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Spherical subgroups in simple algebraic groups

Friedrich Knop and Gerhard Röhrle

In memory of Tonny Springer

Abstract

Let G be a simple algebraic group. A closed subgroup H of G is said to be spherical if it has a dense orbit on the flag variety G/B of G. Reductive spherical subgroups of simple Lie groups were classified by Krämer in 1979. In 1997, Brundan showed that each example from Krämer's list also gives rise to a spherical subgroup in the corresponding simple algebraic group in any positive characteristic. Nevertheless, up to now there has been no classification of all such instances in positive characteristic. The goal of this paper is to complete this classification. It turns out that there is only one additional instance (up to isogeny) in characteristic 2 which has no counterpart in Krämer's classification. As one of our key tools, we prove a general deformation result for subgroup schemes that allows us to deduce the sphericality of subgroups in positive characteristic from the same property for subgroups in characteristic zero.

1. Introduction

Let G be a simple algebraic group defined over an algebraically closed field k of characteristic $p \geqslant 0$. A closed subgroup H of G is said to be *spherical* if it has a dense orbit on the flag variety G/B of G. Alternatively, B acts on G/H with an open dense orbit. Accordingly, a G-variety with this property is also referred to as *spherical*.

The purpose of this paper is to classify connected reductive spherical subgroups of simple groups in arbitrary characteristic, thereby generalizing Krämer's classification [Krä79] in characteristic zero.

The class of reductive spherical subgroups is of particular importance. This is evident from the fact that Krämer's list permeates much of the theory of spherical varieties in characteristic zero. In particular, this kind of subgroup provides many of the building blocks for arbitrary spherical subgroups (see, e.g., [BP11]). We expect reductive spherical subgroups to play a similar role for arbitrary p. In fact, the results of the present paper were already used in [Kno14] to list all spherical subgroups of rank 1, which is crucial for the theory of general spherical varieties.

For $p \neq 2$, the class of reductive spherical subgroups includes all symmetric subgroups, i.e. subgroups which are fixed points of an involutory automorphism of G (see, e.g, [Spr85]). On the other hand, for $p = 2$ symmetric subgroups are not well behaved at all. Thus, reductive spherical subgroups seem to be the correct replacement.

Note that the requirement of having an open orbit in G/B entails that H has only finitely many orbits (see, e.g., [Kno95]). Therefore, our classification theorem can also be viewed as a

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contribution to the program of Seitz [Sei98] to classify all pairs of subgroups (X, Y) of a reductive group G such that there are only finitely many (X, Y)-double cosets in G (see also [Bru98] and [Duc04]).

The most important previous work is the aforementioned paper [Krä79] by Krämer. Not only do we use Krämer's list as a guideline but, more importantly, it enters our computations crucially even in positive characteristic. This is because we employ extensively the technique of reduction mod p that was first used by Brundan in this context [Bru98] in that he showed that all items from Krämer's list descend to arbitrary positive characteristic. To this end, Brundan proved, [Bru98, Theorem 4.3], that if H and G are defined over $\mathbb{Z}$ and an additional technical condition holds, then H is spherical for any $p > 0$ if and only if H is spherical for $p = 0$. In Theorem 3.4 of the present paper, we remove the technical condition, making the reduction mod p technique much more flexible to use. In particular, we barely ever need to check sphericality of a given subgroup; instead, we just have to look it up in Krämer's list.

Note that for the purpose of classifying spherical subgroups, we may replace G with an isogenous group (using Lemma 2.7). Therefore, the simply connected Spin groups do not make an appearance in Table 1, for instance, but rather their isogenous counterparts do.

We now describe our results in detail. The only surprise is that, up to isogeny, there is only one case, namely in characteristic 2, which is genuinely unique to positive characteristic, i.e. which has no counterpart in Krämer's classification.

Theorem. *Let G be a simple algebraic group and let $H \subset G$ be a closed connected reductive subgroup of G. Then H is spherical in G if and only if H is one of the groups in Table 1 (G classical) or Table 2 (G exceptional).*

Our classification is actually a bit more comprehensive, since we classify the connected reductive spherical subgroups of all classical groups up to not only outer but even inner automorphisms of G. Here, by a *classical group* we mean one of the groups $\mathrm{SL}(n)$ $(n \geqslant 2)$, $\mathrm{SO}(n)$ $(n \geqslant 1)$ and $\mathrm{Sp}(n)$ $(n \geqslant 2$ even), which includes also the non-simple groups $\mathrm{SO}(2)$ and $\mathrm{SO}(4)$. In positive characteristic, the latter group contains infinitely many 'new' spherical subgroups, namely the images of Δ_q where $\Delta_q : \mathrm{SL}(2) \to \mathrm{GL}(4)$ denotes the irreducible representation of $\mathrm{SL}(2)$ of highest weight $(q+1)\omega_1$, with $q = p^m > 1$. Since Δ_q is self-dual, its image lies in $\mathrm{SO}(4)$. We note that the images of Δ_q are special cases of finite orbit modules involving Frobenius twists; cf. [GLMS97, Lemma 2.6].

Note that the left columns of Tables 1 and 2 just reproduce Krämer's results. The cases in the right columns are new in the positive-characteristic setting. They are arranged in such a way that each case on the right can be obtained from the corresponding one on the left by a non-central isogeny of G. Thus, the only new case which has no counterpart in Krämer's table is

$$H = \mathrm{G}_2 \times \mathrm{Sp}(2) \subset \mathrm{Sp}(6) \times \mathrm{Sp}(2) \subset G = \mathrm{Sp}(8)$$

for $p = 2$. Of course, there is also $\mathrm{G}_2 \times \mathrm{SO}(3) \subset \mathrm{SO}(9)$, which is isogenous to this case.

In Table 2, $\widetilde{\mathrm{A}}_1$ and $\widetilde{\mathrm{A}}_2$ refer to subgroups of G of types A_1 and A_2, respectively, whose root systems consist only of short roots.

2. Preliminaries

2.1 Notation

Throughout, G is a simple algebraic group and B denotes a Borel subgroup of G. We denote the rank of G by $\mathrm{rk}\, G$. Let H be a closed subgroup of G. Then $R_u(H)$ denotes the unipotent radical

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Table 1. Spherical pairs $H \subset G$ with G classical.

Cases for all $p \geqslant 0$		Additional cases for $p > 0$		
H	G	H	G	
$\mathrm{SO}(n)^{(1)}$	$\mathrm{SL}(n)$ $(n \geqslant 2)$			
$S(\mathrm{GL}(m)\times \mathrm{GL}(n))$	$\mathrm{SL}(m+n)$ $(m \geqslant n \geqslant 1)$			
$\mathrm{SL}(m) \times \mathrm{SL}(n)$	$\mathrm{SL}(m+n)$ $(m > n \geqslant 1)$			
$\mathrm{Sp}(2n)$	$\mathrm{SL}(2n)$ $(n \geqslant 2)$			
$\mathbb{G}_m \cdot \mathrm{Sp}(2n)$	$\mathrm{SL}(2n+1)$ $(n \geqslant 1)$			
$\mathrm{Sp}(2n)$	$\mathrm{SL}(2n+1)$ $(n \geqslant 1)$			
$\mathrm{GL}(n)$	$\mathrm{Sp}(2n)$ $(n \geqslant 1)$			
$\mathbb{G}_m \times \mathrm{Sp}(2n-2)$	$\mathrm{Sp}(2n)$ $(n \geqslant 2)$			
$\mathrm{Sp}(m)\times \mathrm{Sp}(n)$	$\mathrm{Sp}(m+n)$	$\mathrm{SO}(m)\times \mathrm{SO}(n)$	$\mathrm{SO}(m+n-1)$	$p = 2$
$m, n \geqslant 2$ even		$m, n \geqslant 3$ odd		
		$\mathrm{G}_2 \times \mathrm{SO}(3)$	$\mathrm{SO}(9)$	$p = 2$
$\mathrm{GL}(n)^{(2)}$	$\mathrm{SO}(2n)$ $(n \geqslant 2)$			
$\mathrm{SL}(n)$	$\mathrm{SO}(2n)$ $(n \geqslant 3$ odd$)$			
$\mathrm{Sp}(4) \otimes \mathrm{Sp}(2)^{(3)(4)}$	$\mathrm{SO}(8)$			
$\mathrm{Spin}(7)^{(3)(5)}$	$\mathrm{SO}(8)$			
G_2	$\mathrm{SO}(8)$			
$\mathrm{SO}(2) \times \mathrm{Spin}(7)$	$\mathrm{SO}(10)$			
$\mathrm{GL}(n)$	$\mathrm{SO}(2n+1)$ $(n \geqslant 2)$			
$\mathrm{SO}(m)\times \mathrm{SO}(n)$	$\mathrm{SO}(m+n)$	$\mathrm{SO}(2m)\times \mathrm{Sp}(2n)$	$\mathrm{Sp}(2m+2n)$	$p = 2$
$m \geqslant n \geqslant 1$		$m \geqslant 1, n \geqslant 0$		
$\mathrm{Spin}(7)$	$\mathrm{SO}(9)$	$\mathrm{Spin}(7)$	$\mathrm{Sp}(8)$	$p = 2$
G_2	$\mathrm{SO}(7)$	G_2	$\mathrm{Sp}(6)$	$p = 2$
		$\mathrm{G}_2 \times \mathrm{Sp}(2)$	$\mathrm{Sp}(8)$	$p = 2$
		$\Delta_q \mathrm{SL}(2)^{(3)}$	$\mathrm{SO}(4)$	$q > 1$

(1) For $p = 2$ and $n \geqslant 3$ odd, there are two classes which are swapped by an outer automorphism of G.

(2) For n even, there are two classes which are swapped by an outer automorphism of G.

(3) There are two conjugacy classes of H in G which are swapped by an outer automorphism of G.

(4) Using triality, $H \subset G$ is equivalent to $\mathrm{SO}(5) \times \mathrm{SO}(3) \subset \mathrm{SO}(8)$.

(5) Using triality, $H \subset G$ is equivalent to $\mathrm{SO}(7) \subset \mathrm{SO}(8)$.

of H. If G acts on the variety X, we denote the H-orbit of x in X by $H \cdot x$ and the stabilizer in H by $C_H(x)$.

In what follows we label the Dynkin diagram of a simple group G according to the tables in Bourbaki [Bou68], and ω_i denotes the ith fundamental dominant weight of G with respect to this labeling.

For a dominant weight χ of G, we denote by $L(\chi)$ the irreducible G-module of highest weight χ and by $H^0(\chi)$ the G-module of global sections of the G-line bundle $\mathcal{L}(k_\chi)$ on G/B afforded by the weight χ, so that $L(\chi) = \mathrm{soc}_G H^0(\chi)$. Note that $H^0(\chi)$ has the same character as the Weyl module of highest weight χ; for details, see [Jan03, II.2].

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Table 2. Spherical pairs $H \subset G$ with G exceptional.

Cases for all $p \geqslant 0$		Additional cases for $p > 0$		
H	G	H	G	
A_2	G_2	$\widetilde{\mathrm{A}}_2$	G_2	$p = 3$
$\mathrm{A}_1 \times \widetilde{\mathrm{A}}_1$	G_2			
B_4	F_4	C_4	F_4	$p = 2$
$\mathrm{C}_3 \times \mathrm{A}_1$	F_4	$\mathrm{B}_3 \times \widetilde{\mathrm{A}}_1$	F_4	$p = 2$
C_4	E_6			
F_4	E_6			
D_5	E_6			
$\mathbb{G}_m \cdot \mathrm{D}_5$	E_6			
$\mathrm{A}_5 \times \mathrm{A}_1$	E_6			
$\mathbb{G}_m \cdot \mathrm{E}_6$	E_7			
A_7	E_7			
$\mathrm{D}_6 \times \mathrm{A}_1$	E_7			
D_8	E_8			
$\mathrm{E}_7 \times \mathrm{A}_1$	E_8			

By a *classical group* we mean one of the groups $\mathrm{SL}(n)$ ($n \geqslant 2$), $\mathrm{SO}(n)$ ($n \geqslant 1$) and $\mathrm{Sp}(n)$ ($n \geqslant 2$ even). Here, $\mathrm{SO}(n)$ is the reduced, connected identity component of $\mathrm{O}(n)$, i.e. the kernel of the determinant character det, unless $p = 2$ and n is even, in which case det has to be replaced by the Dickson invariant.

2.2 Basic results for spherical subgroups

While elementary, one of our main tools for identifying spherical subgroups (apart from Theorem 3.4 below) is the following necessary condition.

Lemma 2.1. *Let $H \subseteq G$ be spherical in G. Then*

$$\dim H \geqslant \dim G/B = \tfrac{1}{2}(\dim G - \operatorname{rk} G). \tag{2.2}$$

Proof. By definition, B has an open orbit in G/H. Hence $\dim B \geqslant \dim G/H$, which is equivalent to (2.2). □

Below we use the following 'transitivity' property for spherical subgroups without further comment.

Lemma 2.3. *Let $H_1 \subseteq H_2 \subseteq G$ be connected reductive subgroups of G. If H_1 is spherical in G, then H_1 is spherical in H_2 and H_2 is spherical in G.*

Proof. Suppose that H_1 is spherical in G. Then H_1 acts on G/B with a dense orbit, and so does H_2; thus H_2 is spherical in G.

Let $B_2 \subset H_2$ be a Borel subgroup of H_2. Then there is a Borel subgroup B of G such that $B_2 = H_2 \cap B$; see, e.g., [BMR05, Corollary 2.5]. Consider the canonical embedding $H_2/B_2 \to G/B$. Thanks to the finiteness result for irreducible, spherical G-varieties in arbitrary characteristic (see [Kno95, Corollary 2.6]), since H_1 is spherical in G, H_1 admits only a finite number of orbits

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in G/B. Thus there is only a finite number of H_1-orbits in H_2/B_2 and, in particular, there is a dense one. Consequently, H_1 is spherical in H_2. □

The following compatibility property of sphericality for direct products is immediate from the definition of a spherical subgroup and is used below without further reference.

Lemma 2.4. *Let $H_i \subseteq G_i$ be a reductive subgroup of G_i for $i = 1, 2$. Then $H_1 \times H_2$ is spherical in $G_1 \times G_2$ if and only if H_i is spherical in G_i for both $i = 1, 2$.*

Sometimes the following stronger bound on $\dim H$ is needed in place of the inequality (2.2).

Lemma 2.5. *Let $H \subseteq G$ be spherical and let $B \cdot x_0 \in G/H$ be the open B-orbit in G/H. Then*

$$\dim H = \dim G/B + \dim C_B(x_0). \tag{2.6}$$

Proof. This follows because $\dim B - \dim C_B(x_0) = \dim B \cdot x_0 = \dim G/H = \dim G - \dim H$. □

We also frequently use the following observation.

Lemma 2.7. *Let G_1 and G_2 be connected reductive groups and $\varphi : G_1 \to G_2$ an isogeny. Then φ induces a bijection between the sets of (conjugacy classes of) connected (reductive) spherical subgroups of G_1 and G_2.*

Lemma 2.7 has several immediate consequences.

Remark 2.8. (i) The triality automorphism of Spin(8) acts on the conjugacy classes of connected reductive spherical subgroups of SO(8) as well. This action is indicated in Table 1.

(ii) In characteristic $p = 2$, there is a bijective non-central isogeny $\mathrm{SO}(2n+1) \to \mathrm{Sp}(2n)$. Thus, if G is a classical group, we can (and will) safely assume that G is *strictly classical* in the sense that G is not isomorphic to $\mathrm{SO}(2n+1)$, where $n \geqslant 1$ when $p = 2$. Equivalently, a classical group is strictly classical if its natural representation is completely reducible.

3. Deformation of spherical subgroups

In this section, we prove that 'sphericality' is invariant under deformations. This enables us to compare spherical subgroups in positive characteristic to those in characteristic zero. This approach reduces most of the classification work to Krämer's paper [Krä79].

For simplicity, we restrict ourselves to base schemes S which are of the form $\operatorname{Spec} A$, where A is a Dedekind domain,[1] i.e. an integrally closed Noetherian domain of dimension 1.

In what follows, let $\mathcal{G} \to S$ be a split reductive group scheme (this entails connected geometric fibers); see, e.g., [SGA3, Exp. I, 4.2]. Let $\mathcal{T}$ be a split maximal torus of $\mathcal{G}$. Using [SGA3, Exp. XXII, Corollaire 5.5.5(iii)], a Borel subgroup scheme $\mathcal{B}$ of $\mathcal{G}$ containing $\mathcal{T}$ has the form $\mathcal{B}_{R^+}$, where R^+ is a system of positive roots for $\mathcal{G}$. Then, thanks to [SGA3, Exp. XXII, Lemme 5.5.6(iii)], $\mathcal{B}$ is the semi-direct product $\mathcal{B} = \mathcal{T} \cdot \mathcal{U}$ for a smooth subgroup scheme $\mathcal{U}$, and $\mathcal{U}_k$ is a maximal connected unipotent subgroup in $\mathcal{G}_k$ for any A-algebra k which is a field. Let Ξ be the character group of $\mathcal{B}$. For an affine S-scheme $\mathcal{X} \to S$, let $\mathcal{O}(\mathcal{X})$ be its ring of regular functions.

We need the following extension property for invariants due to Seshadri [Ses77]. See also [FvdK10] for a simplified approach.

[1] Our main assertion is surely valid in greater generality, but due to technical difficulties stemming from the construction of coset schemes in [Ana73] and closures of subgroup schemes in [BT84], we shall stick to Dedekind rings.

LEMMA 3.1. *Let $\mathcal{X} \to S$ be an affine $\mathcal{G}$-scheme and $\mathcal{Y} \subseteq \mathcal{X}$ a closed $\mathcal{G}$-invariant subscheme of $\mathcal{X}$. Then, for any $\mathcal{G}$-invariant function $f \in \mathcal{O}(\mathcal{Y})^{\mathcal{G}}$, there is an $n \geqslant 1$ such that f^n extends to a $\mathcal{G}$-invariant function $\overline{f}$ on $\mathcal{X}$.*

Next, we prove that the extension property from Lemma 3.1 also holds for $\mathcal{B}$-semi-invariants.

LEMMA 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Lemma 3.1. Let $f \in \mathcal{O}(\mathcal{Y})$ be a $\mathcal{B}$-semi-invariant function for a character $\chi \in \Xi$. Then there is an exponent $n \geqslant 1$ such that f^n extends to a $\mathcal{B}$-semi-invariant function $\overline{f} \in \mathcal{O}(\mathcal{X})$ for the character $n\chi$.*

Proof. Let $\mathcal{G}/\!\!/\mathcal{U}$ be the basic affine space of $\mathcal{G}$; it is the spectrum of $\bigoplus_{\chi\in\Xi} H^0(\chi)$. (Note that $H^0(\chi)$ is a free A-module, thanks to [Jan03, II 8.8].) Thus $\mathcal{G}/\!\!/\mathcal{U}$ is an affine scheme over S which contains $\mathcal{G}/\mathcal{U}$ as dense open subset. In particular, $\mathcal{G}/\!\!/\mathcal{U}$ contains an S-point e. The Ξ-grading of $\mathcal{O}(\mathcal{G}/\!\!/\mathcal{U})$ induces an action of $\mathcal{T} = \mathcal{B}/\mathcal{U}$ which commutes with the $\mathcal{G}$-action.

Consider the closed embedding $\mathcal{X} \xrightarrow{\mathrm{id}\times e} \mathcal{X} \times_S \mathcal{G}/\!\!/\mathcal{U}$. It is well known [FvdK10, proof of Lemma 24] that restriction to $\mathcal{X}$ induces a $\mathcal{T}$-equivariant isomorphism

$$\mathcal{O}(\mathcal{X} \times_S \mathcal{G}/\!\!/\mathcal{U})^{\mathcal{G}} \xrightarrow{\sim} \mathcal{O}(\mathcal{X})^{\mathcal{U}}.$$

Thus our assertion follows from Lemma 3.1 applied to $\mathcal{Y} \times_S \mathcal{G}/\!\!/\mathcal{U} \subseteq \mathcal{X} \times_S \mathcal{G}/\!\!/\mathcal{U}$ and the fact that $\mathcal{T}$ is linearly reductive. □

Remark 3.3. If $\mathcal{Y}$ is actually defined over a prime field, say $\mathbb{Q}$ or $\mathbb{F}_p$, then the exponent n in Lemmas 3.1 and 3.2 can be chosen to be $n = 1$ and $n \in p^{\mathbb{N}}$, respectively.

Now we are in a position to prove our main deformation statement.

THEOREM 3.4. *Let $\mathcal{H} \subseteq \mathcal{G}$ be a subgroup scheme which is flat over S. Assume that for some geometric point x of S, the geometric fiber $\mathcal{H}_x$ is a spherical subgroup of $\mathcal{G}_x$. Then all geometric fibers of $\mathcal{H}$ are spherical.*

Proof. Since S is the spectrum of a Dedekind ring, the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ in $\mathcal{G}$ is also a flat closed subgroup scheme; cf. [BT84, 1.2.6, 1.2.7, 2.1.6, 2.2.2]. Moreover, $\mathcal{H}_x$ is spherical in $\mathcal{G}_x$ if and only if $\overline{\mathcal{H}}_x$ is (since the former is open and hence of finite index in the latter). Thus, after replacing $\mathcal{H}$ by $\overline{\mathcal{H}}$, we may assume that $\mathcal{H}$ is closed in $\mathcal{G}$.

In that case, it is known that the homogeneous space $\mathcal{X}' := \mathcal{G}/\mathcal{H}$ exists as a scheme which is flat and of finite type over S (see [Ana73]). Moreover, by Sumihiro ([Sum75]; see also [Tho87]), this scheme is equivariantly quasiprojective over S. This means that there is a $\mathcal{G}$-vector bundle $\mathcal{V}$ over S and an equivariant embedding of $\mathcal{X}'$ in the projective space $\mathbb{P}_S(\mathcal{V})$. Let $\mathcal{X}'' \subseteq \mathbb{P}_S(\mathcal{V})$ be the closure of $\mathcal{X}'$. This is a scheme which is projective and flat over S. Moreover, each geometric fiber $\mathcal{X}'_x = \mathcal{G}_x/\mathcal{H}_x$ is an open subset of the fiber $\mathcal{X}''_x$.

Now let $\mathcal{X} \subseteq \mathrm{A}_S(\mathcal{V}) := \operatorname{Spec} S^\bullet\mathcal{V}$ be the affine cone of $\mathcal{X}''$. The affine scheme $\mathcal{X}$ affords an action of $\widetilde{\mathcal{G}} := \mathcal{G} \times_S (\mathbb{G}_m)_S$. Moreover, an irreducible subvariety of $\mathcal{X}''_x$ is spherical as a $\mathcal{G}_x$-variety if and only if its affine cone in $\mathcal{X}_x$ is a spherical $\widetilde{\mathcal{G}}_x$-variety. Thus, by replacing $\mathcal{G}$ with $\widetilde{\mathcal{G}}$ we may assume that $\mathcal{X}' = \mathcal{G}/\mathcal{H}$ is an open dense subscheme of an affine scheme $\mathcal{X}$.

Suppose that $\mathcal{X}_x$ has a spherical irreducible component. Let $y \in S$ be a second geometric point. We have to prove that every component of $\mathcal{X}_y$ is spherical as well. Let η be the generic geometric point of S. We will show that $\mathcal{X}_\eta$ and, subsequently $\mathcal{X}_y$, is spherical. This amounts to assuming that either $y = \eta$ or $x = \eta$.

Assume first that $y = \eta$. Let $Y \subseteq \mathcal{X}_x$ be a spherical irreducible component. This means that somewhere on Y the dimension of the isotropy subgroup of $\mathcal{B}$ is as small as possible, namely

$\dim \mathcal{B}_x - \dim \mathcal{X}_x = \dim \mathcal{B} - \dim \mathcal{X}$. By semi-continuity, this holds on a non-empty open subset $\mathcal{X}^0$ of $\mathcal{X}$. Because then $\mathcal{X}^0 \cap \mathcal{X}_\eta \neq \emptyset$, we conclude that $\mathcal{X}_\eta$ is spherical (for this, observe that $\mathcal{X}_\eta$ is irreducible since it contains an open dense $\mathcal{G}_\eta$-orbit).

Finally, let $x = \eta$ and suppose that some component Y of $\mathcal{X}_y$ is not spherical. Then, by [Ros56], Y admits a non-constant rational $\mathcal{B}_y$-invariant function f. Because Y is affine, this function can be written as $f = f_1/f_2$, where $f_1, f_2 \in \mathcal{O}(Y)$ are $\mathcal{B}_y$-semi-invariants for the same character $\chi \in \Xi$. By Lemma 3.2, there is an $n \in \mathbb{N}$ such that f_1^n and f_2^n extend to $\mathcal{B}$-semi-invariants $\overline{f}_1$ and $\overline{f}_2$ for the same character $n\chi$ on all of $\mathcal{X}$. Now, since $\mathcal{X}$ is integral, we obtain a $\mathcal{B}$-invariant rational function $\overline{f} = \overline{f}_1/\overline{f}_2$ on $\mathcal{X}$ which is not a constant, i.e. a pull-back of a function on S. Thus, in particular, the generic fiber $\mathcal{X}_x$ is not spherical, contrary to our assumption. □

Upon applying Theorem 3.4 to $S = \operatorname{Spec} \mathbb{Z}$, we get the following result, which was previously obtained by Brundan [Bru98, Theorem 4.3] using a representation-theoretic approach and which is partially based on case-by-case considerations.

Corollary 3.5. *Let $H_\mathbb{R} \subseteq G_\mathbb{R}$ be a pair of compact Lie groups in Krämer's list, i.e. with $H_\mathbb{R}$ spherical in $G_\mathbb{R}$. Then the complexification $H_\mathbb{C} \subseteq G_\mathbb{C}$ has a $\mathbb{Z}$-form $H_\mathbb{Z} \subseteq G_\mathbb{Z}$. Moreover, for any field k, the induced pair $H_k \subseteq G_k$ is spherical.*

Proof. The first statement follows by inspection of Krämer's list. The second follows from the first together with Theorem 3.4 for $S = \operatorname{Spec} \mathbb{Z}$. □

In the reverse direction, we recover a classification of Duckworth [Duc04, Theorem 2], which can be formulated as follows.

Corollary 3.6. *Assume that $p \neq 2$ if G is of type B_n, C_n or F_4 and that $p \neq 3$ if G is of type G_2. Then the classification of pairs (G, H), where G is a simple group and H is a spherical subgroup of G with $\operatorname{rk} H = \operatorname{rk} G$, is independent of p.*

Proof. Under the given restrictions, H corresponds to an additively closed subroot system. Therefore it lifts to characteristic zero. Then apply Theorem 3.4 for $S = \operatorname{Spec} \mathbb{Z}$. □

4. Special cases of spherical subgroups

For an arbitrary G-variety X, let $\Xi(X)$ be the group of characters of B-semi-invariant rational functions on X. We define the rank of X to be the $\mathbb{Z}$-rank of $\Xi(X)$. Let S_0 be the set of simple roots α such that the coroot $\alpha^\vee$ is orthogonal to $\Xi(X)$. Then, attached to S_0 is a parabolic subgroup $P = P(X)$ of G such that $\Xi(X) \subseteq \Xi(P)$, where $\Xi(P)$ is the character group of P. We define the subgroup P_0 of P by $P_0 = \{y \in P \mid \chi(y) = 1 \text{ for all } \chi \in \Xi(X)\}$.

Theorem 4.1. *Let X be a quasiaffine G-variety. Let $P = P(X)$ as above. Then there is a P-invariant dense open subset X_0 of X such that $C_P(x)R_u(P) = P_0$ and $C_P(x) \cap R_u(P)$ is finite for all $x \in X_0$. In particular, $C_P(x)$ is a reductive group which is isogenous to a Levi subgroup of P_0.*

Proof. According to [Kno93, Satz 2.10], there is a parabolic subgroup P of G and a P-stable dense open subset $X_0 \subseteq X$ such that:

(i) the action of $R_u(P)$ on X_0 is proper;
(ii) the orbit space $Y := X_0/R_u(P)$ exists;
(iii) if P_1 is the kernel of the P-action on Y, then P/P_1 is a torus;
(vi) the action of P/P_1 on Y is free.

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Let $\pi : X_0 \to Y$ be the quotient map and let $x \in X_0$. Then $C_P(x) \cap R_u(P)$ is finite by (i). We have $C_P(x)R_u(P) \subseteq C_P(y)$ with $y = \pi(x)$. Moreover, for $z \in C_P(y)$ there is $u \in R_u(P)$ with $zx = ux$. Thus $C_P(x)R_u(P) = C_P(y)$. Finally, $C_P(y) = P_0$ by (iv).

It remains to show that $P = P(X)$ and $P_1 = P_0$, as defined above. For this we use the fact that X_0 is constructed as the non-vanishing set of a B-semi-invariant section σ of a sufficiently high power $\mathcal{L}^n$ of any ample line bundle on X. Since X is quasiaffine, we can take $\mathcal{L} = \mathcal{O}_X$. Moreover, P is the stabilizer of the line $k\sigma$. Since σ is a regular function on X, the G-module $M := \langle G\sigma\rangle_k$ generated by σ is finite-dimensional, and σ is a highest weight vector in M with weight denoted by χ. This implies that P is the parabolic subgroup attached to the set S_1 of simple roots α with $\langle \chi, \alpha^\vee\rangle = 0$. From the construction it is easy to see that χ can be chosen such that $S_1 = S_0$. This shows that indeed $P = P(X)$. Finally, observe that $\Xi(X) = \Xi(Y)$. Thus properties (iii) and (iv) ensure that P_1 is the common kernel of all $\chi \in \Xi(X)$, i.e. $P_1 = P_0$. □

Lemma 4.2. *Let $X = \mathrm{SO}(n)/\mathrm{SO}(n-m)$ or $X = \mathrm{Sp}(2n)/\mathrm{Sp}(2n-2m)$ with $2m \leqslant n$. Then $\Xi(X) \subseteq \langle \omega_1, \ldots, \omega_{2m}\rangle_{\mathbb{Z}}$.*

Proof. For $X = \mathrm{SO}(n)/\mathrm{SO}(n-m)$ or $X = \mathrm{Sp}(2n)/\mathrm{Sp}(2n-2m)$ with $2m \leqslant n$, let $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-m)$ or $G = \mathrm{Sp}(2n)$ and $H = \mathrm{Sp}(2n-2m)$, respectively. Write $X = G/H$. First, observe that X lifts to characteristic zero, thanks to Corollary 3.5. Since the character group $\Xi(X)$ is the same in characteristic zero and in positive characteristic p (after inversion of p), we may assume from the outset that $\mathrm{char}\, k = 0$.

Since X is affine, every rational B-semi-invariant is the ratio of two regular ones. Moreover, a regular B-semi-invariant with character χ corresponds to a non-zero H-fixed vector in the dual irreducible H-module $L(\chi)^*$. Now it follows readily from classical branching laws (see, e.g., [GW09, ch. 8]) that χ is a linear combination of the first $2m$ fundamental weights. □

Lemma 4.3. (i) *Let $H \subset \mathrm{SO}(m)$ be a proper, reductive subgroup of $\mathrm{SO}(m)$ such that the group $H \times \mathrm{SO}(n-m)$ is spherical in $\mathrm{SO}(n)$. Then $2m > n$.*

(ii) *Let $H \subseteq \mathrm{Sp}(2m)$ be a reductive subgroup such that $H \times \mathrm{Sp}(2n-2m)$ is spherical in $\mathrm{Sp}(2n)$. Assume that $2m \leqslant n$. Then $\dim H \geqslant \dim \mathrm{SO}(2m) = m(2m-1)$.*

Proof. (i) Suppose that $\widetilde{H} := H \times \mathrm{SO}(n-m)$ is spherical in $\mathrm{SO}(n)$ and $2m \leqslant n$. Let $x_0 \in G/\widetilde{H}$ be in the open B-orbit in $G/\widetilde{H}$. Then, by (2.6), we have

$$\dim \widetilde{H} = \dim G/B + \dim C_B(x_0).$$

By Theorem 4.1 and Lemma 4.2, the generic isotropy group of B on $\mathrm{SO}(n)/\mathrm{SO}(n-m)$ contains a subgroup which is isogenous to a Borel subgroup, say B_2, of $\mathrm{SO}(n-2m)$. Thus Lemma 2.5 implies $\dim C_B(x_0) \geqslant \dim B_2$. To keep the dependence on the parity of n to a minimum, observe that $\dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$ and $\mathrm{rk}\,\mathrm{SO}(n) - \mathrm{rk}\,\mathrm{SO}(n-2m) = m$ for all n and m. Hence we arrive at the following contradiction:

$$\begin{aligned}\dim H &\geqslant \tfrac{1}{2}(\dim \mathrm{SO}(n) - \mathrm{rk}\,\mathrm{SO}(n)) + \tfrac{1}{2}(\dim \mathrm{SO}(n-2m) + \mathrm{rk}\,\mathrm{SO}(n-2m)) - \dim \mathrm{SO}(n-m)\\ &= \tfrac{1}{2}m(m-1) = \dim \mathrm{SO}(m).\end{aligned}$$

For (ii) we argue in the same way and get

$$\dim H \geqslant n^2 + (n-2m)(n-2m+1) - (n-m)(2n-2m+1) = m(2m-1).$$ □

Corollary 4.4. *Let $p = 2$ and $n \geqslant 5$. Then $H = \mathrm{Spin}(7) \times \mathrm{Sp}(2n-8) \subset \mathrm{Sp}(8) \times \mathrm{Sp}(2n-8)$ is not spherical in $G = \mathrm{Sp}(2n)$.*

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Proof. For $n = 5$, 6 and 7, the result follows from (2.2). Now let $n \geqslant 8$. Noting that $21 = \dim \mathrm{Spin}(7) < \dim \mathrm{SO}(8) = 28$, it follows from Lemma 4.3(ii) that H is not spherical. □

Proposition 4.5. *Let $p = 2$ and $n \geqslant 4$. Then $H := \mathrm{G}_2 \times \mathrm{Sp}(2n-6) \subset \mathrm{Sp}(6) \times \mathrm{Sp}(2n-6)$ is spherical in $G = \mathrm{Sp}(2n)$ if and only if $n = 4$. In that case, $\Xi(G/H) = \langle \omega_1 + \omega_4, \omega_2, \omega_3 \rangle_{\mathbb{Z}}$.*

Proof. For $n = 5$, the result follows from (2.2). Let $n \geqslant 6$. Since $\dim \mathrm{G}_2 = 14 < \dim \mathrm{SO}(6) = 15$, the assertion follows from Lemma 4.3(ii).

It remains to check that H is spherical if $n = 4$. Let $\widetilde{H} := \mathrm{Sp}(6) \times \mathrm{Sp}(2)$ and write $\widetilde{H}^0(\chi)$ for the corresponding $\widetilde{H}$-module and $\widetilde{L}(\chi)$ for the simple $\widetilde{H}$-module of highest weight χ.

We first show that $A := \{\omega_1 + \omega_4, \omega_2, \omega_3\} \subseteq \Xi(G/H)$, which is equivalent to the G-modules $H^0(\chi)$ with $\chi \in A$ having an H-fixed vector.

For $\chi = \omega_2$ this follows from the fact that even $\widetilde{H}$ has a fixed vector. Moreover, it is known that G_2 fixes a vector in the $\widetilde{H}$-module $\widetilde{H}^0(\omega_3)$, which in turn is contained in $H^0(\omega_3)$.

For $\chi = \omega_1 + \omega_4$, it suffices to show that the irreducible G-module $L(\chi) \subset H^0(\chi)$ contains the $\widetilde{H}$-module $\widetilde{H}^0(\omega_3)$. Using the known characters of Weyl modules and the dimensions of the irreducible modules in [Lüb01], one easily computes that, as an $\widetilde{H}$-module, $L(\chi)$ has the composition factors $\widetilde{L}(\omega_1 + \omega_2 + \omega_1')$, $\widetilde{L}(2\omega_3 + 2\omega_1')$ and $\widetilde{L}(\omega_3)$, the first two occurring with multiplicity one and the third with multiplicity two. Since $L(\chi)$ is self-dual, (at least) one of the two copies of $\widetilde{L}(\omega_3)$ has to appear in the socle. This concludes the proof that $H^0(\omega_1+\omega_4)^H \neq \{0\}$.

Since there is no simple coroot which is orthogonal to all the weights in A, we infer from Theorem 4.1 that the connected B-isotropy group of a generic point $x \in G/H$ is a torus of dimension at most 1. Thus $\dim B \cdot x \geqslant 19$, whereas $\dim G/H = 36 - 14 - 3 = 19$. This shows that G/H is spherical of rank 3. In particular, $\Xi(G/H)$ is spanned by A, as claimed. □

5. Irreducible spherical subgroups of classical groups

Let G be a classical group with natural representation V. A subgroup $H \subseteq G$ is said to be *irreducible* if V is irreducible as an H-module; otherwise, H is said to be *reducible*. Clearly, irreducible subgroups only exist if G itself is irreducible, i.e. strictly classical and not equal to $\mathrm{SO}(2)$. It is well known that connected irreducible subgroups are necessarily semi-simple.

In preparation for determining the non-simple irreducible spherical subgroups, we consider some very special cases.

Lemma 5.1. *Of the following pairs $H \subset G$,*

$$\begin{aligned} \mathrm{SL}(m) \otimes \mathrm{SL}(n) &\subset \mathrm{SL}(mn), && m \geqslant n \geqslant 2,\\ \mathrm{SO}(m) \otimes \mathrm{SO}(n) &\subset \mathrm{SO}(mn), && m \geqslant n \geqslant 2,\ m \text{ and } n \text{ even if } p = 2,\\ \mathrm{Sp}(m) \otimes \mathrm{Sp}(n) &\subseteq \mathrm{SO}(mn), && m \geqslant n \geqslant 2,\ m \text{ and } n \text{ even},\\ \mathrm{Sp}(m) \otimes \mathrm{SO}(n) &\subset \mathrm{Sp}(mn), && m, n \geqslant 2,\ m \text{ even},\ n \text{ even if } p = 2, \end{aligned}$$

only the following are spherical:

$$\begin{aligned} \mathrm{SL}(2) \otimes \mathrm{SL}(2) &\subset \mathrm{SL}(4),\\ \mathrm{SO}(2) \otimes \mathrm{SO}(2) &\subset \mathrm{SO}(4),\\ \mathrm{Sp}(2) \otimes \mathrm{Sp}(2) &= \mathrm{SO}(4),\\ \mathrm{Sp}(4) \otimes \mathrm{Sp}(2) &\subset \mathrm{SO}(8),\\ \mathrm{Sp}(2) \otimes \mathrm{SO}(2) &\subset \mathrm{Sp}(4). \end{aligned}$$

Proof. There are two possible proofs. First, observe that all subgroups lift to characteristic zero. Hence, the assertion follows (apart from the trivial case $G = \mathrm{SO}(4)$) from Corollary 3.5 and Krämer's classification [Krä79]. The second proof consists in directly using the inequality (2.2), which is easy and left to the reader. □

Next, we determine the irreducible, spherical subgroups which are not simple.

Lemma 5.2. *Let G be a classical group and $H \subset G$ a proper, connected, irreducible, spherical subgroup which is not simple. Then the pair $H \subset G$ is one of the following:*

$$\begin{aligned} \mathrm{SO}(4) &\subset \mathrm{SL}(4), \\ \mathrm{Sp}(4) \otimes \mathrm{Sp}(2) &\subset \mathrm{SO}(8), \\ \mathrm{SO}(4) &\subset \mathrm{Sp}(4) \quad (\textit{if } p = 2). \end{aligned}$$

Proof. By assumption, there are decompositions $H = H_1 \cdot H_2$ and $V = V_1 \otimes V_2$, where V_i is an irreducible H_i-module. For $G = \mathrm{SL}(n)$, Lemma 5.1 shows that $H_1, H_2 \subseteq \mathrm{SL}(2)$, which implies that $H_1 = H_2 = \mathrm{SL}(2)$, and hence $H = \mathrm{SO}(4)$.

Now let $G = \mathrm{SO}(V)$ or $G = \mathrm{Sp}(V)$, and assume first that $p \neq 2$. Since $V = V_1 \otimes V_2$ is self-dual, the same holds for the factors V_i. Thus, H_i is either symplectic or orthogonal. Since $H \neq G$, we have $G \neq \mathrm{SO}(4)$. Therefore the only case to consider, according to Lemma 5.1, is $H_1 \times H_2 \subseteq \widetilde{H} := \mathrm{Sp}(4) \times \mathrm{Sp}(2)$ and $G = \mathrm{SO}(8)$. But in that case $\dim G/B = 12$ while $\dim \widetilde{H} = 13$. This implies $H = \widetilde{H}$, since a semi-simple group does not contain a reductive subgroup of codimension 1.

Now assume that $p = 2$ and that V is self-dual. Then each factor V_i is still self-dual, and we claim that it is even symplectic. To show this, let $\beta : V_i \times V_i \to k$ be a non-zero H_i-invariant pairing. Schur's lemma implies that β is unique up to a scalar. It is symmetric, since otherwise $\beta'(u, v) = \beta(u, v) + \beta(v, u)$ is non-zero and symmetric. But then $\ell(v) := \sqrt{\beta(v, v)}$ is an H_i-invariant linear form. The irreducibility of V_i implies $\ell = 0$. Thus $\beta(v, v) \equiv 0$, proving the claim.

Consequently, we have

$$H \subseteq \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2) \subseteq \mathrm{SO}(V) \subset \mathrm{Sp}(V).$$

According to Lemma 5.1, we are left with the following cases. If $G = \mathrm{SO}(V)$, then $H = \mathrm{Sp}(4) \otimes \mathrm{Sp}(2)$ as before. If $G = \mathrm{Sp}(V)$, then H is spherical in $\mathrm{SO}(V)$ as well. Thus, either $H = \mathrm{SO}(4) \subset G = \mathrm{Sp}(4)$ (which is spherical) or $H = \mathrm{Sp}(4) \otimes \mathrm{Sp}(2) \subset G = \mathrm{Sp}(8)$ (which is not spherical by (2.2), because $\dim G/B = 16$ and $\dim H = 13$). □

To determine all simple irreducible spherical subgroups, we need the following estimate to bound the dimension of a simple H-module. The proof of the result follows easily by inspection.

Lemma 5.3. *Let H be a simple group with Weyl group W_H, and let ω be a fundamental dominant weight of H with*

$$|W_H \cdot \omega| \leqslant 2\sqrt{\dim H + 1/4} + 1.$$

Then the pair (H, ω) appears in Table (5.4).

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H	ω
A_1	ω_1
A_2	ω_1, ω_2
A_3	$\omega_1, \omega_2, \omega_3$
A_4	$\omega_1, \omega_2, \omega_3, \omega_4$
$\mathrm{A}_n, n \geqslant 5$	ω_1, ω_n
B_3	ω_1, ω_3
$\mathrm{B}_n, n \geqslant 4$	ω_1

H	ω
C_2	ω_1, ω_2
C_3	ω_1, ω_3
$\mathrm{C}_n, n \geqslant 4$	ω_1
D_4	$\omega_1, \omega_3, \omega_4$
$\mathrm{D}_n, n \geqslant 5$	ω_1
G_2	ω_1, ω_2

(5.4)

Now we determine the simple, irreducible, spherical subgroups H of a classical group G.

Lemma 5.5. *Let G be a classical group and $H \subset G$ a proper, connected, irreducible, spherical subgroup. Then, up to conjugacy in G, the pair (G, H) appears in Table (5.6).*

H	Weight	G		Conditions on p
$\mathrm{SO}(n)^{(1)}$	ω_1	$\mathrm{SL}(n)$	$n \geqslant 3$	$p \neq 2$ for n odd
$\mathrm{Sp}(2n)$	ω_1	$\mathrm{SL}(2n)$	$n \geqslant 2$	
G_2	ω_1	$\mathrm{SO}(7)$		$p \neq 2$
$\Delta_q \mathrm{SL}(2)^{(2)}$	$(q+1)\omega_1$	$\mathrm{SO}(4)$		$q = p^m > 1$
$\mathrm{Spin}(7)^{(2)}$	ω_3	$\mathrm{SO}(8)$		
$\mathrm{Sp}(4) \otimes \mathrm{Sp}(2)^{(2)}$	$\omega_1 + \omega_1'$	$\mathrm{SO}(8)$		
$\mathrm{SO}(2n)$	ω_1	$\mathrm{Sp}(2n)$	$n \geqslant 2$	$p = 2$
G_2	ω_1	$\mathrm{Sp}(6)$		$p = 2$
$\mathrm{Spin}(7)^{(1)}$	ω_3	$\mathrm{Sp}(8)$		$p = 2$

(5.6)

(1) Not maximal for $p = 2$.

(2) There are two conjugacy classes.

Proof. In view of Lemma 5.2, we may assume that H is simple. Let ω be the highest weight of H in the defining representation V of G. If $p > 0$, recall that ω is said to be *p-restricted* if $\langle \omega, \alpha^\vee \rangle < p$ for all simple roots α of H. In any case, there is a unique expansion

$$\omega = \sum_{i=0}^{m} p^i \omega^{(i)} \quad \text{with } \omega^{(m)} \neq 0,$$

where each $\omega^{(i)}$ is p-restricted. We may assume that $\omega^{(0)} \neq 0$ as well, since otherwise $H \to G$ factors through a Frobenius morphism. Steinberg's tensor product theorem asserts that

$$V = V_0 \otimes \cdots \otimes V_m,$$

where V_i is simple with highest weight $p^i \omega^{(i)}$. If $m \geqslant 1$, then the embedding $H \to G$ factors through one of the subgroups in Lemma 5.1. It follows easily that $G = \mathrm{SO}(4)$ and $H = \Delta_q \mathrm{SL}(2)$ for $q = p^m > 1$.

Thus, we may assume from now on that $\omega = \omega^{(0)}$ is p-restricted. The inequality (2.2) implies the following upper bounds on $\dim V$:

$$\dim V \leqslant \begin{cases} \sqrt{2}\sqrt{\dim H + 1/8} + \frac{1}{2} & \text{if } G = \mathrm{SL}(n), \\ 2\sqrt{\dim H} + 1 & \text{if } G = \mathrm{SO}(2n+1), \\ 2\sqrt{\dim H + 1/4} + 1 & \text{if } G = \mathrm{SO}(2n), \\ 2\sqrt{\dim H} & \text{if } G = \mathrm{Sp}(2n). \end{cases} \tag{5.7}$$

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Thus,

$$\dim V \leqslant 2\sqrt{\dim H + 1/4} + 1$$

in all cases. Now we use the trivial dimension estimate $\dim V \geqslant |W_H \cdot \omega|$ to conclude that H is one of the groups in Table (5.4) and ω is a linear combination of the fundamental weights in the right column of that table.

Assume first that ω is not a fundamental weight. Then we claim that the inequalities in (5.7) leave only two cases to consider, namely $(H, \omega) = (\mathrm{A}_1, 2\omega_1)$ and $(H, \omega) = (\mathrm{A}_1, 3\omega_1)$.

For groups of small rank ($\mathrm{rk}\, H \leqslant 4$ will do), this can be checked using the tables of Lübeck [Lüb01]. So let $\mathrm{rk}\, H \geqslant 5$ and suppose that ω is not a multiple of a fundamental weight. Then, according to Lemma 5.3, $H = \mathrm{A}_n$ and $\omega = a\omega_1 + b\omega_n$ with $a, b \geqslant 1$. In that case, it is readily checked that the Weyl group orbit of ω is too big.

Next, we consider the case where $\omega = a\omega_1$ with $2 \leqslant a < p$. Then

$$\omega' := \omega - \alpha_1 = (a-2)\omega_1 + b\omega_2$$

is also a weight of V, where $b = -\langle \alpha_1, \alpha_2^\vee \rangle > 0$. But ω_2 does not occur in Table (5.4), excluding this possibility. This finishes the proof of the claim.

Finally, it remains to check whether the representations of H with highest weight ω define a proper spherical subgroup of a classical group where ω is one of the fundamental weights of Lemma 5.3 or one of the exceptional cases $(\mathrm{A}_1, 2\omega_1)$ or $(\mathrm{A}_2, 3\omega_1)$. To make it easier, some remarks are in order: first, it clearly suffices to check the ω up to an automorphism of H. Second, the representations (C_2, ω_2) for $p = 2$, (C_3, ω_3) for $p = 2$ and (G_2, ω_2) for $p = 3$ factor through (C_2, ω_1), (B_3, ω_3) and (G_2, ω_1), respectively, and therefore they can be omitted. The result is summarized in the following table.

H	Weight	$G = \mathrm{SL}$	$G = \mathrm{SO}$	$G = \mathrm{Sp}$
A_{n-1} $(n \geqslant 2)$	ω_1	$=$	$-$	$-$
B_n $(n \geqslant 3, p \neq 2)$	ω_1	$\mathrm{SO}(2n{+}1) \subset \mathrm{SL}(2n{+}1)$	$=$	$-$
C_n $(n \geqslant 2)$	ω_1	$\mathrm{Sp}(2n) \subset \mathrm{SL}(2n)$	$-$	$=$
D_n $(n \geqslant 4)$	ω_1	$\mathrm{SO}(2n) \subset \mathrm{SL}(2n)$	$=$	$\mathrm{SO}(2n) \underset{p=2}{\subset} \mathrm{Sp}(2n)$
A_1 $(p \neq 2)$	$2\omega_1$	$\mathrm{SO}(3) \subset \mathrm{SL}(3)$	$=$	$-$
A_1 $(p \neq 2, 3)$	$3\omega_1$	$\times$	$-$	$\times$
A_3	ω_2	$\mathrm{SO}(6) \subset \mathrm{SL}(6)$	$=$	$\mathrm{SO}(6) \underset{p=2}{\subset} \mathrm{Sp}(6)$
A_4	ω_2	$\times$	$-$	$-$
B_3	ω_3	$\times$	$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	$\mathrm{Spin}(7) \underset{p=2}{\subset} \mathrm{Sp}(8)$
C_2 $(p \neq 2)$	ω_2	$\mathrm{SO}(5) \subset \mathrm{SL}(5)$	$=$	$-$
C_3 $(p \neq 2)$	ω_3	$\times$	$-$	$\times$
G_2	ω_1	$\times$	$\mathrm{G}_2 \underset{p\neq 2}{\subset} \mathrm{SO}(7)$	$\mathrm{G}_2 \underset{p=2}{\subset} \mathrm{Sp}(6)$
G_2 $(p \neq 3)$	ω_2	$\times$	$\times$	$\times$

Here the notation is as follows: '$-$' means that H is not a subgroup of G; '$=$' means that H equals G; and '$\times$' means that H is not spherical in G in all cases, because (2.2) is violated. □

6. G-completely reducible, spherical subgroups of classical groups

Following [Ser05], we say that a subgroup H of a reductive group G is *G-completely reducible* if whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P. Thanks to [Ser05, Proposition 4.1], a G-completely reducible subgroup of G is reductive.

Suppose that G is classical with natural module V. Note that for $G = \mathrm{SL}(V)$, a subgroup H of G is G-completely reducible if and only if V is semi-simple as an H-module [Ser05, Exemples 3.2.2(a)]. If $p \neq 2$, then this also holds for $G = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$; see [Ser05, Exemples 3.2.2(b)]. However, if $p = 2$, these two notions differ, and for $G = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ both implications may fail. For example, if $p = 2$, then $H = \mathrm{SO}(2n-1)$ is G-completely reducible in $G = \mathrm{SO}(2n)$ (in fact, H is not contained in any proper parabolic subgroup) but V is not a semi-simple H-module for $n \geqslant 2$. See [BMR05, Example 3.45] for an example of a simple subgroup of $\mathrm{Sp}(V)$ which is semi-simple on V but not G-completely reducible when $p = 2$.

In the following, we always assume that G is strictly classical, i.e. we exclude the case of $G = \mathrm{SO}(n)$ when $p = 2$ and n is odd.

Lemma 6.1. *Let G be a strictly classical group and let $H \subset G$ be maximal among connected spherical, G-completely reducible subgroups. Then H is contained in the following table.*

H	G		
$S(\mathrm{GL}(m) \times \mathrm{GL}(n))$	$\mathrm{SL}(m+n)$	$m \geqslant n \geqslant 1$	
$\mathrm{GL}(n)$	$\mathrm{SO}(2n)$	$n \geqslant 1$	
$\mathrm{GL}(n)$	$\mathrm{Sp}(2n)$	$n \geqslant 1$	$p \neq 2$
$\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$	$\mathrm{Sp}(2m+2n)$	$m \geqslant n \geqslant 1$	
$\mathrm{SO}(m) \times \mathrm{SO}(n)$	$\mathrm{SO}(m+n)$	$m \geqslant n \geqslant 1$	$p \neq 2$
$\mathrm{SO}(2m) \times \mathrm{SO}(2n)$	$\mathrm{SO}(2m+2n)$	$m \geqslant n \geqslant 1$	$p = 2$
$\mathrm{SO}(2n-1)$	$\mathrm{SO}(2n)$	$n \geqslant 2$	$p = 2$

(6.2)

Proof. Let ω be the defining symplectic form of $\mathrm{Sp}(2n)$ and let q be the defining quadratic form of $\mathrm{SO}(n)$.

Choose a non-zero H-invariant subspace $U \subseteq V$ of minimal dimension. If $G = \mathrm{SL}(n)$ or U is isotropic, i.e. $\omega|_U = 0$ in the case $G = \mathrm{Sp}(2n)$ and $q|_U = 0$ in the case $G = \mathrm{SO}(n)$, then the stabilizer of U is a parabolic subgroup P of G. The G-complete reducibility of H implies that H is contained in a Levi complement L of P. Since H is spherical, so is L by Lemma 2.3. The maximality of H implies $H = L$. Since all Levi subgroups lift to characteristic zero, it is easy to derive a list of spherical Levi subgroups from Krämer's list (cf. [Bru98, Theorem 4.1]). One checks that all of them are contained in the table except for $\mathrm{GL}(n) \subset \mathrm{SO}(2n+1)$ and $\mathbb{G}_m \times \mathrm{Sp}(2n-2) \subset \mathrm{Sp}(2n)$, which are not maximal.

Now assume that U is anisotropic. Then, in particular, $G \neq \mathrm{SL}(V)$.

If $G = \mathrm{Sp}(2n)$, then $U \cap U^\perp \subsetneq U$, and therefore $U \cap U^\perp = 0$ by the minimality of U. This means that U is completely anisotropic and that H is contained in a conjugate of $\mathrm{Sp}(2m) \times \mathrm{Sp}(2n-2m)$. The same reasoning works for $p \neq 2$ and $G = \mathrm{SO}(n)$.

So let $G = \mathrm{SO}(2n)$ and $p = 2$. Then the associated bilinear form

$$\omega_q(u, v) = q(u+v) - q(u) - q(v)$$

is actually a symplectic form on V. Again, if $U \cap U^{\perp_{\omega_q}} = 0$, then U is necessarily even-dimensional and $H \subseteq \mathrm{SO}(2m) \times \mathrm{SO}(2n-2m)$. But there is another possibility: U is isotropic with respect to ω_q but $q|_U \neq 0$. Then $\omega|_U = 0$ implies $q|_U = \ell^2$, where ℓ is an H-invariant linear

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form on U. By minimality of U we have $\ker \ell = 0$ and therefore $\dim U = 1$. Thus H is a subgroup of $\mathrm{SO}(U^{\perp_{\omega_q}}) \cong \mathrm{SO}(2n-1)$. □

Corollary 6.3. *Let G be strictly classical and $H \subset G$ a subgroup which is maximal among connected spherical G-completely reducible proper subgroups. Then either:*

(i) *H is a maximal irreducible subgroup in Table (5.6); or*

(ii) *H is contained in Table (6.2).*

In the final lemmas of this section, we classify all spherical G-completely reducible subgroups of the classical groups.

Lemma 6.4. *Let $G = \mathrm{SL}(n)$ with $n \geqslant 2$, and let $H \subset G$ be spherical, G-completely reducible and reducible. Then H is listed in Table 1.*

Proof. The assumptions on H and Lemma 6.1 imply that $H \subseteq \mathbb{G}_m \cdot \mathrm{SL}(m) \cdot \mathrm{SL}(n-m)$ for $1 \leqslant m \leqslant n-1$. We consider first the case $H = \mathbb{G}_m \cdot H_1 \cdot \mathrm{SL}(n-m)$ where $H_1 \subset \mathrm{SL}(m)$. Then, by induction on $\dim G$, we may assume that H_1 is contained in Table 1. One then checks that $H_1 \subset \mathrm{SL}(m)$ lifts to characteristic zero. Hence H is spherical if and only if it is in Krämer's list. This happens in a single case, namely $H = \mathbb{G}_m \times \mathrm{Sp}(n-1) \subset \mathrm{SL}(n)$ with $n \geqslant 3$ odd.

By symmetry, we do not need to consider subgroups of the form $\mathbb{G}_m \cdot \mathrm{SL}(m) \cdot H_2$. Also, no subgroups of the form $H = \mathbb{G}_m \cdot H_1 \cdot H_2$ with $H_1 \subset \mathrm{SL}(m)$ and $H_2 \subset \mathrm{SL}(n-m)$ are spherical. Thus it remains to check $H = \mathbb{G}_m \cdot \mathrm{SL}(m)$ where the $\mathrm{SL}(m)$ factor is diagonally embedded into $\mathrm{SL}(m) \cdot \mathrm{SL}(m)$ with $n = 2m \geqslant 4$. However, in that case H is not spherical, by (2.2).

Finally, assume that $H = H'$ is semi-simple. Then $\mathbb{G}_m \cdot H'$ is one of the instances above. As H' lifts, it is contained in Krämer's table and thus covered by Corollary 3.5. The only cases of that form are $H = \mathrm{SL}(m) \cdot \mathrm{SL}(n-m) \subset \mathrm{SL}(n)$ with $m \neq n-m$ and $H = \mathrm{Sp}(n-1) \subset \mathrm{SL}(n)$ with n odd. □

Lemma 6.5. *Let $G = \mathrm{Sp}(2n)$ with $n \geqslant 2$, and let $H \subset G$ be spherical, G-completely reducible and reducible. Then H is listed in Table 1.*

Proof. The assumptions on H and Lemma 6.1 imply that either $H \subseteq \mathrm{GL}(n)$ or $H \subseteq \mathrm{Sp}(2m) \times \mathrm{Sp}(2n-2m)$ for $0 < m < n$.

In the first instance, the inequality (2.2) shows that $H = \mathrm{GL}(n)$. In the second case, we first consider subgroups of the form $H = H_0 \times \mathrm{Sp}(2n-2m)$ with $H_0 \subset \mathrm{Sp}(2m)$. Then, by induction on $\dim G$, we may assume that H_0 is contained in Table 1. Moreover, if $H_0 \subset \mathrm{Sp}(2m)$ lifts to characteristic zero, then H is spherical if and only if it is in Krämer's list. One can check that there is a single case of that form, namely $H = \mathbb{G}_m \times \mathrm{Sp}(2n-2)$.

Next, we consider those $H_0 \subset \mathrm{Sp}(2m)$ which do not lift. This means that $p = 2$ and we have to deal with the following cases.

(i) $H = \mathrm{SO}(2l) \cdot \mathrm{Sp}(2m-2l) \cdot \mathrm{Sp}(2n-2m)$ with $1 \leqslant l \leqslant m < n$. Then H is contained in the liftable subgroup $\mathrm{Sp}(2l) \cdot \mathrm{Sp}(2m-2l) \cdot \mathrm{Sp}(2n-2m)$, which is spherical if and only if one of the factors is trivial. Thus, for H to be spherical, we need $l = m$. In that case, H is indeed spherical, because then $H \subset G$ is isogenous to the liftable subgroup $\mathrm{SO}(2m) \cdot \mathrm{SO}(2n-2m+1) \subset \mathrm{SO}(2n+1)$.

(ii) $H = \mathrm{Spin}(7) \cdot \mathrm{Sp}(2n-8)$ with $n \geqslant 5$ is never spherical, by Corollary 4.4.

(iii) $H = \mathrm{G}_2 \cdot \mathrm{Sp}(2n-6)$ with $n \geqslant 4$ is spherical only for $n = 4$, by Corollary 4.5.

(iv) $H = \mathrm{G}_2 \cdot \mathrm{Sp}(2) \cdot \mathrm{Sp}(2n-8)$ with $n \geqslant 5$ is never spherical, since it is contained in the non-spherical subgroup $\mathrm{Sp}(6) \cdot \mathrm{Sp}(2) \cdot \mathrm{Sp}(2n-8)$.

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Now we discuss groups of the form $H = H_1 \cdot H_2 \subset \mathrm{Sp}(2m) \cdot \mathrm{Sp}(2n-2m)$. It follows from the discussion above that H_i is one of $\mathbb{G}_m \subset \mathrm{Sp}(2)$ for $p \neq 2$, $\mathrm{SO}(2m) \subset \mathrm{Sp}(2m)$ for $p = 2$, or $\mathrm{G}_2 \subset \mathrm{Sp}(6)$ for $p = 2$. This leads to the following possibilities for H.

(i) $p \neq 2$ and $H = \mathbb{G}_m \cdot \mathbb{G}_m \subset \mathrm{Sp}(4)$, which is not spherical, by (2.2).

(ii) $p = 2$ and $H = \mathrm{SO}(2m) \cdot \mathrm{SO}(2n-2m)$ with $1 \leqslant m < n$, in which case $H \subset G$ is isogenous to the liftable and non-spherical subgroup $\mathrm{SO}(2m) \cdot \mathrm{SO}(2n-2m) \subset \mathrm{SO}(2n+1)$.

(iii) $p = 2$ and $H = \mathrm{G}_2 \cdot \mathrm{SO}(2) \subset \mathrm{Sp}(8)$, which is not spherical, by (2.2).

Finally, a general subgroup H is obtained from a group of the form $H_1 \cdot H_2$ by replacing one or several isogenous factors with diagonal subgroups. This is only possible in the following cases.

(i) $H \subset \mathrm{Sp}(2m) \cdot \mathrm{Sp}(2m) \subseteq \mathrm{Sp}(4m)$ with $m \geqslant 1$, in which case H is not spherical, by (2.2).

(ii) $H \subset \mathrm{SO}(4) \cdot \mathrm{Sp}(2) \subset \mathrm{Sp}(6)$; again, H is not spherical, by (2.2).

This finishes the proof of the lemma. □

Lemma 6.6. *Let $G = \mathrm{SO}(n)$ where $n \geqslant 2$ (with n even if $p = 2$), and let $H \subset G$ be spherical, G-completely reducible and reducible. Then H is listed in Table 1.*

Proof. Thanks to Lemma 6.1, either $H \subseteq \mathrm{GL}(m) \subset \mathrm{SO}(n)$ for $n = 2m \geqslant 2$, or $H \subseteq \mathrm{SO}(m) \cdot \mathrm{SO}(n-m) \subset \mathrm{SO}(n)$ where $1 \leqslant m < n$. For $p = 2$, we may assume in the latter case that either both m and n are even or n is even and $m = 1$.

Assume first that $H \subseteq \mathrm{GL}(m) \subset \mathrm{SO}(2m)$. Then the dimension estimate (2.2) implies that the codimension of H in $\mathrm{GL}(n)$ is at most n. Thus, the codimension of $H_0 = (H \cap \mathrm{SL}(n))^\circ$ in $\mathrm{SL}(n)$ is also at most n. The list of maximal spherical subgroups of $\mathrm{SL}(n)$ (see Corollary 6.3) shows that $H_0 = \mathbb{G}_m \subset \mathrm{SL}(2)$. Thus, the only instance is $H = \mathrm{SO}(2) \cdot \mathrm{SO}(2) \subset \mathrm{SO}(4)$.

Now we treat the case where $H \subseteq \mathrm{SO}(m) \cdot \mathrm{SO}(n-m)$ for $p \neq 2$ or $p = 2$ and m, n are both even. First, let $H = H_0 \cdot \mathrm{SO}(n-m) \subset \mathrm{SO}(n)$, where $H_0 \subset \mathrm{SO}(m)$. By induction, we may assume that H_0 is contained in Table 1. If H_0 is liftable, then sphericality can be checked with Krämer's table. It turns out that there is no instance of this type. On the other hand, there is only one non-liftable case, namely $H = \Delta_q \mathrm{SL}(2) \cdot \mathrm{SO}(n-4) \subset \mathrm{SO}(n)$ with $n \geqslant 5$ and $q = p^s > 1$. None of these subgroups is spherical: use inequality (2.2) for $n = 5, 6, 7$ and Lemma 4.3 for $n \geqslant 8$.

The remaining case to consider is where H is obtained from $\mathrm{SO}(m) \cdot \mathrm{SO}(n-m)$ by replacing some isogenous factors with a diagonal subgroup. Then either $H \subset \mathrm{SO}(m) \cdot \mathrm{SO}(m) \subset \mathrm{SO}(2m)$ with $m \geqslant 2$ or $H \subset \mathrm{SO}(3) \cdot \mathrm{SO}(4) \subset \mathrm{SO}(7)$. None of these subgroups can be spherical, by (2.2).

Now we treat the case where $p = 2$, $n = 2d$ is even and $m = 1$, i.e. $H \subset \mathrm{SO}(2d-1) \subset \mathrm{SO}(2d)$. There is a bijective isogeny $\mathrm{SO}(2d-1) \to \mathrm{Sp}(2d-2)$, and all G-completely reducible, spherical subgroups of $\mathrm{Sp}(2d-2)$ are known, by Lemma 6.5. Thus we arrive at the following cases:

(i) $H = \mathrm{GL}(2d-1) \subset \mathrm{SO}(2d)$ lifts and is not spherical;

(ii) $H = \mathrm{SO}(2) \cdot \mathrm{SO}(2d-3) \subset \mathrm{SO}(2d)$ lifts and is not spherical;

(iii) $H = \mathrm{SO}(2l-1) \cdot \mathrm{SO}(2d-2l+1) \subset \mathrm{SO}(2d)$ lifts and is spherical for all $1 \leqslant l \leqslant d$;

(iv) $H = \mathrm{SO}(2l) \cdot \mathrm{SO}(2d-2l-1) \subset \mathrm{SO}(2d)$ lifts and is not spherical;

(v) $H = \mathrm{Spin}(7) \subset \mathrm{SO}(10)$ lifts and is not spherical;

(vi) $H = \mathrm{G}_2 \subset \mathrm{SO}(8)$ lifts and is spherical;

(vii) $H = \mathrm{G}_2 \cdot \mathrm{SO}(3) \subset \mathrm{SO}(10)$ is not spherical, by (2.2).

This finishes the proof of the lemma. □

This concludes our classification of the spherical, G-completely reducible subgroups of strictly classical groups.

7. G-completely reducible, spherical subgroups of exceptional groups

Throughout this section, let G be a simple group of exceptional type.

Lemma 7.1. *Let G be a simple group of exceptional type and $H \subset G$ a subgroup which is maximal among proper, connected, G-completely reducible, spherical subgroups of G. Then one of the following holds:*

$$\begin{aligned} &G = \mathrm{E}_6 \quad \text{and} \quad H \in \{\mathrm{A}_1\mathrm{A}_5, \mathbb{G}_m \cdot \mathrm{D}_5, \mathrm{F}_4, \mathrm{C}_4\ (p \neq 2)\},\\ &G = \mathrm{E}_7 \quad \text{and} \quad H \in \{\mathrm{A}_1\mathrm{D}_6, \mathrm{A}_7, \mathbb{G}_m \cdot \mathrm{E}_6\},\\ &G = \mathrm{E}_8 \quad \text{and} \quad H \in \{\mathrm{A}_1\mathrm{E}_7, \mathrm{D}_8\},\\ &G = \mathrm{F}_4 \quad \text{and} \quad H \in \{\mathrm{A}_1\mathrm{C}_3\ (p \neq 2), \mathrm{B}_4, \mathrm{C}_4\ (p = 2)\},\\ &G = \mathrm{G}_2 \quad \text{and} \quad H \in \{\mathrm{A}_1\widetilde{\mathrm{A}}_1, \mathrm{A}_2, \widetilde{\mathrm{A}}_2\ (p = 3)\}. \end{aligned}$$

Proof. Assume first that $\operatorname{rk} H = \operatorname{rk} G$. If $p \neq 2$ for $G = \mathrm{F}_4$ and $p \neq 3$ for $G = \mathrm{G}_2$, then H is given by an additively closed subroot system. In particular, H lifts to characteristic zero and the spherical cases can be read off Krämer's list. Observe that $H = \mathrm{A}_1\mathrm{C}_3$ in $G = \mathrm{F}_4$ is no longer maximal for $p = 2$, since it is contained in a subgroup of type C_4.

Now suppose that $G = \mathrm{F}_4$ and $p = 2$ or $G = \mathrm{G}_2$ and $p = 3$, that $\operatorname{rk} H = \operatorname{rk} G$, and that H does not lift. Then the remaining possibilities for H have been determined by Liebeck and Seitz (see [LS04, Table 10.4]), namely $(G, H) = (\mathrm{F}_4, \mathrm{C}_4)$ or $(G, H) = (\mathrm{G}_2, \widetilde{\mathrm{A}}_2)$. Using the inseparable isogeny of G in both cases, H is mapped to a subgroup which lifts and is spherical. So H itself is spherical in both instances.

Finally, assume that $\operatorname{rk} H < \operatorname{rk} G$. Then we claim that H is a maximal connected subgroup of G. Indeed, if H were contained in a proper parabolic subgroup P of G, then the G-complete reducibility of H implies that H lies in a Levi subgroup L of P. Since L is G-completely reducible as well, we get $H = L$ by maximality of H, which contradicts our assumption on the rank of H. But H cannot be a proper subgroup of a connected proper subgroup K of G either, since K would then be G-completely reducible and hence also reductive. In fact, by the argument above, K would not be contained in any proper parabolic subgroup of G. This finishes the proof of the claim.

Now we know that H is one of the subgroups of [LS04, Table 1]. None of them is spherical for dimension reasons, except $(G, H) = (\mathrm{E}_6, \mathrm{F}_4)$ and $(G, H) = (\mathrm{E}_6, \mathrm{C}_4)$ $(p \neq 2)$. In both cases, H lifts and is spherical; cf. [Spr85] and [Bru98]. Observe that for $p = 2$, the group $H = \mathrm{C}_4$ is not maximal in $G = E_6$, because then it is contained in a subgroup of type F_4. □

Lemma 7.2. *Let G be a simple group of exceptional type and $H \subset G$ a proper, connected, non-maximal, G-completely reducible, spherical subgroup of G. Then one of the following holds:*

$$\begin{aligned} &G = \mathrm{E}_6 \quad \text{and} \quad H \in \{\mathrm{D}_5, \mathrm{C}_4\ (p = 2)\},\\ &G = \mathrm{F}_4 \quad \text{and} \quad H \in \{\mathrm{A}_1\mathrm{B}_3\ (p = 2), \mathrm{A}_1\mathrm{C}_3\ (p = 2)\}. \end{aligned}$$

Proof. Since H is spherical in G, it satisfies the inequalities

$$\operatorname{rk} H \leqslant \operatorname{rk} G, \quad \dim H \geqslant \tfrac{1}{2}(\dim G - \operatorname{rk} G).$$

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First we claim that, except for $G = \mathrm{F}_4$ and $p = 2$ or $G = \mathrm{G}_2$ and $p = 3$, we may assume that $\operatorname{rk} H < \operatorname{rk} G$. Indeed, if this were not the case, then H lifts and would therefore be in Krämer's list; but one can easily check that all maximal-rank spherical subgroups there are in fact maximal.

Another constraint on H is that it must be a proper subgroup of one of the groups in Lemma 7.1. It is now easy, though somewhat tedious, to list all possible types for H which match the requirements. We wind up with very few cases, as follows.

(i) $G = \mathrm{E}_6$ and $H \in \{\mathrm{D}_5, \mathbb{G}_m \cdot \mathrm{B}_4,\ \mathrm{B}_4, \mathrm{C}_4\ (p = 2)\}$. We claim that all subgroups of these types lift to characteristic zero. The subgroup $H = \mathrm{D}_5$ has to be the second factor in $\mathbb{G}_m \cdot \mathrm{D}_5$; thus it lifts and is spherical. One checks that a group of type D_5 contains a unique conjugacy class of subgroups of type B_4, namely $\mathrm{SO}(9) \subset \mathrm{SO}(10)$. Thus $\mathbb{G}_m \cdot \mathrm{B}_4$ inside $\mathbb{G}_m \cdot \mathrm{D}_5$ lifts and is not spherical. Also $H = \mathrm{C}_4$ lifts (cf. [Bru98]) and is spherical. There are two possibilities for $H = \mathrm{B}_4$: either H is inside $\mathbb{G}_m \cdot \mathrm{D}_5$, or it is inside F_4. In both cases, H lifts and is not spherical.

(ii) $G = \mathrm{E}_7$ and $H \in \{\mathrm{D}_6, \mathrm{E}_6\}$. Here $H = \mathrm{D}_6$ is normal in $\mathrm{A}_1\mathrm{D}_6$; hence it lifts and is not spherical. Likewise, $H = \mathrm{E}_6$ is normal in $\mathbb{G}_m \cdot \mathrm{E}_6$; hence it also lifts and is not spherical.

(iii) $G = \mathrm{E}_8$ and $H = \mathrm{E}_7$. Here $H = \mathrm{E}_7$ is normal in $\mathrm{A}_1\mathrm{E}_7$; hence it lifts and is not spherical.

(iv) $G = \mathrm{F}_4$, $H \in \{\mathrm{A}_1\mathrm{C}_3, \widetilde{\mathrm{A}}_1\mathrm{B}_3, \mathrm{D}_4\}$ and $p = 2$. Let $H = \mathrm{A}_1\mathrm{C}_3$ or $H = \widetilde{\mathrm{A}}_1\mathrm{B}_3$. Without loss of generality we may assume that the positive root α in the A_1 factor is a dominant weight of F_4. Thus it is either the highest long root or the highest short root. The roots orthogonal to α form a root system of type C_3 or B_3, respectively. Thus $H = \mathrm{A}_1\mathrm{C}_3$ lifts to characteristic zero, while $H = \widetilde{\mathrm{A}}_1\mathrm{B}_3$ differs from the former by an inseparable isogeny of F_4; so both are unique and spherical. There are two subgroups of type D_4 that correspond to the two root subsystems, consisting of all the long roots and all the short roots, respectively. Stemming from a closed root subsystem, the first subgroup lifts, and hence so does the second, as it is obtained from the first by the isogeny of G. Thus, none of them is spherical. □

8. Non-G-completely reducible, reductive spherical subgroups

Now we complete the classification by considering the non-G-completely reducible subgroups of G. Throughout this section let G be a connected reductive group over k and let $H \subseteq G$ be a non-G-completely reducible subgroup of G. Then there exists a parabolic subgroup P of G containing H so that H is in no Levi subgroup of P. Indeed, there is a canonical such parabolic subgroup P which depends only on H, called the *optimal destabilizing parabolic subgroup* associated with H, obtained by means of geometric invariant theory; cf. [BMRT13, § 5.2].

It is convenient to use the characterization of parabolic subgroups of G in terms of one-parameter subgroups of G; see, e.g., [Ric88, 2.1–2.3] and [Spr98, Proposition 8.4.5].

Lemma 8.1. *Given a parabolic subgroup P of G and any Levi subgroup L of P, there exists a one-parameter subgroup λ of G such that the following hold:*

(i) $P = P_\lambda := \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ *exists*$\}$;

(ii) $L = L_\lambda := C_G(\lambda(\mathbb{G}_m))$;

(iii) *the map* $\pi = \pi_\lambda : P_\lambda \to L_\lambda$ *defined by*

$$\pi_\lambda(g) := \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$$

is a surjective homomorphism of algebraic groups; moreover, L_λ is the set of fixed points of π_λ and $R_u(P_\lambda)$ is the kernel of π_λ.

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Remark 8.2. We note that $H \subset G$ is G-completely reducible if and only if for every one-parameter subgroup λ of G with $H \subset P_\lambda$, H is G-conjugate to $\pi_\lambda(H)$; see [BMR05, Lemma 2.17, Theorem 3.1] or [BMRT13, Theorem 5.8(ii)].

Our first result shows that we can reduce the question of non-G-completely reducible, spherical subgroups of G to subgroups that are G-completely reducible and spherical. For this, we again use the deformation theorem, Theorem 3.4, this time with $S = \mathbb{A}^1_k = \operatorname{Spec} k[t]$.

Proposition 8.3. *Let G be a connected reductive group over k, and let $H \subseteq G$ be a reductive subgroup of G lying in the parabolic subgroup $P = P_\lambda$ for some one-parameter subgroup λ of G. Then H is spherical in G if and only if $\pi_\lambda(H)$ is.*

Proof. Define $\mathcal{H}$ to be the closure of $\{(t, g) \mid t \in \mathbb{G}_m, \lambda(t)^{-1} g \lambda(t) \in H\}$ in $\mathbb{A}^1_k \times G$. This is a flat subgroup scheme of the trivial group scheme $\mathcal{G} = \mathbb{A}^1_k \times G \to \mathbb{A}^1_k$ whose fiber $\mathcal{H}_t$ is conjugate to H for $t \neq 0$; cf. [BT84, 1.2.6, 1.2.7, 2.1.6]. Since $\pi_\lambda(h) = \lim_{t\to 0} \lambda(t) h \lambda(t)^{-1}$ for all $h \in H$, we see that $\pi_\lambda(H) \subseteq \mathcal{H}_0$. Since H is reductive, $\ker \pi_\lambda|_H = R_u(P) \cap H$ is finite and therefore $\dim \pi_\lambda(H) = \dim H$. Thus, we also have $\dim \pi_\lambda(H) = \dim \mathcal{H}_0$, which implies that $\pi_\lambda(H)^\circ = \mathcal{H}_0^\circ$. Hence, our assertion boils down to showing that $\mathcal{H}_0$ is spherical if and only if $\mathcal{H}_1 = H$ is spherical, and this follows immediately from Theorem 3.4 with $S = \mathbb{A}^1_k$. □

We analyze the situation of Proposition 8.3 further.

Proposition 8.4. *Let $H \subseteq P = P_\lambda \subseteq G$ be as in Proposition 8.3, and assume that $H^* := \pi_\lambda(H) \subseteq L = L_\lambda$ is not conjugate to H inside P. Let $Z := Z(L)^\circ$ be the connected center of L. Then $Z \not\subseteq H^*$. In particular, if H is spherical, then ZH^* is a reductive, non-semi-simple, spherical subgroup of G.*

Proof. Suppose $Z \subseteq H^*$. Then, by Lemma 8.1(ii), $C^* := \lambda(\mathbb{G}_m) \subseteq H^*$. Let $C \subseteq H$ be the preimage of C^* in H. Since $H \to H^*$ is an isogeny, C is a one-dimensional torus lying in the center of H. Moreover, C is a maximal torus of $C^* R_u(P)$ and hence conjugate to C^*. So we may assume $C = C^*$. But then $H \subseteq C_G(C^*) = L$ and thus $H = H^*$, contradicting our assumptions. □

In the following lemma, we denote by P_m the standard maximal parabolic subgroup of the simple group G corresponding to the mth simple root in the labeling of the Dynkin diagram of G according to [Bou68]. Let $U = R_u(P)$ be the unipotent radical of P_m.

Lemma 8.5. *Let G be a simple group and H a connected, reductive, non-G-completely reducible, spherical subgroup of G which is contained in the parabolic subgroup P of G. Then there are the following possibilities for $H^* = \pi_\lambda(H)$, P and G as in Proposition 8.4.*

H^*	P	G	U	$H^1_{\mathrm{gen}}(H', U)$
$\mathrm{SL}(m) \times \mathrm{SL}(n)$	P_m, P_n	$\mathrm{SL}(m+n)$, $m > n \geqslant 1$	$k^m \otimes k^n$	$\begin{cases} k, & m = 2 \\ 0, & m > 2 \end{cases}$
$\mathrm{Sp}(2n)$	P_1, P_{2n}	$\mathrm{SL}(2n+1)$, $n \geqslant 2$	k^{2n}	k
$\mathrm{SL}(2n+1)$	P_{2n}, P_{2n+1}	$\mathrm{SO}(4n+2)$, $n \geqslant 2$	$\wedge^2 k^{2n+1}$	0
D_5	P_1, P_6	E_6	k^{16} (half-spin reps)	0

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In each case, the unipotent radical U of P is a vector group on which H^* acts linearly and irreducibly according to this table. The last column lists the first generic cohomology group in the sense of [CPSvdK77].

Proof. The subgroups H^* are just those G-completely reducible, spherical subgroups which are centralized by a non-trivial torus, because this is a necessary condition by Proposition 8.4. The cohomology groups have been calculated in, for instance, [CPS75]. □

We keep the notation of Lemma 8.5. We know from the proof of Proposition 8.3 that the projection $\pi : H \to H^*$ is an isogeny. Its kernel $U \cap H$ is therefore a finite normal and hence central subgroup of H. Moreover, $U \cap H$ is a p-group, since it is a subgroup of U. We conclude that $U \cap H = 1$, i.e. that $H \to H^*$ is bijective.

Now let $Q := H \cdot U = H^* \ltimes U$. Our goal is to determine all conjugacy classes of subgroups $H \subseteq Q$ such that the induced projection $\pi : H \to H^*$ is bijective. If this bijection is even an isomorphism, then it is well known that this task is accomplished by the cohomology group $H^1(H^*, U)$.

In general, we use the fact that each H^* of interest is defined over $\mathbb{F}_p$. This means that H^* admits a Frobenius endomorphism $F : H^* \to H^*$. Because π is purely inseparable, it factors through a sufficiently high power F^s of F, i.e. there is an $s \geqslant 0$ and an isogeny $\psi : H^* \to H$ such that $F^s = \pi \circ \psi$.

Now let $\widetilde{Q}$ be the fiber product of Q over $F^s : H^* \to H^*$. Then we have a cartesian diagram as follows.

$$\begin{array}{ccc} \widetilde{Q} & \xrightarrow{\tilde{\pi}} & H^* \\ \downarrow & & \downarrow{\scriptstyle F^s} \\ Q & \xrightarrow{\pi} & H^* \end{array}$$

Moreover, ψ defines a section $\tilde{\psi}$ of $\tilde{\pi}$ such that H is the image of $\tilde{\psi}(H^*)$ in Q. Now observe that $\widetilde{Q} = H^* \ltimes U^{(p^s)}$, where $U^{(p^s)}$ is the sth Frobenius twist of U. Therefore, the conjugacy class of $\tilde{\psi}$ and hence of H is determined by an element of $H^1(H^*, U^{(p^s)})$. By definition, the generic cohomology group $H^1_{\mathrm{gen}}(H^*, U)$ is the inductive limit of the system

$$H^1(H^*, U) \longrightarrow H^1(H^*, U^{(p)}) \longrightarrow H^1(H^*, U^{(p^2)}) \longrightarrow H^1(H^*, U^{(p^3)}) \longrightarrow \cdots$$

(cf. [CPSvdK77]). It is well known that elements of $H^1(H^*, U^{(p^s)})$ classify conjugacy classes of (scheme-theoretic) complements of $U^{(p^s)}$ in $\widetilde{Q}$. Thus, the conjugacy classes of the subgroups H are classified by elements of $H^1_{\mathrm{gen}}(H^*, U)$.

Corollary 8.6. *Let G be a simple group and $H \subseteq G$ a connected, reductive, spherical subgroup which is not G-completely reducible in G. Then $G = \mathrm{SL}(2n+1)$ for some $n \geqslant 1$ and $H = \mathrm{SO}(2n+1)$ or its dual $\mathrm{SO}(2n+1)^\vee$.*

Proof. By the definition of G-complete reducibility, there is a parabolic subgroup $P \subseteq G$ containing H such that H is not conjugate to $H^* = \pi_\lambda(H)$. From the discussion above we infer that $H^* \subset P$ is one of the cases in Lemma 8.5 with $H^1_{\mathrm{gen}}(H^*, U) \neq 0$. Thus $G = \mathrm{SL}(2n+1)$ and $H^* = \mathrm{Sp}(2n)$ with $n \geqslant 1$. Because the centralizer $C_G(H^*) \cong \mathbb{G}_m$ of $\mathrm{Sp}(2n)$ acts non-trivially on $H^1_{\mathrm{gen}}(H^*, U) \cong k$, there exists only one conjugacy class of H in G, depending, though, on the choice of P. Thereby we obtain the two cases above. □

This concludes the proof of our main classification theorem: using Remark 2.8, we may assume that G is either strictly classical or exceptional. Then the G-completely reducible, connected, spherical subgroups are determined in Corollary 6.3 and Lemmas 7.1 and 7.2, respectively. Finally, Corollary 8.6 lists all non-G-completely reducible subgroups.

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