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SOLUTION BRANCHES OF NONLINEAR EIGENVALUE PROBLEMS ON RESTRICTED DOMAINS

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Abstract

We extend bifurcation results of nonlinear eigenvalue problems from real Banach spaces to any neighbourhood of a given point. For points of odd multiplicity on these restricted domains, we establish that the component of solutions through the bifurcation point either is unbounded, admits an accumulation point on the boundary, or contains an even number of odd-multiplicity points. In the simple-multiplicity case, we show that branches of solutions in the directions of corresponding eigenvectors satisfy similar conditions on such restricted domains.

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1. Introduction

For Banach spaces X and Y, any subset B of X and any function $G : B \to Y$, we say that G is *compact* (or *completely continuous*) if it is continuous and maps bounded closed subsets of X contained in B to relatively compact sets. Let X be an arbitrary Banach space, $\mathfrak{X} = X \times \mathbb{R}$, $\lambda_0 \in \mathbb{R}$ and $\Omega \subseteq \mathfrak{X}$ a neighbourhood of $(0, \lambda_0)$. We consider the *nonlinear eigenvalue problem* on Ω of the form

$$0 = x - \lambda K x - H(x, \lambda) =: F(x, \lambda), \tag{1.1}$$

where $K : \mathfrak{X} \to X$ is a compact linear operator and $H : \Omega \to X$ is compact. We suppose that H is such that the function $h : \Omega \to X$ given by

$$h(x, \lambda) = \begin{cases} \|x\|^{-1} H(x, \lambda), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous. Note that, when $\Omega = \mathfrak{X}$, this condition on H is equivalent to the conditions on H given in [4, page 487] and [1, page 1069].

We say that $(0, \lambda_0) \in \Omega$ is a *bifurcation point* of $F(x, \lambda) = 0$ (with respect to the 'curve' of trivial solutions $x = 0$) if every neighbourhood of $(0, \lambda_0)$ contains a nontrivial solution of $F(x, \lambda) = 0$. It is well known that if $(0, \lambda_0)$ is a bifurcation point, then

λ_0^{-1} is an eigenvalue of K [2, Proposition 28.1]. This motivates the definition of a *characteristic value* of K: any $\lambda \in \mathbb{R}$ such that λ^{-1} is an eigenvalue of K. We denote the set of characteristic values of K by $\mathrm{char}(K)$.

We take the *multiplicity* of a characteristic value λ_0 to be the algebraic multiplicity of λ_0^{-1} as an eigenvalue of K. It was proved in the pioneering paper by Rabinowitz [4, Theorem 1.3] that if λ_0 is of odd multiplicity, then $(0, \lambda_0)$ is a bifurcation point. Moreover, assuming that $\Omega = \mathfrak{X}$, he showed that, for such λ_0, the connected component C_{λ_0} containing $(0, \lambda_0)$ of the closure of nontrivial solutions to $F(x, \lambda) = 0$ either is unbounded or contains some $(0, \mu) \neq (0, \lambda_0)$, where $\mu \in \mathrm{char}(K)$ is of odd multiplicity. A strengthened version of this result by Dancer [1, Corollary 1] states that if C_{λ_0} is bounded, then it contains an even number of $(0, \mu)$ with $\mu \in \mathrm{char}(K)$ of odd multiplicity.

Of particular importance is the special case where λ_0 is of multiplicity 1 (that is, it is *simple*) with corresponding eigenvector v. Then we can express C_{λ_0} as the union of $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$: closures of the unions of all *branches of solutions* going from $(0, \lambda_0)$ in the directions of v and $-v$, respectively. Dancer [1, Theorem 2] proved that either $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ are both unbounded or they intersect away from $(0, \lambda_0)$.

The aim of our paper is to generalise the above results by Rabinowitz and Dancer from $\Omega = \mathfrak{X}$ to any neighbourhood of $(0, \lambda_0)$. There have already been some considerations of different domains for odd multiplicity. In his original paper, Rabinowitz mentioned closures of bounded open sets as a 'weaker' result [4, Corollary 1.12]. Turner further investigated these domains, proving that if $\partial\Omega$ is sufficiently nice and λ_0 is the only characteristic value μ with $(0, \mu)$ in $\overline{\Omega}$, then $\partial\Omega$ admits either two solutions or one solution of multiplicity 2 [5, Theorem 2.4]. However, his result assumes that F is globally defined. A generalisation of Rabinowitz's result to any $\Omega = \overline{\mathrm{int}\,\Omega}$ has also been found (presented in [2, Theorem 29.1], for example), but this is insufficient, say, when Ω is open. Consequently, it fails to handle the cases where H has singularities or is unbounded on a bounded domain. The author is not aware of an existing analogue of the simple-multiplicity result for arbitrary neighbourhoods.

2. Characteristic values of odd multiplicity

We start with the generalisation of the odd-multiplicity result. Denote the closure in Ω of nontrivial solutions of $F(x, \lambda) = 0$ by $\mathfrak{S}(F)$ and, for any $\lambda_0 \in \mathrm{char}(K)$ of odd multiplicity, the connected component of $(0, \lambda_0)$ in $\mathfrak{S}$ by $C_{\lambda_0}(F)$. We omit F when it is clear from the context. Our aim is to prove the following theorem for any neighbourhood Ω of $(0, \lambda_0)$.

Theorem 2.1. *Let λ_0, Ω, K and F be as given in the introduction and $\mathfrak{S}$ be as given above. If $\lambda_0 \in \mathrm{char}(K)$ has odd multiplicity, then the connected component C_{λ_0} of $(0, \lambda_0)$ in $\mathfrak{S}$ either is unbounded, admits a limit point on $\partial\Omega$ or contains an even number of trivial solutions $(0, \mu)$ of $F(x, \lambda) = 0$ with $\mu \in \mathrm{char}(K)$ of odd multiplicity.*

We remark that all three alternatives for C_{λ_0} are possible. A simple example of the first is $H(x, \lambda) \equiv 0$. The second case is guaranteed when Ω is a bounded neighbourhood

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of $(0, \lambda_0)$ such that $\mu = \lambda_0$ is the only element of char(K) with $(0, \mu) \in \Omega$. An instance of the final case can be found in [4, pages 492–493].

In the special case where $\Omega = \mathfrak{X}$, the above theorem is simply [1, Corollary 1]. Our approach is to reduce the theorem from general Ω to $\Omega = \mathfrak{X}$. The main step is the following lemma, which will also be useful when we consider bifurcations at $(0, \lambda_0)$ for simple λ_0.

Lemma 2.2. *Let $\Omega_1 \subseteq \Omega_2$ be neighbourhoods of $(0, \lambda_0)$ contained in the domain of F and let $F_i = F|_{\Omega_i}$ for $i = 1, 2$. For any closed $V \subseteq \mathfrak{X}$ containing $(0, \lambda_0)$, let $C_V(F_i)$ denote the connected component of $V \cap \mathfrak{S}(F_i)$ containing $(0, \lambda_0)$. Then*

$$C_V(F_1) \subseteq C_V(F_2),$$

with equality if Ω_1 is closed in $\mathfrak{X}$ and $C_V(F_1) \cap \partial\Omega_1 = \emptyset$.

For this proof and for later results, we need to invoke a special case of a result by Whyburn [6, (9.3)].

Lemma 2.3. *Let M be a compact metric space. Let A_1 and A_2 be disjoint closed subsets of M, with A_1 a connected component of M. Then there exist disjoint compact subsets M_1 and M_2 of M such that $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ and $M = M_1 \cup M_2$.*

Proof of Lemma 2.2. For every $r > 0$, we define

$$\mathfrak{X}_{\lambda_0}(r) = \{(x, \mu) \in \mathfrak{X} \mid \|x\| + |\lambda_0 - \mu| < r\}$$

and denote the closure of $\mathfrak{X}_{\lambda_0}(r)$ in $\mathfrak{X}$ by $\overline{\mathfrak{X}}_{\lambda_0}(r)$.

We see that $V \cap \mathfrak{S}(F_1) \subseteq V \cap \mathfrak{S}(F_2)$ and so, by considering connected components containing $(0, \lambda_0)$, we get $C_V(F_1) \subseteq C_V(F_2)$. To prove the equality case, suppose Ω_1 is closed in $\mathfrak{X}$ and $C_V(F_1) \cap \partial\Omega_1 = \emptyset$. Let $N > 0$ be such that $C_V(F_1) \subseteq \mathfrak{X}_{\lambda_0}(N)$. We note that bounded closed subsets of $\mathfrak{X}$ contained in $F^{-1}(0)$ are compact since $\lambda K x + H(x, \lambda)$ is a compact map, and so $\mathfrak{S}(F_1) \cap \overline{\mathfrak{X}}_{\lambda_0}(N) \cap V$ is compact. By Lemma 2.3, $\mathfrak{S}(F_1) \cap \overline{\mathfrak{X}}_{\lambda_0}(N) \cap V$ can be expressed as the union of disjoint compact sets M_1 and M_2 such that $C_V(F_1) \subseteq M_1$ and $\mathfrak{S}(F_1) \cap \overline{\mathfrak{X}}_{\lambda_0}(N) \cap V \cap [\partial\mathfrak{X}_{\lambda_0}(N) \cup \partial\Omega_1] \subseteq M_2$. Since M_1 and M_2 are compact, we can find an open neighbourhood U of M_1 with $\overline{U} \subseteq \Omega_1 \cap \mathfrak{X}_{\lambda_0}(N)$ such that ∂U and $\mathfrak{S}(F_1) \cap \overline{\mathfrak{X}}_{\lambda_0}(N) \cap V$ are disjoint. We observe that ∂U and $\mathfrak{S}(F_2) \cap V$ are disjoint.

We see that $C_V(F_2)$ is contained in U, since if $C_V(F_2)$ intersected $\mathfrak{X} \setminus \overline{U}$ nontrivially, then connectedness would imply that $C_V(F_2)$ and ∂U are not disjoint. Thus $C_V(F_2)$ must coincide with the connected component of $\mathfrak{S}(F_2) \cap V \cap U$. Since

$$\mathfrak{S}(F_2) \cap V \cap U = \mathfrak{S}(F_2) \cap V \cap U \cap \Omega_1 = \mathfrak{S}(F_1) \cap V \cap U \subseteq \mathfrak{S}(F_1) \cap V,$$

by looking at the respective connected components containing $(0, \lambda_0)$ we conclude that $C_V(F_2) \subseteq C_V(F_1)$ and so $C_V(F_1) = C_V(F_2)$. □

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Now we are ready to reduce Theorem 2.1 to the case $\Omega = \mathfrak{X}$. We recall that $(0, \lambda_0) \in \operatorname{int} \Omega$ and so the set

$$\Omega_\delta := \{(x, \lambda) \in \Omega \mid \operatorname{dist}((x, \lambda), \partial\Omega) > \delta\}$$

is open and nonempty for all $\delta > 0$ sufficiently small. Let $F_\delta = F|_{\overline{\Omega}_\delta}$ for all $\delta > 0$ and define $h_\delta : \overline{\Omega}_\delta \cup (\{0\} \times \mathbb{R}) \to X$ by

$$h_\delta(x, \lambda) = \begin{cases} \|x\|^{-1} H(x, \lambda), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

By our assumption on H, it follows that h_δ is continuous. Let $\widetilde{h}_\delta$ be an extension of h_δ to $\mathfrak{X}$ as given by Dugundji's extension theorem [3, Ch. IX, Theorem 6.1]. For all $(x, \lambda) \in \mathfrak{X}$, let $\widetilde{H}_\delta(x, \lambda) = \|x\| \widetilde{h}_\delta(x, \lambda)$ and $\widetilde{F}_\delta(x, \lambda) = x - \lambda K x - \widetilde{H}_\delta(x, \lambda)$.

Assume that the proposition holds when $\Omega = \mathfrak{X}$ and suppose that $C_{\lambda_0}(F)$ is bounded with no accumulation points on $\partial\Omega$. Then, since $C_{\lambda_0}(F)$ is compact and disjoint from $\partial\Omega$, for some $\delta > 0$ sufficiently small, we have $C_{\lambda_0}(F) \subseteq \Omega_\delta$. Applying Lemma 2.2 with $\Omega_1 = \overline{\Omega}_\delta$, $\Omega_2 = \Omega$ and $V = \mathfrak{X}$ gives $C_{\lambda_0}(F_\delta) \subseteq C_{\lambda_0}(F) \subseteq \Omega_\delta$ and so $C_{\lambda_0}(F_\delta) \cap \partial\Omega_\delta = \emptyset$. Now, by applying Lemma 2.2 twice, both times with Ω_1 and V as before, once with F and $\Omega_2 = \Omega$, and once with $\widetilde{F}_\delta$ and $\Omega_2 = \mathfrak{X}$, we obtain

$$C_{\lambda_0}(F) = C_{\lambda_0}(F_\delta) = C_{\lambda_0}(\widetilde{F}_\delta).$$

Consequently, since $C_{\lambda_0}(F)$ is bounded and $\widetilde{F}_\delta$ is defined on all of $\mathfrak{X}$, we may apply Theorem 2.1 to show that $C_{\lambda_0}(F) = C_{\lambda_0}(\widetilde{F}_\delta)$ contains an even number of trivial solutions $(0, \mu)$ of $\widetilde{F}_\delta(x, \lambda) = 0$ with μ of odd multiplicity. Since $C_{\lambda_0}(F) \subseteq \Omega_\delta$ and $F = \widetilde{F}_\delta$ on $\overline{\Omega}_\delta$, we conclude that $C_{\lambda_0}(F)$ contains an even number of trivial solutions $(0, \mu)$ of $F(x, \lambda) = 0$ with μ of odd multiplicity. Thus we have reduced Theorem 2.1 to the known case $\Omega = \mathfrak{X}$.

3. Simple characteristic values

Now we consider the special case where λ_0 is a simple characteristic value. We start by giving the definition of a branch of solutions in the direction of v or $-v$, where v is a unit-length λ_0^{-1}-eigenvector of K. Let X' be the dual space of X and let $l \in X'$ be the λ_0^{-1}-eigenvector of the dual of K such that $\langle l, v\rangle = 1$. For $0 \leq y < 1$, define

$$\mathfrak{C}_y = \{(x, \lambda) \in \mathfrak{X} \mid |\langle l, x\rangle| > y\|x\|\}.$$

Let $\mathfrak{C}_y^+$ and $\mathfrak{C}_y^-$ be the subsets of $\mathfrak{C}_y$ consisting of the elements with $\langle l, x\rangle > y\|x\|$ and $\langle l, x\rangle < -y\|x\|$, respectively. We say that $F(x, \lambda) = 0$ admits a branch of solutions at $(0, \lambda_0)$ in the direction of v if there exists a connected set $Q^+ \subseteq C_{\lambda_0}$ containing $(0, \lambda_0)$ such that for every $y \in (0, 1)$, there exists $\epsilon_y^+ > 0$ for which

$$\emptyset \neq Q^+ \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon) \subseteq \mathfrak{C}_y^+$$

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for all ϵ with $0 < \epsilon < \epsilon_y$. We then call Q^+ a branch of solutions in the direction of v. We replace v with $-v$ and swap $+$ with $-$ to get the definition of a branch of solutions in the direction of $-v$.

Denote by $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ the closures in Ω of the unions of all branches of solutions in the directions of v and $-v$, respectively. Our desired result is the following theorem.

THEOREM 3.1. *Let λ_0, Ω, K and F be as given in the introduction. Suppose that λ_0 is a simple characteristic value, v is a unit-length λ_0^{-1}-eigenvector of K, and $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ are the closures in Ω of the unions of all branches of solutions of (1.1) in the directions of v and $-v$, respectively. Then at least one of the following alternatives holds:*

(1) *each of $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ is unbounded or admits a limit point on $\partial\Omega$;*
(2) $C^+_{\lambda_0} \cap C^-_{\lambda_0} \neq \{(0, \lambda_0)\}$.

Similarly to Theorem 2.1, both alternatives of Theorem 3.1 can occur. Moreover, the first alternative cannot be strengthened to say that $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ are both unbounded or both admit an accumulation point on $\partial\Omega$. A counterexample is $H(x, \lambda) \equiv 0$ on the domain $\Omega = \overline{\mathfrak{C}^+_0} \cup (\overline{\mathfrak{C}^-_0} \cap \mathfrak{X}_{\lambda_0}(N))$, for any $N > 0$. In this case, $x \mapsto F(x, \lambda)$ is a linear map for every fixed λ, with kernel the λ^{-1}-eigenspace of K for $\lambda \neq 0$. It follows that $C^-_{\lambda_0}$ is bounded, $C^+_{\lambda_0}$ does not have an accumulation point on $\partial\Omega$ and $C^+_{\lambda_0} \cap C^-_{\lambda_0} = \{(0, \lambda_0)\}$.

To avoid duplication, we will use κ to denote one of $+$ and $-$ and will interpret $-\kappa$ in the obvious way. Fix $0 < y < 1$. By [4, Lemma 1.2], there exists $S > 0$ such that $\overline{\mathfrak{X}}_{\lambda_0}(S) \subseteq \operatorname{int}\Omega$ and

$$\overline{\mathfrak{X}}_{\lambda_0}(S) \cap \mathfrak{S} \setminus \{(0, \lambda_0)\} \subseteq \mathfrak{C}_y. \tag{3.1}$$

For every ϵ with $0 < \epsilon < S$, let $C^\kappa_{\lambda_0,\epsilon}$ be the connected component of $C_{\lambda_0} \setminus (\mathfrak{X}_{\lambda_0}(\epsilon) \cap \mathfrak{C}^{-\kappa}_y)$ containing $(0, \lambda_0)$. We notice that $C^\kappa_{\lambda_0,\epsilon} \supseteq C^\kappa_{\lambda_0,\epsilon'}$ whenever $0 < \epsilon < \epsilon' < S$, and so

$$\bigcup_{0<\epsilon<S} C^\kappa_{\lambda_0,\epsilon} = \bigcup_{0<\epsilon<\epsilon'} C^\kappa_{\lambda_0,\epsilon}$$

for $0 < \epsilon' < S$. Also, we deduce from (3.1) that, regardless of y, every branch of solutions is contained in some $C^\kappa_{\lambda_0,\epsilon}$. Thus $C^\kappa_{\lambda_0}$ is the closure of $\bigcup_{0<\epsilon<S} C^\kappa_{\lambda_0,\epsilon}$. We note that $C^\kappa_{\lambda_0}$ is connected as the closure of a union of connected sets sharing a point [3, Ch. V, Theorems 1.5 and 1.6].

Rather than proving directly that $C^+_{\lambda_0}$ and $C^-_{\lambda_0}$ satisfy at least one of the alternatives in Theorem 3.1, we will establish the following stronger result.

PROPOSITION 3.2. *If $C^\kappa_{\lambda_0}$ is bounded and disjoint from $\partial\Omega$ for $\kappa \in \{\pm\}$, then the connected component T^κ_ϵ of $C^\kappa_{\lambda_0} \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y}$ containing $(0, \lambda_0)$ intersects $\partial\mathfrak{X}_{\lambda_0}(\epsilon)$ nontrivially for all $\epsilon > 0$ sufficiently small.*

We verify that this proposition does in fact imply Theorem 3.1. If the first alternative of the theorem does not hold, then $C^\kappa_{\lambda_0}$, and so T^κ_ϵ for all ϵ with $0 < \epsilon < S$, is bounded and disjoint from $\partial\Omega$ for some $\kappa \in \{\pm\}$. Since T^κ_ϵ is connected and $(0, \lambda_0) \in T^\kappa_\epsilon$, by definition of $C^{-\kappa}_{\lambda_0,\epsilon}$ we have $T^\kappa_\epsilon \subseteq C^{-\kappa}_{\lambda_0,\epsilon} \subseteq C^{-\kappa}_{\lambda_0}$. From the proposition, T^κ_ϵ intersects

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$\partial\mathfrak{X}_{\lambda_0}(\epsilon)$ nontrivially for all $\epsilon > 0$ sufficiently small, and so we conclude that the second alternative of the theorem holds.

To prove Proposition 3.2, we first reduce it to the case $\Omega = \mathfrak{X}$. Assume that the proposition holds when $\Omega = \mathfrak{X}$. Recall the definitions of Ω_δ, F_δ and $\widetilde{F}_\delta$ from earlier. Suppose that $C^\kappa_{\lambda_0}(F)$ is bounded without any accumulation points on $\partial\Omega$. By adapting the reduction argument of Theorem 2.1 to use $C^\kappa_{\lambda_0}(F)$ instead of $C_{\lambda_0}(F)$, we find $C^\kappa_{\lambda_0}(\widetilde{F}_\delta) = C^\kappa_{\lambda_0}(F)$ for all $\delta > 0$ sufficiently small. We apply Proposition 3.2 to $\widetilde{F}_\delta$ to see that the connected component of

$$C^\kappa_{\lambda_0}(\widetilde{F}_\delta) \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y} = C^\kappa_{\lambda_0}(F) \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y}$$

containing $(0, \lambda_0)$ intersects $\partial\mathfrak{X}_{\lambda_0}(\epsilon)$ for all ϵ sufficiently small. Thus we have reduced the proposition to the case $\Omega = \mathfrak{X}$.

Finally, we adapt the proof of [1, Theorem 2] to show that Proposition 3.2 holds when $\Omega = \mathfrak{X}$. We will invoke the following result due to Dancer [1, Lemma 3].

Proposition 3.3. *Let $\Omega = \mathfrak{X}$ and take S as in (3.1). Then for $0 < \epsilon < S$, the set $C^\kappa_{\lambda_0,\epsilon}$ either is unbounded or intersects $\partial\mathfrak{X}_{\lambda_0}(\epsilon) \cap \mathfrak{C}^{-\kappa}_y$ nontrivially.*

We also need the following lemma.

Lemma 3.4. *Let $\kappa \in \{\pm\}$. For $0 < \epsilon_1 < S$, if $C^\kappa_{\lambda_0,\epsilon_1} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y} \neq \emptyset$ then the set*

$$(C^\kappa_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y) \cup (\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y})$$

is connected.

Proof. Fix $0 < \epsilon_1 < S$ and let $Y = (C^\kappa_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y) \cup (\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y})$. To prove that Y is connected, we only need to show that

$$Y_\epsilon := (C^\kappa_{\lambda_0,\epsilon} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y) \cup (\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y})$$

is connected for $0 < \epsilon < \epsilon_1$. Then $\bigcup_{0<\epsilon<S} Y_\epsilon$ is connected as the union of connected sets sharing a point [3, Ch. V, Theorem 1.5]. Since $A \cap \overline{B} \subseteq \overline{A \cap B}$ for all sets A and B with A open, by taking $A = \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y$ and $B = \bigcup_{0<\epsilon<\epsilon_1} C^\kappa_{\lambda_0,\epsilon}$, we see that

$$\bigcup_{0<\epsilon<\epsilon_1} Y_\epsilon \subseteq Y \subseteq \overline{\Big(\Big(\bigcup_{0<\epsilon<\epsilon_1} C^\kappa_{\lambda_0,\epsilon}\Big) \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y\Big)} \cup (\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y}) = \overline{\bigcup_{0<\epsilon<\epsilon_1} Y_\epsilon}$$

and so Y is connected as a set contained between a connected set and its closure [3, Ch. V, Theorem 1.6].

Now we show that Y_ϵ is connected. Let V be a closed and open subset of Y_ϵ for some ϵ with $0 < \epsilon < \epsilon_1$ fixed. Since $\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y}$ is connected, V is either disjoint from it or contains it. Swapping V with its complement in Y_ϵ if needed, we may assume that the former case is true. Thus V is a subset of $C^\kappa_{\lambda_0,\epsilon}$. We see from $\mathfrak{S} \setminus \{(0, \lambda_0)\} \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon_1) \subseteq \mathfrak{C}_y$ and the definition of $C^\kappa_{\lambda_0,\epsilon}$ that

$$C^\kappa_{\lambda_0,\epsilon} \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y = C^\kappa_{\lambda_0,\epsilon} \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y} \setminus \mathfrak{X}_{\lambda_0}(\epsilon)$$

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and so V is a closed subset of a closed set in $\mathfrak{X}$. Also, V is open in $C^{\kappa}_{\lambda_0,\epsilon}$ since V is open in $C^{\kappa}_{\lambda_0,\epsilon} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y$, an open subset of $C^{\kappa}_{\lambda_0,\epsilon}$. From the connectedness of $C^{\kappa}_{\lambda_0,\epsilon}$, either $V = \emptyset$ or $V = C^{\kappa}_{\lambda_0,\epsilon}$. Since V is disjoint from $\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y}$, it follows from $C^{\kappa}_{\lambda_0,\epsilon_1} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y} \neq \emptyset$ that $V = \emptyset$ and so Y_ϵ is connected. □

Now we can prove Proposition 3.2 and so conclude that Theorem 3.1 holds.

Proof of Proposition 3.2. As shown earlier, the proposition reduces to the case $\Omega = \mathfrak{X}$, and so we assume $\Omega = \mathfrak{X}$. Suppose that $C^{\kappa}_{\lambda_0}$ is bounded. Suppose for a contradiction that T^{κ}_{ϵ} is disjoint from $\partial\mathfrak{X}_{\lambda_0}(\epsilon)$ for some $\epsilon \in (0, S)$. By Lemma 2.3 we can express $C^{\kappa}_{\lambda_0} \cap \overline{\mathfrak{X}}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y}$ as the disjoint union of compact sets M_1 and M_2 with $T^{\kappa}_{\epsilon} \subseteq M_1$ and $C^{\kappa}_{\lambda_0} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y} \subseteq M_2$. We see that $M_1 \subseteq \mathfrak{X}_{\lambda_0}(\epsilon)$ and so, by compactness of M_1, there exists an open neighbourhood of M_1 contained in $\mathfrak{X}_{\lambda_0}(\epsilon')$ for some $\epsilon' \in (0, \epsilon)$, with boundary disjoint from $C^{\kappa}_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon) \cap \overline{\mathfrak{C}^{-\kappa}_y}$. To obtain a contradiction, we show that whenever $0 < \epsilon_0 < \epsilon_1 < S$, the boundary of every open neighbourhood of $(0, \lambda_0)$ contained in $\mathfrak{X}_{\lambda_0}(\epsilon_0)$ intersects $C^{\kappa}_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y}$ nontrivially.

Fix $0 < \epsilon_0 < \epsilon_1 < S$ and let $U \subseteq \mathfrak{X}_{\lambda_0}(\epsilon_0)$ be an open neighbourhood of $(0, \lambda_0)$. Since $C^{\kappa}_{\lambda_0,\epsilon} \subseteq C^{\kappa}_{\lambda_0}$ for $0 < \epsilon < S$, Proposition 3.3 yields

$$\emptyset \neq C^{\kappa}_{\lambda_0,\epsilon} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon) \cap \mathfrak{C}^{-\kappa}_y \subseteq C^{\kappa}_{\lambda_0} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon) \cap \mathfrak{C}^{-\kappa}_y.$$

Using this and $C^{\kappa}_{\lambda_0} \cap \partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \subseteq \mathfrak{X} \setminus U$, we see that

$$Y := (C^{\kappa}_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \mathfrak{C}^{-\kappa}_y) \cup (\partial\mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y})$$

intersects $\mathfrak{X} \setminus U$ nontrivially and $(0, \lambda_0) \in \overline{Y} \cap U$. From Lemma 3.4, Y and so also $\overline{Y}$ are connected. Thus $\partial U \cap \overline{Y} \neq \emptyset$ and so, since $U \subseteq \mathfrak{X}_{\lambda_0}(\epsilon_0)$, we conclude that ∂U intersects $C^{\kappa}_{\lambda_0} \cap \mathfrak{X}_{\lambda_0}(\epsilon_1) \cap \overline{\mathfrak{C}^{-\kappa}_y}$ nontrivially, as required. □

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References

[1] E. N. Dancer, 'On the structure of solutions of non-linear eigenvalue problems', *Indiana Univ. Math. J.* **23** (1973), 1069–1076.

[2] K. Deimling, *Nonlinear Functional Analysis* (Springer, Berlin, 1985).

[3] J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1978), reprint of the 1966 original, Allyn and Bacon Series in Advanced Mathematics.

[4] P. H. Rabinowitz, 'Some global results for nonlinear eigenvalue problems', *J. Funct. Anal.* **7** (1971), 487–513.

[5] R. E. L. Turner, 'Transversality in nonlinear eigenvalue problems', in: *Contributions of Nonlinear Functional Analysis*, Proceedings of a Symposium by the Mathematics Research Center (ed. E. H. Zarantonello) (Academic Press, New York, 1971), 37–68.

[6] G. T. Whyburn, *Topological Analysis*, second, revised edn, Princeton Mathematical Series, 23 (Princeton University Press, Princeton, NJ, 1964).

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