Ergod. Th. & Dynam. Sys. (2009), **29**, 73–109
doi:10.1017/S0143385708000138

On Khintchine exponents and Lyapunov exponents of continued fractions

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(*Received* 20 *June* 2007 *and accepted in revised form* 28 *January* 2008)

Abstract. Assume that $x \in [0, 1)$ admits its continued fraction expansion $x = [a_1(x), a_2(x), \ldots]$. The Khintchine exponent $\gamma(x)$ of x is defined by $\gamma(x) := \lim_{n\to\infty}(1/n) \sum_{j=1}^{n} \log a_j(x)$ when the limit exists. The Khintchine spectrum $\dim E_\xi$ is studied in detail, where $E_\xi := \{x \in [0, 1) : \gamma(x) = \xi\}$ $(\xi \geq 0)$ and dim denotes the Hausdorff dimension. In particular, we prove the remarkable fact that the Khintchine spectrum $\dim E_\xi$, as a function of $\xi \in [0, +\infty)$, is neither concave nor convex. This is a new phenomenon from the usual point of view of multifractal analysis. Fast Khintchine exponents defined by $\gamma^{\varphi}(x) := \lim_{n\to\infty}(1/(\varphi(n))) \sum_{j=1}^{n} \log a_j(x)$ are also studied, where $\varphi(n)$ tends to infinity faster than n does. Under some regular conditions on φ, it is proved that the fast Khintchine spectrum $\dim(\{x \in [0, 1] : \gamma^{\varphi}(x) = \xi\})$ is a constant function. Our method also works for other spectra such as the Lyapunov spectrum and the fast Lyapunov spectrum.

1. *Introduction and statements*

The continued fraction of a real number can be generated by the Gauss transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod 1 \quad \text{for } x \in (0, 1) \tag{1.1}$$

in the sense that every irrational number x in $[0, 1)$ can be written uniquely as an infinite expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n(x) + T^n(x)}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}} \tag{1.2}$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$ are called *partial quotients* of x ($\lfloor x \rfloor$ denoting the integral part of x). For simplicity, we will denote the second term in (1.2) by $[a_1, a_2, \ldots, a_n + T^n(x)]$ and the third term by $[a_1, a_2, a_3, \ldots]$.

It was known to Borel [**4**] (1909) that for Lebesgue almost all $x \in [0, 1)$, there exists a subsequence $\{a_{n_r}(x)\}$ of $\{a_n(x)\}$ such that $a_{n_r}(x) \to \infty$. A more explicit result due to Borel and Bernstein (see [**2, 4, 5**]) is the 0–1 law which hints that for almost all $x \in [0, 1]$, $a_n(x) > \varphi(n)$ holds for infinitely many n's or finitely many n's depending on whether $\sum_{n \geq 1}(1/(\varphi(n)))$ diverges or converges. This naturally led to the question of quantifying the exceptional sets in terms of Hausdorff dimension (denoted by dim). The first published work on this was due to Jarnik [**20**] (1928) who was concerned with the set E of continued fractions with bounded partial quotients and with the sets $E_2, E_3, \ldots$, where E_α is the set of continued fractions whose partial quotients do not exceed α. He established that the set E is of full Hausdorff dimension, but he did not find the exact dimensions of $E_2, E_3, \ldots$. Later, many authors worked on estimating $\dim E_2$, including Good [**15**], Bumby [**8**], Hensley [**18, 19**], Jenkinson and Pollicott [**21**], and Mauldin and Urbański [**28**] (see also references therein). Up to now, the optimal approximation of $\dim E_2$ is the result given by Jenkinson [**22**] (2004):

$$\dim E_2 = 0.531280506277205141624468647368471785493059109018398779\ldots$$

which is claimed to be accurate to 54 decimal places.

In the present paper, we study the Khintchine exponents and Lyapunov exponents of continued fractions. For any $x \in [0, 1)$ with continued fraction (1.2), we define its *Khintchine exponent* $\gamma(x)$ and *Lyapunov exponent* $\lambda(x)$ respectively by

$$\gamma(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j(x))|,$$

if the limits exist. The Khintchine exponent of x stands for the average (geometric) growth rate of the partial quotients $a_n(x)$, and the Lyapunov exponent, which has been extensively studied from dynamical systems point of view, stands for the expansion rate of T. Their common feature is that both are Birkhoff averages.

Let $\varphi : \mathbb{N} \to \mathbb{R}_+$. Assume that $\lim_{n\to\infty}((\varphi(n))/n) = \infty$. The *fast Khintchine exponent* and *fast Lyapunov exponent* of $x \in [0, 1]$, relative to φ, are respectively defined by

$$\gamma^{\varphi}(x) := \lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{j=1}^{n} \log a_j(x) = \lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log a_1(T^j(x)),$$

$$\lambda^{\varphi}(x) := \lim_{n\to\infty} \frac{1}{\varphi(n)} \log |(T^n)'(x)| = \lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{j=0}^{n-1} \log |T'(T^j(x))|.$$

It is well known (see [**3, 35**]) that the transformation T is measure preserving and ergodic with respect to the Gauss measure μ_G defined as

$$d\mu_G = \frac{dx}{(1 + x)\log 2}.$$

An application of the Birkhoff ergodic theorem yields that for Lebesgue almost all $x \in [0, 1)$,

$$\gamma(x) = \xi_0 = \int \log a_1(x)\, d\mu_G = \frac{1}{\log 2} \sum_{n=1}^{\infty} \log n \cdot \log\left(1 + \frac{1}{n(n+2)}\right) = 2.6854\ldots,$$

$$\lambda(x) = \lambda_0 = \int \log |T'(x)|\, d\mu_G = \frac{\pi^2}{6 \log 2} = 2.37314\ldots.$$

Here ξ_0 is called the *Khintchine constant* and λ_0 the *Lyapunov constant*. Both constants are relative to the Gauss measure.

For real numbers $\xi, \beta \geq 0$, we are interested in the level sets of Khintchine exponents and Lyapunov exponents:

$$E_\xi := \{x \in [0, 1) : \gamma(x) = \xi\},$$
$$F_\beta := \{x \in [0, 1) : \lambda(x) = \beta\}.$$

We are also interested in the level sets of fast Khintchine exponents and fast Lyapunov exponents:

$$E_\xi(\varphi) := \{x \in [0, 1) : \gamma^\varphi(x) = \xi\},$$
$$F_\beta(\varphi) := \{x \in [0, 1) : \lambda^\varphi(x) = \beta\}.$$

The *Khintchine spectrum* and the *Lyapunov spectrum* are the dimensional functions

$$t(\xi) := \dim E_\xi, \quad \tilde{t}(\beta) := \dim F_\beta.$$

The functions

$$t^\varphi(\xi) := \dim E_\xi(\varphi), \quad \tilde{t}^\varphi(\beta) := \dim F_\beta(\varphi)$$

are called the *fast Khintchine spectrum* and the *fast Lyapunov spectrum* relative to φ.

Pollicott and Weiss [**34**] initially studied the level set of F_β and obtained some partial results about the function $\tilde{t}(\beta)$. In the present work, we will present a complete study on the Khintchine spectrum and the Lyapunov spectrum. The fast Khintchine spectrum and fast Lyapunov spectrum are considered here for the first time. We shall see that both functions $t^\varphi(\xi)$ and $\tilde{t}^\varphi(\beta)$ are equal.

We start with the statement of our results on fast spectra.

THEOREM 1.1. *Suppose that* $(\varphi(n+1) - \varphi(n)) \uparrow \infty$ *and* $\lim_{n\to\infty}((\varphi(n+1))/(\varphi(n))) := b \geq 1$. *Then* $E_\xi(\varphi) = F_{2\xi}(\varphi)$ *and* $\dim E_\xi(\varphi) = 1/(b+1)$ *for all* $\xi \geq 0$.

In order to state our results on the Khintchine spectrum, let us first introduce some notation. Let

$$D := \{(t, q) \in \mathbb{R}^2 : 2t - q > 1\}, \quad D_0 := \{(t, q) \in \mathbb{R}^2 : 2t - q > 1, 0 \leq t \leq 1\}.$$

For $(t, q) \in D$, define

$$P(t, q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} \omega_j^q([\omega_j, \ldots, \omega_n + x])^{2t}\right).$$

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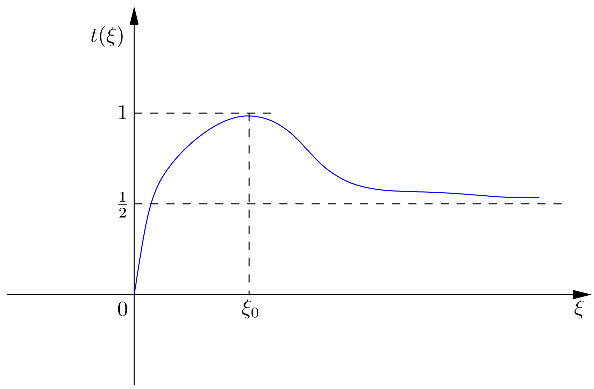

FIGURE 1. Khintchine spectrum.

It will be proved that $P(t, q)$ is an analytic function in D (Proposition 4.6). Moreover, for any $\xi \geq 0$, there exists a unique solution $(t(\xi), q(\xi)) \in D_0$ to the equation

$$\begin{cases} P(t, q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t, q) = \xi \end{cases}$$

(Proposition 4.13).

THEOREM 1.2. *Let* $\xi_0 = \int \log a_1(x)\, d\mu_G(x)$. *For* $\xi \geq 0$, *the set* E_ξ *is of Hausdorff dimension* $t(\xi)$. *Furthermore, the dimension function* $t(\xi)$ *has the following properties:*

(1) $t(\xi_0) = 1$ *and* $t(+\infty) = 1/2$;

(2) $t'(\xi) < 0$ *for all* $\xi > \xi_0$, $t'(\xi_0) = 0$, *and* $t'(\xi) > 0$ *for all* $\xi < \xi_0$;

(3) $t'(0+) = +\infty$ *and* $t'(+\infty) = 0$;

(4) $t''(\xi_0) < 0$, *but* $t''(\xi_1) > 0$ *for some* $\xi_1 > \xi_0$, *so* $t(\xi)$ *is neither convex nor concave.*

See Figure 1 for the graph of $t(\xi)$.

It should be noticed that the above fourth (non-convexity) property of $t(\xi)$ shows a new phenomenon for the multifractal analysis in our settings.

Let

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\}, \qquad \tilde{D}_0 := \{(\tilde{t}, q) : \tilde{t} - q > 1/2,\ 0 \leq \tilde{t} \leq 1\}.$$

For $(\tilde{t}, q) \in \tilde{D}$, define

$$P_1(\tilde{t}, q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\left(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\omega_j, \ldots, \omega_n + x])^{2(\tilde{t}-q)} \right).$$

In fact, $P_1(\tilde{t}, q) = P(\tilde{t} - q, 0)$, thus $P_1(\tilde{t}, q)$ is analytic in $\tilde{D}$.

Denote $\gamma_0 := 2\log((1 + \sqrt{5})/2)$. For any $\beta \in (\gamma_0, \infty)$, the system

$$\begin{cases} P_1(\tilde{t}, q) = q\beta, \\ \dfrac{\partial P_1}{\partial q}(\tilde{t}, q) = \beta \end{cases}$$

admits a unique solution $(\tilde{t}(\beta), q(\beta)) \in \tilde{D}_0$ (Proposition 6.3).

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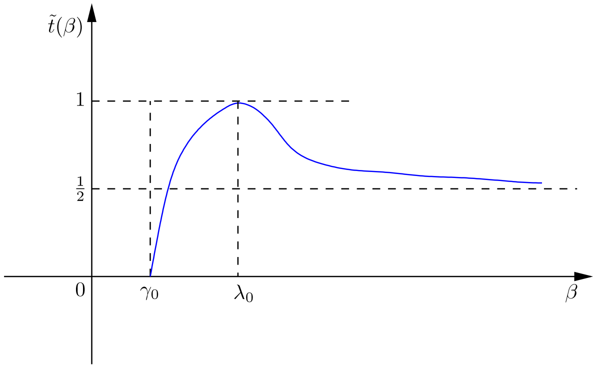

FIGURE 2. Lyapunov spectrum.

THEOREM 1.3. *Let* $\lambda_0 = \int \log |T'(x)|\, d\mu_G$ *and* $\gamma_0 = 2\log((1+\sqrt{5})/2)$. *For any* $\beta \in [\gamma_0, \infty)$, *the set* F_β *is of Hausdorff dimension* $\tilde{t}(\beta)$. *Furthermore, the dimension function* $\tilde{t}(\xi)$ *has the following properties:*

(1) $\tilde{t}(\lambda_0) = 1$ *and* $\tilde{t}(+\infty) = 1/2$;

(2) $\tilde{t}'(\beta) < 0$ *for all* $\beta > \lambda_0$, $\tilde{t}'(\lambda_0) = 0$, *and* $\tilde{t}'(\beta) > 0$ *for all* $\beta < \lambda_0$;

(3) $\tilde{t}'(\gamma_0+) = +\infty$ *and* $\tilde{t}'(+\infty) = 0$;

(4) $\tilde{t}''(\lambda_0) < 0$, *but* $\tilde{t}''(\beta_1) > 0$ *for some* $\beta_1 > \lambda_0$, *so* $\tilde{t}(\beta)$ *is neither convex nor concave.*

See Figure 2 for the graph of $\tilde{t}(\beta)$.

The last two theorems are concerned with special Birkhoff spectra. In general, let (X, T) be a dynamical system (T being a map from a metric space X into itself). The Birkhoff average of a function $\phi : X \to \mathbb{R}$, defined by

$$\overline{\phi}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)), \qquad x \in X$$

(if the limit exists), has been widely studied. From the point of view of multifractal analysis, one is often interested in determining the Hausdorff dimension of the set $\{x \in X : \overline{\phi}(x) = \alpha\}$ for a given $\alpha \in \mathbb{R}$. The function

$$f(\alpha) := \dim(\{x \in X : \overline{\phi}(x) = \alpha\})$$

is called the *Birkhoff spectrum* for the function ϕ. When X is compact, T and ϕ are continuous, the Birkhoff spectrum has been extensively studied (see [**1, 13, 14, 33**] and the references therein).

The main tool of our study is the Ruelle–Perron–Frobenius operator with potential function

$$\Phi_{t,q}(x) = -t \log |T'(x)| + q \log a_1(x), \qquad \Psi_t(x) = -t \log |T'(x)|,$$

where (t, q) are suitable parameters. The classical way to obtain the spectrum through Ruelle theory usually fixes q and finds $T(q)$ as the solution of $P(T(q), q) = 0$. (Here $P(t, q)$ is the pressure corresponding to the potential function of two parameters.)

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By focusing on the curve $T(q)$, one can only get some partial results [**34**]. In the present paper, we look for multifractal information from the whole two-dimensional surface defined by the pressure $P(t, q)$ rather than the single curve $T(q)$. This leads us to obtain complete graphs of the Khintchine spectrum and Lyapunov spectrum.

For the Gauss dynamics, there exist several works on pressure functions associated with different potentials. For a detailed study on pressure functions associated with one potential function, we refer to work by Mayer [**30–32**], and for pressure functions associated with two potential functions, we refer to work by Pollicott and Weiss [**34**], Walters [**36, 37**] and Hanus *et al* [**16**]. We will use the theory developed in [**16**].

The paper is organized as follows. In §2, we collect and establish some basic results that will be used later. Section 3 is devoted to proving the results about the fast Khintchine spectrum and fast Lyapunov spectrum (Theorem 1.1). In §4, we present a general Ruelle operator theory developed in [**16**] and then apply it to the Gauss transformation. Based on §4, we establish Theorem 1.2 in §5. The last section is devoted to the study of the Lyapunov spectrum (Theorem 1.3).

The present paper is a part of the second author's Ph.D. thesis.

2. *Preliminary*

In this section, we collect some known facts and establish some elementary properties of continued fractions that will be used later. For a wealth of classical results about continued fractions, see the books by Cassels [**9**] and Hardy and Wright [**17**]. The books by Billingsley [**3**] and Cornfeld *et al* [**10**] contain an excellent introduction to the dynamics of the Gauss transformation and its connection with Diophantine approximation.

2.1. *Elementary properties of continued fractions.*

Denote by p_n/q_n the usual nth *convergent* of the continued fraction $x = [a_1(x), a_2(x), \ldots] \in [0, 1) \setminus \mathbb{Q}$, defined by

$$\frac{p_n}{q_n} := [a_1(x), \ldots, a_n(x)] := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots + \cfrac{1}{a_n(x)}}}.$$

It is known (see [**25**, p. 9]) that p_n, q_n can be obtained by the following recursive relation:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 2), \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 2). \end{aligned}$$

Furthermore, the following lemma holds.

LEMMA 2.1. [**26**, p. 5] *Let* $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R}^+$. *Define inductively*

$$\begin{gathered} Q_{-1} = 0, \quad Q_0 = 1, \\ Q_n(\varepsilon_1, \ldots, \varepsilon_n) = \varepsilon_n Q_{n-1}(\varepsilon_1, \ldots, \varepsilon_{n-1}) + Q_{n-2}(\varepsilon_1, \ldots, \varepsilon_{n-2}). \end{gathered}$$

(Q_n is commonly called a continuant.) Then:

(i) $Q_n(\varepsilon_1, \ldots, \varepsilon_n) = Q_n(\varepsilon_n, \ldots, \varepsilon_1)$;
(ii) $q_n = Q_n(a_1, \ldots, a_n)$, $p_n = Q_{n-1}(a_2, \ldots, a_n)$.

As consequences, we have the following results.

LEMMA 2.2. [**25**] *For any $a_1, a_2, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{N}$, let $q_n = q_n(a_1, \ldots, a_n)$ and $p_n = p_n(a_1, \ldots, a_n)$. Then:*
(i) $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$;
(ii) $q_{n+m}(a_1, \ldots, a_n, b_1, \ldots, b_m) = q_n(a_1, \ldots, a_n)q_m(b_1, \ldots, b_m) + q_{n-1}(a_1, \ldots, a_{n-1})p_{m-1}(b_1, \ldots, b_{m-1})$;
(iii) $q_n \geq 2^{((n-1)/2)}$, $\prod_{k=1}^n a_k \leq q_n \leq \prod_{k=1}^n (a_k + 1)$.

LEMMA 2.3. [**39**] *For any $a_1, a_2, \ldots, a_n, b \in \mathbb{N}$,*

$$\frac{b+1}{2} \leq \frac{q_{n+1}(a_1, \ldots, a_j, b, a_{j+1}, \ldots, a_n)}{q_n(a_1, \ldots, a_j, a_{j+1}, \ldots, a_n)} \leq b + 1 \quad (\text{for all } 1 \leq j < n).$$

For any $a_1, a_2, \ldots, a_n \in \mathbb{N}$, let

$$I_n(a_1, a_2, \ldots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\}, \tag{2.1}$$

which is called an *nth-order cylinder*.

LEMMA 2.4. [**26**, p. 18] *For any $a_1, a_2, \ldots, a_n \in \mathbb{N}$, the nth-order cylinder $I_n(a_1, a_2, \ldots, a_n)$ is the interval with the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. As a consequence, the length of $I_n(a_1, \ldots, a_n)$ is equal to*

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{2.2}$$

We will denote by $I_n(x)$ the nth-order cylinder that contains x, i.e. $I_n(x) = I_n(a_1(x), \ldots, a_n(x))$. Let $B(x, r)$ denote the ball centered at x with radius r. For any $x \in I_n(a_1, \ldots, a_n)$, the following relationship holds between the ball $B(x, |I_n(a_1, \ldots, a_n)|)$ and $I_n(a_1, \ldots, a_n)$, which is called the *regular property* in [**6**].

LEMMA 2.5. [**6**] *Let $x = [a_1, a_2, \ldots]$.*
(i) *If $a_n \neq 1$, then $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_n(a_1, \ldots, a_n + j)$.*
(ii) *If $a_n = 1$ and $a_{n-1} \neq 1$, then $B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_{n-1}(a_1, \ldots, a_{n-1} + j)$.*
(iii) *If $a_n = 1$ and $a_{n-1} = 1$, then $B(x, |I_n(x)|) \subset I_{n-2}(a_1, \ldots, a_{n-2})$.*

The Gauss transformation T admits the following *Jacobian estimate*.

LEMMA 2.6. *There exists a positive number $K > 1$ such that for all irrational x in* $[0, 1)$,

$$0 < \frac{1}{K} \leq \sup_{n \geq 0} \sup_{y \in I_n(x)} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq K < \infty.$$

Proof. Assume that $x = [a_1, \ldots, a_n, \ldots] \in [0, 1) \setminus \mathbb{Q}$. For any $n \geq 0$ and $y \in I_n(x) = I_n(a_1, \ldots, a_n)$, by the fact that $T'(x) = -(1/x^2)$ we get

$$\sum_{j=0}^{n-1} |\log |T'(T^j(x))| - \log |T'(T^j(y))|| = 2 \sum_{j=0}^{n-1} |\log T^j(x) - \log T^j(y)|.$$

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Applying the mean-value theorem, we obtain

$$|\log T^j(x) - T^j(y)| = \left|\frac{T^j(x) - T^j(y)}{T^j(z)}\right| \leq \frac{a_{j+1}}{q_{n-j}(a_{j+1}, \ldots, a_n)},$$

where the assertion follows from the fact that all three points $T^j(x)$, $T^j(y)$ and $T^j(z)$ belong to $I_{n-j}(a_{j+1}, \ldots, a_n)$. By Lemma 2.2,

$$\sum_{j=0}^{n-1} |\log T^j(x) - \log T^j(y)| \leq \sum_{j=0}^{n-1} \frac{1}{q_{n-j-1}(a_{j+2}, \ldots, a_n)} \leq \sum_{j=0}^{n-1} \left(\frac{1}{2}\right)^{n-j-2} \leq 4.$$

Thus the result is proved with $K = e^4$. □

The above Jacobian estimate property of T enables us to control the length of $I_n(x)$ by $|(T^n)'(x)|^{-1}$, through the fact that $\int_{I_n(x)} |(T^n)'(y)|\, dy = 1$.

LEMMA 2.7. *There exists a positive constant $K > 0$ such that for all irrational numbers x in* $[0, 1)$,

$$\frac{1}{K} \leq \frac{|I_n(x)|}{|(T^n)'(x)|^{-1}} \leq K.$$

We remark that from Lemmas 2.4 and 2.7,

$$\frac{1}{2K} q_n^2(x) \leq |(T^n)'(x)| \leq K q_n^2(x).$$

So the Lyapunov exponent $\lambda(x)$ is nothing but the growth rate of $q_n(x)$ up to a multiplicative constant 2:

$$\lambda(x) = \lim_{n\to\infty} \frac{2}{n} \log q_n(x).$$

For any irrational number x in $[0, 1)$, let $N_n(x) := \{j \leq n : a_j(x) \neq 1\}$. Set

$$A := \left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \log q_n(x) = \frac{\gamma_0}{2}\right\},$$

$$B := \left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = 0\right\},$$

$$C := \left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sharp N_n(x) = 0\right\},$$

where $\sharp$ stands for the cardinality of a set. Then the following relationship holds.

LEMMA 2.8. *In the notation given above,*

$$A = B \subset C.$$

Proof. It is clear that $A \subset C$ and $B \subset C$. Let us prove that $A = B$. First observe that, by Lemma 2.3,

$$\begin{aligned} \frac{1}{n} \log q_n(x) &\geq \frac{1}{n} \sum_{j\in N_n(x)} \log \frac{a_j(x) + 1}{2} + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1) \\ &\geq \frac{1}{n} \sum_{j\in N_n(x)} \log a_j(x) - \frac{1}{n} \sum_{j\in N_n(x)} \log 2 + \frac{1}{n} \log q_{n-\sharp N_n}(1, \ldots, 1). \end{aligned}$$

Assume $x \in A$. Since $A \subset C$,

$$-\frac{1}{n}\sum_{j\in N_n(x)}\log 2+\frac{1}{n}\log q_{n-\sharp N_n}(1,\ldots,1)\longrightarrow 0+\frac{\gamma_0}{2}\quad(n\to\infty).$$

Now by the assumption $x \in A$, it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\log a_j(x)=\frac{1}{n}\sum_{j\in N_n(x)}\log a_j(x)=0.$$

Therefore we have proved $A \subset B$. For the inverse inclusion, notice that

$$\frac{1}{n}\log q_n(x)\le\frac{1}{n}\sum_{j\in N_n(x)}\log(a_j(x)+1)+\frac{1}{n}\log q_{n-\sharp N_n}(1,\ldots,1).$$

Let $x \in B$. Since $B \subset C$,

$$\lim_{n\to\infty}\frac{1}{n}\log q_{n-\sharp N_n}(1,\ldots,1)=\frac{\gamma_0}{2}.$$

Therefore by the assumption $x \in B$, we get

$$\limsup_{n\to\infty}\frac{1}{n}\log q_n(x)\le\frac{\gamma_0}{2}.$$

Thus $B \subset A$. □

2.2. *Exponents $\gamma(x)$ and $\lambda(x)$.* In this subsection, we make a quick examination of the Khintchine exponent $\gamma(x)$ and compare it with the Lyapunov exponent $\lambda(x)$. Our main concern is the possible values of both exponent functions.

A first observation is that for any $x \in [0, 1)$, $\gamma(x) \ge 0$ and $\lambda(x) \ge \gamma_0 = 2\log((\sqrt{5}+1)/2)$. By the Birkhoff ergodic theorem, we know that the Khintchine exponent $\gamma(x)$ attains the value ξ_0 for almost all points x with respect to the Lebesgue measure. We will show that every positive number is the Khintchine exponent $\gamma(x)$ of some point x.

PROPOSITION 2.9. *For any $\xi \ge 0$, there exists a point $x_0 \in [0, 1)$ such that $\gamma(x_0) = \xi$.*

Proof. Assume that $\xi > 0$ (for $\xi = 0$, we take $x_0 = (1+\sqrt{5})/2$ corresponding to $a_n \equiv 1$). Take an increasing sequence of integers $\{n_k\}_{k\ge 1}$ satisfying

$$n_0=1,\quad n_{k+1}-n_k\to\infty\quad\text{and}\quad\frac{n_k}{n_{k+1}}\to 1\quad\text{as } k\to\infty.$$

Let $x_0 \in (0, 1)$ be a point whose partial quotients satisfy

$$e^{(n_k-n_{k-1})\xi}\le a_{n_k}\le e^{(n_k-n_{k-1})\xi}+1;\qquad a_n=1\text{ otherwise.}$$

Since, for $n_k \le n < n_{k+1}$,

$$\frac{1}{n_{k+1}}\sum_{i=1}^{k}\log e^{(n_i-n_{i-1})\xi}\le\frac{1}{n}\sum_{j=1}^{n}\log a_j\le\frac{1}{n_k}\sum_{i=1}^{k}\log(e^{(n_i-n_{i-1})\xi}+1),$$

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then

$$\gamma(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \xi. \qquad \square$$

In the following, we will show that the sets E_ξ and F_β are never equal. So it is two different problems to study $\gamma(x)$ and $\lambda(x)$. However, as we will see, $E_\xi(\phi) = F_{2\xi}(\phi)$ when ϕ grows faster than n.

PROPOSITION 2.10. *For any $\xi \geq 0$ and $\beta \geq 2\log((\sqrt{5}+1)/2)$, we have $E_\xi \neq F_\beta$.*

Proof. Given $\xi \geq 0$, it suffices to construct two numbers with the same Khintchine exponent ξ but different Lyapunov exponents.

For the first number, take just the number x_0 constructed in the proof of Proposition 2.9. We claim that

$$\lambda(x_0) = 2\xi + 2\log\frac{\sqrt{5}+1}{2}. \tag{2.3}$$

In fact, by Lemma 2.3,

$$\prod_{j=1}^{k}\left(\frac{a_{n_j}+1}{2}\right) q_{n_k-k}(1,\ldots,1) \leq q_{n_k}(a_1,\ldots,a_{n_k}) \leq \prod_{j=1}^{k}(a_{n_j}+1) q_{n_k-k}(1,\ldots,1). \tag{2.4}$$

Then, by the assumption on n_k,

$$\lambda(x_0) = \lim_{n\to\infty}\frac{2}{n}\log q_n(x_0) = 2\left(\xi + \log\frac{\sqrt{5}+1}{2}\right).$$

Now construct the second number. Fix $k \geq 1$. Define $x_1 = [\varsigma_1, \ldots, \varsigma_n, \ldots]$, where

$$\varsigma_n = \left(\underbrace{\overbrace{1,\ldots,1,\lfloor e^{k\xi}\rfloor}^{k},\ldots,1,\ldots,1,\lfloor e^{k\xi}\rfloor}_{kn}, \left\lfloor\left(\frac{e^{(k+1)\xi}}{[e^{k\xi}]}\right)^n\right\rfloor\right).$$

Notice that there are n small vectors $(1,\ldots,1,\lfloor e^{k\xi}\rfloor)$ in ς_n and that the length of ς_n is equal to $N_k := kn+1$. We can prove that

$$\gamma(x_1) = \xi, \quad \lambda(x_1) = \lambda([\overline{1,\ldots,\lfloor e^{k\xi}\rfloor}]) + 2\xi - \frac{2}{k}\log\lfloor e^{k\xi}\rfloor,$$

by the same arguments as in proving the similar result for x_0. It is clear that $\lambda(x_0) \neq \lambda(x_1)$ for large $k \geq 1$. $\square$

It is evident that Proposition 2.9 and the formula (2.3) yield the following result due to Pollicott and Weiss [**34**].

COROLLARY 2.11. [**34**] *For any $\beta \geq 2\log((\sqrt{5}+1)/2)$, there exists a point $x_0 \in [0,1)$ such that $\lambda(x_0) = \beta$.*

2.3. *Pointwise dimension.* We will compare the pointwise dimension and the Markov pointwise dimension (corresponding to the continued fraction system) of a Borel probability measure.

Let μ be a Borel probability measure on $[0, 1)$. Define the pointwise dimension and the Markov pointwise dimension respectively by

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r}, \quad \delta_\mu(x) := \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|},$$

if the limits exist, where $B(x, r)$ is the ball centered at x with radius r.

For two series $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, we write $u_n \asymp v_n$ which means that there exist absolute positive constants c_1, c_2 such that $c_1 v_n \leq u_n \leq c_2 v_n$ for n large enough. Sometimes, we need the condition

$$\mu(B(x, |I_n(x)|)) \asymp \mu(I_n(x)) \tag{2.5}$$

at a point x. The following relationship holds between $\delta_\mu(x)$ and $d_\mu(x)$.

LEMMA 2.12. *Let μ be a Borel measure.*

(a) *Assume that condition (2.5) holds. If $d_\mu(x)$ exists, then $\delta_\mu(x)$ exists and $\delta_\mu(x) = d_\mu(x)$.*

(b) *If $\delta_\mu(x)$ and $\lambda(x)$ both exist, then $d_\mu(x)$ exists and $\delta_\mu(x) = d_\mu(x)$.*

Proof. (a) If the limit defining $d_\mu(x)$ exists, then the limit

$$\lim_{n\to+\infty} \frac{\log \mu(B(x, |I_n(x)|))}{\log |I_n(x)|}$$

exists and is equal to $d_\mu(x)$. Thus, by (2.5), the limit defining $\delta_\mu(x)$ also exists and is equal to $d_\mu(x)$.

(b) Since $\lambda(x)$ exists,

$$\lim_{n\to\infty} \frac{\log |I_n(x)|}{\log |I_{n+1}(x)|} = \lim_{n\to\infty} \frac{1}{n} \log |I_n(x)| \Big/ \frac{1}{n+1} \log |I_{n+1}(x)| = 1 \tag{2.6}$$

by Lemma 2.7. For any $r > 0$, there exists an n such that $|I_{n+1}(x)| \leq r < |I_n(x)|$. Then, by Lemma 2.5, $I_{n+1}(x) \subset B(x, r) \subset I_{n-2}(x)$. Thus

$$\frac{\log \mu(I_{n-2}(x))}{\log |I_{n+1}(x)|} \leq \frac{\log \mu(B(x, r))}{\log r} \leq \frac{\log \mu(I_{n+1}(x))}{\log |I_n(x)|}. \tag{2.7}$$

Combining (2.6) and (2.7), we get the desired result. □

Let us give some measures for which condition (2.5) is satisfied. These measures will be used in §5.1. The existence of these measures $\mu_{t,q}$ will be discussed in Proposition 4.6 and §5.1.

LEMMA 2.13. *Suppose $\mu_{t,q}$ is a measure satisfying*

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t, q))|I_n(x)|^t \prod_{j=1}^{n} a_j^q,$$

where $P(t, q)$ is a constant. Then (2.5) is satisfied by $\mu_{t,q}$.

Proof. Notice that when $a_n(x) = 1$, $\mu_{t,q}(I_n(x)) \asymp \mu_{t,q}(I_{n-1}(x))$. Then in the light of Lemma 2.5, we can show that (2.5) is satisfied by $\mu_{t,q}$. □

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3. *Fast growth rate: proof of Theorem 1.1*

3.1. *Lower bound.* We start with the mass distribution principle (see [**11**, Proposition 4.2]), which will be used to estimate the lower bound of the Hausdorff dimension of a set.

LEMMA 3.1. [**11**] *Let $E \subset [0, 1)$ be a Borel set and μ be a measure with $\mu(E) > 0$. Suppose that*

$$\liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s \quad \textit{for all } x \in E,$$

where $B(x, r)$ denotes the open ball with center at x and radius r. Then $\dim E \geq s$.

Next we give a formula for computing the Hausdorff dimension for a class of Cantor sets related to continued fractions.

LEMMA 3.2. *Let $\{s_n\}_{n\geq 1}$ be a sequence of positive integers tending to infinity with $s_n \geq 3$ for all $n \geq 1$. Then for any positive number $N \geq 2$,*

$$\dim\{x \in [0, 1) : s_n \leq a_n(x) < Ns_n \ \forall n \geq 1\} = \liminf_{n\to\infty} \frac{\log(s_1 s_2 \ldots s_n)}{2\log(s_1 s_2 \ldots s_n) + \log s_{n+1}}.$$

Proof. Let F be the set in question and s_0 be the lim inf in the statement. We call

$$J(a_1, a_2, \ldots, a_n) := \mathrm{Cl} \bigcup_{a_{n+1}\geq s_{n+1}} I_{n+1}(a_1, \ldots, a_n, a_{n+1})$$

a *basic CF-interval* of order n with respect to F (or simply basic interval of order n), where $s_k \leq a_k < Ns_k$ for all $1 \leq k \leq n$. Here Cl stands for the closure. Then it follows that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{s_k\leq a_k<Ns_k, 1\leq k\leq n} J(a_1, \ldots, a_n). \tag{3.1}$$

By Lemma 2.4,

$$J(a_1, \ldots, a_n) = \left[\frac{p_n}{q_n}, \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}}\right] \quad \text{or} \quad \left[\frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}}, \frac{p_n}{q_n}\right] \tag{3.2}$$

depending on whether n is even or odd. Then by Lemmas 2.4, 2.2 and the assumption on a_k that $s_k \leq a_k < Ns_k$ for all $1 \leq k \leq n$,

$$\frac{1}{2N^n} \frac{1}{s_{n+1}(s_1 \ldots s_n)^2} \leq |J(a_1, \ldots, a_n)| = \frac{1}{q_n(s_{n+1}q_n + q_{n-1})} \leq \frac{1}{s_{n+1}(s_1 \ldots s_n)^2}. \tag{3.3}$$

Since $s_k \to \infty$ as $k \to \infty$, then

$$\lim_{n\to\infty} \frac{\log s_1 + \cdots + \log s_n}{n} = \infty.$$

This, together with the definition of s_0, implies that for any $s > s_0$, there exists a sequence $\{n_\ell : \ell \geq 1\}$ such that for all $\ell \geq 1$,

$$(N-1)^{n_\ell} < (s_{n_\ell+1}(s_1 \ldots s_{n_\ell})^2)^{((s-s_0)/2)}, \quad \prod_{k=1}^{n_\ell} s_k \leq (s_{n_\ell+1}(s_1 \ldots s_{n_\ell})^2)^{((s+s_0)/2)}.$$

Then, by (3.1), together with (3.3),

$$H^s(F) \le \liminf_{\ell\to\infty} \sum_{s_k \le a_k < N s_k, 1\le k\le n_\ell} |J(a_1, \ldots, a_{n_\ell})|^s$$
$$\le \liminf_{\ell\to\infty} \bigg((N-1)^{n_\ell} \prod_{k=1}^{n_\ell} s_k\bigg)\bigg(\frac{1}{s_{n_\ell+1}(s_1\ldots s_{n_\ell})^2}\bigg)^s \le 1.$$

Since $s > s_0$ is arbitrary, $\dim F \le s_0$.

For the lower bound, we define a measure μ such that for any basic *CF*-interval $J(a_1, a_2, \ldots, a_n)$ of order n,

$$\mu(J(a_1, a_2, \ldots, a_n)) = \prod_{j=1}^{n} \frac{1}{(N-1)s_j}.$$

By the Kolmogorov extension theorem, μ can be extended to a probability measure supported on F. In the following, we will check the mass distribution principle with this measure.

Fix $s < s_0$. By the definition of s_0 and the fact that $s_k \to \infty$ $(k \to \infty)$ and that N is a constant, there exists an integer n_0 such that for all $n \ge n_0$,

$$\prod_{k=1}^{n}(N-1)s_k \ge \bigg(s_{n+1}\bigg(\prod_{k=1}^{n} N s_k\bigg)^2\bigg)^s. \tag{3.4}$$

We take

$$r_0 = (1/(2N^{n_0}))\,(1/(s_{n_0+1}(s_1\ldots s_{n_0})^2)).$$

For any $x \in F$, there exists an infinite sequence $\{a_1, a_2, \ldots\}$ with $s_k \le a_k < N s_k$, for all $k \ge 1$ such that $x \in J(a_1, \ldots, a_n)$, for all $n \ge 1$. For any $r < r_0$, there exists an integer $n \ge n_0$ such that

$$|J(a_1, \ldots, a_{n+1})| \le r < |J(a_1, \ldots, a_n)|.$$

We claim that the ball $B(x, r)$ can intersect only one nth basic interval, which is just $J(a_1, \ldots, a_n)$. We establish this only for the case where n is even, since the argument is similar for n odd.

Case (1). $s_n < a_n < N s_n - 1$. The left and right adjacent nth-order basic intervals to $J(a_1, \ldots, a_n)$ are $J(a_1, \ldots, a_n - 1)$ and $J(a_1, \ldots, a_n + 1)$, respectively. Then by (3.2) and the condition that $s_n \ge 3$, the gap between $J(a_1, \ldots, a_n)$ and $J(a_1, \ldots, a_n - 1)$ is

$$\frac{p_n}{q_n} - \frac{s_{n+1}(p_n - p_{n-1}) + p_{n-1}}{s_{n+1}(q_n - q_{n-1}) + q_{n-1}} = \frac{s_{n+1} - 1}{q_n(s_{n+1}(q_n - q_{n-1}) + q_{n-1})} \ge |J(a_1, \ldots, a_n)|.$$

Hence $B(x, r)$ cannot intersect $J(a_1, \ldots, a_n - 1)$. On the other hand, the gap between $J(a_1, \ldots, a_n)$ and $J(a_1, \ldots, a_n + 1)$ is

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1} - 1}{(q_n + q_{n-1})(s_{n+1}q_n + q_{n-1})} \ge |J(a_1, \ldots, a_n)|.$$

Hence $B(x, r)$ cannot intersect $J(a_1, \ldots, a_n + 1)$ neither.

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Case (2). $a_n = s_n$. The right adjacent nth-order basic interval to $J(a_1, \ldots, a_n)$ is $J(a_1, \ldots, a_n + 1)$. The same argument as in case (1) shows that $B(x, r)$ cannot intersect $J(a_1, \ldots, a_n + 1)$. On the other hand, the gap between the left endpoint of $J(a_1, \ldots, a_n)$ and that of $I_{n-1}(a_1, \ldots, a_{n-1})$ is

$$\frac{p_n}{q_n} - \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} = \frac{s_n - 1}{(q_{n-1} + q_{n-2})q_n} \geq |J(a_1, \ldots, a_n)|.$$

It follows that $B(x, r)$ cannot intersect any nth-order CF-basic intervals on the left of $J(a_1, \ldots, a_n)$. In general, $B(x, r)$ can intersect no other nth-order CF-basic intervals than $J(a_1, \ldots, a_n)$.

Case (3). $a_n = Ns_n - 1$. From case (1), we know that $B(x, r)$ cannot intersect any nth-order CF-basic intervals on the left of $J(a_1, \ldots, a_n)$. On the right, the gap between the right endpoint of $J(a_1, \ldots, a_n)$ and that of $I_{n-1}(a_1, \ldots, a_{n-1})$ is

$$\frac{p_{n-1}}{q_{n-1}} - \frac{s_{n+1}p_n + p_{n-1}}{s_{n+1}q_n + q_{n-1}} = \frac{s_{n+1}}{(s_{n+1}q_n + q_{n-1})q_{n-1}} \geq |J(a_1, \ldots, a_n)|.$$

It follows that $B(x, r)$ cannot intersect any nth-order CF-basic intervals on the right of $J(a_1, \ldots, a_n)$. In general, $B(x, r)$ can intersect no other nth-order CF-basic intervals than $J(a_1, \ldots, a_n)$.

Now we distinguish two cases to estimate the measure of $B(x, r)$.

Case (i). $|J(a_1, \ldots, a_{n+1})| \leq r < |I_{n+1}(a_1, \ldots, a_{n+1})|$. By Lemma 2.5 and the fact that $a_{n+1} \neq 1$, $B(x, r)$ can intersect at most five $(n + 1)$th-order basic intervals. As a consequence, by (3.4),

$$\mu(B(x, r)) \leq 5 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k} \leq 5\left(\frac{1}{s_{n+2}(N^{n+1}s_1 \ldots s_{n+1})^2}\right)^s. \tag{3.5}$$

Since

$$r > |J(a_1, \ldots, a_{n+1})| = \frac{1}{q_{n+1}(s_{n+2}q_{n+1} + q_n)} \geq \frac{1}{2s_{n+2}(N^{n+1}s_1 \ldots s_{n+1})^2},$$

it follows that

$$\mu(B(x, r)) \leq 10r^s.$$

Case (ii). $|I_{n+1}(a_1, \ldots, a_{n+1})| \leq r < |J(a_1, \ldots, a_n)|$. In this case,

$$I_{n+1}(a_1, \ldots, a_{n+1}) = \frac{1}{q_{n+1}(q_{n+1} + q_n)} \geq \frac{1}{2q_{n+1}^2} \geq \frac{1}{2N^{2(n+1)}}\left(\prod_{k=1}^{n+1} s_k\right)^2.$$

So $B(x, r)$ can intersect at most a number $8rN^{2(n+1)}(s_1 \ldots s_{n+1})^2$ of $(n + 1)$th-order basic intervals. As a consequence,

$$\begin{aligned}\mu(B(x, r)) &\leq \min\left\{\mu(J(a_1, \ldots, a_n)), 8rN^{2(n+1)}(s_1 \ldots s_{n+1})^2 \prod_{k=1}^{n+1} \frac{1}{(N-1)s_k}\right\} \\ &\leq \prod_{k=1}^{n} \frac{1}{(N-1)s_k} \min\left\{1, 8rN^{2(n+1)}(s_1 \ldots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}}\right\}.\end{aligned}$$

By (3.4) and the elementary inequality $\min\{a, b\} \le a^{1-s}b^s$ which holds for any $a, b > 0$ and $0 < s < 1$,

$$\mu(B(x, r)) \le \left(\frac{1}{s_{n+1}(N^n s_1 \ldots s_n)^2}\right)^s \cdot \left(8rN^{2(n+1)}(s_1 \ldots s_{n+1})^2 \frac{1}{(N-1)s_{n+1}}\right)^s \le 16Nr^s.$$

Combining these two cases, together with mass distribution principle, we have $\dim F \ge s_0$. □

Let

$$E' = \{x \in [0, 1) : e^{\varphi(n)-\varphi(n-1)} \le a_n(x) \le 2e^{\varphi(n)-\varphi(n-1)},\ \forall n \ge 1\}.$$

It is evident that $E' \subset E_\xi(\varphi)$. Then, applying Lemma 3.2,

$$E_\xi(\varphi) \ge \liminf_{n\to\infty} \frac{\varphi(n)}{\varphi(n+1)+\varphi(n)} = \frac{1}{b+1}.$$

3.2. *Upper bound.* We first give a lemma which is a little bit more than the upper bound for the case $b = 1$. Its proof uses a family of Bernoulli measures with an infinite number of states.

LEMMA 3.3. *If* $\lim_{n\to\infty}((\varphi(n))/n) = \infty$, *then* $\dim E_\xi(\varphi) \le 1/2$.

Proof. For any $t > 1$, we introduce a family of Bernoulli measures μ_t:

$$\mu_t(I_n(a_1, \ldots, a_n)) = e^{-nC(t)-t\sum_{j=1}^n \log a_j(x)} \tag{3.6}$$

where $C(t) = \log \sum_{n=1}^\infty (1/n^t)$.

Fix $x \in E_\xi(\varphi)$ and $\epsilon > 0$. If n is sufficiently large, then

$$(\xi - \epsilon)\varphi(n) < \sum_{j=1}^n \log a_j(x) < (\xi + \epsilon)\varphi(n). \tag{3.7}$$

So

$$E_\xi(\varphi) \subset \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty E_n(\epsilon),$$

where

$$E_n(\epsilon) = \left\{x \in [0, 1) : (\xi - \epsilon)\varphi(n) < \sum_{j=1}^n \log a_j(x) < (\xi + \epsilon)\varphi(n)\right\}.$$

Now let $\mathcal{I}(n, \epsilon)$ be the family of all nth-order cylinders $I_n(a_1, \ldots, a_n)$ satisfying (3.7). For each $N \ge 1$, we select all those cylinders in $\bigcup_{n=N}^\infty \mathcal{I}(n, \epsilon)$ which are maximal ($I \in \bigcup_{n=N}^\infty \mathcal{I}(n, \epsilon)$ is maximal if there is no other I' in $\bigcup_{n=N}^\infty \mathcal{I}(n, \epsilon)$ such that $I \subset I'$ and $I \ne I'$). We denote by $\mathcal{J}(N, \epsilon)$ the set of all maximal cylinders in $\bigcup_{n=N}^\infty \mathcal{I}(n, \epsilon)$. It is evident that $\mathcal{J}(N, \epsilon)$ is a cover of $E_\xi(\varphi)$. Let $I_n(a_1, \ldots, a_n) \in \mathcal{J}(N, \epsilon)$. Then

$$\mu_t(I_n(a_1, \ldots, a_n)) = e^{-nC(t)-t\sum_{j=1}^n \log a_j} \ge e^{-nC(t)-t(\xi+\epsilon)\varphi(n)}.$$

On the other hand,

$$|I_n(a_1, \ldots, a_n)| \le e^{-2\log q_n} \le e^{-2\sum_{j=1}^{n} \log a_j} \le e^{-2(\xi-\epsilon)\varphi(n)}.$$

Since $\lim_{n\to\infty}((\varphi(n))/n) = \infty$, for each $s > t/2$ and N large enough, then

$$|I_n(a_1, \ldots, a_n)|^s \le \mu_t(I_n(a_1, \ldots, a_n)).$$

This implies that $\dim E_\xi(\varphi) \le 1/2$. □

Now we return to the proof of the upper bound.

Case (i). $b = 1$. Since $(\varphi(n+1) - \varphi(n)) \uparrow \infty$, Lemma 3.3 immediately implies that $\dim E_\xi(\varphi) \le 1/2$.

Case (ii). $b > 1$. By (3.7), for each $x \in E_\xi(\varphi)$ and n sufficiently large,

$$(\xi - \epsilon)\varphi(n+1) - (\xi + \epsilon)\varphi(n) \le \log a_{n+1}(x) \le (\xi + \epsilon)\varphi(n+1) - (\xi - \epsilon)\varphi(n).$$

Take

$$L_{n+1} = e^{(\xi-\epsilon)\varphi(n+1)-(\xi+\epsilon)\varphi(n)}, \quad M_{n+1} = e^{(\xi+\epsilon)\varphi(n+1)-(\xi-\epsilon)\varphi(n)}.$$

Define

$$F_N = \{x \in [0, 1] : L_n \le a_n(x) \le M_n, \forall n \ge N\}.$$

Then

$$E_\xi(\varphi) \subset \bigcup_{N=1}^{\infty} F_N.$$

We can only estimate the upper bound of $\dim F_1$. Since F_N can be written as a countable union of sets with the same form as F_1, then $\dim F_N = \dim F_1$ by the σ-stability of Hausdorff dimension. We can further assume that $M_n \ge L_n + 2$.

For any $n \ge 1$, define

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : L_k \le \sigma_k \le M_k, 1 \le k \le n\}.$$

It follows that

$$F_1 = \bigcap_{n\ge 1} \bigcup_{(\sigma_1,\ldots,\sigma_n)\in D_n} J(\sigma_1, \ldots, \sigma_n),$$

where

$$J(\sigma_1, \ldots, \sigma_n) := \mathrm{Cl} \bigcup_{\sigma \ge L_{n+1}} I(\sigma_1, \ldots, \sigma_n, \sigma)$$

(called an admissible cylinder of order n). For any $n \ge 1$ and $s > 0$,

$$\sum_{(\sigma_1,\ldots,\sigma_n)\in D_n} |J(\sigma_1, \ldots, \sigma_n)|^s \le \sum_{(\sigma_1,\ldots,\sigma_n)\in D_n} \left|\frac{1}{q_n^2 L_{n+1}}\right|^s \le \frac{M_1 \cdots M_n}{((L_1 \cdots L_n)^2 L_{n+1})^s}.$$

It follows that

$$\dim F_1 \le \liminf_{n\to\infty} \frac{\log M_1 + \cdots + \log M_n}{\sum_{k=1}^{n} \log L_k + \sum_{k=1}^{n+1} \log L_k} = \frac{\xi + \epsilon + 2\epsilon/(b-1)}{(\xi - \epsilon)(b+1) - 2\epsilon - 4\epsilon/(b-1)}.$$

Letting $\epsilon \to 0$, we get

$$\dim E_\xi(\varphi) \le \frac{1}{b+1}.$$

https://doi.org/10.1017/S0143385708000138 Published online by Cambridge University Press

4. *Ruelle operator theory*

There have been various works on the Ruelle transfer operator for the Gauss dynamics; see Mayer [**30–32**], Jenkinson [**22**], Jenkinson and Pollicott [**21**], Pollicott and Weiss [**34**] and Hanus *et al* [**16**]. In this section we present a general Ruelle operator theory for conformal infinite iterated function systems which was developed in [**16**] and then apply it to the Gauss dynamics. We also prove some properties of the pressure function in the case of Gauss dynamics, which will be used later.

4.1. *Conformal infinite iterated function systems.*

In this subsection, we present the conformal infinite iterated function systems which were studied by Hanus *et al* [**16**]; see also the book of Mauldin and Urbański [**29**].

Let X be a non-empty compact connected subset of $\mathbb{R}^d$ equipped with a metric ρ. Let I be an index set with at least two elements and at most countably many elements. An *iterated function system* $S = \{\phi_i : X \to X : i \in I\}$ is a collection of injective contractions for which there exists $0 < s < 1$ such that for each $i \in I$ and all $x, y \in X$,

$$\rho(\phi_i(x), \phi_i(y)) \le s\rho(x, y). \tag{4.1}$$

Before further discussion, we give a list of notation.

- $I^n := \{\omega : \omega = (\omega_1, \ldots, \omega_n), \omega_k \in I, 1 \le k \le n\}$.
- $I^* := \bigcup_{n\ge1} I^n$.
- $I^\infty := \prod_{i=1}^{\infty} I$.
- $\phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$, for $\omega = \omega_1\omega_2 \ldots \omega_n \in I^n$, $n \ge 1$.
- $|\omega|$ denotes the length of $\omega \in I^* \cup I^\infty$.
- $\omega|_n = \omega_1\omega_2 \ldots \omega_n$, if $|w| \ge n$.
- $[\omega|_n] = [\omega_1 \ldots \omega_n] = \{x \in I^\infty : x_1 = \omega_1, \ldots, x_n = \omega_n\}$.
- $\sigma : I^\infty \to I^\infty$ is the shift transformation.
- $\|\phi'_\omega\| := \sup_{x\in X} |\phi'_\omega(x)|$ for $\omega \in I^*$.
- $C(X)$ is the space of continuous functions on X.
- $\|\cdot\|_\infty$ denotes the supremum norm on the Banach space $C(X)$.

For $\omega \in I^\infty$, the set

$$\pi(\omega) = \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton. We also denote its only element by $\pi(\omega)$. This thus defines a coding map $\pi : I^\infty \to X$. The limit set J of the iterated function system is defined by

$$J := \pi(I^\infty).$$

Denote by ∂X the boundary of X and by $\mathrm{Int}(X)$ the interior of X.

We say that the iterated function system $S = \{\phi_i\}_{i\in I}$ satisfies the *open set condition* if there exists a non-empty open set $U \subset X$ such that $\phi_i(U) \subset U$ for each $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for each pair $i, j \in I$, $i \ne j$.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to be *conformal* if the following conditions are met:

(1) The open set condition is satisfied for $U = \mathrm{Int}(X)$.
(2) There exists an open connected set V with $X \subset V \subset \mathbb{R}^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V.
(3) There exist $h, \ell > 0$ such that for each $x \in \partial X \subset \mathbb{R}^d$, there exists an open cone $\mathrm{Con}(x, h, \ell) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure h and altitude ℓ.
(4) There exists $K \geq 1$ such that $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$ for every $\omega \in I^*$ and every pair of points $x, y \in V$ (Bounded distortion property).

The *topological pressure function* for a conformal iterated function systems $S = \{\phi_i : X \to X : i \in I\}$ is defined as

$$\mathcal{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \|\phi'_\omega\|^t.$$

The system S is said to be *regular* if there exists $t \geq 0$ such that $\mathcal{P}(t) = 0$.

Let $\beta > 0$. A *Hölder family of functions* of order β is a family of continuous functions $F = \{f^{(i)} : X \to \mathbb{C} : i \in I\}$ such that

$$V_\beta(F) = \sup_{n\geq 1} V_n(F) < \infty,$$

where

$$V_n(F) = \sup_{\omega\in I^n} \sup_{x,y\in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\}e^{\beta(n-1)}.$$

A family of functions $F = \{f^{(i)} : X \to \mathbb{R},\ i \in I\}$ is said to be *strong* if

$$\sum_{i\in I} \|e^{f^{(i)}}\|_\infty < \infty.$$

Define the *Ruelle operator* on $C(X)$ associated with F as

$$\mathcal{L}_F(g)(x) := \sum_{i\in I} e^{f^{(i)}(x)} g(\phi_i(x)).$$

Denote by $\mathcal{L}_F^*$ the dual operator of $\mathcal{L}_F$.

The *topological pressure* of F is defined by

$$P(F) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\bigg(\sup_{x\in X} \sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j\omega}(x)\bigg).$$

A measure ν is called F*-conformal* if the following conditions are satisfied:
(1) ν is supported on J.
(2) For any Borel set $A \subset X$ and any $\omega \in I^*$,

$$\nu(\phi_\omega(A)) = \int_A \exp\bigg(\sum_{j=1}^n f^{(\omega_j)} \circ \phi_{\sigma^j\omega} - P(F)|\omega|\bigg)\, d\nu.$$

(3) $\nu(\phi_\omega(X) \cap \phi_\tau(X)) = 0,\ \ \omega, \tau \in I^n, \omega \neq \tau, n \geq 1.$

Two functions $\phi, \varphi \in C(X)$ are said to be *cohomologous* with respect to the transformation T, if there exists $u \in C(X)$ such that

$$\varphi(x) = \phi(x) + u(x) - u(T(x)).$$

The following two theorems are due to Hanus *et al* [**16**].

THEOREM 4.1. [**16**] *For a conformal iterated function system $S = \{\phi_i : X \to X : i \in I\}$ and a strong Hölder family of functions $F = \{f^{(i)} : X \to \mathbb{C} : i \in I\}$, there exists a unique F-conformal probability measure ν_F on X such that $\mathcal{L}_F^* \nu_F = e^{P(F)} \nu_F$. There exists a unique shift invariant probability measure $\tilde{\mu}_F$ on I^∞ such that $\mu_F := \tilde{\mu}_F \circ \pi^{-1}$ is equivalent to ν_F with bounded Radon–Nikodym derivative. Furthermore, the Gibbs property is satisfied:*

$$\frac{1}{C} \le \frac{\tilde{\mu}_F([\omega|_n])}{\exp\left(\sum_{j=1}^n f^{(\omega_j)}(\pi(\sigma^j \omega)) - nP(F)\right)} \le C.$$

Let $\Psi = \{\psi^{(i)} : X \to \mathbb{R} : i \in I\}$ and $F = \{f^{(i)} : X \to \mathbb{R} : i \in I\}$ be two families of real-valued Hölder functions. We define the *amalgamated functions* on I^∞ associated with Ψ and F as follows:

$$\tilde{\psi}(\omega) := \psi^{(\omega_1)}(\pi(\sigma\omega)), \quad \tilde{f}(\omega) := f^{(\omega_1)}(\pi(\sigma\omega)) \quad \text{for all } \omega \in I^\infty.$$

THEOREM 4.2. ([**16**], see also [**29**, pp. 43–48]) *Let Ψ and F be two families of real-valued Hölder functions. Suppose the sets $\{i \in I : \sup_x(\psi^{(i)}(x)) > 0\}$ and $\{i \in I : \sup_x(f^{(i)}(x)) > 0\}$ are finite. Then the function $(t, q) \mapsto P(t, q) = P(t\Psi + qF)$, is real analytic with respect to $(t, q) \in \mathrm{Int}(D)$, where*

$$D = \left\{(t, q) : \sum_{i\in I} \exp\left(\sup_x (t\psi^{(i)}(x) + qf^{(i)}(x))\right) < \infty\right\}.$$

Furthermore, if $t\Psi + qF$ is a strong Hölder family for $(t, q) \in D$ and

$$\int (|\tilde{f}| + |\tilde{\psi}|)\, d\tilde{\mu}_{t,q} < \infty,$$

where $\tilde{\mu}_{t,q} := \tilde{\mu}_{t\Psi+qF}$ is obtained by Theorem 4.1, then

$$\frac{\partial P}{\partial t} = \int \tilde{\psi}\, d\tilde{\mu}_{t,q} \quad \text{and} \quad \frac{\partial P}{\partial q} = \int \tilde{f}\, d\tilde{\mu}_{t,q}.$$

If $t\tilde{\psi} + q\tilde{f}$ is not cohomologous to a constant function, then $P(t, q)$ is strictly convex and

$$H(t, q) := \begin{pmatrix} \dfrac{\partial^2 P}{\partial t^2} & \dfrac{\partial^2 P}{\partial t \partial q} \\ \dfrac{\partial^2 P}{\partial t \partial q} & \dfrac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

is positive definite.

4.2. *Continued fraction dynamical system.* We apply the theory in the previous subsection to the continued fraction dynamical system. Let $X = [0, 1]$ and $I = \mathbb{N}$. The continued fraction dynamical system can be viewed as an iterated function system:

$$S = \left\{ \psi_i(x) = \frac{1}{i + x} : i \in \mathbb{N} \right\}.$$

Recall that the projection mapping $\pi : I^\infty \to X$ is defined by

$$\pi(\omega) := \bigcap_{n=1}^{\infty} \psi_{\omega|_n}(X) \quad \text{for all } \omega \in I^\infty.$$

Notice that $\psi_1'(0) = -1$, thus (4.1) is not satisfied. However, this is not a real problem, since we can consider the system of second-level maps and replace S by $\tilde{S} := \{\psi_i \circ \psi_j : i, j \in \mathbb{N}\}$. In fact, for any $x \in [0, 1)$,

$$(\psi_i \circ \psi_j)'(x) = \left(\frac{1}{i + 1/(j + x)} \right)' = \left(\frac{1}{i(j + x) + 1} \right)^2 \leq \frac{1}{4}.$$

In the following, we will collect or prove some facts on the continued fraction dynamical system, which will be useful for applying Theorems 4.1 and 4.2.

LEMMA 4.3. [**27**] *The continued fraction dynamical system S is regular and conformal.*

For the investigation in the present paper, our problems are tightly connected to the following two families of Hölder functions:

$$\Psi = \{\log |\psi_i'| : i \in \mathbb{N}\} \quad \text{and} \quad F = \{-\log i : i \in \mathbb{N}\}.$$

Remark 4.4. We mention that the method used here is also applicable to other potentials than the two special families introduced here.

The families Ψ and F are Hölder families and their amalgamated functions are equal to

$$\tilde{\psi}(\omega) = -2 \log(\omega_1 + \pi(\sigma\omega)), \quad \tilde{f}(\omega) = -\log \omega_1 \quad \text{for all } \omega \in \mathbb{N}^\infty.$$

For our convenience, we will consider the function $t\Psi - qF$ instead of $t\Psi + qF$.

LEMMA 4.5. *Let $D := \{(t, q) : 2t - q > 1\}$. For any $(t, q) \in D$:*

(i) *the family $t\Psi - qF := \{t \log |\psi_i'| + q \log i : i \in \mathbb{N}\}$ is Hölder and strong;*

(ii) *the topological pressure P associated with the potential $t\Psi - qF$ can be written as*

$$P(t, q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1, \ldots, \omega_n} \exp \left(\sup_x \log \prod_{j=1}^{n} \omega_j^q ([\omega_j, \ldots, \omega_n + x])^{2t} \right).$$

Proof. The assertion on the domain D follows from

$$\frac{1}{4^t} \zeta(2t - q) = \mathcal{L}_{t\Psi - qF} 1 = \sum_{i=1}^{\infty} \frac{i^q}{(i + x)^{2t}} \leq \sum_{i=1}^{\infty} i^{q-2t} = \zeta(2t - q).$$

where ζ is the Riemann zeta function, defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for all } s > 1.$$

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(i) For $(t, q) \in D$, write $(t\Psi - qF)^{(i)} := t \log |\psi_i'| + q \log i$. Then

$$\sum_{i \in I} \|\exp\{(t\Psi - qF)^{(i)}\}\|_\infty = \sum_{i=1}^{\infty} \left\| \frac{i^q}{(i+x)^{2t}} \right\|_\infty = \sum_{i=1}^{\infty} i^{q-2t} = \zeta(2t-q) < \infty,$$

Thus $t\Psi - qF$ is strong.

(ii) It suffices to notice that

$$\sup_x \left(\sum_{j=1}^{n} (t|\psi'_{\omega_j}| + q \log \omega_j) \circ \psi_{\sigma^j \omega}(x) \right) = \sup_x \log \prod_{j=1}^{n} \omega_j^q([\omega_j, \ldots, \omega_n + x])^{2t}. \quad \square$$

Denote by $\mathcal{L}^*_{t\Psi - qF}$ the conjugate operator of $\mathcal{L}_{t\Psi-qF}$. Applying Theorem 4.1 with the help of Lemmas 4.3 and 4.5, we get the following results.

PROPOSITION 4.6. *For each $(t, q) \in D$, there exist a unique $(t\Psi - qF)$-conformal probability measure $\nu_{t,q}$ on $[0, 1]$ such that $\mathcal{L}^*_{t\Psi-qF}\nu_{t,q} = e^{P(t,q)}\nu_{t,q}$, and a unique shift invariant probability measure $\tilde{\mu}_{t,q}$ on $\mathbb{N}^\infty$ such that $\mu_{t,q} := \tilde{\mu}_{t,q} \circ \pi^{-1}$ on $[0, 1]$ is equivalent to $\nu_{t,q}$ and*

$$\frac{1}{C} \le \frac{\tilde{\mu}_{t,q}([\omega|_n])}{\exp\left(\sum_{j=1}^{n} (t\Psi - qF)^{(\omega_j)}(\pi(\sigma^j \omega)) - nP(t,q) \right)} \le C \quad \textit{for all } \omega \in \mathbb{N}^\infty.$$

LEMMA 4.7. *For the amalgamated functions $\tilde{\psi}(\omega) = -2\log(\omega_1 + \pi(\sigma\omega))$ and $\tilde{f}(\omega) = -\log \omega_1$,*

$$-\int \log |T'(x)| \mu_{t,q} = \int \tilde{\psi} \, d\tilde{\mu}_{t,q} \quad \textit{and} \quad \int \log a_1(x)\, d\mu_{t,q} = -\int \tilde{f}\, d\tilde{\mu}_{t,q}, \tag{4.2}$$

and $t\tilde{\psi} - q\tilde{f}$ is not cohomologous to a constant.

Proof. (i) Assertion (4.2) is just a consequence of

$$-\log |T'(\pi(\omega))| = \tilde{\psi}(\omega), \quad \log a_1(\pi(\omega)) = -\tilde{f}(\omega) \quad \text{for all } \omega \in I^\infty.$$

Suppose that $t\tilde{\psi} - q\tilde{f}$ was not cohomologous to a constant. Then there would be a bounded function g such that $t\tilde{\psi} - q\tilde{f} = g - g \circ T + C$, which implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})\,(\sigma^j\omega) = \lim_{n\to\infty} \frac{g - g\circ\sigma^n}{n} + C = C$$

for all $\omega \in I^\infty$. On the other hand, if we take $\omega^1 = [1, 1, \ldots,]$, $\omega^2 = [2, 2, \ldots]$ and $\omega^3 = [3, 3, \ldots]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (t\tilde{\psi} - q\tilde{f})\,(\sigma^j\omega^i) = C_i,$$

where

$$C_1 = 2t \log\left(\frac{\sqrt{5}-1}{2}\right), \quad C_2 = 2t\log(\sqrt{2}-1) + q\log 2,$$

$$C_3 = 2t\log\left(\frac{\sqrt{13}-3}{2}\right) + q \log 3.$$

Thus we get a contradiction. $\square$

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By Theorem 4.2 and the proof of Lemma 4.5, we know that $D = \{(t, q) : 2t - q > 1\}$ is the analytic area of the pressure $P(t, q)$. Applying Lemma 4.7 and Theorem 4.2, we get more from the following proposition.

PROPOSITION 4.8. *The following statements hold on* $D = \{(t, q) : 2t - q > 1\}$.

(1) $P(t, q)$ *is analytic and strictly convex.*

(2) $P(t, q)$ *is strictly decreasing and strictly convex with respect to* t. *In other words,* $(\partial P/\partial t)\,(t, q) < 0$ *and* $(\partial^2 P/\partial t^2)\,(t, q) > 0$. *Furthermore,*

$$\frac{\partial P}{\partial t}(t, q) = -\int \log |T'(x)|\, d\mu_{t,q}. \tag{4.3}$$

(3) $P(t, q)$ *is strictly increasing and strictly convex with respect to* q. *In other words,* $(\partial P/\partial q)\,(t, q) > 0$ *and* $(\partial^2 P/\partial q^2)\,(t, q) > 0$. *Furthermore,*

$$\frac{\partial P}{\partial q}(t, q) = \int \log a_1(x)\, d\mu_{t,q}. \tag{4.4}$$

(4)

$$H(t, q) := \begin{pmatrix} \dfrac{\partial^2 P}{\partial t^2} & \dfrac{\partial^2 P}{\partial t \partial q} \\ \dfrac{\partial^2 P}{\partial t \partial q} & \dfrac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

is positive definite.

We conclude this subsection with the following result due to Mayer [**32**] (see also Pollicott and Weiss [**34**]).

PROPOSITION 4.9. [**32**] *Let* $P(t) := P(t, 0)$ *and* $\mu_t := \mu_{t,0}$. *Then* $P(t)$ *is defined on* $(1/2, \infty)$ *and we have* $P(1) = 0$ *and* $\mu_1 = \mu_G$. *Furthermore,*

$$P'(t) = -\int \log |T'(x)|\, d\mu_t(x). \tag{4.5}$$

In particular,

$$P'(0) = -\int \log |T'(x)|\, d\mu_G(x) = -\lambda_0. \tag{4.6}$$

Remark 4.10. Since $\mu_{1,0} = \mu_1 = \mu_G$, by (4.4), we obtain

$$\frac{\partial P}{\partial q}(1, 0) = \int \log a_1(x)\, d\mu_G = \xi_0. \tag{4.7}$$

4.3. *Further study on* $P(t, q)$. We will use the following simple known fact about convex functions.

Fact 4.11. Suppose f is a convex continuously differentiable function on an interval I. Then $f'(x)$ is increasing and

$$f'(x) \le \frac{f(y) - f(x)}{y - x} \le f'(y), \qquad x, y \in I,\ x < y.$$

First we give an estimate for the pressure $P(t, q)$ and show some behaviors of $P(t, q)$ when q tends to $-\infty$ and $2t - 1$ (t being fixed).

PROPOSITION 4.12. *For* $(t, q) \in D$,

$$-t \log 4 + \log \zeta(2t - q) \leq P(t, q) \leq \log \zeta(2t - q). \tag{4.8}$$

Consequently:

(1) $P(0, q) = \log \zeta(-q)$, *and for any point* (t_0, q_0) *on the line* $2t - q = 1$,

$$\lim_{(t,q)\to(t_0,q_0)} P(t, q) = \infty;$$

(2) *for fixed* $t \in \mathbb{R}$,

$$\lim_{q\to 2t-1} \frac{\partial P}{\partial q}(t, q) = +\infty; \tag{4.9}$$

(3) *for fixed* $t \in \mathbb{R}$,

$$\lim_{q\to -\infty} \frac{P(t, q)}{q} = 0, \quad \lim_{q\to -\infty} \frac{\partial P}{\partial q}(t, q) = 0. \tag{4.10}$$

Proof. Notice that $1/(\omega_j + 1) \leq [\omega_j, \ldots, \omega_n + x] \leq 1/\omega_j$ for $x \in [0, 1)$ and $1 \leq j \leq n$. Thus

$$\frac{1}{4^{nt}} \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n \leq \sum_{\omega_1,\ldots,\omega_n} \prod_{j=1}^{n} \omega_j^q [\omega_j, \ldots, \omega_n + x]^{2t} \leq \sum_{\omega=1}^{\infty} (\omega^{q-2t})^n.$$

Hence by Lemma 4.5(ii), we get (4.8).

We get (1) immediately from (4.8).

Look at (2). For all $q > q_0$, by the convexity of $P(t, q)$ and Fact 4.11,

$$\frac{\partial P}{\partial q}(t, q) \geq \frac{P(t, q) - P(t, q_0)}{q - q_0}.$$

Thus

$$\lim_{q\to 2t-1} \frac{\partial P}{\partial q}(t, q) \geq \lim_{q\to 2t-1} \frac{P(t, q_0) - P(t, q)}{q_0 - q} = \infty.$$

Here we use the fact that $\lim_{q\to 2t-1} P(t, q) = +\infty$. Hence we get (4.9).

In order to show (3), we consider $P(t, q)/q$ as function of q on $(-\infty, 2t - 1) \setminus \{0\}$. Noticed that for fixed $t \in \mathbb{R}$, $\lim_{q\to -\infty} \zeta(2t - q) = 1$. Thus

$$\lim_{q\to -\infty} \frac{\log \zeta(2t - q)}{q} = 0.$$

Then the first formula in (4.10) follows from (4.8).

Fix $q_0 < 2t - 1$. Then for all $q < q_0$, by the convexity of $P(t, q)$ and Fact 4.11,

$$\frac{\partial P}{\partial q}(t, q) \leq \frac{P(t, q_0) - P(t, q)}{q_0 - q}.$$

Thus

$$\lim_{q\to -\infty} \frac{\partial P}{\partial q}(t, q) \leq \lim_{q\to -\infty} \frac{P(t, q_0) - P(t, q)}{q_0 - q} = 0.$$

Hence by Proposition 4.8(3), we get the second formula in (4.10). □

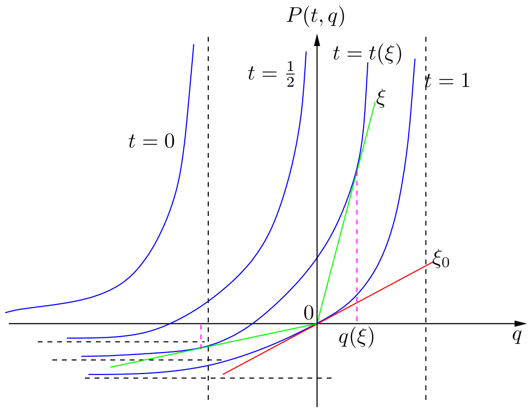

FIGURE 3. Solution of (4.11).

4.4. *Properties of* $(t(\xi), q(\xi))$. Recall that $\xi_0 = \int \log a_1(x)\mu_G$ and $D_0 := \{(t, q) : 2t - q > 1, 0 \le t \le 1\}$.

PROPOSITION 4.13. *For any* $\xi \in (0, \infty)$*, the system*

$$\begin{cases} P(t, q) = q\xi, \\ \dfrac{\partial P}{\partial q}(t, q) = \xi \end{cases} \tag{4.11}$$

admits a unique solution $(t(\xi), q(\xi)) \in D_0$*. For* $\xi = \xi_0$*, the solution is* $(t(\xi_0), q(\xi_0)) = (1, 0)$*. The functions* $t(\xi)$ *and* $q(\xi)$ *are analytic.*

Proof. Existence and uniqueness of solution $(t(\xi), q(\xi))$. Recall that $P(1, 0) = 0$ and $P(0, q) = \log \zeta(-q)$ (Proposition 4.12).

We start with a geometric argument which will be followed by a rigorous proof. Consider $P(t, q)$ as a family of functions of q with parameter t. It can be seen from Figure 3 that for any $\xi > 0$, there exists a unique $t \in (0, 1]$, such that the line ξq is tangent to $P(t, \cdot)$. This $t = t(\xi)$ can be described as the unique point such that

$$\inf_{q < 2t(\xi) - 1} (P(t(\xi), q) - q\xi) = 0. \tag{4.12}$$

We denote by $q(\xi)$ the point where the infimum in (4.12) is attained. Then the tangent point is $(q(\xi), P(t(\xi), q(\xi)))$ and the derivative of $P(t(\xi), q) - q\xi$ (with respect to q) at $q(\xi)$ equals 0, i.e.

$$(P(t(\xi), q) - q\xi)'|_{q(\xi)} = 0.$$

Thus $(\partial P/\partial q)\,(t(\xi), q(\xi)) = \xi$. By (4.12), $P(t(\xi), q(\xi)) - q(\xi)\xi = 0$. Therefore $(t(\xi), q(\xi))$ is a solution of (4.11). The uniqueness of $q(\xi)$ follows by the fact that $(\partial P/\partial q)$ is monotonic with respect to q (Proposition 4.8).

Let us give a rigorous proof. By (4.9), (4.10) and the mean-value theorem, for fixed $t \in \mathbb{R}$ and any $\xi > 0$, there exists a $q(t, \xi) \in (-\infty, 2t - 1)$ such that

$$\frac{\partial P}{\partial q}(t, q(t, \xi)) = \xi. \tag{4.13}$$

The monotonicity of $\partial P/\partial q$ with respect to q implies the uniqueness of $q(t, \xi)$ (Proposition 4.8).

Since $P(t, q)$ is analytic, the implicit $q(t, \xi)$ is analytic with respect to t and ξ. Fix ξ and set

$$W(t) := P(t, q(t, \xi)) - \xi q(t, \xi).$$

Since

$$\begin{aligned} W'(t) &= \frac{\partial P}{\partial t}(t, q(t, \xi)) + \frac{\partial P}{\partial q}(t, q(t, \xi))\frac{\partial q}{\partial t}(t, \xi) - \xi \frac{\partial q}{\partial t}(t, \xi) \\ &= \frac{\partial P}{\partial t}(t, q(t, \xi)) \quad \text{(by (4.13))} \\ &< 0 \quad \text{(by Proposition 4.8(2))}, \end{aligned}$$

$W(t)$ is strictly decreasing.

Since $P(0, q) = \log \zeta(-q) > 0$ $(q < -1)$, for $\xi > 0$,

$$W(0) = P(0, q(0, \xi)) - \xi q(0, \xi) > 0.$$

Since $P(1, q)$ is convex and $P(1, 0) = 0$, the following inequalities are a consequence of Fact 4.11:

$$\frac{P(1, q(1, \xi)) - 0}{q(1, \xi) - 0} \leq \frac{\partial P}{\partial q}(1, q(1, \xi)) = \xi \quad \text{if } q(1, \xi) > 0,$$

and

$$\frac{0 - P(1, q(1, \xi))}{0 - q(1, \xi)} \geq \frac{\partial P}{\partial q}(1, q(1, \xi)) = \xi \quad \text{if } q(1, \xi) < 0.$$

If $q(1, \xi) = 0$, we obtain in fact $\xi = \xi_0$ and $P(1, q(1, \xi)) = 0$. Hence, in any case,

$$P(1, q(1, \xi)) - \xi q(1, \xi) \leq 0. \tag{4.14}$$

Therefore, $W(1) = P(1, q(1, \xi)) - \xi q(1, \xi) \leq 0$.

Thus by the mean-value theorem and the monotonicity of $W(t)$, there exists a unique $t = t(\xi) \in (0, 1]$ such that $W(t(\xi)) = 0$, i.e.

$$P(t(\xi), q(t(\xi), \xi)) = \xi q(t(\xi), \xi). \tag{4.15}$$

If we write $q(t(\xi), \xi)$ as $q(\xi)$, both (4.13) and (4.15) show that $(t(\xi), q(\xi))$ is the unique solution of (4.11). For $\xi = \xi_0$, the assertion in Proposition 4.9 that $P(0, 1) = 0 = 0 \cdot \xi_0$ and the assertion of Remark 4.10 that $(\partial P/\partial q)\,(1, 0) = \xi_0$ imply that $(0, 1)$ is a solution of (4.11). Then the uniqueness of the solution to (4.11) implies that $(t(\xi_0), q(\xi_0)) = (0, 1)$.

Analyticity of $(t(\xi), q(\xi))$. Consider the map

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} P(t, q) - q\xi \\ \dfrac{\partial P}{\partial q}(t, q) - \xi \end{pmatrix}.$$

Then the Jacobian of F is equal to

$$J(F) =: \begin{pmatrix} \dfrac{\partial F_1}{\partial t} & \dfrac{\partial F_1}{\partial q} \\ \dfrac{\partial F_2}{\partial t} & \dfrac{\partial F_2}{\partial q} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial P}{\partial t} & \dfrac{\partial P}{\partial q} - \xi \\ \dfrac{\partial^2 P}{\partial t \partial q} & \dfrac{\partial^2 P}{\partial q^2} \end{pmatrix}.$$

Consequently,

$$\det(J(F))|_{t=t(\xi),q=q(\xi)} = \frac{\partial P}{\partial t} \cdot \frac{\partial^2 P}{\partial q^2} \neq 0.$$

Thus by the implicit function theorem, $t(\xi)$ and $q(\xi)$ are analytic. □

Now let us present some properties on $t(\xi)$. Recall that $\xi_0 = (\partial P/\partial q)\,(1, 0)$.

PROPOSITION 4.14. *$q(\xi) < 0$ for $\xi < \xi_0$; $q(\xi_0) = 0$; $q(\xi) > 0$ for $\xi > \xi_0$.*

Proof. Since $P(1, q)$ is convex and $P(1, 0) = 0$, then

$$\frac{P(1, q) - 0}{q - 0} \geq \frac{\partial P}{\partial q}(1, 0) = \xi_0 \quad (q > 0); \qquad \frac{0 - P(1, q)}{0 - q} \leq \frac{\partial P}{\partial q}(1, 0) = \xi_0 \quad (q < 0).$$

by Fact 4.11. Hence for all $q < 1$,

$$P(1, q) \geq \xi_0 q. \tag{4.16}$$

We recall that $(t(\xi_0), q(\xi_0)) = (1, 0)$ is the unique solution of the system (4.11) for $\xi = \xi_0$. By the above discussion of the existence of $t(\xi)$, $t(\xi) = 1$ if and only if $\xi = \xi_0$. Now we suppose that $t \in (0, 1)$. For $\xi > \xi_0$, using (4.16),

$$P(t, q) > P(1, q) \geq q\xi_0 \geq q\xi \quad (\text{for all } q \leq 0).$$

Thus $q(\xi) > 0$. For $\xi < \xi_0$, again using (4.16),

$$P(t, q) > P(1, q) \geq q\xi_0 \geq q\xi \quad (\text{for all } q \geq 0).$$

Thus $q(\xi) < 0$. □

PROPOSITION 4.15. *For $\xi \in (0, +\infty)$,*

$$t'(\xi) = \frac{q(\xi)}{(\partial P/\partial t)\,(t(\xi), q(\xi))}. \tag{4.17}$$

Proof. Recall that

$$\begin{cases} P(t(\xi), q(\xi)) = q(\xi)\xi, \\ \dfrac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi. \end{cases} \tag{4.18}$$

By differentiating the first equation in (4.18) with respect to ξ, we get

$$t'(\xi)\frac{\partial P}{\partial t}(t(\xi), q(\xi)) + q'(\xi)\frac{\partial P}{\partial q}(t(\xi), q(\xi)) = q'(\xi)\xi + q(\xi).$$

Taking into account the second equation in (4.18), we get

$$t'(\xi)\frac{\partial P}{\partial t}(t(\xi), q(\xi)) = q(\xi). \tag{4.19}$$

□

PROPOSITION 4.16. *$t'(\xi) > 0$ for $\xi < \xi_0$, $t'(\xi_0) = 0$, and $t'(\xi) < 0$ for $\xi > \xi_0$. Furthermore,*

$$t(\xi) \to 0 \quad (\xi \to 0), \tag{4.20}$$

$$t(\xi) \to 1/2 \quad (\xi \to +\infty). \tag{4.21}$$

Proof. By Propositions 4.14 and 4.15 and the fact that $\partial P/\partial t > 0$, $t(\xi)$ is increasing on $(0, \xi_0)$ and decreasing on (ξ_0, ∞). Then by the analyticity of $t(\xi)$, we can obtain two analytic inverse functions on the two intervals respectively. For the first inverse function, write $\xi_1 = \xi_1(t)$. Then $\xi_1'(t) > 0$ and

$$\xi_1(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t))$$

(equations (4.11) are considered as equations on t). By Proposition 4.14, $q(\xi_1(t)) < 0$; then $P(t, q(\xi_1(t))) < 0$. By Proposition 4.12(1), $\lim_{q \to 2t-1} P(t, q) = \infty$. Thus there exists $q_0(t)$ such that $q_0(t) > q(t)$ and $P(t, q_0(t)) = 0$. Therefore

$$\xi_1(t) = \frac{\partial P}{\partial q}(t, q(t)) < \frac{\partial P}{\partial q}(t, q_0(t)).$$

Since $P(0, q) = \log \zeta(-q)$, then $\lim_{t \to 0} q_0(t) = \infty$. Thus

$$\lim_{t \to 0} \frac{\partial P}{\partial q}(t, q_0(t)) = \lim_{q \to -\infty} \frac{\partial P}{\partial q}(0, q) = 0.$$

Hence by $\xi_1(t) \geq 0$, we obtain $\lim_{t \to 0} \xi_1(t) = 0$ which implies (4.20).

Write $\xi_2 = \xi_2(t)$ for the second inverse function. Then $\xi_2'(t) < 0$ and

$$\xi_2(t) = \frac{P(t, q(t))}{q(t)} = \frac{\partial P}{\partial q}(t, q(t)) > \frac{\partial P}{\partial q}(t, 0) \to \infty \quad (t \to 1/2).$$

This implies (4.21). □

Let us summarize. We have proved that $t(\xi)$ is analytic on $(0, \infty)$, $\lim_{\xi \to 0} t(\xi) = 0$ and $\lim_{\xi \to \infty} t(\xi) = 1/2$. We have also proved that $t(\xi)$ is increasing on $(0, \xi_0)$ and decreasing on (ξ_0, ∞), and that $t(\xi_0) = 1$.

5. *Khintchine spectrum*

We are now ready to study the Hausdorff dimensions of the level set

$$E_\xi = \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \xi \right\}.$$

Since $\mathbb{Q}$ is countable, we need only consider

$$\left\{ x \in [0, 1) \setminus \mathbb{Q} : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \xi \right\}$$

which admits the same Hausdorff dimension with E_ξ and is still denoted by E_ξ.

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5.1. *Proof of Theorem 1.2(1) and (2).* Let $(t, q) \in D$ and $\mu_{t,q}$, $\tilde{\mu}_{t,q}$ be the measures in Proposition 4.6. For $x \in [0, 1) \setminus \mathbb{Q}$, let $x = [a_1, \ldots, a_n, \ldots]$ and $\omega = \pi^{-1}(x)$. Then $\omega = a_1 \ldots a_n \ldots \in \mathbb{N}^{\mathbb{N}}$ and

$$\mu_{t,q}(I_n(x)) = \mu_{t,q}(I_n(a_1, \ldots, a_n)) = \tilde{\mu}_{t,q}([\omega|_n]).$$

By the Gibbs property of $\tilde{\mu}_{t,q}$,

$$\tilde{\mu}_{t,q}(\pi([\omega|_n])) \asymp \exp(-nP(t, q)) \prod_{j=1}^{n} \omega_j^q(\omega_j + \pi(\sigma^j\omega))^{-2t}.$$

In other words,

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t, q)) \prod_{j=1}^{n} a_j^q [a_j, \ldots, a_n, \ldots]^{2t}.$$

By Lemma 2.7, $|I_n(x)| \asymp |(T^n)'(x)|^{-1} = \prod_{j=0}^{n-1} |T^j(x)|^2$. Thus we have the following Gibbs property of $\mu_{t,q}$:

$$\mu_{t,q}(I_n(x)) \asymp \exp(-nP(t, q))|I_n(x)|^t \prod_{j=1}^{n} a_j^q. \tag{5.1}$$

It follows that

$$\delta_{\mu_{t,q}}(x) = \lim_{n\to\infty} \frac{\log \mu_{t,q}(I_n(x))}{\log |I_n(x)|} = t + \lim_{n\to\infty} \frac{q \cdot (1/n) \sum_{j=1}^n \log a_j - P(t, q)}{(1/n) \log |I_n(x)|}.$$

The Gibbs property of $\tilde{\mu}_{t,q}$ implies that $\mu_{t,q}$ is ergodic. Therefore,

$$\delta_{\mu_{t,q}}(x) = t + \frac{q \int \log a_1(x)\, d\mu_{t,q} - P(t, q)}{-\int \log |T'(x)|\, d\mu_{t,q}} \quad \mu_{t,q}\text{-a.e.}$$

Using the formula (4.3) and (4.4) in Proposition 4.8, we have

$$\delta_{\mu_{t,q}}(x) = t + \frac{q(\partial P/\partial q)(t, q) - P(t, q)}{(\partial P/\partial t)(t, q)} \quad \mu_{t,q}\text{-a.e.} \tag{5.2}$$

Moreover, the ergodicity of $\tilde{\mu}_{t,q}$ also implies that the Lyapunov exponents $\lambda(x)$ exist for $\mu_{t,q}$-almost every x in [0, 1). Thus by (5.1), Lemmas 2.12 and 2.13, we obtain

$$d_{\mu_{t,q}}(x) = \delta_{\mu_{t,q}}(x) = t + \frac{q(\partial P/\partial q)\,(t, q) - P(t, q)}{(\partial P/\partial t)\,(t, q)} \quad \mu_{t,q}\text{-a.e.} \tag{5.3}$$

For $\xi \in (0, \infty)$, choose $(t, q) = (t(\xi), q(\xi)) \in D_0$ to be the unique solution of (4.11). Then (5.3) gives

$$d_{\mu_{t(\xi),q(\xi)}}(x) = t(\xi) \quad \mu_{t,q}\text{-a.e.}$$

By the ergodicity of $\tilde{\mu}_{t(\xi),q(\xi)}$ and (4.4), we have for $\mu_{t(\xi),q(\xi)}$-almost every x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = \int \log a_1(x)\, d\mu_{t(\xi),q(\xi)} = \frac{\partial P}{\partial q}(t(\xi), q(\xi)) = \xi.$$

So $\mu_{t(\xi),q(\xi)}$ is supported on E_ξ. Hence

$$\dim(E_\xi) \geq \dim \mu_{t(\xi),q(\xi)} = t(\xi). \tag{5.4}$$

In the following we will show that

$$\dim(E_\xi) \leq t \quad (\text{for all } t > t(\xi)). \tag{5.5}$$

This will imply that $\dim(E_\xi) = t(\xi)$ for any $\xi > 0$. For any $t > t(\xi)$, take an $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 < \frac{P(t(\xi), q(\xi)) - P(t, q(\xi))}{q(\xi)} \quad \text{if } q(\xi) > 0,$$

and

$$0 < \epsilon_0 < \frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} \quad \text{if } q(\xi) < 0.$$

(For the special case $q(\xi) = 0$, i.e., $\xi = \xi_0$, the result $\dim E_\xi = 1$ is well known.) Such an ϵ_0 exists, for $P(t, q)$ is strictly decreasing with respect to t. For all $n \geq 1$, set

$$E_\xi^n(\epsilon_0) := \left\{ x \in [0, 1) \setminus \mathbb{Q} : \xi - \epsilon_0 < \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0 \right\}.$$

Then

$$E_\xi \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_\xi^n(\epsilon_0).$$

Let $\mathcal{I}(n, \xi, \epsilon_0)$ be the collection of all nth-order cylinders $I_n(a_1, \ldots, a_n)$ such that

$$\xi - \epsilon_0 < \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0.$$

Then

$$E_\xi^n(\epsilon_0) = \bigcup_{J \in \mathcal{I}(n,\xi,\epsilon_0)} J.$$

Hence $\{J : J \in \mathcal{I}(n, \xi, \epsilon_0), n \geq 1\}$ is a cover of E_ξ. When $q(\xi) > 0$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} |J|^t &\leq \sum_{n=1}^{\infty} \sum_{(a_1 \ldots a_n) > e^{n(\xi-\epsilon_0)}} \frac{e^{nP(t,q(\xi))}}{(a_1 \ldots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1 \ldots a_n)^{q(\xi)}}{e^{nP(t,q(\xi))}} \\ &\leq C \cdot \sum_{n=1}^{\infty} e^{n(P(t,q(\xi)) - (\xi-\epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} \mu_{t,q(\xi)}(J) < \infty\end{aligned}$$

by (5.1), where C is a constant. When $q(\xi) < 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} |J|^t &\leq \sum_{n=1}^{\infty} \sum_{(a_1 \ldots a_n) < e^{n(\xi+\epsilon_0)}} \frac{e^{nP(t,q(\xi))}}{(a_1 \ldots a_n)^{q(\xi)}} \cdot \frac{|J|^t (a_1 \ldots a_n)^{q(\xi)}}{e^{nP(t,q(\xi))}} \\ &\leq C \cdot \sum_{n=1}^{\infty} e^{n(P(t,q(\xi)) - (\xi+\epsilon_0)q(\xi))} \cdot \sum_{J \in \mathcal{I}(n,\xi,\epsilon_0)} \mu_{t,q(\xi)}(J) < \infty.\end{aligned}$$

Hence we get (5.5).

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For the special case $\xi = 0$, we need only show that $\dim(E_0) = 0$. This can be induced by the same process. For any $t > 0$, since $\lim_{\xi\to 0} t(\xi) = 0$, there exists $\xi > 0$ such that $0 < t(\xi) < t$. We can also choose $\epsilon_0 > 0$ such that

$$\frac{P(t, q(\xi)) - P(t(\xi), q(\xi))}{q(\xi)} > \epsilon_0.$$

For $n \geq 1$, set

$$E_0^n(\epsilon_0) := \left\{x \in [0, 1) \setminus \mathbb{Q} : \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) < \xi + \epsilon_0\right\}.$$

Then

$$E_0 \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_0^n(\epsilon_0).$$

By the same calculation, we get $\dim(E_0) \leq t$. Since t can be arbitrary small, we obtain $\dim(E_0) = 0$.

By the discussion in the preceding subsection, we have proved Theorem 1.2(1) and (2).

5.2. *Proof of Theorem 1.2(3) and (4).* We investigate more properties of the functions $q(\xi)$ and $t(\xi)$.

PROPOSITION 5.1. *As ξ goes to zero and infinity respectively,*

$$\lim_{\xi\to 0} q(\xi) = -\infty, \quad \lim_{\xi\to\infty} q(\xi) = 0.$$

Proof. We prove the first limit by contradiction. Suppose that there exists a subsequence $\xi_\delta \to 0$ such that $q(\xi_\delta) \to M > -\infty$. Then by (4.20) and Proposition 4.8(3),

$$\lim_{\xi_\delta\to 0} \frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \frac{\partial P}{\partial q}(0, M) > 0.$$

This is in contradiction to

$$\frac{\partial P}{\partial q}(t(\xi_\delta), q(\xi_\delta)) = \xi_\delta \to 0.$$

On the other hand, we know that $q(\xi) \geq 0$ when $\xi \geq \xi_0$, and $0 \leq q(\xi) < 2t(\xi) - 1$. Then by (4.21), $\lim_{\xi\to\infty} q(\xi) = 0$. □

Applying this proposition and (4.17), combining (4.9) and (4.10),

$$\lim_{\xi\to 0} t'(\xi) = +\infty, \quad \lim_{\xi\to\infty} t'(\xi) = 0.$$

This is assertion (3) of Theorem 1.2.

We now prove the last assertion of Theorem 1.2, that $t''(\xi_0) < 0$ and that there exists $\xi_1 > \xi_0$ such that $t''(\xi_1) > 0$, based on the following proposition.

PROPOSITION 5.2. *For $\xi \in (0, +\infty)$,*

$$q'(\xi) = \frac{1 - t'(\xi)\,(\partial^2 P/\partial t\partial q)\,(t(\xi), q(\xi))}{(\partial^2 P/\partial q^2)\,(t(\xi), q(\xi))}, \tag{5.6}$$

$$t''(\xi) = \frac{t'(\xi)^2(\partial^2 P/\partial t^2)\,(t(\xi), q(\xi)) - q'(\xi)^2(\partial^2 P/\partial q^2)\,(t(\xi), q(\xi))}{-(\partial P/\partial t)\,(t(\xi), q(\xi))}. \tag{5.7}$$

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Proof. Taking the derivative of (4.19) with respect to ξ, we get

$$t'(\xi)^2 \frac{\partial^2 P}{\partial t^2}(t(\xi), q(\xi)) + q'(\xi)t'(\xi)\frac{\partial^2 P}{\partial q \partial t}(t(\xi), q(\xi)) + t''(\xi)\frac{\partial P}{\partial t}(t(\xi), q(\xi)) = q'(\xi). \tag{5.8}$$

Taking the derivative of the second equation of (4.18) with respect to ξ, we get

$$t'(\xi)\frac{\partial^2 P}{\partial t \partial q}(t(\xi), q(\xi)) + q'(\xi)\frac{\partial^2 P}{\partial q^2}(t(\xi), q(\xi)) = 1, \tag{5.9}$$

which immediately gives (5.6).

Subtracting (5.9) multiplied by $q'(\xi)$ from (5.8) gives (5.7). □

We divide the proof of the assertion (4) of Theorem 1.2 into two parts.

Proof of $t''(\xi_0) < 0$. By Proposition 4.8, $(\partial P/\partial t)\,(1, 0) < 0$. Since $q(\xi_0) = 0$, then $t'(\xi_0) = 0$ by (4.17). Also by Proposition 4.8, we get

$$0 < \frac{\partial^2 P}{\partial t^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial t^2}(1, 0) < +\infty$$

and

$$0 \le \frac{\partial^2 P}{\partial q^2}(t(\xi_0), q(\xi_0)) = \frac{\partial^2 P}{\partial q^2}(1, 0) < +\infty.$$

By (5.6) and (5.7),

$$t''(\xi) = \frac{t'(\xi)^2(\partial^2 P/\partial t^2)\,(t(\xi), q(\xi))\,(\partial^2 P/\partial q^2)\,(t(\xi), q(\xi)) - (1 - t'(\xi)\,(\partial^2 P/\partial t \partial q)\,(t(\xi), q(\xi)))^2}{-(\partial P/\partial t)\,(t(\xi), q(\xi))\,(\partial^2 P/\partial q^2)\,(t(\xi), q(\xi))}. \tag{5.10}$$

Thus, by $t'(\xi_0) = 0$, $t''(\xi_0) < 0$. □

Proof of $t''(\xi_1) > 0$. Proposition 5.1 shows that $\lim_{\xi \to \infty} q(\xi) = 0$ and we know that $q(\xi_0) = 0$. However, $q(\xi)$ is not always equal to 0, so there exists a $\xi_1 \in [\xi_0, +\infty)$, such that $q'(\xi_1) < 0$. Writing

$$H(t, q) := \begin{pmatrix} \dfrac{\partial^2 P}{\partial t^2} & \dfrac{\partial^2 P}{\partial t \partial q} \\ \dfrac{\partial^2 P}{\partial t \partial q} & \dfrac{\partial^2 P}{\partial q^2} \end{pmatrix}$$

and adding (5.9) multiplied by $q'(\xi)$ to (5.8), we get

$$(t'(\xi), q'(\xi))H(t, q)(t'(\xi), q'(\xi))^T + \frac{\partial P}{\partial t}(t(\xi), q(\xi))t''(\xi) = 2q'(\xi). \tag{5.11}$$

Since $H(t, q)$ is positive definite, $(\partial P/\partial t)\,(t, q) < 0$ and $q'(\xi_1) < 0$, then $t''(\xi_1) > 0$. This completes the proof. □

6. *Lyapunov spectrum*

In this last section, we follow the same procedure as in §§4–5 to deduce the Lyapunov spectrum of the Gauss map.

Kesseböhmer recently pointed out to us that the Lyapunov spectrum was also studied by Kesseböhmer and Stratmann [**24**].

Take

$$F = \Psi = \{\log |\psi_i'| : i \in \mathbb{N}\}$$

instead of $F = \{-\log i : i \in \mathbb{N}\}$ and $\Psi = \{\log |\psi_i'| : i \in \mathbb{N}\}$. Then the strong Hölder family becomes $(\tilde{t} - q)\Psi$, and D should be changed to

$$\tilde{D} := \{(\tilde{t}, q) : \tilde{t} - q > 1/2\}.$$

Here and in the rest of this section we will use $\tilde{t}$ instead of t to distinguish the present situation from that of Khintchine exponents. What we have done in §4 is still useful. Denote by $P_1(\tilde{t}, q)$ the pressure $P((\tilde{t} - q)\Psi)$. Then

$$P_1(\tilde{t}, q) = P(\tilde{t} - q) \quad \text{with } P(\cdot) = P(\cdot, 0),$$

where $P(\cdot, \cdot)$ is the pressure function studied in §4. Hence $P_1(\tilde{t}, q)$ is analytic and similar equations (4.3) and (4.4) are obtained just with $\log |T'(x)|$ instead of $\log a_1(x)$.

To determine the Lyapunov spectrum, we begin with the following proposition which takes the place of Proposition 4.12.

PROPOSITION 6.1. *For $(\tilde{t}, q) \in \tilde{D}$, we have*

$$-(\tilde{t} - q) \log 4 + \log \zeta(2\tilde{t} - 2q) \le P_1(\tilde{t}, q) \le \log \zeta(2\tilde{t} - 2q). \tag{6.1}$$

Consequently:

(1) *for any point $(\tilde{t}_0, q_0)$ on the line $\tilde{t} - q = 1/2$,*

$$\lim_{(\tilde{t},q)\to(\tilde{t}_0,q_0)} P(\tilde{t}, q) = \infty;$$

(2) *for fixed $\tilde{t} \in \mathbb{R}$,*

$$\lim_{q\to\tilde{t}-1/2} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = +\infty;$$

(3) *recalling that $\gamma_0 = 2 \log((1 + \sqrt{5})/2)$, for fixed $\tilde{t} \in \mathbb{R}$,*

$$\lim_{q\to-\infty} \frac{P_1(\tilde{t}, q)}{q} = \gamma_0, \qquad \lim_{q\to-\infty} \frac{\partial P_1}{\partial q}(\tilde{t}, q) = \gamma_0.$$

Proof. $P_1(\tilde{t}, q)$ is defined as

$$P_1(\tilde{t}, q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega_1=1}^{\infty} \cdots \sum_{\omega_n=1}^{\infty} \exp\bigg(\sup_{x\in[0,1]} \log \prod_{j=1}^{n} ([\omega_j, \ldots, \omega_n + x])^{2(\tilde{t}-q)} \bigg).$$

The proofs of (1) and (2) are the same as in the proof of Proposition 4.12.

To get (3), we follow another method. Since $P_1(\tilde{t}, q) = P(\tilde{t} - q)$, we need only show that

$$\lim_{\tilde{t}\to\infty} P'(\tilde{t}) = -\gamma_0, \qquad P(\tilde{t}) + \tilde{t}\gamma_0 = o(\tilde{t}) \quad (\tilde{t} \to \infty).$$

By Proposition 4.9, $P(\tilde{t})$ is analytic on $(1/2, \infty)$. Let $E := \{P'(\tilde{t}) : \tilde{t} > 1/2\}$, and denote by $\mathrm{Int}(E)$ and $\mathrm{Cl}(E)$ the interior and closure of E. By a result in [**23**],

$$\mathrm{Int}(E) \subset \left\{ -\int \log |T'(x)| \, d\mu : \mu \in \mathcal{M} \right\} \subset \mathrm{Cl}(E),$$

where $\mathcal{M}$ is the set of the invariant measures on [0, 1]. By Birkhoff's theorem, for any $\mu \in \mathcal{M}$,

$$\int \lambda(x) \, d\mu = \int \log |T'(x)| \, d\mu.$$

However, we know that $\lambda(x) \geq \gamma_0 = 2 \log((1 + \sqrt{5})/2)$. Thus

$$-\int \log |T'(x)| \, d\mu \leq -\gamma_0 \quad \text{for all } \mu \in \mathcal{M}. \tag{6.2}$$

Let $\theta_0 = (\sqrt{5} - 1)/2$. Then $T(\theta_0) = \theta_0$ and the Dirac measure $\mu = \delta_{\theta_0}$ is invariant, and

$$-\int \log |T'(x)| \, d\delta_{\theta_0} = -\log |T'(\theta_0)| = -\gamma_0.$$

However, by the continuity of P', we know that E is an interval. Therefore $-\gamma_0$ is the right endpoint of E. Since $P'(\tilde{t})$ is increasing, we get

$$\lim_{\tilde{t} \to \infty} P'(\tilde{t}) = -\gamma_0.$$

Let $\{\beta_n\}_{n \geq 1}$ be such that $\beta_n < -\gamma_0$ and $\lim_{n \to \infty} \beta_n = -\gamma_0$. There exist $t_n \in \mathbb{R}$ such that $t_n \to \infty$ and $P'(t_n) = \beta_n$. By the variational principle ([**38**]; see also [**32**]), there exists an ergodic measure μ_{t_n} such that

$$P(t_n) = h_{\mu_{t_n}} - t_n \int \log |T'|(x) \, d\mu_{t_n},$$

where $h_{\mu_{t_n}}$ stands for the metric entropy of μ_{t_n}. By the compactness of $\mathcal{M}$ there exists an invariant measure μ_∞ which is the weak limit of μ_{t_n}(more precisely some subsequence of μ_{t_n}, but, without loss of generality, we write it as μ_{t_n}). By the semi-continuity of metric entropy, $h_{\mu_{t_n}} \leq h_{\mu_\infty} + \epsilon$ for any $\epsilon > 0$ when t_n is large enough. Thus by (6.2),

$$P(t_n) \leq h_{\mu_\infty} + \epsilon - t_n \gamma_0.$$

We will show that $h_{\mu_\infty} = 0$ (see the next lemma), which will imply that

$$P(t_n) \leq \epsilon - t_n \gamma_0.$$

However, by the definition of $P_1(\tilde{t}, q)$, $P(\tilde{t})$ can be written as

$$P(\tilde{t}) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega_1 = 1}^{\infty} \cdots \sum_{\omega_n = 1}^{\infty} \exp\left(\sup_{x \in [0,1]} \log \prod_{j=1}^{n} ([\omega_j, \ldots, \omega_n + x])^{2\tilde{t}} \right).$$

Thus if we just take one term in the summation, then

$$P(\tilde{t}) \geq \lim_{n \to \infty} \frac{1}{n} \log \exp\left(\sup_{x \in [0,1]} \log \prod_{j=1}^{n} ([\underbrace{1, \ldots, 1}_{n-j}, 1 + x])^{2\tilde{t}} \right) = -\tilde{t}\gamma_0.$$

Hence we get

$$P(\tilde{t}) + \tilde{t}\gamma_0 = o(\tilde{t}) \quad (\tilde{t} \to \infty).$$

□

We now need to prove the following lemma.

LEMMA 6.2. $h_{\mu_\infty} = 0$.

Proof. Let $h_{\mu_\infty}(x)$ be the local entropy of μ_∞ at x which is defined by

$$h_{\mu_\infty}(x) = \lim_{n\to\infty} \frac{\log \mu_\infty(I_n(x))}{n},$$

if the limit exists. Let $\underline{D}_{\mu_\infty}(x)$ be the lower local dimension of μ_∞ at x which is defined by

$$\underline{D}_{\mu_\infty}(x) := \liminf_{r\to 0} \frac{\log \mu_\infty(B(x, r))}{\log r}.$$

By the Shannon–McMillan–Breiman theorem, $h_{\mu_\infty}(x)$ exists μ_∞-almost everywhere. It is also known that $\lambda(x)$ exists almost everywhere (by Birkhoff's theorem). So, by the definitions,

$$h_{\mu_\infty}(x) = \underline{D}_{\mu_\infty}(x)\lambda(x) \quad \mu_\infty\text{-a.e.}$$

By Birkhoff's theorem and (4.5),

$$\begin{aligned}\int \lambda(x)\, d\mu_\infty(x) &= \int \log |T'|(x)\, d\mu_\infty(x)\\ &= \lim_{n\to\infty} \int \log |T'|(x)\, d\mu_{t_n}\\ &= -\lim_{n\to\infty} P'(t_n) = \gamma_0 < \infty.\end{aligned}$$

Hence $\lambda(x)$ is almost everywhere finite. Recall that [**7**]

$$h_{\mu_\infty} = \int h_{\mu_\infty}(x)\, d\mu_\infty(x).$$

Thus it suffices to prove

$$\underline{D}_{\mu_\infty}(x) = 0 \quad \mu_\infty\text{-a.e.}$$

This means [**12**] that the upper dimension of μ_∞ is zero, i.e., μ_∞ is supported by a zero-dimensional set.

Since $\int \lambda(x)\, d\mu_\infty(x) = \gamma_0$ and $\lambda(x) \geq \gamma_0$ for any x, we have for μ_∞ almost everywhere $\lambda(x) = \gamma_0$. Thus by Birkhoff's theorem, μ_∞ is supported by the set

$$\left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j x)| = \gamma_0\right\}. \tag{6.3}$$

Thus we need only show that the Hausdorff dimension of this set is zero.

Recall that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |T'(T^j x)| = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(x).$$

By Lemma 2.8, (6.3) is in fact the set

$$\left\{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_j(x) = 0\right\}. \tag{6.4}$$

However, the Hausdorff dimension of (6.4) is nothing but $t(0)$, the special case $\xi = 0$ discussed in §5.1, which was proved to be zero. This concludes the proof. □

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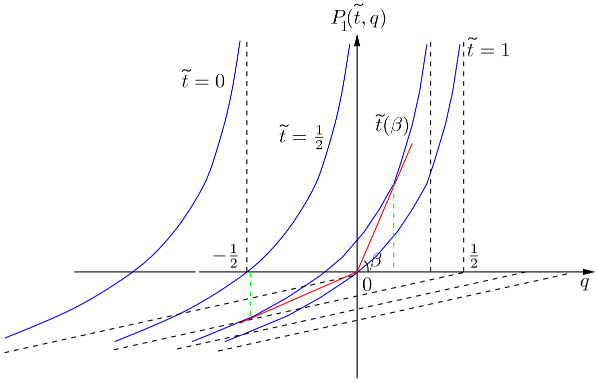

FIGURE 4. Solution of (6.5).

Recall that $\lambda_0 = \int \log |T'(x)|\, d\mu_G$. Let $\tilde D_0 := \{(\tilde t, q) : \tilde t - q > 1/2,\ 0 \le \tilde t \le 1\}$. The following proposition, similar to Proposition 4.13, holds. (See Figure 4, similar to Figure 3, for searching for the solutions.)

PROPOSITION 6.3. *For any $\beta \in (\gamma_0, \infty)$, the system*

$$\begin{cases} P_1(\tilde t, q) = q\beta, \\ \dfrac{\partial P_1}{\partial q}(\tilde t, q) = \beta. \end{cases} \tag{6.5}$$

admits a unique solution $(\tilde t(\beta), q(\beta)) \in \tilde D_0$. For $\beta = \lambda_0$, the solution is $(\tilde t(\lambda_0), q(\lambda_0)) =$ $(1, 0)$. The functions $\tilde t(\beta)$ and $q(\beta)$ are analytic.

With the same argument, we can prove that $\tilde t(\beta)$ is the Lyapunov exponent spectrum. It is analytic, increasing on $(\gamma_0, \lambda_0]$ and decreasing on (λ_0, ∞). It is also neither concave nor convex. In other words, Theorem 1.3 can be similarly proved.

We conclude the paper with the observation that the Lyapunov spectrum can be stated as follows, which is similar to the classic formula, but with the difference that we have to divide the Legendre transform by β.

PROPOSITION 6.4.

$$\tilde t(\beta) = \frac{P(-q(\beta))}{\beta} - q(\beta) = \frac{1}{\beta} \inf_q \{P(-q) - q\beta\}. \tag{6.6}$$

Proof. In fact, the family of functions $P_1(\tilde t, q)$ with parameter $\tilde t$ are just right translation of the function $P(-q)$ with the length $\tilde t$. Write the system (6.5) as

$$\begin{cases} P(\tilde t - q) = q\beta, \\ \dfrac{dP}{dq}(\tilde t - q) = \beta. \end{cases} \tag{6.7}$$

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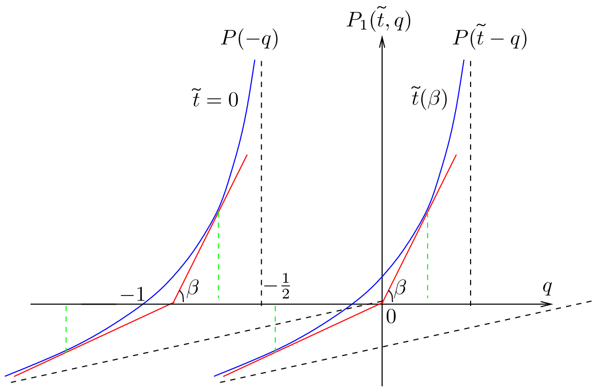

FIGURE 5. The other way to search for $\tilde{t}(\beta)$.

If we denote by μ_q the Gibbs measure with respect to potential $q\Psi$, then by a left translation the system (6.7) can be written as

$$\begin{cases} P(-q) = (\tilde{t} + q)\beta, \\ \dfrac{dP}{dq}(-q) = \beta. \end{cases}$$

Thus

$$\begin{cases} \tilde{t} = \dfrac{P(-q)}{\beta} - q, \\ \dfrac{dP}{dq}(-q) = \beta. \end{cases}$$

By using the second equation, we can write q as a function of β, hence we get (6.6). (See Figure 5 for the other way to search for $\tilde{t}(\beta)$.) □

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