Combinatorics, Probability and Computing (2014) 23, 248–268. © Cambridge University Press 2014
doi:10.1017/S096354831300062X

Invariant Measure Under the Affine Group Over $\mathbb{Z}$

DANIELE MUNDICI

Department of Mathematics and Computer Science, University of Florence, Viale Morgagni 67/A, 50134 Florence, Italy
(e-mail: mundici@math.unifi.it)

Received 8 November 2011; revised 16 November 2013; first published online 2 January 2014

A *rational polyhedron* $P \subseteq \mathbb{R}^n$ is a finite union of simplexes in $\mathbb{R}^n$ with rational vertices. P is said to be $\mathbb{Z}$-*homeomorphic* to the rational polyhedron $Q \subseteq \mathbb{R}^m$ if there is a piecewise linear homeomorphism η of P onto Q such that each linear piece of η and η^{-1} has integer coefficients. When $n = m$, $\mathbb{Z}$-homeomorphism amounts to continuous $\mathcal{G}_n$-equidissectability, where $\mathcal{G}_n = GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^n$ is the affine group over the integers, *i.e.*, the group of all affinities on $\mathbb{R}^n$ that leave the lattice $\mathbb{Z}^n$ invariant. $\mathcal{G}_n$ yields a geometry on the set of rational polyhedra. For each $d = 0, 1, 2, \ldots$, we define a rational measure λ_d on the set of rational polyhedra, and show that any two $\mathbb{Z}$-homeomorphic rational polyhedra $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ satisfy $\lambda_d(P) = \lambda_d(Q)$. $\lambda_n(P)$ coincides with the n-dimensional Lebesgue measure of P. If $0 \leqslant \dim P = d < n$ then $\lambda_d(P) > 0$. For rational d-simplexes T lying in the same d-dimensional affine subspace of $\mathbb{R}^n$, $\lambda_d(T)$ is proportional to the d-dimensional Hausdorff measure of T. We characterize λ_d among all unimodular invariant valuations.

2010 *Mathematics subject classification*: Primary 20H25
Secondary 20H05, 51M25, 51N25, 52B20, 14E05, 52A38

1. Introduction: statement of the main results

1.1. Prologue: Markov unrecognizability theorem

Following [14] and [26], we use the term *polyhedron* P in $\mathbb{R}^n$ $(n = 1, 2, \ldots)$ to mean the union of a finite set of (always closed) simplexes T_i in $\mathbb{R}^n$. If the vertices of each T_i are in $\mathbb{Q}^n$, P is said to be a *rational polyhedron.* By Markov's unrecognizability theorem [25, and references therein], no Turing machine can decide whether there is a PL-homeomorphism θ of two polyhedra P and Q. For the decision problem to make sense, P and Q are assumed to be rational, so that they can be effectively presented as finite strings of symbols. Without loss of generality, one may insist that each linear piece of θ and θ^{-1} has *rational* coefficients [11, p. 55], thus showing that the set of pairs of PL-homeomorphic rational polyhedra is *Gödel incomplete* (*i.e.*, recursively enumerable but not decidable). If we further assume that all coefficients are *integers*, we obtain what in [17] is called a $\mathbb{Z}$-*homeomorphism*. Any $\mathbb{Z}$-homeomorphism preserves (least common) denominators of rational points, thus taking due care of the amount of data needed to specify P and Q. Two rational polyhedra $R, S \subseteq \mathbb{R}^n$ are $\mathbb{Z}$-homeomorphic if and only if they are continuously $\mathcal{G}_n$-equidissectable [14], where $\mathcal{G}_n = GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^n$ is the group of all affinities on $\mathbb{R}^n$ that leave the lattice $\mathbb{Z}^n$ invariant. $\mathcal{G}_n$ induces a geometry on the family of rational polyhedra, and equips them with many invariants. The present paper deals with one such invariant, the d-dimensional rational measure λ_d, for $d = 0, 1, 2, \ldots$.

1.2. Fans and regular triangulations of rational polyhedra

For any $n = 1, 2, \ldots$ and rational point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we let $\mathrm{den}(x)$ denote the least common denominator of the coordinates of x. The integer vector

$$\tilde{x} = \mathrm{den}(x)(x_1, \ldots, x_n, 1) \in \mathbb{Z}^{n+1}$$

is called the *homogeneous correspondent* of x. For $m = 0, 1, \ldots$, an m-simplex

$$T = \mathrm{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$$

is said to be *rational* if all its vertices are rational. We use the notation $T^{\uparrow} = \mathbb{R}_{\geqslant 0}\,\tilde{v}_0 + \cdots + \mathbb{R}_{\geqslant 0}\,\tilde{v}_m \subseteq \mathbb{R}^{n+1}$ for the positive span in $\mathbb{R}^{n+1}$ of the homogeneous correspondents of the vertices of T. We say that $T^{\uparrow}$ is the (*rational simplicial*) *cone* of T. The *generators* $\tilde{v}_0, \ldots, \tilde{v}_m$ of $T^{\uparrow}$ are *primitive*, in the sense that each $\tilde{v}_i$ is minimal as a non-zero integer vector along its *ray* $\mathbb{R}_{\geqslant 0}\,\tilde{v}_i$. $T^{\uparrow}$ uniquely determines the set of its primitive generators, just as T uniquely determines the set $\mathrm{ext}(T)$ of its vertices. Following [8] we say that $T^{\uparrow}$ is *regular* if its primitive generators are part of a basis of the free abelian group $\mathbb{Z}^{n+1}$. By definition, a rational m-simplex $T = \mathrm{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$ is (*Farey*) *regular* if $T^{\uparrow}$ is regular. (*Warning*. In the literature one also finds the term 'regular' simplex T when all edge lengths of T are equal. Regular simplexes in this sense will have no role in this paper.) The mth Farey sequence, $m = 1, 2, \ldots$, yields the vertices of a (Farey) regular triangulation of the unit interval. More generally, any (Farey) regular triangulation of the unit interval consists of intervals that appear in some Farey sequence.

By a (*polyhedral*) *complex* in $\mathbb{R}^n$ we mean a finite set Λ of convex polyhedra P_i in $\mathbb{R}^n$, closed under taking faces, and having the further property that any two elements of Λ intersect in a common face. The complex Λ is said to be *rational* if the vertices of all $P_i \in \Lambda$ are rational. If all P_i are simplexes then Λ is said to be a *simplicial complex*. For every complex Λ, its *support* $|\Lambda| \subseteq \mathbb{R}^n$ is the pointset union of all polyhedra of Λ. Let Δ be a rational simplicial complex. Instead of saying that the support of Δ is the rational polyhedron $|\Delta|$, we say that Δ is a *triangulation* of $|\Delta|$. The set $\Delta^{\uparrow} = \{T^{\uparrow} \mid T \in \Delta\}$ is a *simplicial fan,* [8, 21]. We say that Δ is *regular* if the simplicial fan $\Delta^{\uparrow}$ is *regular* (= *non-singular* in [21]), meaning that every cone $T^{\uparrow} \in \Delta^{\uparrow}$ is regular. Lemma 2.1 ensures that every rational polyhedron P is the support of some regular complex.

1.3. The rational measure λ_d

For $n > 0$ a fixed integer, let $Q \subseteq \mathbb{R}^n$ be a (not necessarily rational) polyhedron. For any triangulation $\mathcal{T}$ of Q and $i = 0, 1, \ldots$ we let $\mathcal{T}^{\max}(i)$ denote the set of maximal i-simplexes

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

of $\mathcal{T}$. The *i-dimensional part $Q^{(i)}$ of Q* is now defined by

$$Q^{(i)} = \bigcup\{T \in \mathcal{T}^{\max}(i)\}. \tag{1.1}$$

Since any two triangulations of Q have a joint subdivision, the definition of $Q^{(i)}$ does not depend on the chosen triangulation $\mathcal{T}$ of Q. If $Q^{(i)}$ is non-empty, then it is an i-dimensional polyhedron whose j-dimensional part $Q^{(j)}$ is empty for each $j \neq i$. Trivially, $Q^{(k)} = \emptyset$ for each integer $k > \dim(Q)$. For every (Farey) regular m-simplex $S = \operatorname{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ we use the notation

$$\operatorname{den}(S) = \prod_{j=0}^{m} \operatorname{den}(v_j),$$

and say that $\operatorname{den}(S)$ is the *denominator* of S. For any rational polyhedron P in $\mathbb{R}^n$, regular triangulation Δ of P, and $i = 0, 1, \dots,$ the rational number $\lambda(n, i, P, \Delta)$ is defined by

$$\lambda(n, i, P, \Delta) = \sum_{T \in \Delta^{\max}(i)} \frac{1}{i!\, \operatorname{den}(T)}, \tag{1.2}$$

with the proviso that $\lambda(n, i, P, \Delta) = 0$ if $\Delta^{\max}(i) = \emptyset$. In particular, this is the case for all $i > \dim(P)$.

Our first main result, Theorem 2.3, shows that the quantity $\lambda(n, i, P, \Delta)$ does not depend on Δ. Thus we can unambiguously write

$$\lambda_d(P) = \lambda(n, d, P, \Delta), \tag{1.3}$$

where Δ is an arbitrary regular triangulation of $P \subseteq \mathbb{R}^n$. We say that λ_d is the *d-dimensional rational measure* of P. Trivially, $\lambda_d(P) = 0$ for each integer $d > \dim(P)$.

As an alternative construction of λ_d for readers having some familiarity with fans [8, 21], let us write $(P, 1)$ as an abbreviation of $\{(x, 1) \in \mathbb{R}^{n+1} \mid x \in P\}$. Let Φ be a regular fan over the set $\{\theta y \in \mathbb{R}^{n+1} \mid 0 \leqslant \theta \in \mathbb{R},\ y \in (P, 1)\}$. Next, let Δ_Φ be the triangulation of $(P, 1)$ obtained by intersecting every cone of Φ with the hyperplane $x_{n+1} = 1$. Then

$$\lambda_d(P) = \sum \left\{ \frac{1}{d!\, \prod_{v \in \operatorname{ext}(T)} \operatorname{den}(v)} \mid T \text{ a maximal } d\text{-simplex of } \Delta_\Phi \right\}.$$

The proof of Theorem 2.3 relies upon the solution of the weak Oda conjecture by Morelli and Włodarczyk [15, 27].

Perusal of the proof of Lemma 2.1 shows that *the map $(P, d) \mapsto \lambda_d(P)$ is Turing-computable.* Some explicit computations are given in Figures 1 and 2.

Recall that $\mathcal{G}_n = GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ denotes the group of transformations of the form $x \mapsto Ax + t \quad (x \in \mathbb{R}^n)$, where $t \in \mathbb{Z}^n$ and A is an $n \times n$ matrix with integer entries and determinant ± 1. Throughout, we will let

$$\mathcal{P}^{(n)}$$

denote the set of all rational polyhedra in $\mathbb{R}^n$. Our second main result is as follows.

Theorem 1.1. *For each $n = 1, 2, \dots$ and $d = 0, 1, \dots,$ the map $\lambda_d : \mathcal{P}^{(n)} \to \mathbb{R}_{\geqslant 0}$ has the following properties, for all $P, Q \in \mathcal{P}^{(n)}$.*

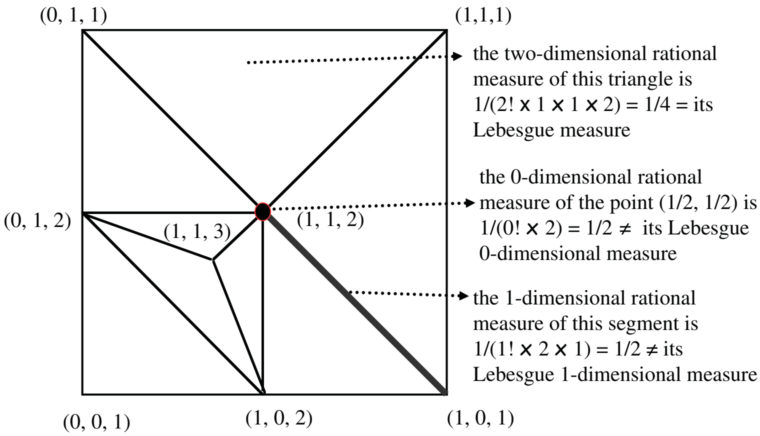

Figure 1. (Colour online) Hironaka's regular triangulation ∇ of the unit square (see [6, pp. 270–271]). The vertices of ∇ are specified by their homogeneous coordinates. Each simplex of ∇ is (Farey) regular. The sum of the two-dimensional measures of the 2-simplexes of ∇ equals 1. Only the rational measure of segments and vertices of ∇ may differ from their Lebesgue measure.

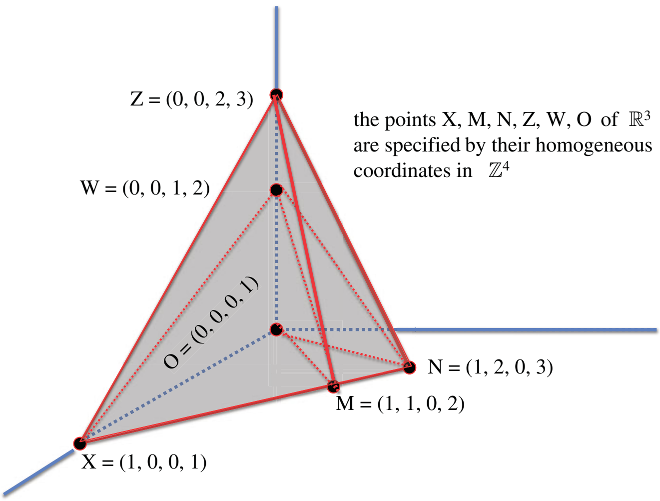

Figure 2. (Colour online) The two-dimensional rational measure 1/9 of the triangle XNZ in $\mathbb{R}^3$ is the sum of the two-dimensional rational measures $1/12 = 1/(2! \times 1 \times 2 \times 3)$ and $1/36 = 1/(2! \times 2 \times 3 \times 3)$ of the (Farey) regular simplexes XMZ and MNZ. The segment XN is not regular. The segments XM and MN, as well as XZ and ZN, are regular. The tetrahedra ZWXM, ZWMN, OWXM, OWMN are regular. Their three-dimensional rational measures are 1/72, 1/216, 1/24, 1/72, respectively, coinciding with their Lebesgue volumes.

(i) Invariance. *If* $P = \gamma(Q)$ *for some* $\gamma \in \mathcal{G}_n$ *then* $\lambda_d(P) = \lambda_d(Q)$.

(ii) Valuation. $\lambda_d(\emptyset) = 0$, $\lambda_d(P) = \lambda_d(P^{(d)})$, *and the restriction of* λ_d *to the set of all rational polyhedra* P, Q *in* $\mathbb{R}^n$ *having dimension* $\leqslant d$ *is a* valuation, *i.e.,*

$$\lambda_d(P) + \lambda_d(Q) = \lambda_d(P \cup Q) + \lambda_d(P \cap Q). \tag{1.4}$$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

(iii) Conservativity. *For any* $P \in \mathcal{P}^{(n)}$*, let*

$$(P,0) = \{(x,0) \in \mathbb{R}^{n+1} \mid x \in P\}.$$

Then $\lambda_d(P) = \lambda_d(P,0)$.

(iv) Pyramid. *For* $k = 1,\ldots,n$*, if* $\mathrm{conv}(v_0,\ldots,v_k)$ *is a (Farey) regular* k*-simplex in* $\mathbb{R}^n$ *with* $v_0 \in \mathbb{Z}^n$*, then*

$$\lambda_k(\mathrm{conv}(v_0,\ldots,v_k)) = \lambda_{k-1}(\mathrm{conv}(v_1,\ldots,v_k))/k. \tag{1.5}$$

(v) Normalization. *Let* $j = 1,\ldots,n$*. Suppose the set* $B = \{w_1,\ldots,w_j\} \subseteq \mathbb{Z}^n$ *is part of a basis of the free abelian group* $\mathbb{Z}^n$*. Let the closed parallelepiped* $P_B \subseteq \mathbb{R}^n$ *be defined by*

$$P_B = \left\{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{j} \gamma_i w_i,\ 0 \leqslant \gamma_i \leqslant 1\right\}. \tag{1.6}$$

Then $\lambda_j(P_B) = 1$.

(vi) Proportionality. *Let* A *be an* m*-dimensional rational affine subspace of* $\mathbb{R}^n$ *for some* $m = 0,\ldots,n$*. Then there is a constant* $\kappa_A > 0$*, depending only on* A*, such that* $\lambda_m(Q) = \kappa_A \cdot \mathcal{H}^m(Q)$ *for every rational* m*-simplex* $Q \subseteq A$*. Here, as usual,* $\mathcal{H}^m$ *denotes the* m*-dimensional Hausdorff measure.*

Conversely, in Theorem 6.2 we will prove that conditions (i)–(vi) uniquely characterize the maps $\lambda_d : \mathcal{P}^{(n)} \to \mathbb{R}_{\geqslant 0}$. As proved in Section 4, the Lebesgue measure on $\mathbb{R}^n$ is obtainable from λ_n via Carathéodory's construction, or using the main result of [20], or even [23]. In contrast to the Lebesgue measure, for each $0 \leqslant d < n$ and rational d-simplex $T \subseteq \mathbb{R}^n$, the rational measure $\lambda_d(T)$ does not vanish. Related measure-theoretic work on *convex* polyhedra, and applications of the λ_i to ordered groups and AF C^*-algebras will be briefly discussed in Section 8.

2. Farey blow-up and $\mathbb{Z}$-homeomorphism

Given two simplicial complexes Λ' and Λ with the same support, we say that Λ' is a *subdivision* of Λ if every simplex of Λ' is contained in a simplex of Λ. For any $c \in |\Lambda|$, the *blow-up* $\Lambda_{(c)}$ *of* Λ *at* c is the subdivision of Λ given by replacing every simplex $C \in \Lambda$ that contains c by the set of all simplexes of the form $\mathrm{conv}(F \cup \{c\})$, where F is any face of C that does not contain c (see [27, p. 376], [8, III, 2.1]).

The inverse of a blow-up is called a *blow-down*.

For any (Farey) regular m-simplex $T = \mathrm{conv}(v_0,\ldots,v_m) \subseteq \mathbb{R}^n$, the *Farey mediant of* T is the rational point v of T whose homogeneous correspondent $\tilde{v}$ coincides with $\tilde{v}_0 + \cdots + \tilde{v}_m$. If T belongs to a regular complex Δ and c is the Farey mediant of T, then the *Farey blow-up* $\Delta_{(c)}$ is regular.

By a *rational (affine) hyperplane* $H \subseteq \mathbb{R}^n$, we mean a subset of $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n \mid a \cdot x = t\}$, where $\cdot$ denotes the scalar product, a is a non-zero vector in $\mathbb{Q}^n$ (equivalently, in $\mathbb{Z}^n$) and $t \in \mathbb{Q}$. When $t = 0$, H is called *homogeneous*. By a *rational affine subspace* of $\mathbb{R}^n$ we mean the intersection $A_{\mathcal{F}}$ of a finite set $\mathcal{F}$ of rational hyperplanes in $\mathbb{R}^n$. In particular,

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

Figure 3. Two $\mathbb{Z}$-homeomorphic rational polyhedra in the unit square $[0,1]^2$.

$A_\emptyset = \mathbb{R}^n$. The *affine hull* $\mathrm{aff}(T)$ of a simplex T in $\mathbb{R}^n$ is the set of all affine combinations of points of T.

Lemma 2.1. *Every rational polyhedron $P \subseteq \mathbb{R}^n$ is the support of a regular complex.*

Proof. By [26, p. 36], P is the support of some simplicial complex Λ. Since P is rational, Λ can be assumed rational. The set $\Lambda^\uparrow = \{T^\uparrow \mid T \in \Lambda\}$ is a simplicial fan in $\mathbb{R}^{n+1}$. The desingularization procedure of [8, VI, 8.5] yields a regular subdivision Λ^* of $\Lambda^\uparrow$. Intersecting each cone of Λ^* with the hyperplane $x_{n+1} = 1$, we obtain a simplicial complex Δ whose support is the set $(P, 1) = \{(x, 1) \in \mathbb{R}^{n+1} \mid x \in P\}$. For each simplex $U \in \Delta$, let U' be the projection of U onto the hyperplane $x_{n+1} = 0$, identified with $\mathbb{R}^n$. Then the regularity of Λ^* ensures that the set $\{U' \mid U \in \Delta\}$ is a regular complex with support P. □

The following notion is of independent interest [17, Proof of Claim 2, pp. 544–545], and will find repeated use in this paper.

Definition 1. Two rational polyhedra $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ are $\mathbb{Z}$-*homeomorphic*, $P \cong_{\mathbb{Z}} Q$, if there is a piecewise linear homeomorphism $\eta = (\eta_1, \ldots, \eta_m)$ of P onto Q (each η_i with a finite number of pieces $l_{i1}, \ldots, l_{ik(i)}$) such that each linear piece of η and η^{-1} is a linear (affine) map with integer coefficients.

The adjective ‘linear’ is understood in the affine sense. Figures 3 and 4 give examples of $\mathbb{Z}$-homeomorphic rational polyhedra in the unit square $[0,1]^2$.

In particular, if $m = n$ and there exists $\gamma \in \mathcal{G}_n$ with $Q = \gamma(P)$, then $P \cong_{\mathbb{Z}} Q$. The converse does not hold: the two 0-simplexes $\{1/5\}$ and $\{2/5\}$ in $\mathbb{R}$ are $\mathbb{Z}$-homeomorphic but there is no $\gamma \in \mathcal{G}_1$ such that $\gamma(1/5) = 2/5$.

Lemma 2.2. *Suppose $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^{n'}$ are rational polyhedra and η is a $\mathbb{Z}$-homeomorphism of P onto P'.*

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

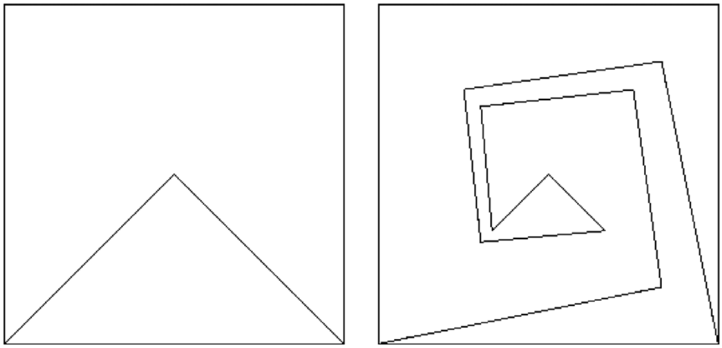

Figure 4. The triangle $T = \mathrm{conv}((0,0),(1,0),(1/2,1/2)) \subseteq [0,1]^2$ and a $\mathbb{Z}$-homeomorphic copy of T.

(i) *A point $z \in P$ is rational if and only if the point $\eta(z) \in P'$ is rational. Further,* $\mathrm{den}(y) = \mathrm{den}(\eta(y))$ *for every rational point $y \in P$.*

(ii) *There is a regular complex Λ with support P such that η is linear (in the affine sense) over every simplex of Λ.*

(iii) *For any regular complex Λ with support P such that η is linear over every simplex of Λ, the set $\Lambda' = \{\eta(S) \mid s \in \Lambda\}$ is a regular complex with support P'.*

Proof. **(i)** This is an immediate consequence of Definition 1.

(ii) Lemma 2.1 yields a regular complex $\mathcal{C}_0$ with support P. Let $\eta_1, \ldots, \eta_{n'}$ be the components of η. Fix $i = 1, \ldots, n'$ and let $l_{i1}, \ldots, l_{ik}$ be the linear pieces of η_i. Letting σ range over all permutations of the set $\{1, \ldots, k\}$, the family of sets $P_\sigma = \{x \in P \mid l_{i\sigma(1)} \leqslant \cdots \leqslant l_{i\sigma(k)}\}$ can be subdivided into a rational (polyhedral) complex $\mathcal{C}_i$ with support P, such that the maps l_{ij} are *stratified* over each polyhedron R of $\mathcal{C}_i$, in the sense that for all $j' \neq j''$ we have either $l_{ij'} \leqslant l_{ij''}$ or $l_{ij'} \geqslant l_{ij''}$ on R. Since every complex can be subdivided into a simplicial complex without adding new vertices [8, III, 2.6], we can assume without loss of generality that all polyhedra in $\mathcal{C}_i$ are simplexes and that $\mathcal{C}_i$ is a subdivision of $\mathcal{C}_0$. Thus η_i is linear over every simplex of $\mathcal{C}_i$. We now routinely construct a common subdivision $\mathcal{C}$ of the rational complexes $\mathcal{C}_1, \ldots, \mathcal{C}_{n'}$, such that every simplex of $\mathcal{C}$ is rational. It follows that η is linear over each simplex of $\mathcal{C}$. The set $\mathcal{C}^\uparrow = \{T^\uparrow \mid T \in \mathcal{C}\}$ is a simplicial fan. The desingularization procedure [8, VI, 8.5] yields a regular fan Φ such that every cone of $\mathcal{C}^\uparrow$ is a union of cones of Φ. Intersecting the cones in Φ with the hyperplane $x_{n+1} = 1$, we have a complex Ξ whose support is the set

$$(P,1) = \{(x,1) \in \mathbb{R}^{n+1} \mid x \in P\}.$$

Dropping the last coordinate from the vertices of the simplexes of Ξ, we obtain a regular complex Λ with support P such that η is linear over every simplex of Λ.

(iii) Λ' is a rational simplicial complex with support P'. Fix a rational j-simplex

$$S = \mathrm{conv}(v_0, \ldots, v_j) \subseteq P,$$

not necessarily belonging to Λ, such that η is linear over S. Let $S' = \eta(S)$. The (affine) linear map $\eta : x \in S \mapsto y \in S'$ determines the homogeneous linear map

$$(x, 1) \in (S, 1) \mapsto (y, 1) \in (S', 1).$$

Let M_S be the $(n' + 1) \times (n + 1)$ integer matrix whose bottom row has the form

$$(0, 0, \dots, 0, 0, 1)$$

(with n zeros), and whose ith row $(i = 1, \dots, n')$ is given by the coefficients of the (affine) linear polynomial $\eta_i \lceil S$. Let $\tilde{v}_0, \dots \tilde{v}_j \in \mathbb{Z}^{n+1}$ be the homogeneous correspondents of the vertices $v_0, \dots, v_j$ of S, and let $S^{\uparrow} = \mathbb{R}_{\geqslant 0}\, \tilde{v}_0 + \cdots + \mathbb{R}_{\geqslant 0}\, \tilde{v}_j \subseteq \mathbb{R}^{n+1}$ be the positive span of $\tilde{v}_0, \dots, \tilde{v}_j$. Similarly, let $S'^{\uparrow}$ be the positive span in $\mathbb{R}^{n'+1}$ of the integer vectors $M_S\tilde{v}_0, \dots, M_S\tilde{v}_j$. By construction, M_S sends integer points of $S^{\uparrow}$ one-to-one into integer points of $S'^{\uparrow}$. Interchanging the roles of S and S', we see that M_S sends integer points of $S^{\uparrow}$ one-to-one *onto* integer points of $S'^{\uparrow}$. Blichfeldt's theorem [13], yields the following characterization:

S is (Farey) regular

$\Leftrightarrow$ the half-open parallelepiped $Q_S = \{\mu_0\tilde{v}_0 + \cdots + \mu_j\tilde{v}_j \mid 0 \leqslant \mu_0, \dots, \mu_j < 1\}$ contains no non-zero integer points

$\Leftrightarrow$ the half-open parallelepiped $Q_{S'}$ contains no non-zero integer points

$\Leftrightarrow$ S' is (Farey) regular.

In particular, if S is a simplex of Λ then the assumed regularity of Λ entails the (Farey) regularity of S, whence of S'. We conclude that Λ' is a regular complex with support P'. □

Recall from (1.2) the definition of $\lambda(n, i, P, \Delta)$.

Theorem 2.3. *For every $n = 1, 2, \dots$, $i = 0, 1, \dots$, polyhedron $P \in \mathcal{P}^{(n)}$ and regular triangulations Δ and Δ' of P, we have $\lambda(n, i, P, \Delta) = \lambda(n, i, P, \Delta')$.*

Proof. We first suppose that Δ' is obtained from Δ by a blow-up at the Farey mediant c of some j-simplex $S = \operatorname{conv}(v_0, \dots, v_j) \in \Delta$, $j = 1, \dots, n$. In symbols, $\Delta' = \Delta_{(c)}$. S is the smallest simplex of Δ containing c as an element. Thus $c \in R \in \Delta \Rightarrow \dim(R) \geqslant j$. Let $d = 0, 1, \dots, n$. If, for no simplex $T \in \Delta^{\max}(d)$, it is the case that $c \in T$, then $\Delta^{\max}(d) = \Delta'^{\max}(d)$. Otherwise, let $T = \operatorname{conv}(v_0, \dots, v_j, \dots, v_d)$ be a simplex of $\Delta^{\max}(d)$ such that $c \in T$. We now define the d-simplexes $S_0, \dots, S_j$ as follows:

$$S_0 = \operatorname{conv}(c, v_1, \dots, v_d),$$
$$S_j = \operatorname{conv}(v_0, v_1, \dots, v_{j-1}, c, \dots, v_d),$$
$$S_t = \operatorname{conv}(v_0, \dots, v_{t-1}, c, v_{t+1}, \dots, v_j, \dots, v_d)$$

for each $t = 1, \dots, j-1$. By the definition of the Farey mediant, $\operatorname{den}(c) = \operatorname{den}(v_0) + \cdots + \operatorname{den}(v_j)$. By the definition of the Farey blow-up, the subcomplex of Δ given by T and its faces is replaced in Δ' by the simplicial complex given by the d-simplexes $S_0, \dots, S_j$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

and their faces. Since T is (Farey) regular, then so is S_u for each $u = 0, \ldots, j$, whence $\mathrm{den}(S_u) = \mathrm{den}(T) \cdot \mathrm{den}(c)/\mathrm{den}(v_u)$. As a consequence,

$$1/\mathrm{den}(T) = \sum_{u=0}^{j} 1/\mathrm{den}(S_u).$$

Since

$$\sum_{T \in \Delta^{\max}(d)} \frac{1}{d!\,\mathrm{den}(T)} = \sum_{U \in \Delta'^{\max}(d)} \frac{1}{d!\,\mathrm{den}(U)},$$

then $\lambda(n, d, P, \Delta) = \lambda(n, d, P, \Delta')$. Thus, in the case $\Delta' = \Delta_{(c)}$, we obtain

$$\lambda(n, i, P, \Delta) = \lambda(n, i, P, \Delta'),$$

for all $i = 0, 1, \ldots$.

In the general case when Δ' is an arbitrary regular triangulation of P, the solution of the weak Oda conjecture [15, 27] yields a sequence of regular triangulations

$$\nabla_0 = \Delta,\ \nabla_1, \ldots, \nabla_{s-1},\ \nabla_s = \Delta',$$

where each ∇_{k+1} is obtained from ∇_k by a Farey blow-up, or, *vice versa*, ∇_k is obtained from ∇_{k+1} by a Farey blow-up. Then the desired conclusion follows by induction on s. □

This theorem enables us to equip the totality of rational polyhedra with the rational d-dimensional measure λ_d defined in (1.3).

3. Proof of Theorem 1.1 (i)–(v)

3.1. *Invariance*

We will actually prove the stronger result that λ_d *is invariant under* $\mathbb{Z}$*-homeomorphisms*. In other words, whenever $P' \subseteq \mathbb{R}^{n'}$ is a rational polyhedron and $P \cong_{\mathbb{Z}} P'$, then $\lambda_d(P) = \lambda_d(P')$ for all $d = 0, 1, \ldots$. Let $\imath$ be a $\mathbb{Z}$-homeomorphism of P onto P'. Let Δ be a regular complex with support P such that $\imath$ is (affine) linear over every simplex of Δ. The existence of Δ is ensured by Lemma 2.2(ii). Let $\Delta' = \{\imath(T) \mid T \in \Delta\}$. By Lemma 2.2(i)–(iii), Δ' is a regular complex with support P', and $\mathrm{den}(\imath(z)) = \mathrm{den}(z)$ for every rational point $z \in P$. It follows that $\lambda(n, d, P, \Delta) = \lambda(n', d, P', \Delta')$. The desired conclusion now follows from Theorem 2.3.

3.2. *Valuation*

The identities $\lambda_d(\emptyset) = 0$, and $\lambda_d(P) = \lambda_d(P^{(d)})$ immediately follow by the definition of rational measure. To prove (1.4), we first observe that both $P \cup Q$ and $P \cap Q$ are rational polyhedra in $\mathbb{R}^n$ whose dimension is at most d. As an application of Lemma 2.1, let the regular complexes $\Delta, \Phi, \Psi, \Omega$ have the following properties:

$$|\Delta| = P \cap Q,\ |\Phi| = P,\ |\Psi| = Q,\ |\Omega| = P \cup Q.$$

Using the extension argument in [8, VI. 9.3], we can assume $\Delta = \Phi \cap \Psi$ and $\Omega = \Phi \cup \Psi$, without loss of generality. For every $X \subseteq \mathbb{R}^n$ we let $\mathrm{cl}(X)$ denote the closure of X in $\mathbb{R}^n$,

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

as usual. By Theorem 2.3 we have

$$
\begin{aligned}
\lambda_d(P) + \lambda_d(Q) &= \lambda(n,d,P,\Phi) + \lambda(n,d,Q,\Psi)\\
&= \tfrac{1}{d!}\left[\textstyle\sum_{T\in\Phi^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{T\in\Psi^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&= \tfrac{1}{d!}\left[\textstyle\sum_{\mathrm{cl}(P\setminus Q)\supseteq T\in\Phi^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{\mathrm{cl}(Q\setminus P)\supseteq T\in\Psi^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&\quad + \tfrac{1}{d!}\left[\textstyle\sum_{P\cap Q\supseteq T\in\Phi^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{P\cap Q\supseteq T\in\Psi^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&= \tfrac{1}{d!}\left[\textstyle\sum_{\mathrm{cl}(P\setminus Q)\supseteq T\in\Phi^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{\mathrm{cl}(Q\setminus P)\supseteq T\in\Psi^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&\quad + \tfrac{2}{d!}\textstyle\sum_{T\in\Delta^{\max}(d)} \mathrm{den}(T)^{-1}\\
&= \tfrac{1}{d!}\left[\textstyle\sum_{\mathrm{cl}(P\setminus Q)\supseteq T\in\Omega^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{\mathrm{cl}(Q\setminus P)\supseteq T\in\Omega^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&\quad + \tfrac{2}{d!}\textstyle\sum_{T\in\Delta^{\max}(d)} \mathrm{den}(T)^{-1}\\
&= \tfrac{1}{d!}\left[\textstyle\sum_{\mathrm{cl}(P\setminus Q)\supseteq T\in\Omega^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{\mathrm{cl}(Q\setminus P)\supseteq T\in\Omega^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&\quad + \tfrac{1}{d!}\left[\textstyle\sum_{P\cap Q\supseteq T\in\Omega^{\max}(d)} \mathrm{den}(T)^{-1} + \sum_{T\in\Delta^{\max}(d)} \mathrm{den}(T)^{-1}\right]\\
&= \lambda(n,d,P\cup Q,\Omega) + \lambda(n,d,P\cap Q,\Delta) = \lambda_d(P\cup Q) + \lambda_d(P\cap Q).
\end{aligned}
$$

3.3. *Conservativity* and *pyramid*

Properties (iii) and (iv) are immediate consequences of the definition of λ_d.

3.4. *Normalization*

To prove property (v), let Π be the set of permutations of the set $\{1,2,\ldots,j\}$. For every permutation $\pi\in\Pi$ we let T_π be the convex hull of the set of points

$$0,\ w_{\pi(1)},\ w_{\pi(1)}+w_{\pi(2)},\ w_{\pi(1)}+w_{\pi(2)}+w_{\pi(3)},\ldots,w_{\pi(1)}+w_{\pi(2)}+\cdots+w_{\pi(j)}.$$

Arguing as in [24, 3.4], it follows that the j-simplexes T_π are the maximal elements of a triangulation Σ of P_B, called the *standard triangulation* Σ. Each simplex T_π is regular and has unit denominator. There are $j!$ such simplexes. By definition, the rational j-dimensional measure of T_π is equal to $1/j!$. A final application of Theorem 2.3 yields $\lambda_j(P_B)=1$.

4. From λ_n to Lebesgue measure on $\mathbb{R}^n$ via Carathéodory's method

In what follows, $\mathcal{L}^n$ will denote the Lebesgue measure on $\mathbb{R}^n$.

Proposition 4.1. *For any $n=1,2,\ldots$ and polyhedron $Q\in\mathcal{P}^{(n)}$, we have $\lambda_n(Q)=\mathcal{L}^n(Q)$.*

Proof. If $\dim(Q)<n$ then $\mathcal{L}^n(Q)=\lambda_n(Q)=0$. If $\dim(Q)=n$, since $\lambda_n(Q)=\lambda_n(Q^{(n)})$ and $\mathcal{L}^n(Q)=\mathcal{L}^n(Q^{(n)})$, without loss of generality we may assume $Q=Q^{(n)}$. Let ∇ be a regular triangulation of Q as given by Lemma 2.1. Since, as we have seen, λ_n is a valuation on $\mathcal{P}^{(n)}$ and $\mathcal{L}^n(Q)=\sum_{S\in\nabla^{\max}(n)}\mathcal{L}^n(S)$, it is enough to prove

$$\lambda_n(S)=\mathcal{L}^n(S) \text{ for every } n\text{-simplex } S=\mathrm{conv}(w_0,\ldots,w_n)\in\nabla. \qquad (4.1)$$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

To this end, let $T \subseteq \mathbb{R}^{n+1}$ be the $(n+1)$-simplex with vertices 0, $(w_0, 1), \ldots, (w_n, 1)$. Then

$$\mathcal{L}^{n+1}(T) = \mathcal{L}^n(S)/(n+1).$$

This is the classical formula for the volume of the $(n+1)$-dimensional pyramid with base S and height 1. Next we observe that T is contained in the closed $(n+1)$-dimensional parallelepiped

$$E = \{\alpha_0(w_0, 1) + \cdots + \alpha_n(w_n, 1) \in \mathbb{R}^{n+1} \mid \alpha_0, \ldots, \alpha_n \in [0, 1]\}.$$

Further,

$$E \subseteq U = \{\alpha_0 \tilde{w}_0 + \cdots + \alpha_n \tilde{w}_n \in \mathbb{R}^{n+1} \mid \alpha_0, \ldots, \alpha_n \in [0, 1]\}.$$

Since S is (Farey) regular, a classical argument in the geometry of numbers ([13] or [8, Proof of VI, 8.5]) yields $\mathcal{L}^{n+1}(U) = 1$. For all $i = 0, \ldots, n$, let $d_i = \mathrm{den}(w_i)$. Since

$$\tilde{w}_0 = d_0(w_0, 1), \ldots, \tilde{w}_n = d_n(w_n, 1),$$

then

$$\mathcal{L}^{n+1}(E) = (d_0 \cdots d_n)^{-1}.$$

The construction of [24, 3.4] now yields a triangulation of E consisting of $(n+1)$-simplexes $T_1, \ldots, T_{(n+1)!}$ and their faces, in such a way that

$$\mathcal{L}^{n+1}(T_i) = \frac{\mathcal{L}^{n+1}(E)}{(n+1)!} \quad \text{for each } i = 1, \ldots, (n+1)!$$

Each T_i is a (Farey) regular simplex. One easily gets a linear (affine) isometry of T_i onto T. Therefore,

$$\mathcal{L}^{n+1}(T) = \frac{\mathcal{L}^{n+1}(E)}{(n+1)!}.$$

Summing up, $\mathcal{L}^n(S) = \mathcal{L}^{n+1}(E)/n! = (n!\, d_0 \cdots d_n)^{-1} = \lambda_n(S)$, and (4.1) is proved. □

Corollary 4.2. *Fix $n = 1, 2, \ldots$ and let $\mathcal{K}^{(n)}$ denote the family of compact subsets of $\mathbb{R}^n$. For any Borel set $E \subseteq \mathbb{R}^n$ let us define*

$$\bar{\lambda}_n(E) = \sup_{E \supseteq K \in \mathcal{K}^{(n)}} \ \inf_{K \subseteq P \in \mathcal{P}^{(n)}} \lambda_n(P).$$

Then $\bar{\lambda}_n(E) = \mathcal{L}^n(E)$.

Proof. We first *claim* that every $K \in \mathcal{K}^{(n)}$ coincides with the intersection of all rational polyhedra of $\mathcal{P}^{(n)}$ containing it.

As a matter of fact, for any $P, Q \in \mathcal{P}^{(n)}$ both $P \cup Q$ and $P \cap Q$ are members of $\mathcal{P}^{(n)}$. Moreover, there exists a rational triangulation $\mathcal{T}$ of $P \cup Q$ such that the set $\{T \in \mathcal{T} \mid T \subseteq P \cap Q\}$ is a triangulation of $P \cap Q$. Thus the set $\{T \in \mathcal{T} \mid T \subseteq \mathrm{cl}(P \setminus Q)\}$ is a triangulation of the set $\mathrm{cl}(P \setminus Q) \subseteq \mathbb{R}^n$, which shows that $\mathrm{cl}(P \setminus Q)$ is a rational polyhedron. For every $x \in \mathbb{R}^n \setminus K$ there is a rational n-simplex T containing x in its interior and such that $T \cap K = \emptyset$. Since K is contained in some rational polyhedron, our claim is settled.

Now let $P_0 \supseteq P_1 \supseteq \cdots$ be a sequence of rational polyhedra such that $\bigcap_i P_i = K$, and for every $R \in \mathcal{P}^{(n)}$ with $K \subseteq R$ there exists $j = 0, 1, \ldots$ such that $P_j \subseteq R$. The existence of this sequence follows from our claim, together with the observation that there are only countably many rational polyhedra. By Proposition 4.1,

$$\lambda_n(P_0) = \mathcal{L}^n(P_0) \geqslant \mathcal{L}^n(P_1) = \lambda_n(P_1) \geqslant \lambda_n(P_2) \geqslant \cdots,$$

whence by construction,

$$\lim_{i\to\infty} \lambda_n(P_i) = \inf\{\lambda_n(R) \mid R \supseteq K,\ R \in \mathcal{P}^{(n)}\} = \bar{\lambda}_n(K).$$

Combining Proposition 4.1 with the countable monotonicity property of $\mathcal{L}^n$, we get

$$\mathcal{L}^n(K) = \lim_{i\to\infty} \mathcal{L}^n(P_i) = \lim_{i\to\infty} \lambda_n(P_i) = \bar{\lambda}_n(K).$$

Having thus proved that $\bar{\lambda}_n$ agrees with $\mathcal{L}^n$ on all compact subsets of $\mathbb{R}^n$, the desired conclusion follows from the regularity properties of the Lebesgue measure. □

Remark. Following [10, 115C], we now routinely extend $\bar{\lambda}_n$ to an outer measure

$$\lambda_n^* : \text{powerset}(\mathbb{R}^n) \to [0, \infty],$$

which, by Corollary 4.2 and [10, 115D], coincides with the Lebesgue outer measure on $\mathbb{R}^n$. As proved in [10, 115E], by applying to λ_n^* Carathéodory's construction [10, 113], we finally obtain the Lebesgue measure on $\mathbb{R}^n$.

Alternatively, one can obtain the Lebesgue measure from $\bar{\lambda}_n$ using the main result of [20], to the effect that if a Borel measure μ on $\mathbb{R}^n$ is invariant under the linear action of $SL(n, \mathbb{Z})$, annihilates the set of rational rays $\{tz \mid t \geqslant 0, z \in \mathbb{Z}^n\}$, and is locally finite at some point x (in the sense that x has some open neighbourhood N with $\mu(N) < \infty$), then μ coincides with a scalar multiple of the Lebesgue n-dimensional measure. This extends a result in [5] concerning locally finite measures which are ergodic under the action of $SL(n, \mathbb{Z})$. Further, see [23].

5. Proof of Theorem 1.1(vi)

5.1. Basic material on Hausdorff measure

In the following proposition we collect a number of well-known consequences of the isodiametric inequality (see [9, 2.10.33]), and of the invariance of the Hausdorff d-dimensional measure under isometries.

Proposition 5.1. *For each $0 < n \in \mathbb{Z}$ we have the following.*

(i) *If $T = \text{conv}(x_0, \ldots, x_n)$ is an n-simplex in $\mathbb{R}^n$, letting M be the $n \times n$ matrix whose ith row is given by the vector $x_i - x_0$ $(i = 1, \ldots, n)$, then $\mathcal{H}^n(T) = |\det(M)|/n! = \mathcal{L}^n(T)$.*

(ii) *If S is an m-simplex in $\mathbb{R}^n$ with $0 < m < n$, and we map S onto a copy S' by means of an isometry $\imath$ sending the affine hull of S onto the linear subspace $\mathbb{R}^m$ of $\mathbb{R}^n$ spanned by the first m standard basis vectors of $\mathbb{R}^n$, then $\mathcal{H}^m(S) = \mathcal{L}^m(S')$. If $\dim(S) = 0$, then $\mathcal{H}^0(S) = 1 =$ number of elements of the singleton S.*

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

(iii) *Suppose Q is a non-empty polyhedron in $\mathbb{R}^n$ and $Q = Q^{(d)}$ for some $d = 0, 1, \ldots, n$. Then, letting $\mathcal{T}$ be an arbitrary triangulation of Q, with its d-simplexes $T_1, \ldots, T_k$, we have*

$$\mathcal{H}^d(Q) = \sum_{j=1}^{k} \mathcal{H}^d(T_j).$$

If $Q = \emptyset$, then $\mathcal{H}^k(Q) = 0$ for all $k = 0, 1, \ldots$.

(iv) *Given integers $0 \leqslant m < n$, suppose $T = \mathrm{conv}(v_0, \ldots, v_m)$ and $T' = \mathrm{conv}(v'_0, \ldots, v'_m)$ are m-simplexes in $\mathbb{R}^n$ with $\mathrm{aff}(T) = \mathrm{aff}(T')$. For v an arbitrary point lying in $\mathbb{R}^n \setminus \mathrm{aff}(T)$, let $U = \mathrm{conv}(T, v)$ and $U' = \mathrm{conv}(T', v)$. Then*

$$\mathcal{H}^{m+1}(U')/\mathcal{H}^{m+1}(U) = \mathcal{H}^m(T')/\mathcal{H}^m(T).$$

(v) *More generally, suppose the points $v_{m+1}, \ldots, v_n \in \mathbb{R}^n$ have the property that*

$$W = \mathrm{conv}(v_0, \ldots, v_m, v_{m+1}, \ldots, v_n)$$

is an n-simplex. Then $W' = \mathrm{conv}(v'_0, \ldots, v'_m, v_{m+1}, \ldots, v_n)$ is also an n-simplex, and we have the identity

$$\mathcal{H}^n(W')/\mathcal{H}^n(W) = \mathcal{H}^m(T')/\mathcal{H}^m(T).$$

□

5.2. Completion of the proof of Theorem 1.1(vi)

It remains to be proved that λ_d has the *proportionality* property (vi). By Lemma 2.1, Q has a regular triangulation. Since λ_m is a valuation, recalling Proposition 5.1(iii) it suffices to consider the case that Q is a (Farey) regular m-simplex. If $m = n$, the result follows from Proposition 4.1 since, by Proposition 5.1(i), $\mathcal{H}^n(Q) = \mathcal{L}^n(Q)$. In this case $\kappa_A = 1$. Next suppose $0 \leqslant m < n$. It suffices to prove that for any two (Farey) regular m-simplexes $T = \mathrm{conv}(v_0, \ldots, v_m)$ and $T' = \mathrm{conv}(v'_0, \ldots, v'_m)$ lying in A,

$$\lambda_m(T)/\lambda_m(T') = \mathcal{H}^m(T)/\mathcal{H}^m(T').$$

To this end, let $U = \mathrm{conv}(v_0, \ldots, v_m, v_{m+1}, \ldots, v_n)$ be a (Farey) regular n-simplex in $\mathbb{R}^n$ having T as a face.

Claim. The simplex $U' = \mathrm{conv}(v'_0, \ldots, v'_m, v_{m+1}, \ldots, v_n)$ is (Farey) regular.

As a matter of fact, the (Farey) regularity of T means that the set $\{\tilde{v}_0, \ldots, \tilde{v}_m\}$ is a basis of the free abelian group $G = \mathbb{Z}^{n+1} \cap (\mathbb{R}\tilde{v}_0 + \cdots + \mathbb{R}\tilde{v}_m)$ of integer points in the $(m+1)$-dimensional linear space spanned by $\tilde{v}_0, \ldots, \tilde{v}_m$ in $\mathbb{R}^{n+1}$. Since $\mathrm{aff}(T') = A = \mathrm{aff}(T)$ and T' is (Farey) regular, $\tilde{v}'_0, \ldots, \tilde{v}'_m$ also constitute a basis of G. Upon writing each $\tilde{v}_i$ and $\tilde{v}'_j$ as a column vector, let M be the $(n+1) \times (m+1)$ matrix whose ith row coincides with $\tilde{v}_i$. Similarly, let M' be the $(n+1) \times (m+1)$ matrix whose jth row equals $\tilde{v}'_j$. Let the $(m+1) \times (m+1)$ integer matrix Z be defined by $MZ = M'$. The $(m+1) \times (m+1)$ integer matrix V defined by $M'V = M$ coincides with Z^{-1}, whence $|\det(Z)| = |\det(Z^{-1})| = 1$.

Let the matrix N be defined by

$$N = \left(\begin{array}{c|c} Z & 0 \\ \hline 0 & \mathrm{I}_{n-m} \end{array}\right),$$

where I_{n-m} denotes the $(n-m)\times(n-m)$ identity matrix. N is a unimodular integer $(n+1)\times(n+1)$ matrix. Let W (resp. W') be the $(n+1)\times(n+1)$ integer matrix whose first $m+1$ columns are those of M (resp. those of M'), and whose last $n-m$ columns are given by the column vectors $\tilde{v}_{m+1},\ldots,\tilde{v}_n$. From $WN = W'$, it follows that the vectors $\tilde{v}'_0,\ldots,\tilde{v}'_m, \tilde{v}_{m+1},\ldots,\tilde{v}_n$ constitute a basis of the free abelian group $\mathbb{Z}^{n+1}$. Therefore,

$$\operatorname{conv}(v'_0,\ldots,v'_m, v_{m+1},\ldots,v_n)$$

is a (Farey) regular n-simplex in $\mathbb{R}^n$, and our claim is settled.

Now let $d_i = \operatorname{den}(v_i)$ $(i = 0,\ldots,n)$ and $d'_j = \operatorname{den}(v'_j)$ $(j = 0,\ldots,m)$. Since both simplexes U and U' are (Farey) regular, we can write the identities

$$\frac{\lambda_m(T)}{\lambda_m(T')} = \frac{(m!\,d_0\cdots d_m)^{-1}}{(m!\,d'_0\cdots d'_m)^{-1}} = \frac{(n!\,d_0\cdots d_m d_{m+1}\cdots d_n)^{-1}}{(n!\,d'_0\cdots d'_m d_{m+1}\cdots d_n)^{-1}} = \frac{\lambda_n(U)}{\lambda_n(U')}.$$

By Propositions 4.1 and 5.1(ii)–(v), we obtain

$$\frac{\lambda_n(U)}{\lambda_n(U')} = \frac{\mathcal{L}^n(U)}{\mathcal{L}^n(U')} = \frac{\mathcal{H}^n(U)}{\mathcal{H}^n(U')} = \frac{\mathcal{H}^m(T)}{\mathcal{H}^m(T')},$$

as required to prove (vi).

The proof of Theorem 1.1 is now complete. □

6. Uniqueness

For every non-empty rational affine subspace F of $\mathbb{R}^n$, let the integer $d_F \geqslant 1$ be defined by

$$d_F = \min\{q \in \mathbb{Z} \mid q = \operatorname{den}(r) \text{ letting } r \text{ range over all rational points of } F\}. \tag{6.1}$$

Lemma 6.1. *Fix $n = 1,2,\ldots$ and $e = 0,\ldots,n$. Let F be a rational e-dimensional affine subspace of $\mathbb{R}^n$ and $d = d_F$.*

(i) *There are rational points $v_0,\ldots,v_e \in F$, all with denominator d, such that* $\operatorname{conv}(v_0,\ldots,v_e)$ *is a (Farey) regular e-simplex.*

(ii) *For any rational point $y \in F$ there is an integer $k = 1,2,\ldots$ such that* $\operatorname{den}(y) = kd$.

Proof. **(i)** For some (Farey) regular e-simplex

$$S_0 = \operatorname{conv}(u_0,\ldots,u_e),$$

we can write

$$F = \operatorname{aff}(u_0,\ldots,u_e).$$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

The (Farey) regularity of S_0 means that the set $B_0 = \{\tilde{u}_0, \ldots, \tilde{u}_e\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$, whence B_0 is a basis of the lattice $\mathbb{Z}^{n+1} \cap F^*$, where $F^* = \mathbb{R}\tilde{u}_0 + \cdots + \mathbb{R}\tilde{u}_e$ is the linear subspace of $\mathbb{R}^{n+1}$ generated by $\tilde{u}_0, \ldots, \tilde{u}_e$.

It is impossible for the *heights* (= last coordinates) of $\tilde{u}_0, \ldots, \tilde{u}_e$ all to be equal to the same integer $h > d$, for otherwise, any primitive vector $\tilde{r}$ in F^* of height d, for r as in (6.1), could not arise as a linear combination of the $\tilde{u}_i$ with integer coefficients, and B_0 would not be a basis of $\mathbb{Z}^{n+1} \cap F^*$.

If the heights of $\tilde{u}_0, \ldots, \tilde{u}_e$ are all equal to d we have nothing to prove. Otherwise, we will construct a finite sequence $B_0, B_1, \ldots$ of bases of $\mathbb{Z}^{n+1} \cap F^*$, and finally obtain a basis $\{\tilde{v}_0, \ldots, \tilde{v}_e\}$ having the property that the height of each $\tilde{v}_i$ is equal to d.

The first step is as follows. Choose a vector $\tilde{u}_i \in B_0$ of greatest height, a vector $\tilde{u}_j \in B_0$ of smaller height, and replace $\tilde{u}_i$ by $\tilde{u}_i - \tilde{u}_j$. We get a new basis B_1 of $\mathbb{Z}^{n+1} \cap F^*$, and a new (Farey) regular e-simplex S_1 in F. Specifically, letting the rational point $w \in F$ be defined by $\tilde{w} = \tilde{u}_i - \tilde{u}_j$, the vertices of S_1 are $u_0, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_e$. Observe that the sum of the heights of the elements of B_1 is strictly smaller than the sum of the heights of the elements of B_0.

Proceeding inductively, and replacing a top vector $\tilde{u}$ of the basis B_t by a vector $\tilde{u} - \tilde{v}$ with $\tilde{v} \in B_t$ of smaller height than $\tilde{u}$, we obtain a new basis B_{t+1} such that the sum of the heights of the elements of B_{t+1} is strictly smaller than the sum of the heights of the elements of B_t. We also get a new (Farey) regular e-simplex S_{t+1} lying in F. The process must terminate with a basis $\{\tilde{v}_0, \ldots, \tilde{v}_e\}$ of $\mathbb{Z}^{n+1} \cap F^*$ where all $\tilde{v}_i$ have the same height, which by our initial discussion must be equal to d. By definition, $\mathrm{conv}(v_0, \ldots, v_e)$ is the desired (Farey) regular e-simplex in F.

(ii) Condition (ii) now follows trivially from (i), since the (Farey) regularity of $\mathrm{conv}(v_0, \ldots, v_e)$ implies that the primitive vector $\tilde{y} \in \mathbb{Z}^{n+1}$ is a linear combination of the $\tilde{v}_i$ with integer coefficients. □

Theorem 6.2. *For each $n = 1, 2, \ldots$, properties* (i)–(vi) *in Theorem 1.1 uniquely characterize the rational measures $\lambda_0, \ldots, \lambda_n$ among all maps from $\mathcal{P}^{(n)}$ to $\mathbb{R}_{\geqslant 0}$.*

Proof. Suppose that for each $n = 1, 2, \ldots$, the maps $\mu_0, \ldots, \mu_n : \mathcal{P}^{(n)} \to \mathbb{R}_{\geqslant 0}$, as well as the maps $\mu'_0, \ldots, \mu'_{n+1} : \mathcal{P}^{(n+1)} \to \mathbb{R}_{\geqslant 0}$, have all properties (i)–(vi). Since by Lemma 2.1 every rational polyhedron has a regular triangulation, and each μ_j and λ_j is a valuation, it suffices to show that $\mu_m(S) = \lambda_m(S)$ for all $m = 0, \ldots, n$, and (Farey) regular m-simplex S in $\mathbb{R}^n$. Let $F = \mathrm{aff}(S)$ be the affine hull of S in $\mathbb{R}^n$. Let $d = d_F$ be the smallest denominator of a rational point of F as in (6.1) above. Let us identify $\mathbb{R}^n$ with the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. Let $T = \mathrm{conv}(v_0, \ldots, v_m) \subseteq F$ be a (Farey) regular m-simplex such that $\mathrm{den}(v_0) = \cdots = \mathrm{den}(v_m) = d$. The existence of T is ensured by Lemma 6.1. Let $T' = \{(x, 1) \in \mathbb{R}^{n+1} \mid x \in T\}$. There exists $\alpha \in \mathcal{G}_{n+1}$ such that $\alpha(T, 0) = T'$. From the *invariance* and *conservativity* properties of all μ_i and μ'_j, we obtain

$$\mu'_m(T') = \mu'_m(T, 0) = \mu_m(T). \tag{6.2}$$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

The (Farey) regularity of T means that the set $B = \{\tilde{v}_0, \ldots, \tilde{v}_m\}$ is part of a basis of the free abelian group $\mathbb{Z}^{n+1}$. As in (1.6) above, let the closed parallelepiped P_B be defined by

$$P_B = \left\{x \in \mathbb{R}^{n+1} \mid x = \sum_{i=0}^{m} \gamma_i \tilde{v}_i,\ 0 \leqslant \gamma_i \leqslant 1\right\}.$$

From the *normalization* property we get $\mu'_{m+1}(P_B) = 1$. Arguing as in Section 3.4, we obtain a triangulation Δ of P_B consisting of $(m+1)$-simplexes $T_1, \ldots, T_{(m+1)!}$ and their faces. Each T_i is (Farey) regular and has denominator 1. A direct verification shows that for any two such simplexes T_i and T_j there exists $\gamma \in \mathcal{G}_{n+1}$ such that $T_i = \gamma(T_j)$. From the *valuation* and *invariance* properties of μ'_{m+1}, it follows that

$$\mu'_{m+1}(T_j) = \frac{\mu'_{m+1}(P_B)}{(m+1)!} = \frac{1}{(m+1)!} \quad \text{for all } j = 1, \ldots, (m+1)!$$

Let $D \subseteq \mathbb{R}^{n+1}$ be the $(m+1)$-simplex with vertices $0, \tilde{v}_0, \ldots, \tilde{v}_m$. It is easily seen that D is (Farey) regular and $\mathrm{den}(D) = 1$. Thus an easy exercise yields an $\eta \in \mathcal{G}_{n+1}$ such that $\eta(T_1) = D$. One more application of the *invariance* property of μ'_{m+1} yields

$$\mu'_{m+1}(D) = \frac{1}{(m+1)!}.$$

Since the $(m+1)$-simplex D' with vertices $0, (v_0, 1), \ldots, (v_m, 1)$ has the same affine hull as D, by the assumed *proportionality* property of μ'_{m+1} we have

$$\mu'_{m+1}(D') = \frac{1}{(m+1)!\, d^{m+1}}.$$

On the other hand, the *pyramid* property is to the effect that

$$\mu'_{m+1}(D') = \frac{\mu'_m(T')}{m+1},$$

whence

$$\mu'_m(T') = \frac{1}{m!\, d^{m+1}}.$$

Recalling (6.2), we obtain

$$\mu_m(T) = \frac{1}{m!\, d^{m+1}} = \lambda_m(T),$$

because T is (Farey) regular and the denominators of its vertices are all equal to d. Since S and T have the same affine hull, a final application of the *proportionality* property of μ_m and λ_m yields

$$\frac{\mu_m(S)}{\mu_m(T)} = \frac{\mathcal{H}^m(S)}{\mathcal{H}^m(T)} = \frac{\lambda_m(S)}{\lambda_m(T)}.$$

In conclusion,

$$\mu_m(S) = \lambda_m(S)\frac{\mu_m(T)}{\lambda_m(T)} = \lambda_m(S).$$

The proof is complete. □

7. The value of the proportionality constant κ_A of Theorem 1.1(vi)

Following [13], for any k-dimensional sub-lattice Λ of $\mathbb{Z}^n$, the *determinant* $\det(\Lambda)$ of Λ is the k-dimensional volume of a fundamental region (also known as a *cell*) for Λ in the k-dimensional rational linear subspace spanned by Λ.

Theorem 7.1. *Let $A = A_0 + t$ be a rational e-dimensional affine subspace of $\mathbb{R}^n$, for $e = 0, 1, \ldots, n$, where the affine rational subspace A_0 is homogeneous and $t \in \mathbb{Q}^n$. Let $d = d_A$ be the smallest denominator of a rational point of A.*

(i) *The ratio k_A between the e-dimensional rational measure and the e-dimensional Hausdorff measure of any e-simplex lying in A is equal to $(d_A \times \det(A_0 \cap \mathbb{Z}^n))^{-1}$.*

(ii) *If the lattice $A_0 \cap \mathbb{Z}^n$ is equipped with a basis $h_1, \ldots, h_e$, where each h_i is a vector in $\mathbb{Z}^n$, then letting M be the matrix with the coordinates of h_i in the ith row, we have the identity $\det(A_0 \cap \mathbb{Z}^n) = (MM')^{1/2}$, where M' is the transpose of M. This holds independently of the chosen basis.*

Proof. **(i)** Lemma 6.1 yields rational points $v_0, \ldots, v_e \in A$, all with the same denominator d, which are the vertices of a (Farey) regular e-simplex $S = \mathrm{conv}(v_0, \ldots, v_e) \subseteq A$. Thus the integer vectors $\tilde{v}_0, \ldots, \tilde{v}_e \in \mathbb{Z}^{n+1}$ are a basis of the lattice $A^* \cap \mathbb{Z}^{n+1}$ of integer points in the linear span A^* of the set $\{\tilde{v}_0, \ldots, \tilde{v}_e\}$ in $\mathbb{R}^{n+1}$.

By the definition of rational measure we immediately have $\lambda_e(S) = (e!\, d^{e+1})^{-1}$.

To compute the e-dimensional Hausdorff measure of S, let the e-simplex $S' = (S, 1) \subseteq \mathbb{R}^{n+1}$ be obtained by vertically lifting S to the hyperplane $x_{n+1} = 1$. Multiplying each vector of S' by the scalar d, we obtain the e-simplex

$$dS' = \{dy \in \mathbb{R}^{n+1} \mid y \in S'\}.$$

All vertices of dS' lie at the same height d in $\mathbb{R}^{n+1}$. From $dS' = \mathrm{conv}(\tilde{v}_0, \ldots, \tilde{v}_e)$ it follows that

$$\mathcal{H}^{(e)}(S) = \frac{\mathcal{H}^{(e)}(dS')}{d^e} = \frac{\mathcal{H}^{(e)}(\mathrm{conv}(\tilde{v}_0, \ldots, \tilde{v}_e))}{d^e}.$$

Translating the e-simplex $\mathrm{conv}(\tilde{v}_0, \ldots, \tilde{v}_e) \subseteq \mathbb{R}^{n+1}$ by a shift of $-\tilde{v}_0$, since the Hausdorff measure is translation-invariant, we have

$$\mathcal{H}^{(e)}(S) = \frac{\mathcal{H}^{(e)}(\mathrm{conv}(0, \tilde{v}_1 - \tilde{v}_0, \ldots, \tilde{v}_e - \tilde{v}_0))}{d^e}.$$

On the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$ we now have an e-simplex with integer vertices $0, \tilde{v}_1 - \tilde{v}_0, \ldots, \tilde{v}_e - \tilde{v}_0$. Let us write w_i for the vector in $\mathbb{R}^n$ obtained by forgetting the last (zero) coordinate of $\tilde{v}_e - \tilde{v}_0$. The points $0, w_1, \ldots, w_e \in \mathbb{Z}^n \cap A_0 \subseteq \mathbb{R}^n$ satisfy

$$\mathcal{H}^{(e)}(S) = \frac{\mathcal{H}^{(e)}(\mathrm{conv}(0, w_1, \ldots, w_e))}{d^e}.$$

Let $\mathcal{P}(w_1, \ldots, w_e)$ be the parallelopiped spanned by the vectors $w_1, \ldots, w_e$ in $\mathbb{R}^n$. As a trivial corollary of the isodiametric inequality ([9, 2.10.33]), we have

$$\mathcal{H}^{(e)}(S) = \frac{\mathcal{H}^{(e)}(\mathcal{P}(w_1, \ldots, w_e))}{d^e\, e!}.$$

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

Note that A_0 coincides with A translated by $-v_0$. Further, $\{w_1,\dots,w_e\}$ is a basis of the lattice $A_0\cap\mathbb{Z}^n$. By the definition of the lattice determinant,

$$\mathcal{H}^{(e)}(S)=\frac{\det(A_0\cap\mathbb{Z}^n)}{d^e\,e!},$$

whence

$$k_A=\frac{\lambda_e(S)}{\mathcal{H}^{(e)}(S)}=\frac{1}{e!\,d^{e+1}}:\frac{\det(A_0\cap\mathbb{Z}^n)}{d^e\,e!}=\frac{1}{d\times\det(A_0\cap\mathbb{Z}^n)},$$

as required to complete the proof of (i).

(ii) See, for example, [3, Theorem 7, p. 308]. □

8. Related work and concluding remarks

In the literature, $\mathcal{G}_n$-invariance is also known as *unimodular* invariance [14, p. 979 and references therein]. Theorems 1.1 and 6.2 uniquely characterize the array of unimodular invariant maps λ_i on the set $\mathcal{P}^{(n)}$ of rational polyhedra in $\mathbb{R}^n$, and relate λ_i to i-dimensional Hausdorff measure. All valuations on classes of polyhedra pre-existing in the literature miss at least one of the conditions in Theorem 1.1.

As a notable example, following [14, p. 979–980], let $\mathcal{P}^n_{\mathbb{Z}}$ be the set of *convex lattice* polyhedra in $\mathbb{R}^n$ (the word 'lattice' meaning here that all vertices have integer coordinates). The Betke–Kneser theorem ([2], [12, 19.6]) states that every additive unimodular invariant function defined on the space $\mathcal{P}^n_{\mathbb{Z}}$ is a linear combination of the $n+1$ functions G_i introduced in [7] by Ehrhart. Here 'additive' means that the *valuation* property holds for $P,Q\in\mathcal{P}^n_{\mathbb{Z}}$ *subject to the condition that both* $P\cup Q$ *and* $P\cap Q$ *belong to* $\mathcal{P}^n_{\mathbb{Z}}$. See [1, 8, 12] for more information on Ehrhart theory.

The convexity of each element of $\mathcal{P}^n_{\mathbb{Z}}$, together with the condition that all its vertices are integers, is indispensable for the Betke–Kneser theorem to hold. In fact, an *infinite-dimensional* space of unimodular invariant (*unconditionally*) additive maps on $\mathcal{P}^{(n)}$ can be immediately constructed as follows. For every prime p consider the function $C_p:\mathcal{P}^{(n)}\to\{0,1,2,\dots\}$, where for every $P\in\mathcal{P}^{(n)}$, $C_p(P)$ counts the number of rational points x in P having $\operatorname{den}(x)=p$. A moment's reflection shows that C_p is a valuation on $\mathcal{P}^{(n)}$. Further, C_p is $\mathcal{G}_n$-invariant, because each $\gamma\in\mathcal{G}_n$ preserves denominators of rational points. For primes $p_1<p_2<\cdots<p_t$ the valuations $C_{p_1},C_{p_2},\dots,C_{p_t}$ form a linearly independent set, which shows that the space of all $\mathcal{G}_n$-invariant valuations on $\mathcal{P}^{(n)}$ is infinite-dimensional. As expected, conditions (iv)–(vi) in Theorem 1.1 do not hold in general for the valuations C_p.

While $\mathcal{P}^n_{\mathbb{Z}}$ is not even closed under unions and intersections, for any $P,Q\in\mathcal{P}^{(n)}$, $P\cap Q$, $P\cup Q$ and the closure of $P\setminus Q$ are still in $\mathcal{P}^{(n)}$, and so is the orthogonal projection of P onto any rational hyperplane of $\mathbb{R}^n$. Cartesian products and Minkowski sums of rational polyhedra are rational polyhedra. The value $\lambda_d(Q)$ exists for all $Q\in\mathcal{P}^{(n)}$ and all $d=0,1,\dots$. Thus $\dim(Q)$ may be strictly smaller than the dimension of the ambient space $\mathbb{R}^n$. With the notation of (1.1), one may have non-empty $Q^{(i)}$ and $Q^{(j)}$ for distinct i and j. Further, the vertices of Q need not belong to the lattice $\mathbb{Z}^n$.

The rational measure λ_d has the additivity property for all rational polyhedra in $\mathbb{R}^n$ whose dimension is $\leqslant d$. This latter restriction cannot be dispensed with: for instance, the 0-dimensional rational measure of the positive and of the negative unit segment in $\mathbb{R}$ with vertex 0 is 0, but the 0-dimensional measure of their intersection is 1. Because the additivity properties of the λ_d and of the Betke–Kneser functionals are so different, the latter can hardly provide an alternative approach to the restriction to $\mathcal{P}^n_{\mathbb{Z}}$ of the rational measures $\lambda_0, \ldots, \lambda_{n-1}$. (Recall that λ_n coincides with the Lebesgue n-dimensional measure, and $0 = \lambda_{n+1} = \lambda_{n+2} = \cdots$).

We conclude this paper with a brief discussion of the invariance property of rational polyhedra under $\mathbb{Z}$-homeomorphism *versus* unimodular invariance and invariance under rational PL-homeomorphism.

(i) The arithmetic–algebraic–geometric structure of the set of rational polyhedra and their piecewise linear maps with integer coefficients makes $\mathcal{P}^n$ into a category tightly connected to certain other categories. In [18], rational polyhedra are classified in terms of weighted abstract simplicial complexes, and a duality (contravariant categorical equivalence) is constructed between rational polyhedra and finitely presented MV-algebras and lattice-ordered abelian groups with a distinguished strong unit (for short, *unital ℓ-groups*). $\mathbb{Z}$-homeomorphic rational polyhedra correspond to *isomorphic* unital ℓ-groups. Every convex combination of the λ_d with coefficients > 0 equips every finitely presented unital ℓ-group L with a faithful invariant *state* σ_L (= normalized order-preserving, unit-preserving homomorphism σ with $\ker \sigma = 0$ of L into the naturally ordered additive group $\mathbb{R}$ equipped with the strong unit 1). See [17, 4.1] for a proof. Via Elliott classification and the Grothendieck K_0-group, this result has an application to AF C*-algebras A whose Murray–von Neumann order of projections is a lattice. Since the invariant tracial states of A are in one-to-one correspondence with the invariant states of $K_0(A)$, from $\sigma_{K_0(A)}$ one constructs an invariant (ergodic, by [22]) state on notable examples of AF C*-algebras existing in the literature, such as the 'free one-generator' AF C*-algebra $\mathfrak{M}_1$ introduced in [16] and recently rediscovered in [4] (as shown in [19]).

(ii) Paraphrasing McMullen [14, p. 939], for any group $\mathcal{A}$ of affinities in $\mathbb{R}^d$, the core of Hilbert's third problem is to characterize $\mathcal{A}$-equidissectability in terms of suitable families of *simple* $\mathcal{A}$-invariant valuations, those $\mathcal{A}$-invariant valuations vanishing on P whenever $\dim(P) < d$. Now, the $\mathbb{Z}$-homeomorphism of two rational polyhedra $R, S \subseteq \mathbb{R}^n$ amounts to their *continuous* $\mathcal{G}_n$-equidissectability (this easily follows from the fact that every rational triangulation has a regular subdivision). Each rational measure λ_d is a simple *Turing-computable* $\mathcal{G}_n$-invariant valuation on all rational polyhedra in $\mathbb{R}^n$, and the conditions $\lambda_d(P) = \lambda_d(Q)$, $d = 0, 1, \ldots$, are necessary for P to be $\mathbb{Z}$-homeomorphic to Q. By considering all sorts of Turing-computable $\mathcal{G}_n$-invariant valuations on $\mathcal{P}_n$, for instance the C_p described above in this section and variants thereof, one might hope to get a better understanding of continuous $\mathcal{G}_n$-equidissectability of rational polyhedra, and of the complexity of the following problem.

Instance. Two rational polyhedra $R, S \subseteq \mathbb{R}^n$.

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

Question. Does there exist a $\mathbb{Z}$-homeomorphism of R onto S?

It turns out that the decidability of this (continuous $\mathcal{G}_n$-equidissectability) problem is open, notwithstanding Markov's unrecognizability theorem for rational polyhedra under rational PL-homeomorphism. The problem has an equivalent algebraic counterpart, known as the isomorphism problem for finitely presented unital ℓ-groups, or MV-algebras. Indeed, as remarked above, rational polyhedra are dually equivalent to finitely presented unital ℓ-groups and MV-algebras. The difficulty of proving that this problem is undecidable may be due in part to the wealth of Turing-computable invariants for $\mathbb{Z}$-homeomorphisms possessed by rational polyhedra, of which the rational measures λ_d are an example, well beyond those of rational polyhedra under rational PL-homeomorphisms. Indeed, $\mathbb{Z}$-homeomorphism is a much finer equivalence relation than PL-homeomorphism.

Acknowledgements

The author is very grateful to the referee and to the editors for their valuable comments and suggestions for improving clarity and comprehensibility, resulting in a substantial reframing of large portions of this paper.

References

[1] Barvinok, A. (2002) *A Course in Convexity*, Vol. 54 of *Graduate Studies in Mathematics*, AMS.
[2] Betke, U. and Kneser, M. (1985) Zerlegungen und Bewertungen von Gitterpolytopen. *J. Reine Angew. Math.* **358** 202–208.
[3] Birkhoff, G. and Mac Lane, S. (1953) *A Survey of Modern Algebra*, revised edition, Macmillan.
[4] Boca, F. (2008) An AF algebra associated with the Farey tessellation. *Canad. J. Math.* **60** 975–1000.
[5] Dani, S. G. (1979) On invariant measures, minimal sets and a lemma of Margulis. *Inventio. Math.* **51** 239–260.
[6] Danilov, V. I. (1983) Birational geometry of toric 3-folds. *Math. USSR Izvestiya* **21** 269–280.
[7] Ehrhart, E. (1962) Sur les polyèdres rationnels homothétiques à n dimensions. *CR Acad. Sci. Paris Sér. A* **254** 616–618.
[8] Ewald, G. (1996) *Combinatorial Convexity and Algebraic Geometry*, Springer.
[9] Federer, H. (1969) *Geometric Measure Theory*, Springer.
[10] Fremlin, D. H. (2011) *Measure Theory*, Vol. 1, second edition. First published in 2000 by Torres Fremlin, 25 Ireton Road, Colchester CO3 3AT, UK. Source files available from: `http://www.essex.ac.uk/maths/people/fremlin/mt1.2011/index.htm`
[11] Glass, A. M. W. and Madden, J. J. (1984) The word problem versus the isomorphism problem. *J. London Math. Soc.* (2) **30** 53–61.
[12] Gruber, P. M. (2007) *Convex and Discrete Geometry*, Vol. 336 of *Grundlehren der Mathematischen Wissenschaften*, Springer.
[13] Lekkerkerker, C. G. (1969) *Geometry of Numbers*, Wolters-Noordhoff.
[14] McMullen, P. (1993) Valuations and dissections. In *Handbook of Convex Geometry*, Vol. 2 (P. M. Gruber and J. M. Wills, eds), Elsevier, pp. 933–988.
[15] Morelli, R. (1996) The birational geometry of toric varieties. *J. Algebraic Geometry* **5** 751–782.
[16] Mundici, D. (1988) Farey stellar subdivisions, ultrasimplicial groups, and K_0 of AF C^*-algebras. *Adv. Math.* **68** 23–39.

https://doi.org/10.1017/S096354831300062X Published online by Cambridge University Press

[17] Mundici, D. (2008) The Haar theorem for lattice-ordered abelian groups with order-unit. *Discrete Continuous Dyn. Syst.* **21** 537–549.
[18] Mundici, D. (2011) Finite axiomatizability in Łukasiewicz logic. *Ann. Pure Applied Logic* **162** 1035–1047.
[19] Mundici, D. (2011) Revisiting the Farey AF algebra. *Milan J. Math.* **79** 643–656.
[20] Nogueira, A. (2002) Relatively prime numbers and invariant measures under the natural action of $SL(n, \mathbb{Z})$ on $\mathbb{R}^n$. *Ergodic Theory Dyn. Syst.* **22** 899–923.
[21] Oda, T. (1988) *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, Springer.
[22] Panti, G. (2009) Invariant measures in free MV-algebras. *Commun. Algebra* **36** 2849–2861.
[23] Panti, G. (2012) Denominator-preserving maps. *Aequationes Math.* **84** 13–25.
[24] Semadeni, Z. (1982) *Schauder Bases in Banach Spaces of Continuous Functions*, Vol. 918 of *Lecture Notes in Mathematics*, Springer.
[25] Shtan'ko, M. A. (2004) Markov's theorem and algorithmically non-recognizable combinatorial manifolds. *Izvestiya RAN, Ser. Math.* **68** 207–224.
[26] Stallings, J. R. (1968) *Lectures on Polyhedral Topology,* Vol. 43 of *Lectures in Mathematics*, Tata Institute of Fundamental Research.
[27] Włodarczyk, J. (1997) Decompositions of birational toric maps in blow-ups and blow-downs. *Trans. Amer. Math. Soc.* **349** 373–411.