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Compact groups with a set of positive Haar measure satisfying a nilpotent law

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Abstract

The following question is proposed by Martino, Tointon, Valiunas and Ventura in [4, question 1·20]:

Let G be a compact group, and suppose that

$$\mathcal{N}_k(G) = \{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\}$$

has positive Haar measure in G^{k+1}. Does G have an open k-step nilpotent subgroup?

We give a positive answer for $k = 2$.

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1. *Introduction and results*

Let G be a (Hausdorff) compact group. Then G has a unique normalised Haar measure denoted by $\mathbf{m}_G$. The following question is proposed by Martino, Tointon, Valiunas and Ventura in [4, question 1·20].

Question 1·1 [4, question 1·20]. Let G be a compact group, and suppose that $\mathcal{N}_k(G) = \{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\}$ has positive Haar measure in G^{k+1}. Does G have an open k-step nilpotent subgroup?

Here $[x, y] := x^{-1}y^{-1}xy$ for elements x,y of a group and $[x_1, \ldots, x_k, x_{k+1}]$ is a left normed commutator defined inductively as $[[x_1, \ldots, x_k], x_{k+1}]$ for $k \geq 2$.

A positive answer to Question 1.1 is known for $k = 1$ (see [3, theorem 1·2]). It follows from [4, theorem 1·19] that Question 1·1 has positive answer for arbitrary k whenever we further assume that G is totally disconnected (i.e., profinite). Here we give a positive answer to Question 1·1 for $k = 2$ (see Theorem 1·2 below).

THEOREM 1·2 *Let G be a compact group, and suppose that* $\mathcal{N}_2(G) = \{(x_1, x_2, x_3) \in G \times G \times G \mid [x_1, x_2, x_3] = 1\}$ *has positive Haar measure in* $G \times G \times G$. *Then G has an open 2-step nilpotent subgroup.*

2. *A preliminary lemma*

We need the following lemma in the proof of our main result.

LEMMA 2·1 *Suppose that* $x_1, x_2, x_3, g_1, g_2, g_3$ *are elements of a group such that* $[x_1u_1, x_2u_2, x_3u_3] = 1$ *for each triple of the following triples* (u_1, u_2, u_3):

$$(1,1,1), (g_1, g_2, g_3), (g_1, g_2, 1), (g_1, 1, g_2);$$
$$(g_1, 1, 1), (g_1, 1, g_3), (1, 1, g_1), (1, g_2, g_1);$$
$$(1, g_2, 1), (1, 1, g_2), (1, g_2, g_3), (1, 1, g_3).$$

Then $[g_1, g_2, g_3] = 1$.

Proof. Note that $[x,y]$ denotes $x^{-1}y^{-1}xy$ and $[x, y, z] := [[x, y], z]$ for arbitrary elements x,y,z of a group. We will throughout use famous commutator calculus identities: $[xy, z] = [x, z]^y[y, z]$ (†) and $[x, yz] = [x, z][x, y]^z$ (††) for all elements x,y,z of a group, where g^h denotes $h^{-1}gh$. In the following (i) refers to the equality $[x_1u_1, x_2u_2, x_3u_3] = 1$, where (u_1, u_2, u_3) is the ith triple counting them from left to right starting at the top.

$$\begin{aligned} 1 &= [x_1g_1, x_2g_2, g_3] = [[x_1g_1, g_2][x_1g_1, x_2]^{g_2}, g_3] \text{ by († †), (2) and (3)} \\ &= [[x_1g_1, g_2][x_1g_1, x_2], g_3] \text{ by (4) and (5)} \\ &= [x_1g_1, g_2, g_3] = [[x_1, g_2]^{g_1}[g_1, g_2], g_3] \text{ by (†), (5) and (6).} \quad \text{(I)} \end{aligned}$$

On the other hand,

$$\begin{aligned} 1 &= [x_1, x_2g_2, g_1] \text{ by (8) and (9)} \\ &= [[x_1, g_2][x_1, x_2]^{g_2}, g_1] = [[x_1, g_2][x_1, x_2], g_1] \text{ by († †), (1) and (10)} \\ &= [x_1, g_2, g_1] \text{ by (1) and (7).} \quad \text{(II)} \end{aligned}$$

Also,

$$\begin{aligned} 1 &= [x_1, x_2g_2, g_3] \text{ by (9) and (11)} \\ &= [[x_1, g_2][x_1, x_2]^{g_2}, g_3] = [[x_1, g_2][x_1, x_2], g_3] \text{ by († †), (1) and (10)} \\ &= [x_1, g_2, g_3] \text{ by (1) and (12).} \quad \text{(III)} \end{aligned}$$

Now it follows from (I), (II) and (III) that $[g_1, g_2, g_3] = 1$.

Remark. The "left version" (g_ix_j instead of x_jg_i) of Lemma 1·2 is not clear to hold. The validity of a similar result to Lemma 1·2 for commutators with length more than 3 is also under question.

3. *Compact groups with many elements satisfying the 2-step nilpotent law*

We need the "right version" of [5, theorem 2·3] as follows.

THEOREM 3·1 *If A is a measurable subset with positive Haar measure in a compact group G, then for any positive integer k there exists an open subset U of G containing 1 such that* $\mathbf{m}_G(A \cap Au_1 \cap \cdots \cap Au_k) > 0$ *for all* $u_1, \ldots, u_k \in U$.

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Proof. Since $\mathbf{m}_G(A) = \mathbf{m}_G(A^{-1})$, it follows from [5, theorem 2·3] that there exists an open subset V of G containing 1 such that

$$\mathbf{m}_G(A^{-1} \cap v_1A^{-1} \cap \cdots \cap v_kA^{-1}) > 0$$

for all $v_1, \ldots, v_k \in V$. Now take $U := V^{-1}$ which is an open subset of G containing 1. Thus for all $u_1, \ldots, u_k \in U$

$$\begin{aligned}\mathbf{m}_G(A \cap Au_1 \cap \cdots \cap Au_k) &= \mathbf{m}_G((A \cap Au_1 \cap \cdots \cap Au_k)^{-1}) \\ &= \mathbf{m}_G(A^{-1} \cap u_1^{-1}A^{-1} \cap \cdots \cap u_k^{-1}A^{-1}) > 0\end{aligned}$$

This completes the proof.

Now we can prove our main result.

Proof of Theorem 1·2. Let $X := \mathcal{N}_2(G)$. It follows from Theorem 3·1 and [2, theorem 4·5] that there exists an open subset $U = U^{-1}$ of G containing 1 such that

$$X \cap X\bar{u}_1 \cap \cdots \cap X\bar{u}_{11} \neq \varnothing \qquad (*)$$

for all $\bar{u}_1, \ldots, \bar{u}_{11} \in U \times U \times U$. Now take arbitrary elements $g_1, g_2, g_3 \in U$ and consider

$$\begin{aligned}\bar{u}_1 &= (g_1^{-1}, g_2^{-1}, g_3^{-1}), \bar{u}_2 = (g_1^{-1}, g_2^{-1}, 1), \bar{u}_3 = (g_1^{-1}, 1, g_2^{-1}) \\ \bar{u}_4 &= (g_1^{-1}, 1, 1), \bar{u}_5 = (g_1^{-1}, 1, g_3^{-1}), \bar{u}_6 = (1, 1, g_1^{-1}), \bar{u}_7 = (1, g_2^{-1}, g_1^{-1}) \\ \bar{u}_8 &= (1, g_2^{-1}, 1), \bar{u}_9 = (1, 1, g_2^{-1}), \bar{u}_{10} = (1, g_2^{-1}, g_3^{-1}), \bar{u}_{11} = (1, 1, g_3^{-1}).\end{aligned}$$

By ($*$), there exists $(x_1, x_2, x_3) \in X$ such that all the following 3-tuples are in X:

$$\begin{aligned}&(x_1g_1, x_2g_2, x_3g_3), (x_1g_1, x_2g_2, x_3), (x_1g_1, x_2, x_3g_2) \\ &(x_1g_1, x_2, x_3), (x_1g_1, x_2, x_3g_3), (x_1, x_2, x_3g_1), (x_1, x_2g_2, x_3g_1) \\ &(x_1, x_2g_2, x_3), (x_1, x_2, x_3g_2), (x_1, x_2g_2, x_3g_3), (x_1, x_2, x_3g_3).\end{aligned}$$

Now Lemma 2·1 implies that $[g_1, g_2, g_3] = 1$. Therefore the subgroup $H := \langle U \rangle$ generated by U is 2-step nilpotent. Since $H = \bigcup_{n \in \mathbb{N}} U^n$, H is open in G. This completes the proof.

We finish with the following open question that would resolve Question 1·1 for arbitrary k:

Question 3·2 Are there finitely many words w_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq k+1)$ in the free group on $k+1$ generators such that if G is a compact group, $(x_1, \ldots, x_{k+1})$, $u = (u_1, \ldots, u_{k+1}) \in G^{k+1}$ and $[x_1w_{i1}(u), \ldots, x_{k+1}w_{i,k+1}(u)] = 1$ for all $i \in \{1, \ldots, n\}$ then $[u_1, \ldots, u_{k+1}] = 1$?

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