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ON A PERTURBED CONSERVATIVE SYSTEM OF SEMILINEAR WAVE EQUATIONS WITH PERIODIC-DIRICHLET BOUNDARY CONDITIONS

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Abstract

In this paper, some existence and uniqueness results for generalized solutions to a periodic-Dirichlet problem for semilinear wave equations are given, using a global inverse function theorem. These results extend those known in the literature.

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1. Introduction

Let $\mathcal{J} = [0, 2\pi] \times [0, \pi]$, let $n \geq 1$ be an integer, let $\mathbb{N}^*$ be the set of nonnegative integers, and let $F : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^n$ be a function of class C^2. Suppose that $V : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}$ is a function of class C^2 whose gradient and Hessian matrix with respect to u are denoted by V' and V'', respectively. Let $h \in \mathcal{H}$ with $\mathcal{H} = (L^2(\mathcal{J}))^n$ be given, with the usual inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. We consider the system of semilinear wave equations

$$u_{tt} - u_{xx} - V'(t, x, u) + F(t, x, u) = h(t, x), \tag{1.1}$$

where subscripts denote the partial derivative, and where $F(t, x, u)$ is called a *perturbing term*. By a generalized solution of the periodic-Dirichlet problem on $\mathcal{J}$ for (1.1) (or GPDS on $\mathcal{J}$ for short) we mean an element $u \in \mathcal{H}$ such that

$$\langle u, v_{tt} - v_{xx} \rangle - \langle v, V'(t, x, u) \rangle + \langle v, F(t, x, u) \rangle = \langle h(t, x), v \rangle,$$

for all $v \in (C^2(\mathcal{J}))^n$ satisfying

$$v(t, 0) = v(t, \pi) = 0, \quad \forall t \in [0, 2\pi];$$
$$v(0, x) = v(2\pi, x), \quad v_t(0, x) = v_t(2\pi, x), \quad \forall x \in [0, \pi].$$

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When the perturbing term $F(t, x, u)$ is 0, it is easy to see that the conservative system

$$u_{tt} - u_{xx} - V'(u) = h(t, x), \tag{1.2}$$

is included in the system (1.1). In [6], Mawhin obtained the following existence and uniqueness theorem for the GPDS of (1.2) on $\mathcal{J}$ using a Galerkin type argument similar to that in Bates and Castro [2] and a global inverse function theorem.

THEOREM 1.1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2 and let $\mathcal{J} = [0, 2\pi] \times [0, \pi]$. Assume that there exist two $n \times n$ symmetric matrices A and B, with respective eigenvalues $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_n$, such that*

$$A \le V''(u) \le B \tag{1.3}$$

for every $u \in \mathbb{R}^n$ and

$$\bigcup_{k=1}^{n} [\alpha_k, \beta_k] \cap \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\} = \varnothing. \tag{1.4}$$

Then (1.2) with the periodic-Dirichlet boundary conditions on $\mathcal{J}$ has a unique generalized solution $u \in (L^2(\mathcal{J}))^n$ for every $h \in (L^2(\mathcal{J}))^n$.

For more results on the existence of GPDS on $\mathcal{J}$ of (1.1), we refer the reader to [1, 4, 5, 7] and the references therein.

In this paper, we establish some new sufficient conditions for the existence of a unique GPDS on $\mathcal{J}$ of (1.1). Our proof is different from those mentioned above, and we use a new global inverse function theorem. Our results extend those in [1, 2, 4–7].

Throughout this paper we use the following assumption.

(A1). The eigenvalues $\lambda_i(V''(t, x, u))$, $i = 1, \ldots, n$, of $V''(t, x, u)$ satisfy

$$\alpha_i + \phi_i(t, x, \|u\|) \le \lambda_i(V'') \le \beta_i - \varphi_i(t, x, \|u\|)$$

on $\mathcal{J} \times \mathbb{R}^n$, where $\alpha_i, \beta_i \in \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\}$, $i = 1, \ldots, n$, are consecutive, $\phi_i(t, x, s)$ and $\varphi_i(t, x, s)$, $i = 1, \ldots, n$, are continuous functions defined from $\mathcal{J} \times [0, \infty)$ to $(0, \infty)$, they are nonincreasing with respect to s, and

$$\int_0^{+\infty} \min_{1 \le i \le n, (t,x) \in \mathcal{J}} \{\phi_i(t, x, s), \ \varphi_i(t, x, s)\} \, ds = +\infty. \tag{1.5}$$

Here we say that $\alpha_i, \beta_i \in \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\}$ are consecutive, for $i = 1, \ldots, n$, if

$$\bigcup_{j=1}^{n} (\alpha_i, \beta_i) \cap \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\} = \emptyset,$$

and $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$, $\beta_1 \le \beta_2 \le \cdots \le \beta_n$, $\alpha_i < \beta_i$ for each i.

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2. Abstract reformulation

If $\{c_k \mid 1 \leq k \leq n\}$ denotes an orthonormal basis in $\mathbb{R}^n$ and if we set

$$v_{lm}(t, x) = \exp(ilt)\sin mx, \quad l \in \mathbb{Z},\, m \in \mathbb{N}^*,$$

then every $u \in \mathcal{H}$ has a Fourier series

$$u = \sum_{k=1}^{n} \sum_{(l,m)\in\mathbb{Z}\times\mathbb{N}^*} u_{klm} v_{lm} c_k, \tag{2.1}$$

where the u_{klm} satisfy $u_{klm} = \overline{u_{k,-l,m}}$ to make the series real. If we define

$$\operatorname{dom} \mathcal{L} = \{u \in \mathcal{H} : u \text{ is given by (2.1)}\} \tag{2.2}$$

with

$$\sum_{k=1}^{n} \sum_{(l,m)\in\mathbb{Z}\times\mathbb{N}^*} (m^2 - l^2)^2 |u_{klm}|^2 < +\infty,$$

and

$$\mathcal{L} : \operatorname{dom} \mathcal{L} \subset \mathcal{H} \to \mathcal{H}, \quad u \mapsto \sum_{k=1}^{n} \sum_{(l,m)\in\mathbb{Z}\times\mathbb{N}^*} (m^2 - l^2) u_{klm} v_{lm} c_k, \tag{2.3}$$

it is easy to check that $\mathcal{L}$ is a self-adjoint operator such that

$$\ker \mathcal{L} = \operatorname{span}\{\cos mt \sin mx c_k, \sin mt \sin mx c_k \mid m \in \mathbb{N}^*, 1 \leq k \leq n\},$$

$$\operatorname{im} \mathcal{L} = (\ker \mathcal{L})^\perp,$$

$$\text{spectrum } \sigma(\mathcal{L}) = \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\}.$$

Moreover, for every $h \in \mathcal{H}$, u is a GPDS on $\mathcal{J}$ of the system

$$u_{tt} - u_{xx} = h$$

if and only if $u \in \operatorname{dom} \mathcal{L}$ and $\mathcal{L}u = h$ (see [6] and references therein). Therefore, if we assume the existence of a constant $C \geq 0$ such that, for all $u \in \mathbb{R}^n$,

$$\|V''(t, x, u)\| \leq C, \tag{2.4}$$

it is well known that the mapping N defined on $\mathcal{H}$ by

$$(N(u))(t, x) = -V'(t, x, u), \quad \text{a.e. on } \mathcal{J}, \tag{2.5}$$

continuously maps $\mathcal{H}$ into itself, and so the existence of GPDS on $\mathcal{J}$ for (1.1) is equivalent to the existence of a solution $u \in \operatorname{dom} \mathcal{L}$ for the equation

$$\mathcal{L}u + N(u) + F(u) = h \tag{2.6}$$

in $\mathcal{H}$, where the perturbing term $F : \operatorname{dom} \mathcal{L} \to \mathcal{H}$ is defined by

$$(F(u))(t, x) = F(t, x, u), \quad \forall (t, x) \in \mathcal{J}.$$

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In the sequel, $\mathcal{B}$ will be the set of all continuous and nondecreasing mappings ω that satisfy

$$\omega:\mathbb{R}_+\to\mathbb{R}_+,\ \omega(t)>0,\ t>0,\qquad \int_0^\infty \frac{1}{\omega(t)}\,dt=\infty. \tag{2.7}$$

LEMMA 2.1 (see [8, 9]). *Assume that $\mathcal{H}$ is a Hilbert space. Let $\mathcal{T}\in C^1(\mathcal{H},\mathcal{H})$, and assume that $\mathcal{T}'(u)$ is everywhere invertible for all $u\in\mathcal{H}$. Then $\mathcal{T}$ is a global diffeomorphism onto $\mathcal{H}$ if there exists $\omega\in\mathcal{B}$ satisfying $\|\mathcal{T}'(u)^{-1}\|\le\omega(\|u\|)$.*

3. Existence and uniqueness

Consider the boundary value problem (1.1). As shown in Section 2, if (2.4) holds, then (1.1) is equivalent to the operator equation

$$\mathcal{L}u+N(u)+F(u)=h,\qquad u\in\operatorname{dom}\mathcal{L}.$$

Let $Q(u)=(V''(t,x,u))$. Then

$$(N'(u)v)(t,x)=-(V''(t,x,u))v(t,x)=-Q(t,x,u)v(t,x),\qquad u,v\in\operatorname{dom},L$$

and $\mathcal{L}+N'(u)=\mathcal{L}-Q(t,x,u)$, where $Q(u)$ is a symmetric matrix.

Let $b_1(t,x,u),\dots,b_n(t,x,u)$ be eigenvalues of $Q(t,x,u)$, and for all $u\in\operatorname{dom}L$,

$$\alpha_i<b_i(t,x,u)<\beta_i,\qquad i=1,\dots,n, \tag{3.1}$$

which shows that (2.4) holds, that is, there exists a constant C such that $\|N'(u)v\|\le C\|v\|$, for all $u,v\in\operatorname{dom}\mathcal{L}$.

For each fixed point $(t,x)\in\mathcal{J}$, consider the eigenvalue problem

$$\mathcal{L}u-Q(t,x,u_0)u=\gamma u, \tag{3.2}$$

where $u_0\in\operatorname{dom}\mathcal{L}$ is fixed. Since $\alpha_i,\beta_i,i=1,\dots,n$, are consecutive and (3.1) holds, it follows that the eigenvalues of $Q(t,x,u_0)$ are ordered according to

$$b_1(t,x,u_0)\le b_2(t,x,u_0)\le\cdots\le b_n(t,x,u_0),$$

and zero is not an eigenvalue of (3.2). Hence, $\mathcal{L}-Q(t,x,u_0)$ is invertible at u_0 for each fixed point $(t,x)\in\mathcal{J}$, and by the spectral theorem [3, 10, 11]

$$\begin{aligned}\|(\mathcal{L}-Q(t,x,u_0))^{-1}\|&=\{\text{distance of 0 from the spectrum of }\mathcal{L}-Q(t,x,u_0)\}^{-1}\\&\le\Big(\min_{1\le i\le n}\{b_i(t,x,u_0)-\alpha_i,\ \beta_i-b_i(t,x,u_0)\}\Big)^{-1}.\end{aligned} \tag{3.3}$$

Let $\delta:\mathbb{R}_+\to\mathbb{R}_+\backslash\{0\}$ be defined by

$$\delta(s)=\max_{\|u\|\le s,(t,x)\in\mathcal{J}}\Big\{\Big(\min_{1\le i\le n}\{b_i(t,x,u)-\alpha_i,\ \beta_i-b_i(t,x,u)\}\Big)^{-1}\Big\}. \tag{3.4}$$

Then δ is continuous and nondecreasing with respect to s. Now since u_0 is arbitrary, we have that $\mathcal{L}+N'(u)$ is invertible on $\mathcal{J}$ for all $u\in D(\mathcal{L})$, and $\|(\mathcal{L}+N'(u))^{-1}\|\le\delta(\|u\|)$.

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LEMMA 3.1. *Assume that there exists $\eta < 1$ with*

$$\|F_u(t, x, u)\| \leq \eta(\delta(\|u\|))^{-1}. \tag{3.5}$$

Then

$$\|[\mathcal{L} + N'(u) + F'(u)]^{-1}\| \leq \frac{\delta(s)}{1-\eta}. \tag{3.6}$$

PROOF. From $(F'(u)v)(t, x) = (F_u)v(t, x)$, for all $u, v \in \operatorname{dom} \mathcal{L}$,

$$\|F'(u)v\| \leq \eta(\delta(\|u\|))^{-1}\|v\|. \tag{3.7}$$

For all $y \in \mathcal{H}$, notice that

$$\|[\mathcal{L} + N'(u)]^{-1}y\| \leq \delta(\|u\|)\|y\|. \tag{3.8}$$

Define the mapping $P = F'(u)[\mathcal{L} + N'(u)]^{-1} : \mathcal{H} \to \mathcal{H}$. Then from (3.7) and (3.8), for all $y \in \mathcal{H}$,

$$\begin{aligned}
\|Py\| &= \|F'(u)[\mathcal{L} + N'(u)]^{-1}y\| \\
&\leq \eta(\delta(\|u\|))^{-1}\|[\mathcal{L} + N'(u)]^{-1}y\| \\
&\leq \eta(\delta(\|u\|))^{-1}\delta(\|u\|)\|y\| = \eta\|y\|.
\end{aligned}$$

Then $I + P$ is invertible and $\|[I + P]^{-1}\| \leq (1 - \eta)^{-1}$. Note that

$$\begin{aligned}
\mathcal{L} + N'(u) + F'(u) &= (I + F'(u)[\mathcal{L} + N'(u)]^{-1}) \cdot (\mathcal{L} + N'(u)) \\
&= (I + P) \cdot (\mathcal{L} + N'(u)).
\end{aligned}$$

Hence, it follows from the invertibility of $I + P$ that $\mathcal{L} + N'(u) + F'(u)$ is invertible and $[\mathcal{L} + N'(u) + F'(u)]^{-1} = [\mathcal{L} + N'(u)]^{-1}(I + P)^{-1}$. This, together with (3.4), yields (3.6). □

THEOREM 3.2. *Assume that (A1) and (3.5) hold. Then (1.1) with the periodic-Dirichlet boundary conditions on $\mathcal{J}$ has a unique generalized solution $u \in (L^2(\mathcal{J}))^n$ for every $h \in (L^2(\mathcal{J}))^n$.*

PROOF. From (3.4),

$$\begin{aligned}
\delta(s) &= \max_{\|u\| \leq s, (t,x) \in \mathcal{J}} \left\{ \left(\min_{1 \leq i \leq n} \{b_i(t, x, u) - \alpha_i, \beta_i - b_i(t, x, u)\} \right)^{-1} \right\} \\
&\leq \max_{\|u\| \leq s, (t,x) \in \mathcal{J}} \left\{ \left(\min_{1 \leq i \leq n} \{\alpha_i + \phi_i(t, x, \|u\|) - \alpha_i, \ \beta_i - \beta_i + \varphi_i(t, x, \|u\|)\} \right)^{-1} \right\} \\
&= \max_{\|u\| \leq s, (t,x) \in \mathcal{J}} \left\{ \left(\min_{1 \leq i \leq n} \{\phi_i(t, x, \|u\|), \varphi_i(t, x, \|u\|)\} \right)^{-1} \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^\infty \frac{1}{\delta(s)}\, ds &\geq \int_0^\infty \left(\max_{\|u\| \leq s, (t,x) \in \mathcal{J}} \left\{ \left(\min_{1 \leq i \leq n} \{\phi_i(t, x, \|u\|), \varphi_i(t, x, \|u\|)\} \right)^{-1} \right\} \right)^{-1} ds \\
&\geq \int_0^\infty \min_{1 \leq i \leq n, (t,x) \in \mathcal{J}} \{\phi_i(t, x, \|s\|), \ \varphi_i(t, x, \|s\|)\}\, ds.
\end{aligned}$$

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Then, by (1.5) in assumption (A1), Lemma 2.1 (with (3.6)) and Lemma 3.1, the system (1.1) has a unique generalized solution $u \in (L^2(\mathcal{J}))^n$ for every $h \in (L^2(\mathcal{J}))^n$. The proof is complete. □

We now use the following assumption.

(A2). There exist two symmetric $n \times n$ matrices A and B such that

$$A + \phi(t, x, \|u\|)I \leq V'' \leq B - \varphi(t, x, \|u\|)I$$

on $\mathcal{J} \times \mathbb{R}^n$, and the eigenvalues of A and B are α_i, β_i, $i = 1, \ldots, n$, respectively, where I is the $n \times n$ identity matrix, $\phi_i(t, x, s)$ and $\varphi_i(t, x, s)$, $i = 1, \ldots, n$, are continuous functions defined from $\mathcal{J} \times [0, \infty)$ to $(0, \infty)$ that are nonincreasing with respect to s, and

$$\int_0^{+\infty} \min_{(t,x)\in\mathcal{J}} \{\phi(t, x, s), \varphi(t, x, s)\}\, ds = +\infty. \tag{3.9}$$

Here α_i, $\beta_i \in \sigma(L)$, $i = 1, \ldots, n$, are consecutive.

Essentially the same reasoning as in Theorem 3.2 yields the following result.

THEOREM 3.3. *Assume that (A2) and (3.5) hold. Then (1.1) with the periodic-Dirichlet boundary conditions on $\mathcal{J}$ has a unique generalized solution $u \in (L^2(\mathcal{J}))^n$ for every $h \in (L^2(\mathcal{J}))^n$.*

REMARK 3.4. Theorems 3.2 and 3.3 allow the eigenvalues of $V''(t, x, u)$, when $\|u\| \to \infty$, to interact with points of the spectral set $\{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\}$. Consider the nonlinear semilinear-wave equation

$$u_{tt} - u_{xx} + V'(t, x, u) = h(t, x), \quad \forall (t, x) \in \mathcal{J}, \tag{3.10}$$

with the periodic-Dirichlet boundary conditions on $\mathcal{J}$. Let

$$V'(t, x, u) = mu - \frac{\sin^2(t) + 1}{4} \ln(u + \sqrt{1 + u^2}), \quad m \in \{1, 2, \ldots\},$$

and let $h : \mathcal{J} \to \mathbb{R}$ be in $L^2(\mathcal{J})$. Theorem 3.2 guarantees the existence of a unique periodic-Dirichlet solution to (3.10) since

$$m - 1 + \frac{1}{2} \leq V''(t, x, u) = m - \frac{\sin^2(t) + 1}{4\sqrt{1 + u^2}} \leq m.$$

Also,

$$\lim_{\|u\|\to\infty} \|V''(t, x, u) - m\| = 0.$$

We now discuss the case where the eigenvalues of $V''(t, x, u)$ do not interact with points of the spectral set $\{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\}$ as $\|u\| \to \infty$.

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COROLLARY 3.5. *Suppose that*

$$A_1 \leq V'' \leq B_1, \quad \alpha_i < \mu_i \leq \nu_i < \beta_i, \tag{3.11}$$

where μ_i and ν_i are eigenvalues of the symmetric $n \times n$ matrices A_1 and B_1, respectively, and $\alpha_i, \beta_i \in \sigma(\mathcal{L})$, $i = 1, \ldots, n$, are consecutive. Assume that (3.5) holds. Then (1.1) with the periodic-Dirichlet boundary conditions on $\mathcal{J}$ has a unique generalized solution $u \in (L^2(\mathcal{J}))^n$ for every $h \in (L^2(\mathcal{J}))^n$.

PROOF. It follows from (3.11) that the eigenvalues λ_i, $i = 1, \ldots, n$, of V'' satisfy

$$\alpha_i + \min_{1\leq i\leq n} (\mu_i - \alpha_i) \leq \lambda_i(V'') \leq \beta_i - \min_{1\leq i\leq n} (\beta_i - \nu_i).$$

If we let $\phi_j(t, x, s) = \min_{1\leq i\leq n} (\mu_i - \alpha_i)$, $\varphi_j(t, x, s) = \min_{1\leq i\leq n} (\beta_i - \nu_i)$, $j = 1, \ldots, n$, then (1.5) holds. The result follows from Theorem 3.2. □

REMARK 3.6. Since $\alpha_i, \beta_i \in \sigma(\mathcal{L})$, $i = 1, \ldots, n$, are consecutive in Corollary 3.5, the respective eigenvalues $\mu_1 \leq \mu_2 \cdots \leq \mu_n$ and $\nu_1 \leq \nu_2 \cdots \leq \nu_n$ of A_1 and B_1 satisfy

$$\bigcup_{k=1}^{n} [\mu_i, \nu_i] \cap \{m^2 - l^2 \mid l \in \mathbb{Z}, m \in \mathbb{N}^*\} = \varnothing. \tag{3.12}$$

Then Theorem 1.1, that is, the main result of [6], is a special case of Corollary 3.5, when the perturbing term $F(t, x, u) = 0$.

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