GAUSSIAN PRIMES IN NARROW SECTORS

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§1. *Introduction.* The purpose of this paper is to show how a sieve method which has had many applications to problems involving rational primes can be modified to derive new results on Gaussian primes (or, more generally, prime ideals in algebraic number fields). One consequence of our main theorem (Theorem 2 below) is the following result on rational primes.

THEOREM 1. *There exist infinitely many primes p with*

$$p = m^2 + n^2, \qquad m, n \in \mathbb{Z}, n < p^{\theta}, \tag{1}$$

and

$$\theta < 0{\cdot}119. \tag{2}$$

Of course, it is conjectured that $m^2 + 1$ is infinitely often prime. Kubilius [**13**] and Ankeny [**1**] have shown that there exist infinitely many primes p with

$$p = m^2 + n^2, \qquad m, n \in \mathbb{Z}, n \ll \log p, \tag{3}$$

assuming the Riemann Hypothesis for Hecke L-functions with Grossencharactere over $\mathbb{Q}(i)$. The best unconditional results in this area are due to M. D. Coleman [**7**] (where references to earlier work may be found), who obtains (1) with $\theta < 0{\cdot}1631$. We use essentially the same arithmetical information as Coleman, but we employ a more efficient sieve procedure. However, it should be noted that one of his important lemmas (Lemma 9 in [**7**]) does not work for $\theta < \frac{1}{8}$. Our main result is as follows.

THEOREM 2. *Let $X > X_0$. Then, for any given β, γ with*

$$0 \leqslant \beta \leqslant \frac{\pi}{2}, \qquad X^{-0{\cdot}381} \leqslant \gamma \leqslant \frac{\pi}{2}, \tag{4}$$

the number of Gaussian primes p satisfying

$$\beta \leqslant \arg p < \beta + \gamma, \qquad |p|^2 \leqslant X, \tag{5}$$

is greater than

$$\frac{cX\gamma}{\log X}, \tag{6}$$

where c is an absolute positive constant.

We have chosen to state only the simplest results which can be obtained by our method. The reader may verify that our approach improves Theorem

[MATHEMATIKA, **48** (2001), 119–135]

2 of [7], replacing 0·33691 there by 0·381. From this one may deduce an analogue to our Theorem 1, with m^2+n^2 replaced by any positive definite primitive quadratic form with integer coefficients, thus improving Theorem 1 in [7]. Recently Matsui [14] has published results on the least Gaussian prime in a narrow sector, apparently unaware of the literature already existing in this field. The results of Kubilius and Ankeny mentioned above imply a better result than Theorem 1 in [14], while from Coleman's result one can immediately deduce a far stronger form of Matsui's Theorem 2. Our main result has the following immediate corollary.

THEOREM 3. *Suppose that* $0 \leqslant \beta \leqslant \pi/2$, $0<\gamma\leqslant\pi/2$. *Then there is Gaussian prime p with*

$$\beta \leqslant \arg p < \beta+\gamma, \qquad |p|^2 \leqslant K\gamma^{-1/0\cdot381}, \tag{7}$$

where K is an absolute constant.

The cognate problem of finding Gaussian primes in small circles will be dealt with elsewhere.

Before commencing the proof of Theorem 2, we mention one further corollary.

THEOREM 4. *There are infinitely many primes p with*

$$\{\sqrt{p}\} < p^{-0\cdot262}. \tag{8}$$

This improves one case of the work of Balog [4] and the first author [10], although, unlike these results, we cannot deduce an inhomogeneous result, nor can we consider p^λ with $\lambda \neq \frac{1}{2}$.

§2. *Notation and outline of the method.* We let $\gamma = X^{\theta-1/2}$ where $\frac{1}{8} > \theta > \frac{1}{9}$. The proof for larger γ becomes progressively easier, but some details differ as γincreases. We eventually show that (6) holds for $\theta = 0\cdot119$. Allowing θ to vary at first makes the relation between the different parameters and parts of the proof easier to comprehend. We write

$$\sigma = \frac{1}{2} - \theta, \qquad \tau = \frac{1}{2} + \theta, \qquad \rho = \frac{1+22\theta}{12},$$

$$\zeta = \frac{1+4\theta}{3}, \qquad \nu = \frac{10\theta+7}{12}, \qquad \chi = \frac{34\theta+7}{24}.$$

For $\theta = 0\cdot119$ the values of these parameters will be

$$\sigma = 0\cdot381, \qquad \tau = 0\cdot619, \qquad \rho = 0\cdot3015,$$

$$\zeta = 0\cdot492, \qquad \nu = 0\cdot6825, \qquad \chi = 0\cdot46205.$$

We write $\eta = (\log X)^{-A}$, where A can be made as large as we like (it usually arises from a term which in reality is $\exp(-(\log X)^\delta)$ for some $\delta > 0$). We write

C for fixed exponents of logarithms, which might arise from an averaged divisor function. We may thus write, for example,

$$\eta(\log X)^C = O(\eta), \qquad (\log X)^C(\log X)^C = O((\log X)^C).$$

Implied constants will depend at most on the A implicit in this notation. We put

$$W = X^\sigma, \qquad T = W\eta^{-1}.$$

We also write (script letters always denoting sets of Gaussian integers here)

$$\mathscr{A} = \{n \in \mathbb{Z}[i]: \beta \leqslant \arg n < \beta + \gamma, \tfrac{1}{2}X \leqslant |n|^2 < X\},$$

$$\mathscr{B} = \{n \in \mathbb{Z}[i]: 0 \leqslant \arg n < \pi/2, \tfrac{1}{2}X \leqslant |n|^2 < X\},$$

$$S(\mathscr{E}, z) = \{n \in \mathscr{E}: p|n \Rightarrow |p|^2 \geqslant z\}, \qquad \text{if } z \text{ is real},$$

$$S(\mathscr{E}, q) = \{n \in \mathscr{E}: p|n \Rightarrow |p| \geqslant |q|\}, \qquad \text{if } q \in \mathbb{Z}[\mathrm{i}],$$

$$\mathscr{E}_m = \{n \in \mathscr{E}: m|n\}.$$

The number of solutions to (5) with $|p|^2 > \frac{1}{2}X$ is then $S(\mathscr{A}, X^{1/2})$.

Our philosophy is to relate $S(\mathscr{A}, X^{1/2})$ to $S(\mathscr{B}, X^{1/2})$. It is well known that

$$S(\mathscr{B}, X^{1/2}) \sim \frac{X}{2\log X} \qquad \text{as } X \to \infty. \tag{9}$$

This can be seen directly, since

$$S(\mathscr{B}, X^{1/2}) = 2 \sum_{\substack{X/2 \leqslant p < X \\ p \equiv 1 \bmod 4}} 1 + \sum_{\substack{X/2 \leqslant p^2 < X \\ p \equiv 3 \bmod 4}} 1.$$

Alternatively, what is required for more general situations, (9) follows from the Prime Ideal Theorem. We would expect that

$$S(\mathscr{A}, X^{1/2}) \sim \delta S(\mathscr{B}, X^{1/2}) \qquad \text{as } X \to \infty, \tag{10}$$

where $\delta = 2\gamma/\pi$. This follows for $\gamma > X^{-0\cdot 3}$ by the work of Ricci **[15]**. Although we are unable to establish (10) for smaller values of γ, nevertheless we can derive formulae of the type

$$\sum_{p \sim P\, q \sim Q} S(\mathscr{A}_{pq}, z) = \delta(1 + \eta) \sum_{p \sim P\, q \sim Q} S(\mathscr{B}_{pq}, z), \tag{11}$$

for various ranges of the parameters P, Q. Here and henceforth we write $b \sim B$ to indicate the condition $B \leqslant |b|^2 < 2B$ with $0 \leqslant \arg\, b < \pi/2$. Henceforth, all Gaussian integers will be assumed to lie in the first quadrant in the complex plane. We thus interpret the inequality $\beta \leqslant \arg n < \beta + \gamma (\bmod \pi/2)$, so that, for example,

$$1 \leqslant \arg n < 2 \quad \text{becomes } 0 \leqslant \arg n < 2 - \frac{\pi}{2} \quad \text{or} \quad 1 \leqslant \arg n < \frac{\pi}{2}.$$

This is permissible since the angular distribution of Gaussian primes has period $\pi/2$.

To obtain (11), we use the first author's method **[9]**, **[11]**, adapted to $\mathbb{Z}[i]$. For this, we require arithmetical information which is obtained, as in earlier work on this problem, from results on Hecke characters. We put, for $n \in \mathbb{Z}[i]$,

$$\lambda^m(n) = \left(\frac{n}{|n|}\right)^{4m}.$$

We may then pick out the condition $\beta \leqslant \arg n < \beta + \gamma$ using Fourier analysis in a familiar way, namely, for any set $\mathscr{E} \subset \mathbb{Z}[i]$, $T \geqslant 1$, and any weights a_n,

$$\sum_{\substack{n \in \mathscr{E} \\ \beta \leqslant \arg n < \beta + \gamma}} a_n = \delta \sum_{n \in \mathscr{E}} a_n + O\left(\frac{1}{T} \sum_{n \in \mathscr{E}} |a_n|\right) + O\left(\gamma \sum_{m=1}^{T} \left| \sum_{n \in \mathscr{E}} a_n \lambda^m(n) \right|\right). \tag{12}$$

This follows from Chapter 2 of **[2]**, for example.

Once we have obtained results like (11), we utilize Buchstab's identity:

$$S(\mathscr{E}, z) = S(\mathscr{E}, w) - \sum_{w \leqslant |p|^2 < z} S(\mathscr{E}_p, p). \tag{13}$$

When we apply Buchstab's identity to obtain expressions like $|q|^2 < |p|^2$, we will mean

$$|p| \leqslant |q| \quad \text{and} \quad \arg p < \arg q \quad \text{if } |p| = |q|.$$

This convention is required to cover our need, for each $p \equiv 1 (\bmod 4)$ with $p = a^2 + b^2$, to sieve by both $a + ib$ and $b + ia$. Applying (13) twice to both $S(\mathscr{A}, X^{1/2})$ and $S(\mathscr{B}, X^{1/2})$, we obtain that

$$S(\mathscr{B}, X^{1/2}) = S(\mathscr{B}, z_1) - \sum_{z_1(p) \leqslant |p|^2 < X^{1/2}} S(\mathscr{B}_p, z_2(p)) + \sum_{z_2(p) \leqslant |q|^2 < |p|^2 < X^{1/2}} S(\mathscr{B}_{pq}, q), \tag{14}$$

and

$$S(\mathscr{A}, X^{1/2}) = S(\mathscr{A}, z_1) - \sum_{z_1 \leqslant |p|^2 < X^{1/2}} S(\mathscr{A}_p, z_2(p)) + \sum_{z_2(p) \leqslant |q|^2 < |p|^2 < X^{1/2}} S(\mathscr{A}_{pq}, q), \tag{15}$$

Here, z_1 and $z_2(p)$ could be any values with $z_1 < X^{1/2}$, $z_2(p) < |p|^2$; the appropriate choices will be made in §4. Now, if we can show that

$$S(\mathscr{A}, z_1) = \delta S(\mathscr{B}, z_1)(1 + O(\eta))$$

and

$$\sum_{z_1 \leqslant |p|^2 < X^{1/2}} S(\mathscr{A}_p, z_2(p)) = \delta(1 + O(\eta)) \sum_{z_1 \leqslant |p|^2 < X^{1/2}} S(\mathscr{B}_p, z_2(p)),$$

then it follows that

$$S(\mathscr{A}, X^{1/2}) = \delta(1+O(\eta))S(\mathscr{B}, X^{1/2}) - \sum_{z_2(p)\leqslant |q|^2<|p|^2<X^{1/2}}' (\delta S(\mathscr{B}_{pq}, q) - S(\mathscr{A}_{pq}, q))$$

$$\geqslant \delta(1+O(\eta))S(\mathscr{B}, X^{1/2}) - \delta \sum_{z_2(p)\leqslant |q|^2<|p|^2<X^{1/2}}' S(\mathscr{B}_{pq}, q),$$

where $'$ indicates that the summation is only over those regions of p and q for which we have been unable to establish (11). If $|p|^2 = X^{\alpha}$, $|q|^2 = X^{\beta}$, and the regions for which we have not obtained (11) correspond to $(\alpha, \beta)\in \mathscr{S}$, then, working in an analogous manner to the case of rational primes (see [3], [9], [11]),

$$\sum_{z_2(p)\leqslant |q|^2<|p|^2<X^{1/2}}' S(\mathscr{B}_{pq}, q)$$

$$= (1+O(\eta))S(\mathscr{B}, X^{1/2}) \int_{(\alpha,\beta)\in\mathscr{S}} \frac{1}{\alpha\beta^2}\,\omega\left(\frac{1-\alpha-\beta}{\beta}\right) d\beta\, d\alpha, \qquad (17)$$

where $\omega(u)$ is Buchstab's function which satisfies

$$\omega(u) = \begin{cases} \dfrac{1}{u}, & \text{if } 1\leqslant u<2,\\[2mm] \dfrac{1+\log(u-1)}{u}, & \text{if } 2\leqslant u\leqslant 3,\end{cases}$$

and

$$\omega(u) \leqslant \frac{1+\log 2}{3}, \qquad \text{if } u>3.$$

All that remains to establish Theorem 2 is to show that the integral in (17) is strictly less than 1. This is a simple exercise in numerical integration. In fact, our method is slightly more complicated than outlined above in that (13) may be applied four times, and so we obtain

$$S(\mathscr{A}, X^{1/2}) \geqslant \delta S(\mathscr{B}, X^{1/2})(1+O(\eta))(1-\mathscr{I}_1-\mathscr{I}_2),$$

where $\mathscr{I}_1$ has the form of the integral in (17), while $\mathscr{I}_2$ is a similar four-dimensional integral.

The same basic philosophy lies behind the recent proof [3] that intervals of the form $[x, x+x^{0\cdot 525}]$ contain primes for all $x>x_0$. However, in our present situation we are hampered by a lack of results analogous to those used in [3]. We have a mean-value estimate, a zero-free region, and a mean-square L-function estimate which correspond to results used for rational primes in short intervals. We lack an analogue of the Halasz–Montgomery–Huxley large values estimate, the fourth power moment, and Watt's mean value theorem. Except for the last mentioned of these results, our problems stem from only having the 'Fourier variable' in play for our auxiliary estimates (compare [7] with [8]—Coleman explains this point on p. 81 of [7]). The L-functions of importance here are the Hecke L-functions, defined for $\Re s > 1$ by

$$L(s, \lambda^m) = \sum_{0 \leq \arg n < \pi/2} |n|^{-2s} \lambda^m(n), \tag{19}$$

which may be continued to the whole complex plane, as was first shown by Hecke [12]. The function is entire except when $m = 0$, when there is a simple pole at $s = 1$.

§3. *The arithmetical information.* In this section, we assemble all the information currently available appropriate for the sieve method described in the next section.

LEMMA 1. *Suppose that $T \geq 1$, $N \geq 1$. Then, for any coefficients $c(n)$,*

$$\sum_{m=1}^{T} \left| \sum_{|n|^2 \leq N} c(n)\lambda^m(n) \right|^2 \ll (T+N) \sum_{|n|^2 \leq N}{}' |c_1(n)|^2, \tag{20}$$

where

$$c_1(n) = \sum_{\arg v = \arg n} c(v),$$

and the ′ on the right hand side of (20) *indicates that the summation is only over primitive Gaussian integers n, that is, $n = x + iy$ with $(x, y) = 1$.*

Proof. See Lemma 6 of [7].

LEMMA 2. *Suppose that $T \geq 2$, $U \geq 2$. Then*

$$\sum_{m=1}^{T} \int_{-U}^{U} |L(\tfrac{1}{2} + it, \lambda^m)|^2 dt \ll TU(\log TU)^C. \tag{21}$$

Proof. This follows from Lemma 10 of [7]. It is important in our situation that the right side of (21) has a factor TU, not $T^2 + U^2$. That is why the known fourth power moment result is unsuitable for our purposes. We give a weaker fourth power result in the following lemma, which will suffice for our purposes.

LEMMA 3. *Suppose that $T \geq 2$, $T^{5/3} \geq U \geq 2$. Then*

$$\sum_{m=1}^{T} \int_{-U}^{U} |L(\tfrac{1}{2} + it, \lambda^m)|^4 \frac{dt}{1+|t|} \ll T^{5/3}(\log T)^C. \tag{22}$$

Proof. This follows from Lemma 2, along with the bound

$$L(\tfrac{1}{2} + it, \lambda^m) \ll (|t| + |m|)^{1/3} \log^4(|t| + |m|)$$

(see (6.16) of [7]), and the known fourth power moment

$$\sum_{m=1}^{T} \int_{-U}^{U} |L(\tfrac{1}{2}+it, \lambda^m)|^4 dt \ll (T^2+U^2)(\log UT)^C.$$

LEMMA 4. *Say that* $R, U \geqslant 2$, $T \geqslant S \geqslant R + \exp((\log UT)^{4/5})$, $1 \leqslant m \leqslant T$, $|t| \leqslant U$. *Then*

$$\left| \sum_{R \leqslant |p|^2 < S} |p|^{2it} \lambda^m(p) \right| \ll S \exp\left(-\frac{\log S}{2(\log TU)^{7/10}} \right). \tag{23}$$

Proof. Let $\Lambda(n)$ be the von Mangoldt function on $\mathbb{Z}[i]$. By partial summation, we need only show that, for $\max(R, S^{1/2}) \leqslant V \leqslant S$,

$$\left| \sum_{|n|^2 \leqslant V} \Lambda(n) |n|^{2it} \lambda^m(n) \right| \ll S \exp\left(-\frac{\log S}{2(\log TU)^{7/10}} \right).$$

Using the Perron formula (Theorem 3.12 in [16]) with $c = 1 + (\log V)^{-1}$, we have

$$\sum_{|n|^2 \leqslant V} \Lambda(n) |n|^{2it} \lambda^m(n) = -\frac{1}{2\pi i} \int_{c-iV}^{c+iV} \frac{L'}{L}(s-it, \lambda^m) \frac{V^s}{s} ds + o(V^{1/2}). \tag{24}$$

Using the zero-free region for $L(s, \lambda^m)$ (see [6]), we can move the line of integration to $\Re s = y$ with

$$y = 1 - (\log UT)^{-7/10}. \tag{25}$$

Using a standard upper bound (see (4.7) of [5]), we have

$$\frac{L'}{L}(s, \lambda^m) \ll (\log UT)^{17/10}$$

for

$$y \leqslant \Re s \leqslant c, \qquad -V-U \leqslant \Im s \leqslant V+U.$$

From this we deduce that the right side of (24) is

$$\ll V^y (\log UT)^{27/10} + o(V^{1/2}) \ll S \exp\left(-\frac{\log S}{2(\log TU)^{7/10}} \right),$$

which completes the proof.

In the following, we write

$$\mu = \exp((\log X)^{4/5}).$$

LEMMA 5. *Suppose that* $U, V \geqslant W$, $UV \leqslant X\mu^{-1}$, *and the coefficients* a_u, b_v *are bounded by divisor functions. Then*

$$\sum_{m=1}^{T} \left| \sum_{u \sim U} \sum_{v \sim V} \sum_{puv \in \mathscr{B}} a_u b_v \lambda^m(uvp) \right| \ll X\eta. \tag{26}$$

Proof. Using Perron's formula, it suffices to establish that, for $|t| \leqslant X$,

$$\sum_{m=1}^{T} \left| \sum_{u \sim U} \sum_{v \sim V} \sum_{|p|^2 \leqslant 4X/|UV|^2} a_u b_v \lambda^m(uvp) |uvp|^{2it} \right| \ll X\eta.$$

Using Cauchy's inequality with Lemmas 1 and 4, the left side above is

$$\ll \frac{X}{UV}(U(U+T)(\log X)^C)^{1/2}(V(V+T)(\log X)^C)^{1/2} \exp\left(-\frac{(\log X)^{4/5}}{2(\log X)^{7/10}}\right)$$

$$\ll X\eta,$$

as required.

LEMMA 6. *Suppose that $U \leqslant X^\tau \mu^{-1}$. If a_u is bounded by a divisor function, then*

$$\sum_{m=1}^{T} \left| \sum_{u \sim U} a_u \sum_{uv \in \mathscr{A}} \lambda^m(uv) \right| \ll X\eta. \tag{27}$$

Proof. Let $M = X^{1/2}$. Write

$$S_m = \sum_{u \sim U} a_u \sum_{uv \in \mathscr{A}} \lambda^m(uv).$$

Using the Perron formula we then have

$$S_m = \sum_{u \sim U} a_u \frac{1}{2\pi i} \int_{c-Mi}^{c+Mi} L(s, \lambda^m) \left(\frac{X}{|u|^2}\right)^s (1 - 2^{-s}) \frac{ds}{s} + O\left(\sum_{u \sim U} |a_u| \frac{X^{1/2}}{|u|^2} \log X\right)$$

$$= \mathscr{S}_m + O(X^{1/2}(\log X)^C),$$

say. We now shift the contour of integration back to $\Re s = \frac{1}{2}$. Writing $\mathscr{S}'_m$ for the new integral and using the bound $|L(s, \lambda^m)| \leqslant (|m| + |s|)^{(1-\sigma)/2}$, we get

$$\mathscr{S}_m = \mathscr{S}'_m + O\left(\max_{1/2 \leqslant \sigma \leqslant 1} \sum_{u \sim U} |a_u| \left(\frac{X}{|u|^2}\right)^\sigma \frac{X^{(1-\sigma)/4}}{M}\right).$$

Using $U \leqslant X^\tau$ the error term above is no more than $X^\varphi (\log X)^C$, where

$$\varphi = \max_{\sigma \leqslant 1}\left(\tau(1-\sigma) + \sigma + \frac{1-\sigma}{4} - \frac{1}{2}\right) = \frac{1}{2}.$$

Thus

$$\sum_{m=1}^{T} |S_m| = \sum_{m=1}^{T} |\mathscr{S}'_m| + O(X\eta).$$

Also,

$$\sum_{m=1}^{T} |\mathscr{S}'_m| \leqslant \sum_{m=1}^{T} X^{1/2} \int_{-M}^{M} |L(\tfrac{1}{2} + it, \lambda^m)| \left| \sum_{u \sim U} |u|^{-1-2it} \lambda^m(u) a_u \right| \frac{dt}{1+|t|}$$

$$\leqslant (XT)^{1/2}(T+U)^{1/2}(\log X)^C,$$

using Cauchy's inequality with Lemmas 1 and 2. The bound (27) then follows, since $T<U$ and $XTU\leqslant X^2\mu^{-1}$.

LEMMA 7. *Suppose that* $U\leqslant X^{\zeta}\mu^{-1}$, $V\leqslant X^{\rho}\mu^{-1}$, $\theta>\frac{1}{14}$, $UV\leqslant X^{\nu}\mu^{-1}$. *If* a_u, b_v *are bounded by divisor functions, then*

$$\sum_{m=1}^{T}\left|\sum_{u\sim U}a_u\sum_{u\sim V}b_v\sum_{uvr\in\mathscr{A}}\lambda^m(uvr)\right|\ll X\eta. \tag{28}$$

Proof. We may work as in the previous lemma, with the final expression to estimate being

$$\sum_{m=1}^{T}\int_{-M}^{M}|L(\tfrac{1}{2}+it,\lambda^m)|\left|\sum_{u\sim U}|u|^{-1-2it}\lambda^m(u)a_u\right|\left|\sum_{u\sim V}|v|^{-1-2it}\lambda^m(v)b_v\right|\frac{dt}{1+|t|}.$$

An appeal to Hölder's inequality and Lemmas 1 and 3 then gives an upper bound

$$\ll(\log X)^C X^{1/2}T^{5/12}(U+T)^{1/2}(V^2+T)^{1/4}.$$

The conditions on U, V, θ ensure that all the following four conditions are satisfied:

$$T^{5/6}UV\ll X\eta,\qquad T^{11/6}V\ll X\eta,$$
$$T^{7/3}\ll X\eta,\qquad T^{4/3}U\ll X\eta.$$

This establishes (28).

§4. *Asymptotic Formulae.* We can use (12) in conjunction with Lemma 5 to obtain some asymptotic formulae immediately. Suppose that we consider numbers

$$p_1\cdots p_j\in\mathscr{A},\qquad \text{with } p_i\sim P_i,$$

where

$$P_1\geqslant\mu,\qquad P_2\cdots P_l\geqslant W,\qquad P_{l+1}\cdots P_j\leqslant W.$$

The number of such numbers, say S', is $\delta S+O(X\gamma\eta)$, where S represents the number of solutions to

$$p_1\cdots p_j\in\mathscr{B},\qquad \text{with } p_i\sim P_i. \tag{29}$$

It is straightforward to obtain an asymptotic formula for S from (9), and thus deduce a formula for S'. Indeed, if a_r, b_s are bounded by divisor functions, then we have

$$\sum_{\substack{rsp\in\mathscr{A}\\ |r|^2>W,\,|s|^2>W,\,|rs|^2\mu\leqslant X}}a_rb_s=\delta\sum_{\substack{rsp\in\mathscr{B}\\ |r|^2>W,\,|s|^2>W,\,|rs|^2\mu\leqslant X}}a_rb_s+O(X\gamma\eta). \tag{30}$$

Now we introduce the sieve method.

LEMMA 8. *Suppose that $W \ll Y \ll X^{\tau-\varepsilon}$ for some $\varepsilon > 0$. Let $l(n)$ be a given function of $n \in \mathbb{Z}[i]$, with $n \sim Y$, of the form $l(n) = R|n|^k$, where $R > 0$, $-2 \leqslant k \leqslant 2$, and*

$$\mu \leqslant l(n) \ll X^\tau Y^{-1}.$$

Then, assuming that a_n is bounded by a divisor function,

$$\sum_{n \sim Y} a_n S(\mathscr{A}_n, l(n)) = \delta \sum_{n \sim Y} a_n S(\mathscr{B}_n, l(n)) + O(X\gamma\eta). \tag{31}$$

Proof. By Buchstab's identity,

$$\sum_{n \sim Y} a_n S(\mathscr{A}_n, l(n)) = \sum_{n \sim Y} a_n S(\mathscr{A}_n, \mu) - \sum_{n \sim Y} a_n \sum_{\mu \leqslant |q|^2 < l(n)} S(\mathscr{A}_{nq}, q)$$
$$= S_1 - S_s,$$

say. Since $|nq|^2 \ll X^\tau$, $|n|^2 \geqslant W$, $|q|^2 \geqslant \mu$, we recover a formula for S_2 from (30), after an appeal, if necessary, to Perron's formula to remove the condition $|q|^2 < l(n)$. The required formula for S_1 may be obtained by appealing to the familiar "Fundamental Lemma" concerning sieve theory. For this we need the arithmetical information from Lemma 3, from which we can deduce that

$$\sum_{n \sim Y} |a_n| \sum_{|d| \leqslant X^{\varepsilon/4}} \left| |\mathscr{A}_{dn}| - \frac{|\mathscr{A}|}{|dn|^2} \right| \ll X\gamma\eta.$$

This suffices to obtain the required formula

$$\sum_{n \sim Y} a_n S(\mathscr{A}_n, \mu) = \delta \sum_{n \sim Y} a_n S(\mathscr{B}_n, \mu) + O(X\gamma\eta),$$

by modifying the argument on page 258 of [**11**] say.

We now extend the result of Lemma 8 to smaller values of Y. To do this, we first write

$$z(y) = \begin{cases} X^\tau / Y, & \text{if } Y \geqslant W, \\ (X^\tau / Y)^{1/(j+2)}, & \text{if } WX^{-(j+1)\theta} \leqslant Y < WX^{-j\theta}, j = 0, 1, 2, \ldots \end{cases}$$

LEMMA 9. *Suppose that $X^\theta \ll Y \ll W$. Let $l(n)$ be a given function of $n \in \mathbb{Z}[i]$, $n \sim Y$, with $l(n) = R|n|^k$, where $R > 0$, $-2 \leqslant k \leqslant 2$, and*

$$l(n) \ll z(y).$$

Then, assuming that a_n is bounded by a divisor function,

$$\sum_{n \sim Y} a_n S(\mathscr{A}_n, l(n)) = \delta \sum_{n \sim Y} a_n S(\mathscr{B}_n, l(n)) + O(X\gamma\eta). \tag{32}$$

Proof. The idea is to decompose the left side of (32) using Buchstab's identity into parts for which either the Fundamental Lemma or a variant of Lemma 8 may be applied. Suppose that

$$WX^{-\theta} \leqslant Y < W.$$

Then, as in the previous lemma, we write the left side of (32) as $S_1 - S_2$, where we can give an asymptotic formula for S_1 using a fundamental lemma. For that part of S_2 with $|pn|^2 \geqslant W$, we note that

$$|pn|^2 \ll X^{\tau/2} W^{1/2} = X^{1/2},$$

and

$$|p|^2 \ll \frac{X^{\tau}}{Y|p|^2}.$$

Although S_2 does not quite have the same form as the left side of (31), the proof of Lemma 8 goes through essentially unchanged for our current sum.

We have that part of S_2 with $|pn|^2 < W$ still to deal with. We apply Buchstab's identity, and again have three cases to deal with: fundamental lemma, a variant of Lemma 8, and a sum

$$\sum_{n \sim Y} a_n \sum_{\substack{\mu \leqslant |q|^2 < |p|^2 < l(n) \\ |npq|^2 < W}} S(\mathscr{A}_{npq}, q),$$

for which a further application of Buchstab's identity is necessary. Since each prime variable has modulus squared at least μ, the iterative process must stop after no more than $(\log X)^{1/5}$ steps. Since the error involved at each stage is $O(X\gamma\eta)$, we thereby obtain (31) upon combining all the sums involving $\mathscr{B}$.

There are no difficulties in extending the range for Y to

$$WX^{-(j+1)\theta} \leqslant Y < WX^{-j\theta},$$

by taking $j = 1, 2, \ldots.$ in turn. This completes the proof.

LEMMA 10. *With the same notation,*

$$S(\mathscr{A}, X^{\theta}) = \delta S(\mathscr{B}, X^{\theta}) + O(X\gamma\eta).$$

Proof. This follows using the same procedures as Lemma 9.

LEMMA 11. *Suppose that $W \leqslant P \ll X^{\chi}$ and that*

$$\max(X^{\tau}/P, PX^{-2\theta}) \ll Q \ll X^{\nu}/(P\mu).$$

Then there is an asymptotic formula for

$$\sum_{p \sim P, q \sim Q} S(\mathscr{A}_{pq}, q). \tag{33}$$

Proof. First, we note that $\chi < \zeta$ and that

$$P > W \Rightarrow \frac{X^{\nu}}{P\mu} \leqslant \frac{X^{\rho}}{\mu}.$$

It follows, using Lemma 7 and a fundamental lemma, that we can give an asymptotic formula for

$$\sum_{p \sim P, q \sim Q} S(\mathscr{A}_{pq}, \mu).$$

Applying Buchstab's identity to (33) leaves us to estimate

$$\sum_{p\sim P,\, q\sim Q}\ \sum_{\mu\leqslant|r|^2<|q|^2} S(\mathscr{A}_{pqr}, r). \tag{34}$$

We note that we can assume that

$$|r|^2 \ll \left(\frac{X}{PQ}\right)^{1/2},$$

for otherwise the inner sum in (34) is empty. Thus

$$|pr|^2 \ll X^{1/2}\left(\frac{P}{Q}\right)^{1/2} \leqslant X^{\tau}.$$

Hence (34) counts numbers $pqrs \in \mathscr{A}$ which satisfy

$$|qs|^2 \gg W, \quad |p|^2 > W, \quad |r|^2 > \mu,$$

and so a formula may be obtained from (30) (after applying Perron's formula to disentangle the relationship between r and s).

LEMMA 12. *Suppose that* $W \leqslant PQ \leqslant X^{\zeta}\mu^{-1}$, $PQR \leqslant X^{\nu}\mu^{-1}$, $P \geqslant Q \geqslant R \geqslant X^{\theta}$. *Then there is an asymptotic formula for*

$$\sum_{\substack{p\sim p,\, q\sim Q\\ r\sim R}} S(\mathscr{A}_{pqr}, X^{\tau}/PQ).$$

Proof. Using a fundamental lemma result as in Lemma 11, it remains to estimate

$$\sum_{\substack{p\sim p,\, q\sim Q,\\ r\sim R}}\ \sum_{\mu\leqslant|s|^2<X^{\tau}/PQ} S(\mathscr{A}_{pqrs}, s).$$

This counts numbers $pqrst$ where

$$|s|^2 \geqslant \mu, \qquad |pq|^2 \geqslant W, \quad \text{and} \quad |rt|^2 \gg \frac{X}{|pqs|^2} \gg \frac{X}{PQX^{\tau}/PQ} = W.$$

Hence (30) is again applicable to complete the proof, after removing the interdependence between s and t.

LEMMA 13. *Suppose that* $P \geqslant Q \geqslant X^{\theta}$, $W \leqslant PQ \leqslant X^{\zeta}\mu^{-1}$, $PQ^2 \leqslant X^{\nu}\mu^{-1}$. *If* $Q^2P < X^{\tau}$, *then there is an asymptotic formula for*

$$S = \sum_{p\sim P, q\sim Q} S(\mathscr{A}_{pq}, q).$$

Otherwise

$$S = \sum_{p\sim P,\, q\sim Q} S(\mathscr{A}_{pq}, X^{\tau}/PQ)$$
$$- \sum_{\substack{p\sim P,\, q\sim Q\\ X^{\tau}/PQ\leqslant|r|^2<|q|^2}} S(\mathscr{A}_{pqr}, X^{\tau}/PQ) + \sum_{\substack{p\sim P,\, q\sim Q\\ X^{\tau}/PQ\leqslant|s|^2<|r|^2<|q|^2}} S(\mathscr{A}_{pqrs}, s), \tag{35}$$

and there is an asymptotic formula for the first two terms on the right side of (35).

Proof. This follows from Buchstab's identity in conjunction with Lemmas 8 and 12.

LEMMA 14. *Suppose that* $P \geqslant Q \geqslant X^{\theta}$, $WX^{-\theta} \leqslant PQ < W$. *If* $PQ^3 < X^{\tau}$, *then there is an asymptotic formula for*

$$S = \sum_{p\sim P,\, q\sim Q} S(\mathscr{A}_{pq}, q).$$

Otherwise,

$$S = \sum_{p\sim P,\, q\sim Q} S(\mathscr{A}_{pq}, (X^{\tau}/PQ)^{1/2}) - \sum_{\substack{p\sim P,\, q\sim Q \\ (X^{\tau}/PQ)^{1/2} \leqslant |r|^2 < |q|^2}} S(\mathscr{A}_{pqr}, X^{\tau}/PQ|r|^2) + {\sum}^* S(\mathscr{A}_{pqrs}, s), \tag{36}$$

with $*$ *representing*

$$p\sim P, \qquad q\sim Q, \qquad \left(\frac{X^{\tau}}{PQ}\right)^{1/2} \leqslant |r|^2 < |q|^2, \qquad \frac{X^{\tau}}{PQ|r|^2} \leqslant |s|^2 < |r|^2,$$

and there are asymptotic formulae for the first two sums on the right of (36).

Proof. This result follows from Buchstab's identity in conjunction with Lemmas 8 and 9. It should be noted that the variable s may take values with modulus squared smaller than X^{θ}. Indeed, for $\theta = 0{\cdot}119$ the value of $|s|^2$ can reduce down to $X^{0{\cdot}0475}$.

§5. *The final decomposition.* Having gathered all the arithmetical information in the previous section, we can now embark on the final part of the proof: to apply Buchstab's identity to $S(\mathscr{A}, X^{1/2})$ in such a way that (18) holds with $\mathscr{I}_1 + \mathscr{I}_2 < 1$. Applying Buchstab's identity twice, we have

$$S(\mathscr{A}, X^{1/2}) = S(\mathscr{A}, X^{\theta}) - \sum_{X^{\theta} \leqslant |p|^2 < X^{1/2}} S(\mathscr{A}_p, z(|p|^2)) + {\sum}^* S(\mathscr{A}_{pq}, Q). \tag{37}$$

Here $*$ represents the conditions

$$X^{\theta} \leqslant |p|^2 < X^{1/2}, \qquad z(|p|^2) \leqslant |q|^2 < \min\left(|p|^2, \frac{X^{1/2}}{|p|}\right).$$

We can give asymptotic formulae for the first two terms on the right side of (37) using Lemmas 8–10. We consider the final sum in sections as follows.

1. $W \leqslant |p|^2 < X^{1/2}$. We can apply Lemma 11 to part of this sum. We discard the remainder of the sum leading to a "loss"

$$\varphi_1 \delta S(\mathscr{B}, X^{1/2}),$$

where

$$\varphi_1 = \left(\int_{\sigma}^{\chi} \int_{\nu-\alpha}^{(1-\alpha)/2} + \int_{1/4+3\theta/2}^{\chi} \int_{\tau-\alpha}^{\alpha-2\theta} + \int_{\chi}^{1/2} \int_{\tau-\alpha}^{(1-\alpha)/2} \right) \frac{\omega((1-\alpha-\beta)/\beta)}{\alpha\beta^2} \, d\beta \, d\alpha.$$

Strictly speaking, $\nu-\alpha$ should be $\nu-\alpha-\exp(-(\log X)^{1/5})$, but this only introduces a further error $O(X\gamma\eta)$. We shall not point out similar details in future. Numerical integration gives, for $\theta = 0{\cdot}119$,

$$\varphi_1 < 0{\cdot}275.$$

2. $X^{1/3} \leqslant |p|^2 < W$. If $|pq|^2 < X^{\xi}$, then $|pq^2|^2 = |pq|^4|p|^{-2} < X^{2/3} < X^{\nu}$. Hence we can apply Lemma 13 to part of the sum in our present case. We thus have a loss from a two-dimensional integral

$$\varphi_2 = \int_{1/3}^{\sigma} \int_{\zeta-\alpha}^{(1-\alpha)/2} \frac{\omega((1-\alpha-\beta)/\beta)}{\alpha\beta^2} \, d\beta \, d\alpha,$$

and a loss from a four-dimensional integral amounting to

$$\varphi_3 = \int_{1/3}^{\sigma} \int_{(\tau-\alpha)/2}^{\zeta-\alpha} \int_{\tau-\alpha-\beta}^{\min(\beta,\alpha+\beta-2\theta)} \int_{\tau-\alpha-\beta}^{\min(\gamma,(1-\alpha-\beta-\gamma)/2)} \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} \, d\delta \, d\gamma \, d\beta \, d\alpha.$$

At $\theta = 0{\cdot}119$, we have

$$\varphi_2 < 0{\cdot}341, \qquad \varphi_3 < 0{\cdot}001.$$

3. $X^{\rho} \leqslant |p|^2 < X^{1/3}$. Again, $|pq|^2 < X^{\zeta} \Rightarrow |pq^2|^2 < X^{\nu}$, so the analysis of this section is identical to the previous one, except that the upper limit for $|q|$ is now $|p|$. Thus we have losses

$$\varphi_4 = \int_{\rho}^{1/3} \int_{\zeta-\alpha}^{\alpha} \frac{\omega((1-\alpha-\beta)/\beta)}{\alpha\beta^2} \, d\beta \, d\alpha,$$

and

$$\varphi_5 = \int_{\rho}^{1/3} \int_{(\tau-\alpha)/2}^{\zeta-\alpha} \int_{\tau-\alpha-\beta}^{\beta} \int_{\tau-\alpha-\beta}^{\gamma} \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} \, d\delta \, d\gamma \, d\beta \, d\alpha.$$

At $\theta = 0{\cdot}119$, we have

$$\varphi_4 < 0{\cdot}156, \qquad \varphi_5 < 0{\cdot}001.$$

4. $WX^{-\sigma} \leqslant |p|^2 < X^{\rho}$. Now the constraint $|pq^2|^2 < X^{\nu}$ is more stringent than $|pq| < X^{\zeta}$. For this case, we have a two-dimensional loss

$$\varphi_6 = \int_{1/2-2\theta}^{\rho} \int_{(\nu-\alpha)/2}^{\alpha} \frac{\omega((1-\alpha-\beta)/\beta)}{\alpha\beta^2} \, d\beta \, d\alpha,$$

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and a four-dimensional loss

$$\varphi_7 = \int_{1/2-2\theta}^{\rho} \int_{(\tau-\alpha)/2}^{(\nu-\alpha)/2} \int_{\tau-\alpha-\beta}^{\beta} \int^{*} \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} d\delta d\gamma d\beta d\alpha.$$

Here the $*$ indicates the conditions

$$\tau-\alpha-\beta \leqslant \delta \leqslant \gamma \quad \text{and} \quad \delta > 1-\alpha-\gamma-\sigma \text{ if } \alpha+\gamma \geqslant \sigma.$$

At $\theta = 0{\cdot}119$, we have

$$\varphi_6 < 0{\cdot}112, \qquad \varphi_7 < 0{\cdot}001.$$

5. $|p|^2 < WX^{-\theta}$, $|pq|^2 > WX^{-\theta}$. In view of Lemma 14, we can always give an asymptotic formula for the part of the sum with $|p|^2 < X^{\tau/4}$, and a formula for that part of the remainder with $|pq^3|^2 < X^{\tau}$, $|pq|^2 < W$. When $|pq|^2 > W$ we have a two-dimensional loss

$$\varphi_8 = \int_{\nu/3}^{\sigma-\theta} \int_{(\nu-\alpha)/2}^{\alpha} \frac{\omega((1-\alpha-\beta)/\beta)}{\alpha\beta^2} d\beta d\alpha,$$

and a four-dimensional loss

$$\varphi_9 = \int_{\tau/3}^{\sigma-\theta} \int_{(\tau-\alpha)/2}^{\min((\nu-\alpha)/2,\alpha)} \int_{\tau-\alpha-\beta}^{\min((\beta,\beta+\alpha-2\theta)} \times \int_{\tau-\alpha-\beta}^{\min(\gamma,(1-\alpha-\beta-\gamma)/2)} \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} d\delta d\gamma d\beta d\alpha.$$

(We omit here a couple of extra conditions on the integral ranges, corresponding to other ways of combining variables to get asymptotic formulae). At $\theta = 0{\cdot}119$ we have

$$\varphi_8 < 0{\cdot}036, \qquad \varphi_9 < 0{\cdot}001.$$

When $|pq|^2 < W$ there is just the four-dimensional loss

$$\phi_{10} = \int_{\tau/4}^{1/2-2\theta} \int_{(\tau-\alpha)/3}^{1/2-2\theta-\alpha} \int_{(\tau-\alpha-\beta)/2}^{\beta} \int_{\tau-\alpha-\beta-\gamma}^{\gamma} \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} d\delta d\gamma d\beta d\alpha.$$

At $\theta = 0{\cdot}119$, we have

$$\varphi_{10} < 0{\cdot}017.$$

6. $|pq|^2 < WX^{-\theta}$. Here we have just the four-dimensional loss

$$\varphi_{11} = \int_{\tau/5}^{1/2-3\theta} \int_{(\tau-\alpha)/4}^{\min(\alpha,1/2-2\theta-\alpha)} \int_{(\tau-\alpha-\beta)/3}^{\beta} \int_{(\tau-\alpha-\beta-\gamma)/2}^{\gamma} \times \frac{\omega((1-\alpha-\beta-\gamma-\delta)/\delta)}{\alpha\beta\gamma\delta^2} d\delta d\gamma d\beta d\alpha.$$

We have $\varphi_{11} < 0{\cdot}001$ at $\theta = 0{\cdot}119$.

Conclusion of Proof. We have

$$\sum_{j=1}^{11} \varphi_j < 0{\cdot}943.$$

Thus

$$S(\mathscr{A}, X^{1/2}) > \delta 0{\cdot}057 S(\mathscr{B}, X^{1/2}) + O(X\gamma\eta).$$

This completes the proof of Theorem 2. The lower bound constant by the above method ceases to be positive for some value of θ between $0{\cdot}117$ and $0{\cdot}118$, but we have not thought it worthwhile to spend a great deal of time on the numerical calculations. The more interesting challenge would be to improve the results of §3.

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