2,0} = -\langle\mathcal{L}_1 P_{3,0}\rangle + \langle\phi f^2\rangle(\partial_{xx} + \mathcal{D}_2\partial_x + \mathcal{D}_2^2/4)P_{0,0}.$$

Calculating the $\langle\mathcal{L}_1 P_{3,0}\rangle$ and using Lemma 3.2, we obtain:

$$\begin{aligned}
F_{2,0} = {} & \frac{B-x}{\bar\sigma^2}[(C_5(z)\mathcal{D}_1^2 - C_6(z)\mathcal{D}_1 - C_7(z)\mathcal{D}_1 + C_8(z))(\partial_x + \mathcal{D}_2/2)P_{0,0}] \\
& - \{\rho_1 C_1(z)\mathcal{D}_1 - C_2(z)\}\left[\frac{1}{\bar\sigma^2}\left(\frac{B-x}{\bar\sigma^2}\right)\mathcal{A}_{1,0}^{\epsilon}P_{0,0} - \frac{1}{2}\left(\frac{B-x}{\bar\sigma^2}\right)^2\mathcal{A}_{1,0}^{\epsilon}\frac{\partial P_{0,0}}{\partial x}\right] \\
& - \frac{1}{4}\left(\frac{B-x}{\bar\sigma^2}\right)^2\{\rho_1 C_1(z)\mathcal{D}_1\mathcal{D}_2\mathcal{A}_{1,0}^{\epsilon} - C_2(z)\mathcal{D}_2\mathcal{A}_{1,0}^{\epsilon}\}P_{0,0} \\
& - \frac{B-x}{\bar\sigma^2}\langle\phi f^2\rangle\{\partial_{xx} + \mathcal{D}_2\partial_x + \mathcal{D}_2^2/4\}P_{0,0},
\end{aligned} \tag{3.19}$$

where $C_1(z)$, $C_2(z)$ are defined in (3.15), and

$$\begin{aligned}
\mathcal{A}_{1,0}^{\epsilon} &= C_1(z)\left(s\frac{\partial^2}{\partial s\partial x} + s^2\frac{\partial^2}{\partial s^2} + \frac{s^3}{2}\frac{\partial^3}{\partial s^3}\right) - C_2(z)\left(\frac{\partial}{\partial x} + \frac{s^2}{2}\frac{\partial^2}{\partial s^2}\right), \\
C_5(z) &= \rho_1^2\langle\beta(\cdot)f(\cdot,z)\phi_1'\rangle,\ C_6(z) = \rho_1\langle\beta(\cdot)\Lambda(\cdot,z)\phi_1'\rangle, \\
C_7(z) &= \rho_1\langle\beta(\cdot)f(\cdot,z)\phi_2'\rangle,\ C_8(z) = \langle\beta(\cdot)\Lambda(\cdot,z)\phi_2'\rangle.
\end{aligned} \tag{3.20}$$

Thus, from (3.14), the second-order term $P_{2,0}$ has been obtained.

3.2.3. First-order slow term $P_{0,1}$ and fast–slow term $P_{1,1}$. The first-order slow term $P_{0,1}$ and fast–slow term $P_{1,1}$ are defined in (2.3). Substituting the expansions (3.2) into (3.5) and collecting terms in powers of $\sqrt{\epsilon}$, we have

$$\begin{aligned}
O(\sqrt{\delta}/\epsilon) &: 0 = \mathcal{L}_0 P_{0,1}, \\
O(\sqrt{\delta}/\sqrt{\epsilon}) &: 0 = \mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \cancel{\mathcal{M}_3 P_{0,0}}.
\end{aligned}$$

We see that $\mathcal{L}_0 P_{0,1} = 0$ is an ordinary differential equation in y, which has constants in solutions (there are also exponentially growing solutions). Just as in the previous sections, we still look for solutions $P_{0,1}$ and $P_{1,1}$, which are independent of y. We

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analyse and obtain Poisson equations of the form in Assumption 2.5:

$$\mathcal{O}(\sqrt{\delta}) : 0 = \mathcal{L}_0 P_{2,1} + \cancel{\mathcal{L}_1 P_{1,1}} + \mathcal{L}_2 P_{0,1} + \cancel{\mathcal{M}_3 P_{1,0}} + \mathcal{M}_1 P_{0,0},$$
$$\mathcal{O}(\sqrt{\delta}\sqrt{\epsilon}) : 0 = \mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} + \mathcal{M}_3 P_{2,0} + \mathcal{M}_1 P_{1,0}. \tag{3.21}$$

Using the solvable condition, we have equations about $\langle \mathcal{L}_2 \rangle$ for $P_{0,1}$ and $P_{1,1}$:

$$\mathcal{O}(\sqrt{\delta}) : 0 = \langle \mathcal{L}_2 \rangle P_{0,1} + \langle \mathcal{M}_1 \rangle P_{0,0}, \tag{3.22}$$

$$\mathcal{O}(\sqrt{\delta}\sqrt{\epsilon}) : 0 = \langle \mathcal{L}_1 P_{2,1} \rangle + \langle \mathcal{L}_2 \rangle P_{1,1} + \langle \mathcal{M}_3 P_{2,0} \rangle + \langle \mathcal{M}_1 \rangle P_{1,0}, \tag{3.23}$$

with terminal conditions $P_{0,1}(B, s, z) = 0$ and $P_{1,1}(B, s, z) = 0$. By Lemma 3.2, we can obtain an expression for $P_{0,1}$:

$$P_{0,1} = \frac{B - x}{\bar{\sigma}^2}(C_3(z)\mathcal{D}_1 \partial_{\bar{\sigma}} + C_4(z)\partial_{\bar{\sigma}})P_{0,0},$$

where

$$C_3(z) = \tfrac{1}{2}\rho_2 g(z)\langle f(\cdot, z)\rangle \bar{\sigma}'(z), \quad C_4(z) = -\tfrac{1}{2} g(z)\langle \Gamma(\cdot, z)\rangle \bar{\sigma}'(z). \tag{3.24}$$

Note that $P_{1,1}$ can be obtained using equation (3.23), if we compute $\langle \mathcal{L}_1 P_{2,1} \rangle$ and $\langle \mathcal{M}_3 P_{2,0} \rangle$. Using (3.21) and (3.22), $P_{2,1}$ is given by

$$P_{2,1} = -\phi(y, z)(\partial_x + \tfrac{1}{2}\mathcal{D}_2)P_{0,1} - \{\rho_2 g(z)\phi_3(y, z)\mathcal{D}_1 \partial_z - g(z)\phi_4(y, z)\partial_z\}P_{0,0} + F_{2,1}(x, s, z),$$

where $F_{2,1}(x, s, z)$ does not depend on y, and $\phi_3(y, z)$ and $\phi_4(y, z)$ satisfy Poisson equations

$$\mathcal{L}_0 \phi_3 = f - \langle f \rangle, \tag{3.25}$$

$$\mathcal{L}_0 \phi_4 = \Gamma - \langle \Gamma \rangle. \tag{3.26}$$

Using (3.14), the above term $P_{2,1}$, and operators $\mathcal{L}_1$ and $\mathcal{M}_3$ in (2.4), we obtain an equation about $\mathcal{L}_2$ for $P_{1,1}$. By Lemma 3.2,

$$\begin{aligned}
P_{1,1} = &-\frac{1}{2}\Big(\frac{B - x}{\bar{\sigma}^2}\Big)^2 \frac{4r}{\bar{\sigma}^3}\{C_1(z)\mathcal{D}_1 + C_2(z)\}\{C_3(z)\mathcal{D}_1 + C_4(z)\}(P_{0,0} - \mathcal{D}_1 P_{0,0}) \\
&+ \frac{1}{3}\Big(\frac{B - x}{\bar{\sigma}^2}\Big)^3 \frac{2r}{\bar{\sigma}}\{C_1(z)\mathcal{D}_1 + C_2(z)\}\{C_3(z)\mathcal{D}_1 + C_4(z)\}(\partial_x - \mathcal{D}_1 \partial_x)P_{0,0} \\
&- \frac{1}{6}\Big(\frac{B - x}{\bar{\sigma}^2}\Big)^3 \frac{2r}{\bar{\sigma}}\{C_1(z)\mathcal{D}_1\mathcal{D}_2 - C_2\mathcal{D}_2\}\{C_3(z)\mathcal{D}_1 + C_4(z)\}(P_{0,0} - \mathcal{D}_1 P_{0,0}) \\
&+ \frac{1}{2}\Big(\frac{B - x}{\bar{\sigma}^2}\Big)^2 \frac{2r}{\bar{\sigma}}\{\tilde{C}_2(z)\mathcal{D}_1^2 + \tilde{C}_1(z)\mathcal{D}_1 + \tilde{C}_0(z)\}(P_{0,0} - \mathcal{D}_1 P_{0,0}) \\
&+ 2\Big(\frac{B - x}{\bar{\sigma}^2}\Big)^2 \{C_3(z)\mathcal{D}_1 + C_4(z)\}\Big[\frac{1}{\bar{\sigma}}\mathcal{A}^{\epsilon}_{1,0} - \frac{1}{3}\mathcal{A}^{\epsilon}_{1,0}\partial_{\bar{\sigma}} - \frac{1}{3}\frac{1}{\bar{\sigma}'(z)}\partial_z \mathcal{A}^{\epsilon}_{1,0}\Big]P_{0,0} \\
&- \frac{B - x}{\bar{\sigma}^2}\{\tilde{C}_3(z) + \tilde{C}_4(z)\partial_{\bar{\sigma}}\}(\partial_x + \mathcal{D}_2/2)P_{0,0},
\end{aligned} \tag{3.27}$$

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where $C_1(z)$ and $C_2(z)$ are defined in (3.15), $C_3(z)$ and $C_4(z)$ are given in (3.24), $\mathcal{A}_{1,0}$ is given in (3.20), and

$$\begin{aligned}
\tilde{C}_0(z) &= -\bar{\sigma}'(z)g(z)\langle\beta(\cdot)\Lambda(\cdot,z)\partial_y\phi_4(\cdot,z)\rangle,\\
\tilde{C}_1(z) &= \rho_2\bar{\sigma}'(z)g(z)\langle\beta(\cdot)\Lambda(\cdot,z)\partial_y\phi_3(\cdot,z)\rangle+\rho_1 g(z)\langle\beta(\cdot)f(\cdot,z)\partial_y\phi_4(\cdot,z)\rangle,\\
\tilde{C}_2(z) &= -\rho_1\rho_2\bar{\sigma}'(z)g(z)\langle\beta(\cdot)f(\cdot,z)\partial_y\phi_3(\cdot,z)\rangle,\\
\tilde{C}_3(z) &= \rho_{12}g(z)\langle\beta(\cdot,z)\partial_{yz}\phi(\cdot,z)\rangle,\\
\tilde{C}_4(z) &= \frac{1}{2}\rho_{12}g(z)\langle\beta(\cdot,z)\partial_y\phi(\cdot,z)\rangle\bar{\sigma}'(z),\\
\frac{\partial\mathcal{A}^{\epsilon}_{1,0}}{\partial z} &= \frac{\partial C_1(z)}{\partial z}\Big(s\frac{\partial^2}{\partial s\partial x}+s^2\frac{\partial^2}{\partial s^2}+\frac{s^3}{2}\frac{\partial^3}{\partial s^3}\Big)-\frac{\partial C_2(z)}{\partial z}\Big(\frac{\partial}{\partial x}+\frac{s^2}{2}\frac{\partial^2}{\partial s^2}\Big).
\end{aligned}$$

3.2.4. Second-order slow term $P_{0,2}$. We insert the expansion (3.2) into (3.6) and obtain equations in powers of $\sqrt{\epsilon}$. Similarly, based on the equations, we seek the solution of $P_{0,2}(x,y,z)$. The $O(\delta)$ equation is

$$O(\delta): 0 = \mathcal{L}_0 P_{2,2} + \cancel{\mathcal{L}_1 P_{1,2}} + \mathcal{L}_2 P_{0,2} + \cancel{\mathcal{M}_3 P_{1,1}} + \mathcal{M}_1 P_{0,1} + \mathcal{M}_2 P_{0,0}. \tag{3.28}$$

Equation (3.28) is a Poisson equation with solvable condition

$$O(\delta): 0 = \langle\mathcal{L}_2\rangle P_{0,2} + \langle\mathcal{M}_1\rangle P_{0,1} + \mathcal{M}_2 P_{0,0}. \tag{3.29}$$

The associated terminal condition is $P_{0,2}(B,s,z)=0$. By computing $\langle\mathcal{M}_1\rangle P_{0,1}$ and $\mathcal{M}_2 P_{0,0}$, and by Lemma 3.2, the solution $P_{0,2}$ is given as

$$\begin{aligned}
P_{0,2} = &-\Big[\frac{4}{3\bar{\sigma}}\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^2 N_1^2 - \frac{2}{3\bar{\sigma}'}\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^2 N_1 N_1'\Big]\partial_{\bar{\sigma}}P_{00} + \frac{1}{2}\Big(\frac{B-x}{\bar{\sigma}^2}\Big)^2 N_1^2\Big(\partial^2_{\bar{\sigma}\bar{\sigma}} - \frac{3}{\bar{\sigma}}\partial_{\bar{\sigma}}\Big)P_{0,0}\\
&+\frac{1}{6}\frac{B-x}{\bar{\sigma}^2}g^2\bar{\sigma}'^2\Big(\partial^2_{\bar{\sigma}\bar{\sigma}} - \frac{3}{2\bar{\sigma}}\partial_{\bar{\sigma}}\Big)P_{0,0} - \Big(\frac{1}{4}\frac{B-x}{\bar{\sigma}^2}g^2\bar{\sigma}'' - \frac{1}{2}\frac{B-x}{\bar{\sigma}^2}c(z)\bar{\sigma}'\Big)\partial_{\bar{\sigma}}P_{0,0},
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
N_1 &= C_3(z)\mathcal{D}_1 + C_4(z), \quad N_1' = C_3'(z)\mathcal{D}_1 + C_4'(z),\\
C_3'(z) &= \partial_z C_3(z), \quad C_4'(z) = \partial_z C_4(z),\\
\bar{\sigma}'(z) &= \partial_z\bar{\sigma}(z), \quad \bar{\sigma}''(z) = \partial^2_{zz}\bar{\sigma}(z).
\end{aligned}$$

3.3. Second-order asymptotics for effective implied volatility The investigation of implied volatility is essential, as option prices rarely conform to the idealized assumption of constant volatility in the Black–Scholes pricing framework, and option prices are quoted in terms of it in practice. For timer options, there is no fixed maturity and hence the "implied volatility" does not exist in the usual sense. Recall that in our pricing formula $(B-x)/\bar{\sigma}^2$ plays the part of the corresponding fixed time to maturity. Therefore, this effective maturity is a reasonable candidate for exploring a similar

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concept of "implied volatility" for timer options, which we mentioned as "effective implied volatility" in the Introduction.

In this section, we convert the expansion of the time option price obtained in the previous section into an expansion of effective implied volatility of the form $I^{\epsilon,\delta} = \sum_{i\geq 0}\sum_{j\geq 0}\sqrt{\epsilon}^i\sqrt{\delta}^j I_{i,j}$ such that $P^{\epsilon,\delta} = P_{BS}(I^{\epsilon,\delta})$. Taking the Taylor expansion of $P_{BS}(I^{\epsilon,\delta})$ about $I_{0,0}$ and rearranging terms yields

$$\begin{aligned}
P_{0,0} &+ \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1} + \sqrt{\epsilon\delta}P_{1,1} + \epsilon P_{2,0} + \delta P_{0,2} + \cdots \\
&= P_{BS}(I_{0,0} + \sqrt{\epsilon}I_{1,0} + \sqrt{\delta}I_{0,1} + \sqrt{\epsilon\delta}I_{1,1} + \epsilon I_{2,0} + \delta I_{0,2} + \cdots) \\
&= P_{BS}(I_{0,0}) + \sqrt{\epsilon}I_{1,0}\partial_{\bar{\sigma}}P_{BS}(I_{0,0}) + \sqrt{\delta}I_{0,1}\partial_{\bar{\sigma}}P_{BS}(I_{0,0}) \\
&\quad + \sqrt{\epsilon\delta}\{I_{1,0}I_{0,1}\partial^2_{\bar{\sigma}\bar{\sigma}}P_{BS}(I_{0,0}) + I_{1,1}\partial_{\bar{\sigma}}P_{BS}(I_{0,0})\} \\
&\quad + \epsilon\{\tfrac{1}{2}I^2_{1,0}\partial^2_{\bar{\sigma}\bar{\sigma}}P_{BS}(I_{0,0}) + I_{2,0}\partial_{\bar{\sigma}}P_{BS}(I_{0,0})\} \\
&\quad + \delta\{\tfrac{1}{2}I^2_{0,1}\partial^2_{\bar{\sigma}\bar{\sigma}}P_{BS}(I_{0,0}) + I_{0,2}\partial_{\bar{\sigma}}P_{BS}(I_{0,0})\} + \cdots .
\end{aligned}$$

Equating the same powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$ and applying $P_{0,0} = P_{BS}(\bar{\sigma})$, we obtain

$$\begin{aligned}
O(1) &: I_{0,0} = \bar{\sigma}, \\
O(\sqrt{\epsilon}) &: I_{1,0} = \frac{P_{1,0}}{\partial_{\bar{\sigma}}P_{0,0}}, \\
O(\epsilon) &: I_{2,0} = \frac{P_{2,0}}{\partial_{\bar{\sigma}}P_{0,0}} - \frac{1}{2}I^2_{1,0}\frac{\partial^2_{\bar{\sigma}\bar{\sigma}}P_{0,0}}{\partial_{\bar{\sigma}}P_{0,0}}, \\
O(\sqrt{\delta}) &: I_{0,1} = \frac{P_{0,1}}{\partial_{\bar{\sigma}}P_{0,0}}, \\
O(\delta) &: I_{0,2} = \frac{P_{0,2}}{\partial_{\bar{\sigma}}P_{0,0}} - \frac{1}{2}I^2_{0,1}\frac{\partial^2_{\bar{\sigma}\bar{\sigma}}P_{0,0}}{\partial_{\bar{\sigma}}P_{0,0}}, \\
O(\sqrt{\epsilon\delta}) &: I_{1,1} = \frac{P_{1,1}}{\partial_{\bar{\sigma}}P_{0,0}} - I_{1,0}I_{0,1}\frac{\partial^2_{\bar{\sigma}\bar{\sigma}}P_{0,0}}{\partial_{\bar{\sigma}}P_{0,0}}.
\end{aligned} \tag{3.31}$$

Therefore, the desired second-order effective implied volatility approximation $I_{0,0} + \sqrt{\epsilon}I_{1,0} + \sqrt{\delta}I_{0,1} + \sqrt{\epsilon\delta}I_{1,1} + \epsilon I_{2,0} + \delta I_{0,2}$ of the effective implied volatility $I^{\epsilon,\delta}$ is established. It is clear that the zero-order term of the approximation is the effective volatility $\bar{\sigma}$; $P_{0,0}$ and its derivatives with respect to $\bar{\sigma}$ appear in the corrected terms. We consider the "forward log-moneyness" with effective maturity for timer options, $d = \log(K/Se^{r(B-x)/\bar{\sigma}^2})$, which is the analogue of the "forward log-moneyness" for vanilla options in [8]. We show an effective implied volatility surface with d and effective maturity in Figure 1 in the next section.

4. Numerical example

In this section, we provide a numerical example illustrating the accuracy of our approach. We perform price approximations in a multiscale stochastic volatility model

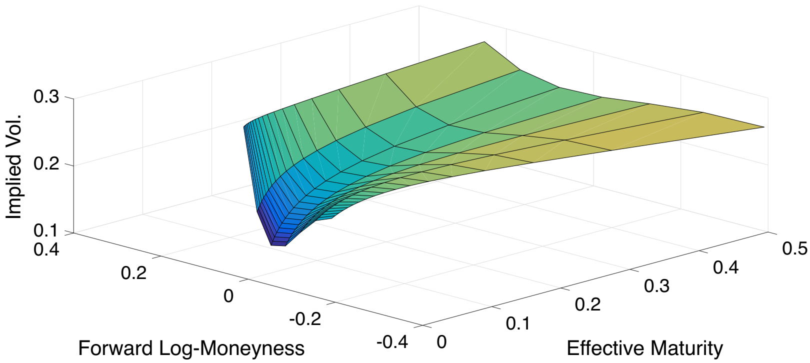

FIGURE 1. Effective implied volatility surface.

where the fast and slow drivers are OU process with $\alpha(Y_t) = m - Y_t$, $\beta(Y_t) = \nu\sqrt{2}$ and $c(Z_t) = c - Z_t$, $g(Z_t) = g$, respectively. Hence the financial market dynamics are

$$\begin{cases} dS_t = rS_t\,dt + f(Y_t, Z_t)S_t\,dW_t^{(0)\star} \\ dY_t = \left(\frac{1}{\epsilon}(m - Y_t) - \frac{1}{\sqrt{\epsilon}}\Lambda(Y_t, Z_t)\nu\sqrt{2}\right)dt + \frac{1}{\sqrt{\epsilon}}\nu\sqrt{2}\,dW_t^{(1)\star} \\ dZ_t = (\delta c(Z_t) - \sqrt{\delta}\Gamma(Y_t, Z_t)g(Z_t))\,dt + \sqrt{\delta}g(Z_t)\,dW_t^{(2)\star} \\ dI_t = f^2(Y_t, Z_t)\,dt, \end{cases} \tag{4.1}$$

where the two-factor volatility function is assumed to be

$$f(y,z) = \begin{cases} 20.01 - 10e^{-(y+z)} & \text{for } y \geq 0, z \geq 0 \\ 10.01 & \text{for } y \geq 0, z < 0 \\ 10.01 & \text{for } y < 0, z \geq 0 \\ 0.01 + 10e^{(y+z)} & \text{for } y < 0, z < 0. \end{cases}$$

We take the market prices of volatility risk $\gamma = 0$ and $\xi = 0$, and adopt the same parameters as those in [9]:

$$\epsilon = \frac{1}{200}, \quad \delta = \frac{1}{200}, \quad m = \log(0.1), \quad \nu = \frac{1}{\sqrt{2}}, \quad \mu = 0.2, \quad \rho_1 = 0,$$
$$r = 0.04, \quad \rho_2 = 0, \quad \tilde{\rho}_{12} = 0, \quad c = \log(0.1), \quad g = 1.$$

A straightforward calculation yields

$$\bar{\sigma}^2 = 0.0855,$$
$$C_1 = 0,\ C_2 = 0.055,\ C_3 = 0,\ C_4 = -0.0027,\ C_5 = 0,\ C_6 = 0,\ C_7 = 0,\ C_8 = 0.1717,$$
$$\tilde{C}_0 = -0.8583,\ \tilde{C}_1 = 0,\ \tilde{C}_2 = 0,\ \tilde{C}_3 = 0,\ \tilde{C}_4 = 0.$$

Regarding the prices generated from Monte Carlo simulations as the true prices, we evaluate the performance of the price approximations. Setting the initial values

TABLE 1. Timer option prices under multiscale stochastic volatility.

Option parameters	Monte Carlo price	P_0	P_1	P_2
T=1/12, k=−200	9.84 (0.074)	9.32 (−5.29%)	9.37 (−4.70%)	9.47 (−3.76%)
T=1/12, k=−100	6.46 (0.064)	6.04 (−6.36%)	6.09 (−5.59%)	6.20 (−3.92%)
T=1/12, k=0	3.79 (0.054)	3.53 (−6.83%)	3.57 (−5.85%)	3.67 (−3.09%)
T=1/12, k=100	2.01 (0.049)	1.84 (−8.73%)	1.86 (−7.54%)	1.93 (−4.01%)
T=1/12, k=200	0.92 (0.0086)	0.85 (−8.16%)	0.86 (−6.71%)	0.89 (−3.91%)
T=1/6, k=−200	14.28 (0.033)	13.26 (−7.16%)	13.37 (−6.37%)	13.46 (−5.70%)
T=1/6, k=−100	9.33 (0.030)	8.63 (−7.48%)	8.73 (−6.43%)	8.85 (−5.10%)
T=1/6, k=0	5.42 (0.025)	5.08 (−6.13%)	5.16 (−4.77%)	5.29 (−2.38%)
T=1/6, k=100	2.892 (0.019)	2.70 (−6.78%)	2.745 (−5.10%)	2.84 (−1.95%)
T=1/6, k=200	1.38 (0.013)	1.289 (−6.37%)	1.32 (−4.33%)	1.35 (−1.91%)
T=1/4, k=−200	16.98 (0.045)	16.30 (−3.96%)	16.467 (−2.99%)	16.57 (−2.41%)
T=1/4, k=−100	10.92 (0.033)	10.64 (−2.57%)	10.78 (−1.25%)	10.92 (0.015%)
T=1/4, k=0	6.53 (0.0462)	6.31 (−1.72%)	6.42 (−3.40%)	6.57 (0.57%)
T=1/4, k=100	3.53 (0.0418)	3.39 (−3.40%)	3.47 (−1.71%)	3.58 (1.34%)
T=1/4, k=200	1.72 (0.0431)	1.67 (−3.43%)	1.71 (−0.92%)	1.75 (1.46%)

$S_0 = 100$, $Y_0 = m$, $Z_0 = m$, $x = 0$, we simulate the SDEs (4.1) using Euler's method. We adapt 10^5 simulations with time step 10^{-3} in the simulation process, with various strike prices and variance budgets. Here, we take $B = \bar{\sigma}^2 T$ for $T = 1/12, 1/6, 1/4$, which means that the corresponding options will be expected to expire in 1 month, 2 months, and 3 months, respectively. For each choice of B, we consider the strike price $K = S_0 + (k/2)\sqrt{B}$ for $k = 0, \pm 100, \pm 200$.

We show the numerical results in Table 1. For Monte Carlo prices, the numbers in parentheses denote the standard errors for prices computed by Monte Carlo simulation. P_0, P_1, and P_2 denote the zero-order term $P_{0,0}$, the first-order approximate price $P_{0,0} + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1}$, and the second-order approximate price $P_{0,0} + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1} + \epsilon P_{2,0} + \delta P_{0,2} + \sqrt{\epsilon\delta}P_{1,1}$, respectively. The numbers in parentheses for the approximate prices represent the relative errors given by $(P_A - P_{MC})/P_{MC}$, where P_A is one of the approximate prices in Table 1 and P_{MC} is the Monte Carlo price. From Table 1, we can see the second-order approximations for option prices work well as a whole. For the expected expiration $T = 1/12$ and $T = 1/4$, the relative errors of principal term $P_{0,0}$ for in-the-money options are smaller than those of out-of-the-money options, while the reverse is true for $T = 1/6$. The approximations P_1 and P_2 perform similarly to P_0, when the options are in the money and out of the money. Generally speaking, the second-order approximate prices P_2 perform better than the first-order approximate prices P_1 and the principal term P_0.

The effective implied volatility is also investigated in this numerical example with the second-order asymptotic approximation. We produce an implied volatility surface

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with forward log-moneyness and effective maturity. In Figure 1, for any fixed value of effective maturity, we observe that the effective implied volatility curve has a skew effect, which is in accordance with the observations of empirical studies of vanilla options [14, 20]. In addition, it changes more rapidly for short effective maturity.

5. Conclusion

Derivatives with random expiration depending on the realized volatility are important owing to the need for volatility trading and hedging. We have established a full second-order asymptotic approximation for the timer option under a class of multi-factor stochastic volatility models, perturbed by fast and slow scale parameters, using the techniques of combined singular-regular perturbation. In addition, to explore the corresponding concept of implied volatility for timer options, we proposed and investigated the effective implied volatility. By inverting our price asymptotic expansion, we achieved a second-order approximation to the effective implied volatility. Both approximation formulas, for the price and for the effective implied volatility, use only the Black–Scholes formula and its derivatives. Numerical experiments were performed to demonstrate the accuracy of the approximations with various variance budgets and strike prices, and to illustrate the behaviour of the effective implied volatility. This study of the effective implied volatility should shed some light for further investigations of the use of implied volatility for timer options in pricing and hedging.

There are several directions of study on timer options in which similar perturbation techniques may provide an effective approach; we mention two of them. First, we focused here on the plain timer option, whereas one may develop asymptotic pricing formulas for other timer-type options, such as the timer barrier option and the compound timer option, where the pricing for their European counterparts have been studied under multiscale stochastic volatility models [7, 12]. Second, from a model point of view, it will be interesting to study the pricing of the timer options under stochastic volatility with regime-switching diffusions [19], which is another class of stochastic volatility models to which singular perturbation is well applied.

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References

[1] P. Carr and D. Madan, “Towards a theory of volatility trading”, in: *Handbooks in mathematical finance: Option pricing, interest rates and risk management* (Eds E. Jouini, J. Cvitanic and M. Musiela), (Cambridge University Press, Cambridge, 2001) 458–476; doi:10.1017/CBO9780511569708.013.

[2] W.-T. Chen and S.-P. Zhu, “Pricing perpetual American puts under multi-scale stochastic volatility”, *Asymptot. Anal.* **80** (2012) 133–148; doi:10.3233/ASY-2012-1110.

[3] M. Chernov, A. R. Gallant, E. Ghysels and G. Tauchen, “Alternative models for stock price dynamics”, *J. Econometrics* **116** (2003) 225–257; doi:10.1016/S0304-4076(03)00108-8.

[4] Z. Cui, J. L. Kirkby, G. Lian and D. Nguyen, “Integral representation of probability density of stochastic volatility models and timer options”, *Int. J. Theor. Appl. Finance* **20** (2017) 1750055; doi:10.1142/S0219024917500558.

[5] R. Durrett, *Brownian motion and Martingales in analysis, Wadsworth Math. Ser.* (Wadsworth International Group, Belmont, CA, 1984); ISBN-10 0534030653.

[6] J.-P. Fouque and C.-H. Han, “Asian options under multiscale stochastic volatility”, in: *Mathematics of finance*, Volume 351 of *Contemp. Math.* (Eds G. Yin and Q. Zhang), (American Mathematical Society, Providence, RI, 2004) 125–138; doi:10.1090/conm/351/06398.

[7] J.-P. Fouque and C.-H. Han, “Evaluation of compound options using perturbation approximation”, *J. Comput. Finance* **9** (2005) 41–61; doi:10.21314/JCF.2005.125.

[8] J.-P. Fouque, M. Lorig and R. Sircar, “Second order multiscale stochastic volatility asymptotics: stochastic terminal layer analysis and calibration”, *Finance Stoch.* **20** (2016) 543–588; doi:10.1007/s00780-016-0298-y.

[9] J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, “Singular perturbations in option pricing”, *SIAM J. Appl. Math.* **63** (2003) 1648–1665; doi:10. 1137/S0036139902401550.

[10] J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, “Multiscale stochastic volatility asymptotics”, *Multiscale Model. Simul.* **2** (2003) 22–42; doi:10.1137/030600291.

[11] J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, “Short time-scale in S&P500 volatility”, *J. Comput. Finance* **6** (2003) 1–24; doi:10.21314/JCF.2003.103.

[12] J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, *Multiscale stochastic volatility for equity, interest rate, and credit derivatives* (Cambridge University Press, Cambridge, 2011); doi:10.1017/CBO9781139020534.

[13] S. L. Heston, “A closed-form solution for options with stochastic volatility with applications to bond and currency options”, *Rev. Finance Stud.* **6** (1993) 327–343; doi:10.1093/rfs/6.2.327.

[14] I.-D. Kuo and K.-L. Wang, “Implied deterministic volatility functions: an empirical test for Euribor options”, *J. Futures Mark.* **29** (2009) 319–347; doi:10.1002/fut.20363.

[15] C. Li, “Managing volatility risk: innovation of financial derivatives, stochastic models and their analytical implementation”, Ph. D. Thesis, Columbia University, 2010, available at http://www.math.columbia.edu/~thaddeus/theses/2010/li.pdf.

[16] L. Liang, D. Lemmens and J. Tempere, “Path integral approach to the pricing of timer options with the Duru–Kleinert time transformation”, *Phys. Rev. E* **83** (2011) 056112; doi:10.1103/PhysRevE.83.056112.

[17] D. Saunders, “Pricing timer options under fast mean-reverting stochastic volatility”, *Can. Appl. Math. Q.* **17** (2009) 737–753, available at http://www.math.ualberta.ca/ami/CAMQ/pdf_files/ vol_17/17_4/17_4f.pdf.

[18] N. Sawyer, “SG CIB launches timer options” (CME Group, 2007), available at https://www.risk.net/derivatives/structured-products/1506473/sg-cib-launches-timer-options.

[19] G. Yin, “Asymptotic expansions of option price under regime-switching diffusions with a fast-varying switching process”, *Asymptot. Anal.* **65** (2009) 203–222; doi:10.3233/ASY-2009-0953.

[20] S. Zeto and Y. Man, “Pricing and hedging American fixed-income derivatives with implied volatility structures in the two-factor Heath–Jarrow–Morton model”, *J. Futures Mark.* **22** (2002) 839–875; doi:10.1002/fut.10031.

[21] J. Zhang, X. Lu and Y. Han, “Pricing perpetual timer option under the stochastic volatility model of Hull–White”, *ANZIAM J.* **58** (2017) 406–416; doi:10.1017/S1446181117000177.

[22] W. Zheng and P. Zeng, “Pricing timer options and variance derivatives with closed-form partial transform under the 3/2 mode”, *Appl. Math. Finance* **23** (2016) 344–373; doi:10.1080/1350486X.2017.1285242.