Combinatorics, Probability and Computing (2012) 21, 374–411.
doi:10.1017/S0963548312000053

Measurable Events Indexed by Trees

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Received 11 January 2011; revised 24 January 2012; first published online 12 March 2012

A tree T is said to be homogeneous if it is uniquely rooted and there exists an integer $b \geqslant 2$, called the branching number of T, such that every $t \in T$ has exactly b immediate successors. We study the behaviour of measurable events in probability spaces indexed by homogeneous trees.

Precisely, we show that for every integer $b \geqslant 2$ and every integer $n \geqslant 1$ there exists an integer $q(b,n)$ with the following property. If T is a homogeneous tree with branching number b and $\{A_t : t \in T\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$, then for every $0 < \theta < \varepsilon$ there exists a strong subtree S of T of infinite height, such that for every finite subset F of S of cardinality $n \geqslant 1$ we have

$$\mu\Big(\bigcap_{t\in F} A_t\Big) \geqslant \theta^{q(b,n)}.$$

In fact, we can take $q(b,n) = \big((2^b-1)^{2n-1}-1\big)\cdot(2^b-2)^{-1}$. A finite version of this result is also obtained.

AMS 2010 *Mathematics subject classification*: Primary 05D10; 05C05

1. Introduction

1.1. Overview

Let (Ω, Σ, μ) be a probability space and let $\{A_i : i \in \mathbb{N}\}$ be a family of measurable events in (Ω, Σ, μ) satisfying $\mu(A_i) \geqslant \varepsilon > 0$ for every $i \in \mathbb{N}$. It is well known (and easy to see)

† Supported by NSF grant DMS-0903558.

that for every $0 < \theta < \varepsilon$ there exist $i, j \in \mathbb{N}$ with $i \neq j$ such that $\mu(A_i \cap A_j) \geqslant \theta^2$. Using the classical Ramsey Theorem [11] and iterating this basic fact, we get the following.

If $\{A_i : i \in \mathbb{N}\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_i) \geqslant \varepsilon > 0$ for every $i \in \mathbb{N}$, then for every $0 < \theta < \varepsilon$ there exists an infinite subset L of $\mathbb{N}$ such that, for every integer $n \geqslant 1$ and every finite subset F of L of cardinality n, we have

$$\mu\Big(\bigcap_{i\in F} A_i\Big) \geqslant \theta^n.$$

In other words, if we are given a sequence of measurable events in a probability space and we are allowed to refine (*i.e.*, to pass to a subsequence), then we may behave as if the events are at least as correlated as if they were independent.

Now suppose that the events are not indexed by the integers but are indexed by another 'structured' set $\mathbb{S}$. A natural problem is to decide whether the aforementioned result is valid in the new setting. Namely, given a family $\{A_s : s \in \mathbb{S}\}$ of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_s) \geqslant \varepsilon > 0$ for every $s \in \mathbb{S}$, is it possible to find a 'substructure' $\mathbb{S}'$ of $\mathbb{S}$ such that the events in the family $\{A_s : s \in \mathbb{S}'\}$ are highly correlated? And if yes, then can we get explicit (and, hopefully, optimal) lower bounds for their joint probability? Of course, what 'substructure' is will depend on the nature of the index set $\mathbb{S}$. From a combinatorial perspective, these questions are of particular importance when the 'structured' set $\mathbb{S}$ is a *Ramsey space*, a notion introduced by T. J. Carlson in [3] and further developed by S. Todorcevic in [15].

Various versions have been studied in the literature and several results have been obtained so far. Undoubtedly, the most well-known and heavily investigated case is when the events are indexed by the Ramsey space $W(\mathbb{A})$ of all finite words over a non-empty finite alphabet $\mathbb{A}$. Specifically, it was shown by H. Furstenberg and Y. Katznelson in [4] that for every $0 < \varepsilon \leqslant 1$ and every integer $b \geqslant 2$ there exists a strictly positive constant $\theta(\varepsilon, b)$ with the following property. If $\mathbb{A}$ is an alphabet with b letters and $\{A_w : w \in W(\mathbb{A})\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_w) \geqslant \varepsilon$ for every $w \in W(\mathbb{A})$, then there exists a combinatorial line $\mathbb{L}$ (see [6]) such that

$$\mu\Big(\bigcap_{w\in \mathbb{L}} A_w\Big) \geqslant \theta(\varepsilon, b).$$

In fact, this statement is equivalent to the density Hales–Jewett Theorem. Although powerful, the arguments in [4] are not effective and give no estimate on the constant $\theta(\varepsilon, b)$. Explicit lower bounds can be extracted from the recent 'polymath' proof of the density Hales–Jewett Theorem [10].

Another version has been studied in [2]. The events in this case were assumed to be of a rather 'canonical' form and the index set $\mathbb{S}$ was the level product of a finite sequence of homogeneous trees; we recall that a tree T is said to be *homogeneous* if it is uniquely rooted and there exists an integer $b \geqslant 2$, called the *branching number* of T, such that every $t \in T$ has exactly b immediate successors. We will not state this result explicitly since this requires a fair amount of terminology. We point out, however, that it was needed as a tool in the proof of the density version of the Halpern–Läuchli Theorem [7].

1.2. The main results

Our goal in this paper is to study the above problem when the index set $\mathbb{S}$ is a (finite or infinite) homogeneous tree and to obtain explicit and fairly 'civilized' lower bounds. Of course, such a problem can be also studied if the events are indexed by a boundedly branching tree or, even more generally, by a finitely branching tree. However, as is shown in Appendix A, the case of boundedly branching trees is essentially reduced to the case of homogeneous trees, while for finitely branching but not boundedly branching trees one can construct examples showing that our results do not hold in this wider category.

In the context of trees the most natural (and practically useful) notion of 'substructure' is that of a *strong subtree*. We recall that a (finite or infinite) subtree S of a uniquely rooted tree T is said to be strong provided that: (a) S is uniquely rooted and balanced (that is, all maximal chains of S have the same cardinality), (b) every level of S is a subset of some level of T, and (c) for every non-leaf node $s \in S$ and every immediate successor t of s in T there exists a unique immediate successor s' of s in S with $t \leqslant s'$. The last condition is the most important one and expresses a basic combinatorial requirement, namely that a strong subtree of T must respect the 'tree structure' of T (it implies, for instance, that a strong subtree of infinite height of a homogeneous tree is also homogeneous and has the same branching number). Although the notion of a strong subtree was isolated in the late 1960s, it was highlighted with the work of K. Milliken [8, 9], who showed that the family of strong subtrees of a uniquely rooted and finitely branching tree is partition-regular.

1.2.1. The infinite case. We are ready to state the first main result of the paper.

Theorem 1.1. *Let T be a homogeneous tree with branching number b. Also, let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$. Then, for every $0 < \theta < \varepsilon$ there exists a strong subtree S of T of infinite height such that for every integer $k \geqslant 1$ and every strong subtree R of S of height k we have*

$$\mu\Big(\bigcap_{t\in R} A_t\Big) \geqslant \theta^{p(b,k)}, \tag{1.1}$$

where

$$p(b,k) = \frac{(2^b-1)^k-1}{2^b-2}. \tag{1.2}$$

Notice that a strong subtree R of height k of a homogeneous tree with branching number b has cardinality $(b^k-1)/(b-1)$. Therefore, the exponent appearing in the right-hand side of inequality (1.1) depends polynomially on the cardinality of R; specifically, if R has cardinality n, then the corresponding exponent is $O(n^{b/\log b})$.

It is shown in Appendix B that every non-empty finite subset F of a homogeneous tree is contained in a strong subtree of height $2|F|-1$. This fact and Theorem 1.1 yield the following.

Corollary 1.2. *Let T be a homogeneous tree with branching number b. Also, let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$. Then, for every $0 < \theta < \varepsilon$ there exists a strong subtree S of T of infinite height such that, for every integer $n \geqslant 1$ and every subset F of S of cardinality n, we have*

$$\mu\Big(\bigcap_{t\in F} A_t\Big) \geqslant \theta^{q(b,n)}, \tag{1.3}$$

where

$$q(b,n) = \frac{(2^b-1)^{2n-1}-1}{2^b-2}. \tag{1.4}$$

1.2.2. Free sets: improving the lower bound. Observe that the integer $q(b,n)$ obtained by Corollary 1.2 depends exponentially on n. We do not know whether it is possible to have polynomial dependence. However, if we restrict our attention to a certain class of finite subsets of homogeneous trees, then we get optimal lower bounds. This class of finite sets, which we call *free*, is defined in Section 6 in the main text. It includes various well-known classes of subsets of trees (such as all finite chains, all doubletons and many more) and is sufficiently rich in the sense that every infinite subset A of a homogeneous tree contains an infinite set B such that every non-empty finite subset of B is free. Related to this concept, we show the following.

Theorem 1.3. *Let T be a homogeneous tree. Also, let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$. Then, for every $0 < \theta < \varepsilon$ there exists a strong subtree S of T of infinite height such that for every integer $n \geqslant 1$ and every free subset F of S of cardinality n we have*

$$\mu\Big(\bigcap_{t\in F} A_t\Big) \geqslant \theta^{n}. \tag{1.5}$$

1.2.3. The finite case. Theorem 1.1 has the following finite counterpart, which is the third main result of the paper.

Theorem 1.4. *For every integer $b \geqslant 2$, every integer $k \geqslant 1$ and every pair of reals $0 < \theta < \varepsilon \leqslant 1$, there exists an integer N with the following property. If T is a finite homogeneous tree with branching number b and of height at least N, and $\{A_t : t \in T\}$ is a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon$ for every $t \in T$, then there exists a strong subtree S of T of height k such that*

$$\mu\Big(\bigcap_{t\in S} A_t\Big) \geqslant \theta^{p(b,k)}, \tag{1.6}$$

where $p(b,k)$ is as in (1.2).

The least integer N with the property described in Theorem 1.4 is denoted by $\mathrm{Cor}(b,k,\theta,\varepsilon)$. It is interesting to point out that Theorem 1.4 does not follow from

Theorem 1.1 via compactness, and one has to appropriately convert the arguments to the finite setting. An advantage of having an effective proof is that we can extract explicit and reasonable upper bounds for the integers $\mathrm{Cor}(b,k,\theta,\varepsilon)$; see, for instance, Proposition 1.5 below.

1.3. Outline of the proofs

As we have already mentioned, the proofs of Theorem 1.1 and Theorem 1.4 are conceptually similar. The main goal is to construct a strong subtree W of T (which is either infinite, or of sufficiently large height) for which we can control the joint probability of the events over all *initial* subtrees of W. Once this is done, both Theorem 1.1 and Theorem 1.4 follow by an application of Milliken's Theorem. The desired strong subtree W is constructed recursively using the following detailed version of the case '$k=2$' of Theorem 1.4, and which is the basic pigeon-hole principle in the 'one-step extension' of the recursive selection.

Proposition 1.5. *There exists a primitive recursive function $\Phi:\mathbb{N}^2\to\mathbb{N}$ such that, for every integer $b\geqslant 2$ and every pair of reals $0<\theta<\varepsilon\leqslant 1$, the following holds. If T is a finite homogeneous tree with branching number b such that*

$$h(T)\geqslant \Phi\left(b,\left\lceil \frac{2^b-1}{\varepsilon^{2^b}-\theta^{2^b}}\right\rceil\right), \tag{1.7}$$

and $\{A_t : t\in T\}$ is a family of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_t)\geqslant\varepsilon$ for every $t\in T$, then there exists a strong subtree S of T of height 2 such that

$$\mu\left(\bigcap_{t\in S}A_t\right)\geqslant \theta^{2^b}. \tag{1.8}$$

In particular,

$$\mathrm{Cor}(b,2,\theta,\varepsilon)\leqslant \Phi\left(b,\left\lceil \frac{2^b-1}{\varepsilon^{2^b}-\theta^{2^b}}\right\rceil\right). \tag{1.9}$$

Proposition 1.5 will be proved in Section 3. The basic ingredient of its proof is an appropriate generalization of the notion of a 'Shelah line', a fundamental tool in Ramsey Theory introduced by S. Shelah in his work [13] on the van der Waerden and the Hales–Jewett numbers. We call these new combinatorial objects *generalized Shelah lines.*

The proof of Theorem 1.3 is somewhat different. In particular, in this case the desired strong subtree S is constructed recursively and *directly*. The 'one-step extension' of the recursive selection is achieved using the following result.

Proposition 1.6. *Let T be a homogeneous tree. Also, let $\{A_t : t\in T\}$ be a family of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_t)\geqslant\varepsilon>0$ for every $t\in T$. Then, for every $0<\theta<\varepsilon$ there exists a strong subtree S of T of infinite height such that for every $s,t\in S$ we have $\mu(A_s\cap A_t)\geqslant\theta^2$.*

The main difficulty in the proof of Proposition 1.6 lies in the fact that the class of doubletons of homogeneous trees is *not* Ramsey; that is, one can find a 2-colouring of the set of all doubletons of, say, the dyadic tree D such that every strong subtree of D of height at least 2 contains doubletons of both colours. These pathologies in Ramsey Theory for trees were observed in the late 1960s by F. Galvin and are reflected in his conjecture about partitions of finite subsets of the reals [5], settled in the affirmative in the early 1980s by A. Blass [1]. The key observation in Blass's work is that, for a fixed integer $n \geqslant 1$, the set of all n-element subsets of certain trees can be categorized in a finite list of classes, each of which has the Ramsey property. A similar observation is also the driving force behind the proof of Proposition 1.6.

1.4. Organization of the paper

The paper is organized as follows. In Section 2 we set up our notation and terminology and we gather some background material needed in the rest of the paper. In the next section we introduce the aforementioned notion of a generalized Shelah line and we give the proof of Proposition 1.5. The proof of Theorem 1.4 is given in Section 4 while the proofs of Theorem 1.1 and Corollary 1.2 are given in Section 5. Finally, in Section 6 we define the class of free subsets of homogeneous trees and we give the proofs of Theorem 1.3 and Proposition 1.6. To assist the interested reader we have also included two appendices. In Appendix A we show that Theorem 1.1 still holds if the tree T is merely assumed to be boundedly branching, and we provide counterexamples for the case of finitely branching but *not* boundedly branching trees. In Appendix B we prove that every finite subset F of a homogeneous tree T is contained in a strong subtree of T of height $2|F|-1$, a result needed for the proof of Corollary 1.2.

2. Background material

We let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the natural numbers. The cardinality of a set X will be denoted by $|X|$.

2.1. Trees

By the term *tree* we mean a non-empty partially ordered set $(T, <)$ such that the set $\{s \in T : s < t\}$ is finite and linearly ordered under $<$ for every $t \in T$. The cardinality of this set is defined to be the *length* of t in T and will be denoted by $\ell_T(t)$. For every $n \in \mathbb{N}$ the *n-level* of T, denoted by $T(n)$, is defined to be the set $\{t \in T : \ell_T(t) = n\}$. The *height* of T, denoted by $h(T)$, is defined as follows. If there exists $k \in \mathbb{N}$ with $T(k) = \varnothing$, then we set $h(T) = \max\{n \in \mathbb{N} : T(n) \neq \varnothing\} + 1$; otherwise, we set $h(T) = \infty$.

For every node t of a tree T, the set of *successors* of t in T is defined by

$$\mathrm{Succ}_T(t) = \{s \in T : t \leqslant s\}. \tag{2.1}$$

The set of *immediate successors* of t in T is the subset of $\mathrm{Succ}_T(t)$ defined by $\mathrm{ImmSucc}_T(t) = \{s \in T : t \leqslant s \text{ and } \ell_T(s) = \ell_T(t) + 1\}$.

A *subtree* of a tree T is a subset of T viewed as a tree equipped with the induced partial ordering. For every $k \in \mathbb{N}$ with $k < h(T)$, we set

$$T \upharpoonright k = T(0) \cup \cdots \cup T(k). \tag{2.2}$$

Notice that $h(T \upharpoonright k) = k+1$. An *initial subtree* of T is a subtree of T of the form $T \upharpoonright k$ for some $k \in \mathbb{N}$. A *chain* of T is a subset C of T such that for every $s,t \in C$ we have that either $s \leqslant t$ or $t \leqslant s$.

A tree T is said to be *pruned* (respectively, *finitely branching*) if for every $t \in T$ the set of immediate successors of t in T is non-empty (respectively, finite). It is said to be *boundedly branching* if there exists an integer $m \geqslant 1$ such that every $t \in T$ has at most m immediate successors, and it is said to be *balanced* if all maximal chains of T have the same cardinality. Finally, a tree T is said to be *uniquely rooted* if $|T(0)| = 1$; the *root* of a uniquely rooted tree T is defined to be the node $T(0)$.

Let T be a uniquely rooted tree. For every $s,t \in T$, the *infimum* of s and t in T, denoted by $s \wedge_T t$, is defined to be the $<$-maximal node $w \in T$ such that $w \leqslant s$ and $w \leqslant t$ (notice that the infimum is well-defined since $T(0) \leqslant t$ for every $t \in T$). More generally, for every non-empty subset F of T the *infimum* of F in T, denoted by $\wedge_T F$, is defined to be the $<$-maximal node $w \in T$ such that $w \leqslant t$ for every $t \in F$. Observe that $s \wedge_T t = \wedge_T \{s,t\}$.

2.2. Vector trees

A *vector tree* $\mathbf{T}$ is a non-empty finite sequence of trees having common height; this common height is defined to be the *height* of $\mathbf{T}$ and will be denoted by $h(\mathbf{T})$. We notice that, throughout the paper, we will start the enumeration of vector trees with 1 instead of 0.

The *level product* of a vector tree $\mathbf{T} = (T_1, \dots, T_d)$, denoted by $\otimes\mathbf{T}$, is defined to be the set

$$\bigcup_{n<h(\mathbf{T})} T_1(n) \times \cdots \times T_d(n). \tag{2.3}$$

We say that a vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is *pruned* (respectively, *finitely branching*, *boundedly branching*, *balanced*, *uniquely rooted*) if for every $i \in \{1, \dots, d\}$ the tree T_i is pruned (respectively, finitely branching, boundedly branching, balanced, uniquely rooted).

2.3. Strong subtrees and vector strong subtrees

A subtree S of a uniquely rooted tree T is said to be *strong* provided that: (a) S is uniquely rooted and balanced, (b) every level of S is a subset of some level of T, and (c) for every non-maximal node $s \in S$ and every $t \in \mathrm{ImmSucc}_T(s)$ there exists a unique node $s' \in \mathrm{ImmSucc}_S(s)$ such that $t \leqslant s'$. The *level set* of a strong subtree S of T is defined to be the set

$$L_T(S) = \{m \in \mathbb{N} : \text{exists } n < h(S) \text{ with } S(n) \subseteq T(m)\}. \tag{2.4}$$

A basic property of strong subtrees is that they preserve infima. That is, if S is a strong subtree of T and F is a non-empty subset of S, then $\wedge_S F = \wedge_T F$.

The concept of a strong subtree is naturally extended to vector trees. Specifically, a *vector strong subtree* of a uniquely rooted vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is a vector tree $\mathbf{S} = (S_1, \dots, S_d)$ such that S_i is a strong subtree of T_i for every $i \in \{1, \dots, d\}$ and $L_{T_1}(S_1) = \cdots = L_{T_d}(S_d)$.

2.4. Homogeneous trees and vector homogeneous trees

Let $b \in \mathbb{N}$ with $b \geqslant 2$. By $b^{<\mathbb{N}}$ we shall denote the set of all finite sequences having values in $\{0, \dots, b-1\}$. The empty sequence is denoted by $\varnothing$ and is included in $b^{<\mathbb{N}}$. We view $b^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of end-extension. Notice that $b^{<\mathbb{N}}$ is a homogeneous tree with branching number b. For every $n \in \mathbb{N}$ we let b^n denote the n-level of $b^{<\mathbb{N}}$. If $n \geqslant 1$, then $b^{<n}$ stands for the initial subtree of $b^{<\mathbb{N}}$ of height n. We let $<_{\mathrm{lex}}$ denote the usual lexicographical order on b^n. For every $t, s \in b^{<\mathbb{N}}$, we shall let $t^\frown s$, or simply ts, denote the concatenation of t and s.

For technical reasons that will become transparent below, we will not work with abstract homogeneous trees but with a concrete subclass. Observe that all homogeneous trees with the same branching number are pairwise isomorphic, so this restriction will have no effect in the generality of our results.

Convention. *In the rest of the paper, we use the term 'homogeneous tree'* (*respectively, 'finite homogeneous tree'*) *to mean a strong subtree of $b^{<\mathbb{N}}$ of infinite* (*respectively, finite*) *height for some integer $b \geqslant 2$. For every, possibly finite, homogeneous tree T, we shall let b_T denote the branching number of T. We follow the same convention for vector trees. In particular, we use the term 'vector homogeneous tree' to mean a vector strong subtree of $(b_1^{<\mathbb{N}}, \dots, b_d^{<\mathbb{N}})$ of infinite height for some integers $b_1, \dots, b_d$ with $b_i \geqslant 2$ for every $i \in \{1, \dots, d\}$.*

The above convention has two basic advantages. Firstly, it enables us to effectively enumerate the set of immediate successors of a given node of a, possibly finite, homogeneous tree T. Specifically, for every $t \in T$ and every $p \in \{0, \dots, b_T - 1\}$, let

$$t^{\frown_T} p = \mathrm{ImmSucc}_T(t) \cap \mathrm{Succ}_{b_T^{<\mathbb{N}}}(t^\frown p), \tag{2.5}$$

and notice that

$$\mathrm{ImmSucc}_T(t) = \{t^{\frown_T} p : p \in \{0, \dots, b_T - 1\}\}. \tag{2.6}$$

Also, observe that for every $p, q \in \{0, \dots, b_T - 1\}$ we have $t^{\frown_T} p <_{\mathrm{lex}} t^{\frown_T} q$ if and only if $p < q$.

Secondly, under the above convention, the infimum operation has a particularly simple description. Namely, the infimum of a non-empty subset F of a, possibly finite, homogeneous tree T is the maximal common initial subsequence of every finite sequence in F. Having this representation in mind, we will drop the subscript in the infimum operation and we will denote it simply by $\wedge$.

2.5. Canonical embeddings and canonical isomorphisms

Let T and S be two, possibly finite, homogeneous trees with the same branching number. We say that a map $f : T \to S$ is a *canonical embedding* if for every $t, t' \in T$ the following conditions are satisfied.

(a) We have $\ell_T(t) = \ell_T(t')$ if and only if $\ell_S\big(f(t)\big) = \ell_S\big(f(t')\big)$.
(b) We have $t \sqsubset t'$ if and only if $f(t) \sqsubset f(t')$.
(c) If $\ell_T(t) = \ell_T(t')$, then $t <_{\mathrm{lex}} t'$ if and only if $f(t) <_{\mathrm{lex}} f(t')$.
(d) We have $f(t \wedge t') = f(t) \wedge f(t')$.

Observe that a canonical embedding $f : T \to S$ is an injection, and its image $f(T)$ is a strong subtree of S. Also, notice that if S and T have the same height, then there exists a *unique* bijection between T and S satisfying the above conditions. This unique bijection will be called the *canonical isomorphism* between T and S and will be denoted by $\mathrm{I}(T, S)$.

2.6. Milliken's Theorem

Let T be a, possibly finite, homogeneous tree. For every integer $k \geqslant 1$, we let $\mathrm{Str}_k(T)$ denote the set of all strong subtrees of T of height k, and let $\mathrm{Str}_\infty(T)$ denote the set of all strong subtrees of T of infinite height. For every vector homogeneous tree $\mathbf{T} = (T_1, \ldots, T_d)$ the sets $\mathrm{Str}_k(\mathbf{T})$ and $\mathrm{Str}_\infty(\mathbf{T})$ are analogously defined. It is easy to see that $\mathrm{Str}_\infty(\mathbf{T})$ is a G_δ (hence Polish) subspace of $2^{T_1} \times \cdots \times 2^{T_d}$. We will need the following result due to K. Milliken.

Theorem 2.1 ([9]). *Let* $\mathbf{T}$ *be a vector homogeneous tree. Then, for every Borel subset* $\mathcal{C}$ *of* $\mathrm{Str}_\infty(\mathbf{T})$ *there exists a vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *of infinite height such that either* $\mathrm{Str}_\infty(\mathbf{S}) \subseteq \mathcal{C}$ *or* $\mathrm{Str}_\infty(\mathbf{S}) \cap \mathcal{C} = \varnothing$.

In particular, for every integer $k \geqslant 1$ *and every subset* $\mathcal{F}$ *of* $\mathrm{Str}_k(\mathbf{T})$ *there exists a vector strong subtree* $\mathbf{R}$ *of* $\mathbf{T}$ *of infinite height such that either* $\mathrm{Str}_k(\mathbf{R}) \subseteq \mathcal{F}$ *or* $\mathrm{Str}_k(\mathbf{R}) \cap \mathcal{F} = \varnothing$.

By Theorem 2.1 and a standard compactness argument, we get the following.

Corollary 2.2. *For every integer* $b \geqslant 2$*, every pair of integers* $m \geqslant k \geqslant 1$ *and every integer* $r \geqslant 2$ *there exists an integer* M *with the following property. For every finite homogeneous tree* T *with branching number* b *and of height at least* M*, and every* r*-colouring of the set* $\mathrm{Str}_k(T)$*, there exists a strong subtree* S *of* T *of height* m *such that the set* $\mathrm{Str}_k(S)$ *is monochromatic. The least integer* M *with this property will be denoted by* $\mathrm{Mil}(b, m, k, r)$.

Notice that the reduction of Corollary 2.2 to Theorem 2.1 via compactness is non-effective and gives no estimate for the numbers $\mathrm{Mil}(b, m, k, r)$. An analysis of the finite version of Milliken's Theorem has been carried out by M. Sokić, yielding explicit and reasonable upper bounds. In particular, we have the following.

Theorem 2.3 ([14]). *For every integer* $k \geqslant 1$ *there exists a primitive recursive function* $\phi_k : \mathbb{N}^3 \to \mathbb{N}$ *belonging to the class* $\mathcal{E}^{5+k}$ *of Grzegorczyk's hierarchy such that, for every integer*

https://doi.org/10.1017/S0963548312000053 Published online by Cambridge University Press

$b \geqslant 2$, every integer $m \geqslant k$ and every integer $r \geqslant 2$, we have

$$\mathrm{Mil}(b,m,k,r) \leqslant \phi_k(b,m,r). \tag{2.7}$$

2.7. Probabilistic preliminaries

We recall the following well-known fact. The proof is sketched for completeness.

Lemma 2.4. *Let $0<\theta<\varepsilon\leqslant 1$ and $N\in\mathbb{N}$ with $N\geqslant(\varepsilon^2-\theta^2)^{-1}$. Also, let $(A_i)_{i=0}^{N-1}$ be a family of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_i)\geqslant\varepsilon$ for every $i\in\{0,\dots,N-1\}$. Then there exist $i,j\in\{0,\dots,N-1\}$ with $i\neq j$ such that $\mu(A_i\cap A_j)\geqslant\theta^2$.*

Proof. For every $i\in\{0,\dots,N-1\}$, let $\mathbf{1}_{A_i}$ be the indicator function of the event A_i and set $X=\sum_{i=0}^{N-1}\mathbf{1}_{A_i}$. Then $\mathbb{E}[X]\geqslant\varepsilon N$ so, by convexity,

$$\sum_{i\in\{0,\dots,N-1\}}\ \sum_{j\in\{0,\dots,N-1\}\setminus\{i\}}\mu(A_i\cap A_j)=\mathbb{E}[X(X-1)]\geqslant\varepsilon N(\varepsilon N-1).$$

Therefore, there exist $i,j\in\{0,\dots,N-1\}$ with $i\neq j$ such that $\mu(A_i\cap A_j)\geqslant\theta^2$. □

Finally, for every probability space (Ω,Σ,μ), every $Y\in\Sigma$ with $\mu(Y)>0$ and every $A\in\Sigma$, we let $\mu(A\mid Y)$ denote the conditional probability of A relative to Y; that is,

$$\mu(A\mid Y)=\frac{\mu(A\cap Y)}{\mu(Y)}. \tag{2.8}$$

The conditional probability measure of μ relative to Y will be denoted by μ_Y. Notice that $\mu_Y(A)=\mu(A\mid Y)$ for every $A\in\Sigma$.

3. Proof of Proposition 1.5

This section is devoted to the proof of Proposition 1.5 stated in the Introduction. It is organized as follows. In Section 3.1 we introduce the class of generalized Shelah lines and we present some of their basic properties. In Section 3.2 we define the primitive recursive function Φ. The proof of Proposition 1.5 is given in Section 3.3. In Section 3.4 we prove a ‘relativized’ version of Proposition 1.5; this ‘relativized’ version is needed for the proof of Theorem 1.4. Finally, in Section 3.5 we make some comments concerning the upper bounds for the numbers $\mathrm{Cor}(b,2,\theta,\varepsilon)$ obtained by Proposition 1.5.

3.1. Generalized Shelah lines

We start with the following definition.

Definition 3.1. Let T be a finite homogeneous tree of height at least 2. Also, let $F\in\mathrm{Str}_2(T)$ and P be a (possibly empty) subset of $\{0,\dots,b_T-1\}$. The *P-restriction* of F, denoted by $F|_P$, is defined to be the set

$$F|_P=\{F(0)\}\cup\{F(0)^{\frown_T}p : p\in P\}. \tag{3.1}$$

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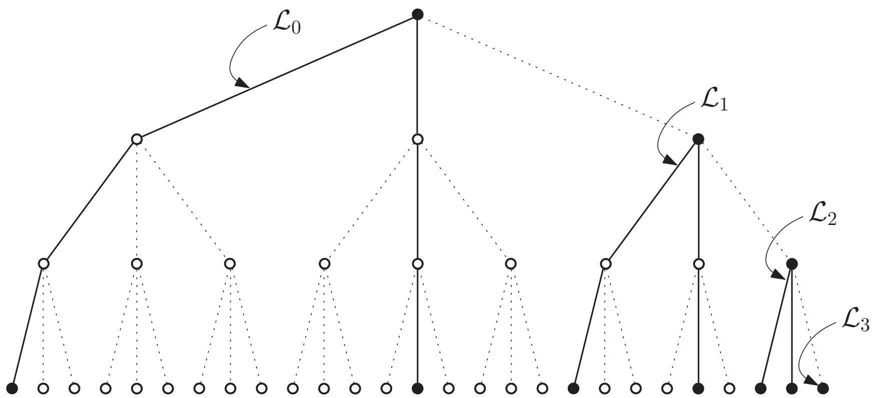

Figure 1. A generalized Shelah line for $b=3$, $i=2$, $P=\{0,1\}$ and $N=4$.

Notice that $F|_{\varnothing}=\{F(0)\}$ and $F|_{\{0,\dots,b_T-1\}}=F$. Moreover, it easy to see that $F|_{P\cup Q}=F|_P\cup F|_Q$ for any pair P and Q of subsets of $\{0,\dots,b_T-1\}$. We are ready to introduce the main object of study in this subsection.

Definition 3.2 (Standard generalized Shelah lines and their components). Let $b\in\mathbb{N}$ with $b\geqslant 2$. Also, let $i\in\{0,\dots,b-1\}$, $P\subseteq\{0,\dots,b-1\}$ and $N\in\mathbb{N}$ with $N\geqslant 1$. The *standard* (b,i,P,N)*-generalized Shelah line*, denoted by $\mathcal{L}(b,i,P,N)$, is the subset of $b^{<N}$ defined by

$$\mathcal{L}(b,i,P,N)=\bigcup_{k=0}^{N-1}\{i^k\}\cup\{i^k p^{N-1-k}:p\in P\}. \tag{3.2}$$

For every $k\in\{0,\dots,N-1\}$, the *k-component* of $\mathcal{L}(b,i,P,N)$ is defined by

$$\mathcal{L}_k(b,i,P,N)=\{i^k\}\cup\{i^k p^{N-1-k}:p\in P\}. \tag{3.3}$$

Next we extend Definition 3.2 to all finite homogeneous trees as follows.

Definition 3.3 (Generalized Shelah lines of finite homogeneous trees). Let T be a finite homogeneous tree and denote its height by N. Also, let $i\in\{0,\dots,b_T-1\}$ and $P\subseteq\{0,\dots,b_T-1\}$. The (i,P)*-generalized Shelah line* of T is defined to be the image of $\mathcal{L}(b_T,i,P,N)$ under the canonical isomorphism $\mathrm{I}(b_T^{<N},T)$ between $b_T^{<N}$ and T (see Section 2.5). Respectively, for every $k\in\{0,\dots,N-1\}$, the *k-component* of the (i,P)-generalized Shelah line of T is defined to be the image of the corresponding k-component $\mathcal{L}_k(b_T,i,P,N)$ of $\mathcal{L}(b_T,i,P,N)$ under the canonical isomorphism $\mathrm{I}(b_T^{<N},T)$.

Below, we isolate some basic properties of all generalized Shelah lines of a finite homogeneous tree T.

($\mathcal{P}1$) Every generalized Shelah line of T is the union of its components.
($\mathcal{P}2$) The last component of every generalized Shelah line of T is a singleton.

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($\mathcal{P}$3) If T has height $N \geqslant 2$ and $k \in \{0,\dots,N-2\}$, then the k-component of the (i,P)-generalized Shelah line of T is the P-restriction of a strong subtree of T of height 2.

Properties ($\mathcal{P}$1) and ($\mathcal{P}$2) are straightforward consequences of the relevant definitions. To see property ($\mathcal{P}$3), consider the k-component $\mathcal{L}_k(b_T,i,P,N)$ of the standard generalized Shelah line $\mathcal{L}(b_T,i,P,N)$ and set

$$F_k = \{i^k\} \cup \{i^k j^{N-1-k} : j \in \{0,\dots,b_T-1\}\}. \tag{3.4}$$

Notice that $F_k \in \mathrm{Str}_2(b_T^{<N})$ and that $F_k|_P = \mathcal{L}_k(b_T,i,P,N)$. Since strong subtrees of height 2 and their restrictions are preserved under canonical isomorphisms, we see that property ($\mathcal{P}$3) is also satisfied. The most important property, however, of generalized Shelah lines is included in the following proposition.

Proposition 3.4. *Let T be a finite homogeneous tree of height at least 2. Further, let $i \in \{0,\dots,b_T-1\}$ and P be a (possibly empty) subset of $\{0,\dots,b_T-1\}$. If $i \notin P$, then the union of any two distinct components of the (i,P)-generalized Shelah line of T contains the $(P \cup \{i\})$-restriction of a strong subtree of T of height 2.*

Proof. Clearly we may assume that T is the tree $b_T^{<N}$, where N is the height of T. We fix $0 \leqslant k_0 < k_1 \leqslant N-1$ and we consider the following cases.

Case 1: $P = \varnothing$. Let $F \in \mathrm{Str}_2(b_T^{<N})$ be defined by

$$F = \{i^{k_0}\} \cup \{i^{k_0} j^{k_1-k_0} : j \in \{0,\dots,b_T-1\}\},$$

and observe that $F|_{\{i\}} = \{i^{k_0}\} \cup \{i^{k_1}\} = \mathcal{L}_{k_0}(b_T,i,\varnothing,N) \cup \mathcal{L}_{k_1}(b_T,i,\varnothing,N)$.

Case 2: $P \neq \varnothing$. We set $p = \min P$. Let $G \in \mathrm{Str}_2(b_T^{<N})$ be defined by

$$G = \{i^{k_0}\} \cup \{i^{k_0} j^{N-1-k_0} : j \in \{0,\dots,b_T-1\} \quad \text{and} \quad j \neq i\} \cup \{i^{k_1} p^{N-1-k_1}\}.$$

Notice that $G|_P = \mathcal{L}_{k_0}(b_T,i,P,N)$ and $G|_{\{i\}} = \{i^{k_0}\} \cup \{i^{k_1} p^{N-1-k_1}\}$. Thus,

$$G|_{P\cup\{i\}} = G|_P \cup G|_{\{i\}} \subseteq \mathcal{L}_{k_0}(b_T,i,P,N) \cup \mathcal{L}_{k_1}(b_T,i,P,N).$$

The proof is complete. □

3.2. The primitive recursive function Φ

For every $b,i,m \in \mathbb{N}$ with $b \geqslant 2$ and $m \geqslant 2$ we define recursively the integer $M^{(i)}(b,m)$ by the rule

$$\begin{aligned} M^{(0)}(b,m) &= m, \\ M^{(i+1)}(b,m) &= \mathrm{Mil}\big(b, M^{(i)}(b,m), 2, 2\big). \end{aligned} \tag{3.5}$$

Inductively, it is easy to show that

$$M^{(i)}(b,m) \geqslant m. \tag{3.6}$$

Moreover, we have the following.

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Fact 3.5. *There exists a primitive recursive function $\Phi : \mathbb{N}^2 \to \mathbb{N}$ belonging to the class $\mathcal{E}^8$ of Grzegorczyk's hierarchy such that, for every integer $b \geqslant 2$ and every integer $m \geqslant 2$, we have*

$$M^{(b-1)}(b,m) \leqslant \Phi(b,m). \tag{3.7}$$

Proof. The result follows easily by Theorem 2.3 and elementary properties of primitive recursive functions (see, *e.g.*, [12]). We will provide the details for the benefit of the reader. To this end, we first need to recall some pieces of notation. For every $j \in \{1,2\}$, we let $\pi_j : \mathbb{N}^2 \to \mathbb{N}$ denote the projection function to the j-coordinate; it belongs to the class $\mathcal{E}^0$. Also, let $\mathrm{ms} : \mathbb{N}^2 \to \mathbb{N}$ be the modified subtraction function defined by $\mathrm{ms}(n,k) = n-k$ if $n \geqslant k$ and $\mathrm{ms}(n,k) = 0$ if $n < k$; it belongs to the class $\mathcal{E}^3$.

Now, let $\phi_2 : \mathbb{N}^3 \to \mathbb{N}$ be the primitive recursive function obtained by Theorem 2.3 for '$k=2$'. Recall that ϕ_2 belongs to the class $\mathcal{E}^7$, and that for every integer $b \geqslant 2$, every integer $m \geqslant 2$ and every integer $r \geqslant 2$ we have $\mathrm{Mil}(b,m,2,r) \leqslant \phi_2(b,m,r)$. Define $\psi : \mathbb{N}^3 \to \mathbb{N}$ by the rule

$$\psi(0,x) = \pi_2(x),$$
$$\psi(i+1,x) = \phi_2\big(\pi_1(x), \psi(i,x), 2\big).$$

Since ϕ_2 belongs to the class $\mathcal{E}^7$, we see that the function ψ belongs to the class $\mathcal{E}^8$. Finally, let $\Phi : \mathbb{N}^2 \to \mathbb{N}$ be defined by

$$\Phi(x) = \psi\big(\mathrm{ms}(\pi_1(x),1), \pi_1(x), \pi_2(x)\big).$$

Clearly the function Φ belongs to the class $\mathcal{E}^8$. It is easy to check that Φ is as desired. □

3.3. Proof of Proposition 1.5

By Fact 3.5, it is enough to show the following.

Lemma 3.6. *Let $0 < \theta < \varepsilon \leqslant 1$. Also, let T be a finite homogeneous tree such that*

$$h(T) \geqslant M^{(b_T-1)}\left(b_T, \left\lceil \frac{2^{b_T}-1}{\varepsilon^{2^{b_T}} - \theta^{2^{b_T}}} \right\rceil\right) \tag{3.8}$$

and let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon$ for every $t \in T$. Then there exists $F \in \mathrm{Str}_2(T)$ such that

$$\mu\left(\bigcap_{t\in F} A_t\right) \geqslant \theta^{2^{b_T}}. \tag{3.9}$$

Proof. In what follows, for notational simplicity, we let b denote the branching number of the tree T. We set

$$\delta = \frac{\varepsilon^{2^b} - \theta^{2^b}}{2^b - 1}. \tag{3.10}$$

Recursively, for every $i \in \{0, \dots, b-1\}$ we will select

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(i) a positive real ε_i,
(ii) a positive integer N_i, and
(iii) a strong subtree R_i of T,

such that the following conditions are satisfied.

(C1) We have $\varepsilon_0 = \varepsilon$ and $\varepsilon_{i+1}^{2^{i+1}} = \varepsilon_i^{2^{i+1}} - \delta$ for every $i \in \{0,\dots,b-2\}$.
(C2) For every $i \in \{0,\dots,b-1\}$ we have $N_i = M^{(b-1-i)}\big(b,\lceil \delta^{-1}\rceil\big)$.
(C3) For every $i \in \{0,\dots,b-2\}$ the tree R_{i+1} is a strong subtree of R_i.
(C4) For every $i \in \{0,\dots,b-1\}$ the height of the tree R_i is N_i.
(C5) For every $i \in \{0,\dots,b-1\}$ and every $F \in \mathrm{Str}_2(R_i)$ we have

$$\mu\Big(\bigcap_{t\in F|_{\{0,\dots,i-1\}}} A_t\Big) \geqslant \varepsilon_i^{2^i}. \tag{3.11}$$

with the convention that $\{0,\dots,i-1\} = \varnothing$ if $i=0$.

We proceed to the recursive selection. For $i=0$ we set '$\varepsilon_0 = \varepsilon$', '$N_0 = M^{(b-1)}(b,\lceil\delta^{-1}\rceil)$' and '$R_0 = T\upharpoonright(N_0-1)$', and we observe that with these choices conditions (C1), (C2) and (C4) are satisfied. Noticing that $F|_\varnothing = \{F(0)\}$ for every $F \in \mathrm{Str}_2(T)$, we see that condition (C5) is also satisfied. Since condition (C3) is meaningless in this case, the first step of the recursive selection is complete.

Let $i \in \{0,\dots,b-2\}$, and assume that the recursive selection has been carried out up to i so that conditions (C1)–(C5) are satisfied. We start the next step of the recursive selection setting '$\varepsilon_{i+1} = \big(\varepsilon_i^{2^{i+1}} - \delta\big)^{1/2^{i+1}}$', and we observe that with this choice condition (C1) is satisfied. Next we set '$N_{i+1} = M^{(b-1-i-1)}\big(b,\lceil\delta^{-1}\rceil\big)$', and we notice that condition (C2) is also satisfied. Now, let

$$\mathcal{F} = \Big\{F \in \mathrm{Str}_2(R_i) : \mu\Big(\bigcap_{t\in F|_{\{0,\dots,i\}}} A_t\Big) \geqslant \varepsilon_{i+1}^{2^{i+1}}\Big\}. \tag{3.12}$$

By our inductive assumptions, the height of the tree R_i is N_i. Moreover,

$$\begin{aligned} N_i &\overset{(C2)}{=} M^{(b-1-i)}\big(b,\lceil\delta^{-1}\rceil\big) \overset{(3.5)}{=} \mathrm{Mil}\big(b, M^{(b-1-i-1)}(b,\lceil\delta^{-1}\rceil),2,2\big)\\ &= \mathrm{Mil}(b,N_{i+1},2,2). \end{aligned}$$

Therefore, by Corollary 2.2, there exists a strong subtree R of R_i of height N_{i+1} such that either $\mathrm{Str}_2(R) \subseteq \mathcal{F}$ or $\mathrm{Str}_2(R)\cap\mathcal{F} = \varnothing$. We set '$R_{i+1} = R$' and we claim that with this choice all the other conditions are satisfied. It is clear that (C3) and (C4) are satisfied, so we only need to check condition (C5). Notice that it is enough to show that $\mathrm{Str}_2(R)\cap\mathcal{F} \neq \varnothing$. To this end we argue as follows. Let $\mathcal{L}$ be the $(i,\{0,\dots,i-1\})$-generalized Shelah line of R (recall that, by convention, we set $\{0,\dots,i-1\} = \varnothing$ if $i=0$). For every $k \in \{0,\dots,N_{i+1}-1\}$, let $\mathcal{L}_k$ be the k-component of $\mathcal{L}$ and set

$$A_k = \bigcap_{t\in\mathcal{L}_k} A_t. \tag{3.13}$$

By property ($\mathcal{P}3$) in Section 3.1, if $k \in \{0,\dots,N_{i+1}-2\}$, then the k-component $\mathcal{L}_k$ of $\mathcal{L}$ is the $\{0,\dots,i-1\}$-restriction of some strong subtree of R of height 2. This fact and condition (C5) of our inductive assumptions yield that $\mu(A_k) \geqslant \varepsilon_i^{2^i}$ if $k \in \{0,\dots,N_{i+1}-2\}$.

On the other hand, if $k = N_{i+1} - 1$, then by property $(\mathcal{P}2)$ in Section 3.1 the k-component of $\mathcal{L}$ is a singleton. Noticing that $\varepsilon \geqslant \varepsilon_i^{2^i}$, we conclude that

$$\mu(A_k) \geqslant \varepsilon_i^{2^i} \tag{3.14}$$

for every $k \in \{0, \ldots, N_{i+1} - 1\}$. Moreover, by the choice of N_{i+1} and ε_{i+1}, we have

$$N_{i+1} = M^{(b-1-i-1)}\big(b, \lceil \delta^{-1} \rceil\big) \overset{(3.6)}{\geqslant} \lceil \delta^{-1} \rceil \geqslant \frac{1}{\delta} = \frac{1}{(\varepsilon_i^{2^i})^2 - (\varepsilon_{i+1}^{2^i})^2}. \tag{3.15}$$

Hence, by Lemma 2.4 applied for '$N = N_{i+1}$', '$\varepsilon = \varepsilon_i^{2^i}$' and '$\theta = \varepsilon_{i+1}^{2^i}$', there exist $0 \leqslant k < k' < N_{i+1}$ such that $\mu(A_k \cap A_{k'}) \geqslant \varepsilon_{i+1}^{2^{i+1}}$. By Proposition 3.4, there exists $G \in \mathrm{Str}_2(R)$ such that $G|_{\{0,\ldots,i-1\}\cup\{i\}} \subseteq \mathcal{L}_k \cup \mathcal{L}_{k'}$. Observing that $G|_{\{0,\ldots,i\}} = G|_{\{0,\ldots,i-1\}\cup\{i\}}$, we see that

$$\mu\Big(\bigcap_{t \in G|_{\{0,\ldots,i\}}} A_t\Big) \geqslant \mu\Big(\bigcap_{t \in \mathcal{L}_k \cup \mathcal{L}_{k'}} A_t\Big) = \mu(A_k \cap A_{k'}) \geqslant \varepsilon_{i+1}^{2^{i+1}}. \tag{3.16}$$

Therefore, $G \in \mathrm{Str}_2(R) \cap \mathcal{F}$. This shows that condition (C5) is also satisfied, and so the recursive selection is complete.

We isolate, for future use, the following consequence of condition (C1). The proof is left to the interested reader.

Fact 3.7. *For every $i \in \{0, \ldots, b-1\}$ we have $\varepsilon_i^{2^i} \geqslant \varepsilon^{2^i} - (2^i - 1)\delta$.*

We are ready for the final step of the argument. Let R_{b-1} be the strong subtree of T obtained above. We will show that there exists $F \in \mathrm{Str}_2(R_{b-1})$ satisfying the estimate in (3.9). This will finish the proof. To this end we set

$$r = \varepsilon_{b-1}^{2^{b-1}} \quad \text{and} \quad \eta = \big(r^2 - \delta\big)^{1/2}. \tag{3.17}$$

By condition (C5) and the choice of r, for every $F \in \mathrm{Str}_2(R_{b-1})$ we have

$$\mu\Big(\bigcap_{t \in F|_{\{0,\ldots,b-2\}}} A_t\Big) \geqslant r. \tag{3.18}$$

Moreover,

$$h(R_{b-1}) \overset{(C4)}{=} N_{b-1} \overset{(C2)}{=} M^{(0)}\big(b, \lceil \delta^{-1} \rceil\big) \overset{(3.5)}{=} \lceil \delta^{-1} \rceil \geqslant \frac{1}{\delta} \overset{(3.17)}{=} \frac{1}{r^2 - \eta^2}. \tag{3.19}$$

Let $\mathcal{G}$ be the $(b-1, \{0, \ldots, b-2\})$-generalized Shelah line of R_{b-1}. Also, for every $k \in \{0, \ldots, h(R_{b-1}) - 1\}$ let $\mathcal{G}_k$ be the k-component of $\mathcal{G}$, and set

$$B_k = \bigcap_{t \in \mathcal{G}_k} A_t. \tag{3.20}$$

Arguing precisely as in the 'one-step extension' of the recursive selection and using the estimates in (3.18) and (3.19), it is possible to find $0 \leqslant k < k' < h(R_{b-1})$ and $F \in \mathrm{Str}_2(R_{b-1})$ such that $\mu(B_k \cap B_{k'}) \geqslant \eta^2$ and $F|_{\{0,\ldots,b-2\}\cup\{b-1\}} \subseteq \mathcal{G}_k \cup \mathcal{G}_{k'}$. Since $F = F|_{\{0,\ldots,b-2\}\cup\{b-1\}}$, we see that

$$\mu\Big(\bigcap_{t\in F} A_t\Big) \geqslant \mu(B_k \cap B_{k'}) \geqslant \eta^2. \tag{3.21}$$

Moreover, by (3.17) and Fact 3.7, we have

$$\eta^2 \geqslant \big(\varepsilon^{2^{b-1}} - (2^{b-1}-1)\delta\big)^2 - \delta \geqslant \varepsilon^{2^b} - (2^b-1)\delta \overset{(3.10)}{=} \theta^{2^b}. \tag{3.22}$$

The proof of Lemma 3.6 is complete. □

As we have already indicated in the beginning of the subsection, having completed the proof of Lemma 3.6, the proof of Proposition 1.5 is also complete.

3.4. Consequences

We have already mentioned that in this subsection we will give a 'relativized' version of Proposition 1.5. To this end we first need to introduce some quantitative invariants closely related to the numbers $\mathrm{Cor}(b, 2, \theta, \varepsilon)$.

Definition 3.8. For every integer $b \geqslant 2$ and every pair of reals $0 < \theta < \varepsilon \leqslant 1$, we let $\mathrm{Rel}(b, \theta, \varepsilon)$ denote the least integer N (if it exists) with the following property. For every finite homogeneous tree T with branching number b and of height at least N, and every family $\{A_t : t \in T\}$ of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon$ for every $t \in T$, there exists $F \in \mathrm{Str}_2(T)$ such that

$$\mu\Big(\bigcap_{t\in F(1)} A_t \mid A_{F(0)}\Big) \geqslant \theta^{2^b-1}. \tag{3.23}$$

We will show that the numbers $\mathrm{Rel}(b, \theta, \varepsilon)$ exist. In fact, we shall obtain upper bounds which are expressed in terms of the Milliken's numbers. Specifically, for every integer $b \geqslant 2$ and every $0 < \theta < \varepsilon \leqslant 1$, we set

$$\lambda(b, \theta, \varepsilon) = \big(\varepsilon \cdot \theta^{-1}\big)^{\frac{2^b-1}{2^b+1}}, \tag{3.24}$$

and we notice that $\lambda(b, \theta, \varepsilon) > 1$. Also, let

$$r(b, \theta, \varepsilon) = \left\lceil \frac{\ln \varepsilon^{-1}}{\ln \lambda(b, \theta, \varepsilon)} \right\rceil. \tag{3.25}$$

Finally, for every $i \in \{0, \dots, r(b, \theta, \varepsilon)\}$, let

$$\varepsilon_i = \varepsilon \cdot \lambda(b, \theta, \varepsilon)^{i-1}, \tag{3.26}$$

and define

$$m(b, \theta, \varepsilon) = \max\left\{ M^{(b-1)}\left(b, \left\lceil \frac{2^b-1}{\varepsilon_i^{2^b} - \varepsilon_{i-1}^{2^b}} \right\rceil\right) : 1 \leqslant i \leqslant r(b, \theta, \varepsilon)\right\}. \tag{3.27}$$

We have the following.

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Corollary 3.9. *For every integer $b \geqslant 2$ and every $0<\theta<\varepsilon\leqslant 1$, we have*

$$\mathrm{Rel}(b,\theta,\varepsilon)\leqslant \mathrm{Mil}\big(b,m(b,\theta,\varepsilon),1,r(b,\theta,\varepsilon)\big). \tag{3.28}$$

Proof. The result follows easily by Lemma 3.6 and a stabilization argument. Let us give the details. For notational simplicity we set $\lambda=\lambda(b,\theta,\varepsilon)$, $r=r(b,\theta,\varepsilon)$ and $m=m(b,\theta,\varepsilon)$. Also, let $\varepsilon_{r+1}=\varepsilon\lambda^r$ and notice that $\varepsilon_{r+1}\geqslant 1$ by the choice of r in (3.25). Since $\lambda>1$, by (3.26), we see that

$$\varepsilon=\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_r<\varepsilon_{r+1}.$$

Let T be a finite homogeneous tree with branching number b and height at least $\mathrm{Mil}(b,m,1,r)$, and let $\{A_t : t\in T\}$ be a family of measurable events in a probability space (Ω,Σ,μ) satisfying $\mu(A_t)\geqslant\varepsilon$ for every $t\in T$. There exist a strong subtree R of T of height m and $i_0\in\{1,\dots,r\}$ such that for every $t\in R$ we have

$$\varepsilon_{i_0}\leqslant\mu\big(A_t\big)\leqslant\varepsilon_{i_0+1}. \tag{3.29}$$

Therefore, $\mu(A_t)\geqslant\varepsilon_{i_0}>\varepsilon_{i_0-1}$ and

$$h(R)=m\overset{(3.27)}{\geqslant}M^{(b-1)}\left(b,\left\lceil\frac{2^b-1}{\varepsilon_{i_0}^{2^b}-\varepsilon_{i_0-1}^{2^b}}\right\rceil\right).$$

By Lemma 3.6 applied for '$\theta=\varepsilon_{i_0-1}$', '$\varepsilon=\varepsilon_{i_0}$' and the family '$\{A_t : t\in R\}$', there exists $F\in\mathrm{Str}_2(R)$ such that

$$\mu\Big(\bigcap_{t\in F}A_t\Big)\geqslant\varepsilon_{i_0-1}^{2^b}. \tag{3.30}$$

By (3.24), (3.26), (3.29) and (3.30), and taking into account that $\lambda>1$ and $i_0\geqslant 1$, we conclude that

$$\begin{aligned}\mu\Big(\bigcap_{t\in F(1)}A_t \mid A_{F(0)}\Big)&=\frac{\mu(\bigcap_{t\in F}A_t)}{\mu(A_{F(0)})}\geqslant\frac{\varepsilon_{i_0-1}^{2^b}}{\varepsilon_{i_0+1}}=\varepsilon^{2^b-1}\cdot\lambda^{2^b i_0-2^{b+1}-i_0}\\&\geqslant\varepsilon^{2^b-1}\cdot\lambda^{-2^b-1}=\theta^{2^b-1}.\end{aligned}$$

This shows that $\mathrm{Rel}(b,\theta,\varepsilon)\leqslant\mathrm{Mil}(b,m,1,r)$, as desired. □

3.5. Comments

By Fact 3.5 and Lemma 3.6, the numbers $\mathrm{Cor}(b,2,\theta,\varepsilon)$ are controlled by a primitive recursive function belonging to the class $\mathcal{E}^8$ of Grzegorczyk's hierarchy. We point out that this upper bound is not optimal and, in fact, we can have significantly better upper bounds. Precisely, by appropriately modifying the arguments in the proof of Proposition 1.5 (avoiding, in particular, the use of Milliken's Theorem), it is possible to show that the estimate in (1.9) is satisfied for the function $\Psi:\mathbb{N}^2\to\mathbb{N}$ defined by

$$\Psi(b,m)=m^{(m+1)^b}. \tag{3.31}$$

Such a modification, however, is technically involved and conceptually less natural to grasp, so we prefer to omit it.

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4. Proof of Theorem 1.4

We fix an integer $b \geqslant 2$ and a pair of reals $0 < \theta < \varepsilon \leqslant 1$. We will define the numbers $\mathrm{Cor}(b,k,\theta,\varepsilon)$ by recursion on k. It is clear that $\mathrm{Cor}(b,1,\theta,\varepsilon) = 1$. The definition of the number $\mathrm{Cor}(b,2,\theta,\varepsilon)$ is the content of Proposition 1.5.

So let $k \in \mathbb{N}$ with $k \geqslant 2$ and assume that the number $\mathrm{Cor}(b,k,\theta,\varepsilon)$ has been defined. Let

$$\eta = \frac{\varepsilon + \theta}{2}, \tag{4.1}$$

and set

$$n(k) = \max\{\mathrm{Rel}(b,\eta,\varepsilon), \mathrm{Cor}(b,k,\theta^{2^b-1},\eta^{2^b-1})\} + 1. \tag{4.2}$$

Claim 4.1. *We have*

$$\mathrm{Cor}(b,k+1,\theta,\varepsilon) \leqslant \mathrm{Mil}(b,n(k),2,2). \tag{4.3}$$

It is, of course, clear that Theorem 1.4 follows by Claim 4.1. So, what remains is to prove Claim 4.1. To this end let T be a finite homogeneous tree with branching number b such that

$$h(T) \geqslant \mathrm{Mil}(b,n(k),2,2) \tag{4.4}$$

and a family $\{A_t : t \in T\}$ of measurable events in a probability measure space (Ω,Σ,μ) satisfying $\mu(A_t) \geqslant \varepsilon$ for every $t \in T$. We need to find a strong subtree S of T of height $k+1$ such that

$$\mu\Big(\bigcap_{t\in S} A_t\Big) \geqslant \theta^{p(b,k+1)}. \tag{4.5}$$

We argue as follows. First we set

$$\mathcal{F} = \Big\{F \in \mathrm{Str}_2(T) : \mu\Big(\bigcap_{t\in F(1)} A_t \mid A_{F(0)}\Big) \geqslant \eta^{2^b-1}\Big\}. \tag{4.6}$$

By Corollary 2.2 and the estimate in (4.4), there exists a strong subtree R of T of height $n(k)$ such that either $\mathrm{Str}_2(R) \subseteq \mathcal{F}$ or $\mathrm{Str}_2(R) \cap \mathcal{F} = \varnothing$. By the choice of $n(k)$ made in (4.2), we have $n(k) \geqslant \mathrm{Rel}(b,\eta,\varepsilon)$. It follows that $\mathrm{Str}_2(R) \subseteq \mathcal{F}$.

Let $\{r_0 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} r_{b-1}\}$ be the lexicographical increasing enumeration of the 1-level $R(1)$ of R. Since the height of R is $n(k)$, for every $i \in \{0,\dots,b-1\}$ we have that $\mathrm{Succ}_R(r_i)$ is a strong subtree of R of height $n(k)-1$. In particular, $\mathrm{Succ}_R(r_i)$ is a finite homogeneous tree with branching number b and of height $n(k)-1$. This observation permits us to consider the canonical isomorphism $\mathrm{I}\big(b^{<n(k)-1}, \mathrm{Succ}_R(r_i)\big)$ between $b^{<n(k)-1}$ and $\mathrm{Succ}_R(r_i)$. For notational simplicity we shall denote it by I_i.

We set

$$Y = A_{R(0)}. \tag{4.7}$$

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Also, for every $u \in b^{<n(k)-1}$ let

$$F_u = \{R(0)\} \cup \{\mathrm{I}_i(u) : i \in \{0, \dots, b-1\}\} \tag{4.8}$$

and define

$$B_u = \bigcap_{t \in F_u} A_t \in \Sigma. \tag{4.9}$$

Observe that $F_u \in \mathrm{Str}_2(R)$ with $F_u(1) = \{\mathrm{I}_0(u), \dots, \mathrm{I}_{b-1}(u)\}$ and $F_u(0) = R(0)$. Since $\mathrm{Str}_2(R) \subseteq \mathcal{F}$, we get that

$$\mu_Y(B_u) = \frac{\mu(B_u \cap A_{R(0)})}{\mu(A_{R(0)})} = \mu\bigg(\bigcap_{t \in F_u(1)} A_t \mid A_{F_u(0)}\bigg) \geqslant \eta^{2^b - 1}. \tag{4.10}$$

Moreover, by (4.2), we have $n(k) - 1 \geqslant \mathrm{Cor}(b, k, \theta^{2^b-1}, \eta^{2^b-1})$. Therefore, applying our inductive assumptions to the probability space '(Ω, Σ, μ_Y)' and the family of measurable events '$\{B_u : u \in b^{<n(k)-1}\}$', we may find a strong subtree U of $b^{<n(k)-1}$ of height k such that

$$\mu_Y\bigg(\bigcap_{u \in U} B_u\bigg) \geqslant \big(\theta^{2^b-1}\big)^{p(b,k)}. \tag{4.11}$$

We are now in a position to define the desired tree S. In particular, let

$$S = \{R(0)\} \cup \{\mathrm{I}_i(u) : u \in U \text{ and } i \in \{0, \dots, b-1\}\}. \tag{4.12}$$

It is easy to see that S is a strong subtree of T of height $k+1$ and with the same root as R. Moreover,

$$\begin{aligned} \mu\bigg(\bigcap_{t \in S} A_t\bigg) &\overset{(4.12)}{=} \mu\bigg(A_{R(0)} \cap \bigcap_{u \in U} \bigcap_{i=0}^{b-1} A_{\mathrm{I}_i(u)}\bigg) \overset{(4.8)}{=} \mu\bigg(\bigcap_{u \in U} \bigcap_{t \in F_u} A_t\bigg) \\ &\overset{(4.9)}{=} \mu\bigg(\bigcap_{u \in U} B_u\bigg) = \mu\bigg(A_{R(0)} \cap \bigcap_{u \in U} B_u\bigg) \\ &= \mu(A_{R(0)}) \cdot \mu_Y\bigg(\bigcap_{u \in U} B_u\bigg) \\ &\overset{(4.11)}{\geqslant} \varepsilon \cdot \theta^{(2^b-1)p(b,k)} \geqslant \theta^{1+(2^b-1)p(b,k)}. \end{aligned} \tag{4.13}$$

Finally, notice that $p(b,k) = \sum_{i=0}^{k-1} (2^b - 1)^i$. Therefore,

$$1 + (2^b - 1)p(b,k) = 1 + \sum_{i=1}^{k} (2^b - 1)^i = \sum_{i=0}^{k} (2^b - 1)^i = p(b, k+1). \tag{4.14}$$

Combining (4.13) and (4.14), we conclude that the estimate in (4.5) is satisfied for the tree S. This completes the proof of Claim 4.1, and as we have already indicated, the proof of Theorem 1.4 is also complete. □

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5. Proof of Theorem 1.1 and its consequences

This section is devoted to the proofs of Theorem 1.1 and Corollary 1.2 stated in the Introduction. We start with the following lemma, which is essentially a multi-dimensional version of Corollary 3.9.

Lemma 5.1. *Let $b \in \mathbb{N}$ with $b \geqslant 2$ and $\mathbf{T} = (T_1, \ldots, T_d)$ be a vector homogeneous tree such that $b_{T_i} = b$ for every $i \in \{1, \ldots, d\}$. Also, let $\{A_t : t \in T_i \text{ and } i \in \{1, \ldots, d\}\}$ be a family of measurable events in a probability space (Ω, Σ, μ) and $Y \in \Sigma$ with $\mu(Y) > 0$ such that, for every $(t_1, \ldots, t_d)$ in the level product of $\mathbf{T}$, we have*

$$\mu\Big(\bigcap_{i=1}^{d} A_{t_i} \mid Y\Big) \geqslant \varepsilon > 0. \tag{5.1}$$

Then, for every $0 < \theta < \varepsilon$ there exists a vector strong subtree $\mathbf{S}$ of $\mathbf{T}$ of infinite height such that, for every $(F_1, \ldots, F_d) \in \mathrm{Str}_2(\mathbf{S})$, we have

$$\mu\Big(\bigcap_{i=1}^{d} \bigcap_{t \in F_i(1)} A_t \mid Y \cap \bigcap_{i=1}^{d} A_{F_i(0)}\Big) \geqslant \theta^{2^b - 1}. \tag{5.2}$$

Proof. We fix $0 < \theta < \varepsilon$. Let $\mathcal{F}$ be the set of all vector strong subtrees of $\mathbf{T}$ of height 2 for which the estimate in (5.2) is satisfied for the fixed constant θ. By Theorem 2.1, there exists vector strong subtree $\mathbf{S} = (S_1, \ldots, S_d)$ of $\mathbf{T}$ of infinite height such that either $\mathrm{Str}_2(\mathbf{S}) \subseteq \mathcal{F}$ or $\mathrm{Str}_2(\mathbf{S}) \cap \mathcal{F} = \varnothing$. The proof will, of course, be complete once we show that $\mathrm{Str}_2(\mathbf{S}) \cap \mathcal{F} \neq \varnothing$.

To this end we argue as follows. For every $u \in b^{<\mathbb{N}}$ we set

$$B_u = \bigcap_{i=1}^{d} A_{\mathrm{I}_i(u)} \cap Y \in \Sigma, \tag{5.3}$$

where I_i stands for the canonical isomorphism $\mathrm{I}(b^{<\mathbb{N}}, S_i)$ between $b^{<\mathbb{N}}$ and S_i for every $i \in \{1, \ldots, d\}$. By (5.1), we have $\mu_Y(B_u) \geqslant \varepsilon$ for every $u \in b^{<\mathbb{N}}$. Therefore, by Corollary 3.9, there exist an integer $N \leqslant \mathrm{Rel}(b, \theta, \varepsilon)$ and $F \in \mathrm{Str}_2(b^{<N})$ such that

$$\mu_Y\Big(\bigcap_{u \in F(1)} B_u \mid B_{F(0)}\Big) \geqslant \theta^{2^b - 1}. \tag{5.4}$$

For every $i \in \{1, \ldots, d\}$ we set $F_i = \mathrm{I}_i(F)$. Notice that $(F_1, \ldots, F_d) \in \mathrm{Str}_2(\mathbf{S})$. Moreover,

$$\begin{aligned}
\mu\Big(\bigcap_{i=1}^{d} \bigcap_{t \in F_i(1)} A_t \mid Y \cap \bigcap_{i=1}^{d} A_{F_i(0)}\Big) &= \frac{\mu\big(\bigcap_{i=1}^{d} \bigcap_{t \in F_i} A_t \cap Y\big)}{\mu\big(Y \cap \bigcap_{i=1}^{d} A_{F_i(0)}\big)} = \frac{\mu\big(\bigcap_{u \in F} B_u\big)}{\mu(B_{F(0)})} \\
&= \frac{\mu_Y\big(\bigcap_{u \in F} B_u\big)}{\mu_Y(B_{F(0)})} = \mu_Y\Big(\bigcap_{u \in F(1)} B_u \mid B_{F(0)}\Big).
\end{aligned}$$

By (5.4) and the above equations, we conclude that $(F_1, \ldots, F_d) \in \mathrm{Str}_2(\mathbf{S}) \cap \mathcal{F}$ and the proof is complete. □

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The following lemma is the final step of the proof of Theorem 1.1. It shows that, under the assumptions of Theorem 1.1, we can control the joint probability of the events over all initially subtrees of an appropriately chosen strong subtree of T.

Lemma 5.2. *Let T be a homogeneous tree. Also, let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_t) \geqslant \varepsilon > 0$ for every $t \in T$. Then, for every $0 < \theta < \varepsilon$ there exists a strong subtree W of T of infinite height such that, for every $k \in \mathbb{N}$, we have*

$$\mu\Big(\bigcap_{t \in W \upharpoonright k} A_t\Big) \geqslant \theta^{p(b_T, k+1)}. \tag{5.5}$$

Proof. We fix $0 < \theta < \varepsilon$. Let us denote by b the branching number of T. We set

$$\alpha = 2^b - 1. \tag{5.6}$$

We select a sequence (δ_k) of reals in the interval $(0, 1)$ satisfying

$$\prod_{k \in \mathbb{N}} (1 - \delta_k) \geqslant \frac{\theta}{\varepsilon}. \tag{5.7}$$

Also, let (ε_k) be the sequence of positive reals defined by the rule

$$\begin{aligned} \varepsilon_0 &= \varepsilon, \\ \varepsilon_{k+1} &= \big(\varepsilon_k (1 - \delta_k)\big)^{\alpha}. \end{aligned} \tag{5.8}$$

We isolate, for future use, the following elementary fact. The proof is left to the interested reader.

Fact 5.3. *For every integer $k \geqslant 1$ we have*

$$\prod_{i=0}^{k} \varepsilon_i = \big(\varepsilon^{\sum_{j=0}^{k} \alpha^j}\big) \cdot \prod_{i=0}^{k-1} \big((1 - \delta_i)^{\sum_{j=1}^{k-i} \alpha^j}\big). \tag{5.9}$$

Recursively we will select a sequence (R_k) of strong subtrees of T of infinite height such that, for every $k \in \mathbb{N}$, the following conditions are satisfied.

(C1) The tree R_{k+1} is a strong subtree of R_k.

(C2) We have $R_{k+1} \upharpoonright k = R_k \upharpoonright k$.

(C3) We have $\mu\big(\bigcap_{t \in R_k \upharpoonright k} A_t\big) \geqslant \prod_{i=0}^{k} \varepsilon_i$.

(C4) If $\{r_1^k <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} r_{b^{k+1}}^k\}$ is the lexicographical increasing enumeration of the $(k+1)$-level $R_k(k+1)$ of R_k, then for every $(t_1, \ldots, t_{b^{k+1}})$ in the level product of

$$\big(\mathrm{Succ}_{R_k}(r_1^k), \ldots, \mathrm{Succ}_{R_k}(r_{b^{k+1}}^k)\big)$$

we have

$$\mu\Big(\bigcap_{i=1}^{b^{k+1}} A_{t_i} \mid \bigcap_{t \in R_k \upharpoonright k} A_t\Big) \geqslant \varepsilon_{k+1}. \tag{5.10}$$

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The recursive selection is somewhat lengthy, and so we will briefly comment on it for the benefit of the reader. Conditions (C1) and (C2) are natural and quite common in constructions of this sort. We are mainly interested in condition (C3). It will be used, later on, to complete the proof of the lemma. Condition (C4) is a technical one. It will be used to show that the recursive selection can be carried out.

We proceed to the details. For $k = 0$ we apply Lemma 5.1 for '$\mathbf{T} = (T)$', '$Y = \Omega$' and '$\theta = \varepsilon(1 - \delta_0)$' and we find a strong subtree S of T of infinite height such that, for every $F \in \mathrm{Str}_2(S)$, we have

$$\mu\Big(\bigcap_{t\in F(1)} A_t \mid A_{F(0)}\Big) \geqslant \big(\varepsilon(1-\delta_0)\big)^{\alpha}. \tag{5.11}$$

We set '$R_0 = S$' and we observe that with this choice condition (C3) is satisfied. To see that condition (C4) is satisfied, let $\{r^0_1 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} r^0_b\}$ be the lexicographical increasing enumeration of $R_0(1)$, and fix an element $(t_1, \dots, t_b)$ in the level product of $\big(\mathrm{Succ}_{R_0}(r^0_1), \dots, \mathrm{Succ}_{R_0}(r^0_b)\big)$. We set $F = \{R_0(0)\} \cup \{t_1, \dots, t_b\}$ and we notice that $F \in \mathrm{Str}_2(R_0) = \mathrm{Str}_2(S)$, $F(0) = R_0(0)$ and $F(1) = \{t_1, \dots, t_b\}$. By (5.11) and taking into account the previous observations and the choice of ε_1 made in (5.8), we conclude that condition (C4) is also satisfied. Since conditions (C1) and (C2) are meaningless in this case, the first step of the recursive selection is complete.

Let $k \in \mathbb{N}$ and assume that the recursive selection has been carried out up to k so that conditions (C1)–(C4) are satisfied. Let $\{r^k_1 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} r^k_{b^{k+1}}\}$ be the lexicographical increasing enumeration of $R_k(k+1)$. Notice that conditions (C3) and (C4) allow us to apply Lemma 5.1 for '$(T_1, \dots, T_d) = \big(\mathrm{Succ}_{R_k}(r^k_1), \dots, \mathrm{Succ}_{R_k}(r^k_{b^{k+1}})\big)$', '$Y = \bigcap_{t\in R_k\upharpoonright k} A_t$', '$\varepsilon = \varepsilon_{k+1}$' and '$\theta = \varepsilon_{k+1}(1 - \delta_{k+1})$'. Hence, there exists a vector strong subtree $\mathbf{S} = (S_1, \dots, S_{b^{k+1}})$ of $\big(\mathrm{Succ}_{R_k}(r^k_1), \dots, \mathrm{Succ}_{R_k}(r^k_{b^{k+1}})\big)$ of infinite height such that for every $(F_1, \dots, F_{b^{k+1}}) \in \mathrm{Str}_2(\mathbf{S})$ we have

$$\mu\Big(\bigcap_{i=1}^{b^{k+1}} \bigcap_{t\in F_i(1)} A_t \mid \bigcap_{t\in R_k\upharpoonright k} A_t \cap \bigcap_{i=1}^{b^{k+1}} A_{F_i(0)}\Big) \geqslant \big(\varepsilon_{k+1}(1-\delta_{k+1})\big)^{\alpha}. \tag{5.12}$$

We set

$$R_{k+1} = (R_k \upharpoonright k) \cup \bigcup_{i=1}^{b^{k+1}} S_i, \tag{5.13}$$

and we claim that with this choice conditions (C1)–(C4) are satisfied. Indeed, it is clear that R_{k+1} is a strong subtree of R_k and $R_{k+1} \upharpoonright k = R_k \upharpoonright k$. Thus, conditions (C1) and (C2) are satisfied. To see that condition (C3) is satisfied, notice first that

$$R_{k+1}(k+1) = \{S_1(0) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} S_{b^{k+1}}(0)\}. \tag{5.14}$$

Since $(S_1, \dots, S_{b^{k+1}})$ is a vector strong subtree of $\big(\mathrm{Succ}_{R_k}(r^k_1), \dots, \mathrm{Succ}_{R_k}(r^k_{b^{k+1}})\big)$, we see that $\big(S_1(0), \dots, S_{b^{k+1}}(0)\big)$ is an element of the level product of the vector tree

$$\big(\mathrm{Succ}_{R_k}(r^k_1), \dots, \mathrm{Succ}_{R_k}(r^k_{b^{k+1}})\big).$$

Therefore, by condition (C4) of our inductive assumptions and (5.14), we get

$$\mu\Big(\bigcap_{t\in R_{k+1}(k+1)} A_t \mid \bigcap_{t\in R_k\upharpoonright k} A_t\Big) \geqslant \varepsilon_{k+1}. \tag{5.15}$$

Since $R_{k+1}\upharpoonright k = R_k\upharpoonright k$, we also have that

$$\mu\Big(\bigcap_{t\in R_{k+1}\upharpoonright k+1} A_t\Big) = \mu\Big(\bigcap_{t\in R_k\upharpoonright k} A_t\Big)\cdot\mu\Big(\bigcap_{t\in R_{k+1}(k+1)} A_t \mid \bigcap_{t\in R_k\upharpoonright k} A_t\Big). \tag{5.16}$$

Therefore, by (5.16), (5.15) and condition (C3) of our inductive assumptions,

$$\mu\Big(\bigcap_{t\in R_{k+1}\upharpoonright k+1} A_t\Big) \geqslant \Big(\prod_{i=0}^{k}\varepsilon_i\Big)\cdot\varepsilon_{k+1} = \prod_{i=0}^{k+1}\varepsilon_i. \tag{5.17}$$

Thus, condition (C3) is satisfied for the tree R_{k+1}. So, what remains is to check that condition (C4) is also satisfied. To this end let $\{r_1^{k+1} <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} r_{b^{k+2}}^{k+1}\}$ be the lexicographical increasing enumeration of the $(k+2)$-level $R_{k+1}(k+2)$ of the tree R_{k+1}. Also, let $(t_1,\dots,t_{b^{k+2}})$ be an arbitrary element of the level product of $\big(\mathrm{Succ}_{R_{k+1}}(r_1^{k+1}),\dots,\mathrm{Succ}_{R_{k+1}}(r_{b^{k+2}}^{k+1})\big)$. For every $i\in\{1,\dots,b^{k+1}\}$ we define

$$F_i = \{S_i(0)\}\cup\{t_{(i-1)b+j} : j\in\{1,\dots,b\}\}. \tag{5.18}$$

Notice that $(F_1,\dots,F_{b^{k+1}})\in \mathrm{Str}_2(\mathbf{S})$. Moreover, for every $i\in\{1,\dots,b^{k+1}\}$,

$$F_i(0)=S_i(0) \quad\text{and}\quad F_i(1)=\{t_{(i-1)b+j} : j\in\{1,\dots,b\}\}. \tag{5.19}$$

By (5.19), we see that

$$\bigcap_{i=1}^{b^{k+1}}\bigcap_{t\in F_i(1)} A_t = \bigcap_{i=1}^{b^{k+2}} A_{t_i} \tag{5.20}$$

while by (5.19) and (5.14) and the fact that $R_k\upharpoonright k = R_{k+1}\upharpoonright k$, we have

$$\bigcap_{t\in R_k\upharpoonright k} A_t \cap \bigcap_{i=1}^{b^{k+1}} A_{F_i(0)} = \bigcap_{t\in R_{k+1}\upharpoonright (k+1)} A_t. \tag{5.21}$$

Since $(F_1,\dots,F_{b^{k+1}})\in\mathrm{Str}_2(\mathbf{S})$, by (5.12) and the identities isolated in (5.20) and (5.21), we conclude that

$$\mu\Big(\bigcap_{i=1}^{b^{k+2}} A_{t_i} \mid \bigcap_{t\in R_{k+1}\upharpoonright (k+1)} A_t\Big) \geqslant \big(\varepsilon_{k+1}(1-\delta_{k+1})\big)^{\alpha} \overset{(5.8)}{=} \varepsilon_{k+2}. \tag{5.22}$$

As $(t_1,\dots,t_{b^{k+2}})$ was arbitrary, we see that condition (C4) is satisfied. Hence, the recursive selection is complete.

We are now in a position to complete the proof of the lemma. We define

$$W = \bigcup_{k\in\mathbb{N}} R_k(k). \tag{5.23}$$

By conditions (C1) and (C2), we see that W is a strong subtree of T of infinite height. It suffices to show that the estimate in (5.5) holds for every $k\in\mathbb{N}$. If $k=0$, then this is

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straightforward. So, let $k \in \mathbb{N}$ with $k \geqslant 1$ and observe that $W \upharpoonright k = R_k \upharpoonright k$. Therefore,

$$\begin{aligned}
\mu\Big(\bigcap_{t \in W \upharpoonright k} A_t\Big) &= \mu\Big(\bigcap_{t \in R_k \upharpoonright k} A_t\Big) \overset{(C3)}{\geqslant} \prod_{i=0}^{k} \varepsilon_i \\
&\overset{(5.9)}{=} \big(\varepsilon^{\sum_{j=0}^{k} \alpha^j}\big) \cdot \prod_{i=0}^{k-1} \big((1-\delta_i)^{\sum_{j=1}^{k-i} \alpha^j}\big) \\
&\geqslant \big(\varepsilon^{\sum_{j=0}^{k} \alpha^j}\big) \cdot \Big(\prod_{i=0}^{k-1} (1-\delta_i)\Big)^{\sum_{j=0}^{k} \alpha^j} \\
&\geqslant \Big(\varepsilon \cdot \prod_{i \in \mathbb{N}} (1-\delta_i)\Big)^{\sum_{j=0}^{k} \alpha^j} \\
&\overset{(5.7)}{\geqslant} \Big(\varepsilon \cdot \frac{\theta}{\varepsilon}\Big)^{\sum_{j=0}^{k} \alpha^j} = \theta^{\sum_{j=0}^{k} \alpha^k} = \theta^{p(b,k+1)}.
\end{aligned}$$

The proof of Lemma 5.2 is thus complete. □

We are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We fix $0 < \theta \leqslant \varepsilon$. Let

$$\mathcal{C} = \Big\{ W \in \mathrm{Str}_\infty(T) : \mu\Big(\bigcap_{t \in W \upharpoonright k} A_t\Big) \geqslant \theta^{p(b,k+1)} \text{ for every } k \in \mathbb{N} \Big\}.$$

It is easy to see that $\mathcal{C}$ is a closed subset of $\mathrm{Str}_\infty(T)$. Therefore, by Theorem 2.1 and Lemma 5.2, there exists a strong subtree S of T of infinite height such that $\mathrm{Str}_\infty(S) \subseteq \mathcal{C}$. The strong subtree S is the desired one. Indeed, let $k \in \mathbb{N}$ with $k \geqslant 1$ and R be an arbitrary strong subtree of S of height k. There exists a strong subtree W of S of infinite height such that $R = W \upharpoonright (k-1)$. Since $W \in \mathrm{Str}_\infty(S) \subseteq \mathcal{C}$ we see that

$$\mu\Big(\bigcap_{t \in R} A_t\Big) = \mu\Big(\bigcap_{t \in W \upharpoonright (k-1)} A_t\Big) \geqslant \theta^{p(b,k)},$$

as desired. □

We proceed to the proof of Corollary 1.2.

Proof of Corollary 1.2. Follows by Theorem 1.1 and Corollary B.2 in Appendix B. □

6. Free sets

This section is organized as follows. In Section 6.1 we introduce the class of free subsets of homogeneous trees and we present some of their properties. In Section 6.2 we give the proof of Proposition 1.6. Finally, in Section 6.3 we give the proof of Theorem 1.3.

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6.1. Definition and basic properties

We start with the following.

Definition 6.1. Let T be a homogeneous tree. Recursively, for every integer $k \geqslant 1$ we define a family $\mathrm{Fr}_k(T)$ of finite subsets of T as follows. First, let $\mathrm{Fr}_1(T)$ and $\mathrm{Fr}_2(T)$ consist of all singletons and all doubletons of T respectively. Let $k \in \mathbb{N}$ with $k \geqslant 2$ and assume that the family $\mathrm{Fr}_k(T)$ has been defined. Then $\mathrm{Fr}_{k+1}(T)$ consists of all subsets of T which can be written in the form $\{t\} \cup G$, where $t \in T$ and $G \in \mathrm{Fr}_k(T)$ are such that $\ell_T(t) < \ell_T(\wedge G)$. We set

$$\mathrm{Fr}(T) = \bigcup_{k \geqslant 1} \mathrm{Fr}_k(T). \tag{6.1}$$

An element of $\mathrm{Fr}(T)$ will be called a *free* subset of T.

We have the following characterization of free sets. The proof is straightforward.

Fact 6.2. *Let T be a homogeneous tree and $k \in \mathbb{N}$ with $k \geqslant 3$. Also, let F be a subset of T of cardinality k. Then F is free if and only if there exists an enumeration $\{t_1, \dots, t_k\}$ of F such that*

(a) $\ell_T(t_1) < \cdots < \ell_T(t_{k-1}) \leqslant \ell_T(t_k)$, *and*
(b) $\ell_T(t_m) < \ell_T\big(\wedge\{t_{m+1}, \dots, t_k\}\big)$ *for every* $m \in \{1, \dots, k-2\}$.

Using Fact 6.2 it is easily seen that the class of free sets includes various well-known classes of finite subsets of homogeneous trees; for instance, all finite chains are free, as well as the class of 'combs' studied in [15, §6.4]. Moreover, we have the following.

Lemma 6.3. *Every infinite subset A of a homogeneous tree T contains an infinite subset B such that every non-empty finite subset of B is free.*

Proof. Recursively, it is possible to select a sequence (t_n) in A such that, for every $m \in \mathbb{N}$ and every non-empty finite subset F of $\mathbb{N}$ with $m < \min F$, we have that $\ell_T(t_m) < \ell_T\big(\wedge\{t_n : n \in F\}\big)$. We set $B = \{t_n : n \in \mathbb{N}\}$. By Fact 6.2, we see that every non-empty finite subset of B is free, as desired. □

Finally, we isolate below some elementary properties of all free subsets of a homogeneous tree T.

($\mathcal{P}1$) If $F \in \mathrm{Fr}_k(T)$, then F has cardinality k.
($\mathcal{P}2$) If $F \in \mathrm{Fr}(T)$ and G is a non-empty subset of F, then $G \in \mathrm{Fr}(T)$.
($\mathcal{P}3$) If $S \in \mathrm{Str}_\infty(T)$ and $F \subseteq S$, then $F \in \mathrm{Fr}(T)$ if and only if $F \in \mathrm{Fr}(S)$.

Properties ($\mathcal{P}1$) and ($\mathcal{P}2$) are immediate consequences of Definition 6.1. Property ($\mathcal{P}3$) follows from the fact that strong subtrees preserve infima.

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6.2. Proof of Proposition 1.6

For the proof of Proposition 1.6 we need to do some preparatory work, which is of independent interest. To motivate the reader let us point out that, by Corollary B.2 in Appendix B, every doubleton of a homogeneous tree is contained in a strong subtree of height 3. The first step in the proof of Proposition 1.6 is to analyse how this embedding is achieved. As a consequence of this analysis and Theorem 2.1, the set of all doubletons of a homogeneous tree will be categorized in a finite list of classes, each of which is partition-regular. This information will be used, later on, to complete the proof of Proposition 1.6.

We proceed to the details. In what follows, T will be a homogeneous tree.

Doubletons of type I

Let $p \in \{0, \dots, b_T - 1\}$, and for every $F \in \mathrm{Str}_3(T)$ we set

$$F[p] = \{F(0), F(0)^{\frown_F} p\}. \tag{6.2}$$

We say that a doubleton of T is of *type* I *with parameter* (p) if it is of the form $F[p]$ for some $F \in \mathrm{Str}_3(T)$. We set

$$\mathcal{D}_{(p)}(T) = \{F[p] : F \in \mathrm{Str}_3(T)\}. \tag{6.3}$$

Doubletons of type II

Let $p, q \in \{0, \dots, b_T - 1\}$ with $p \neq q$, and for every $F \in \mathrm{Str}_3(T)$ we set

$$F[p,q] = \{F(0)^{\frown_F} p, F(0)^{\frown_F} q\}. \tag{6.4}$$

We say that a doubleton of T is of *type* II *with parameters* (p, q) if it is of the form $F[p, q]$ for some $F \in \mathrm{Str}_3(T)$. As above, we set

$$\mathcal{D}_{(p,q)}(T) = \{F[p,q] : F \in \mathrm{Str}_3(T)\}. \tag{6.5}$$

Doubletons of type III

Let $p, q, r \in \{0, \dots, b_T - 1\}$ with $p \neq q$. For every $F \in \mathrm{Str}_3(T)$ we set

$$F[p,q,r] = \{F(0)^{\frown_F} p, (F(0)^{\frown_F} q)^{\frown_F} r\}. \tag{6.6}$$

We say that a doubleton of T is of *type* III *with parameters* (p, q, r) if it is of the form $F[p, q, r]$ for some $F \in \mathrm{Str}_3(T)$, and we set

$$\mathcal{D}_{(p,q,r)}(T) = \{F[p,q,r] : F \in \mathrm{Str}_3(T)\}. \tag{6.7}$$

Observe that for every $p \in \{0, \dots, b_T - 1\}$ the class $\mathcal{D}_{(p)}(T)$ is hereditary when passing to strong subtrees; that is, if $S \in \mathrm{Str}_\infty(T)$, then $\mathcal{D}_{(p)}(S) \subseteq \mathcal{D}_{(p)}(T)$. Also, notice that, by Theorem 2.1, for every finite colouring of the set $\mathcal{D}_{(p)}(T)$ there exists $S \in \mathrm{Str}_\infty(T)$ such that the set $\mathcal{D}_{(p)}(S)$ is monochromatic. Of course, these properties are also shared by the classes $\mathcal{D}_{(p,q)}(T)$ and $\mathcal{D}_{(p,q,r)}(T)$. Moreover, we have the following.

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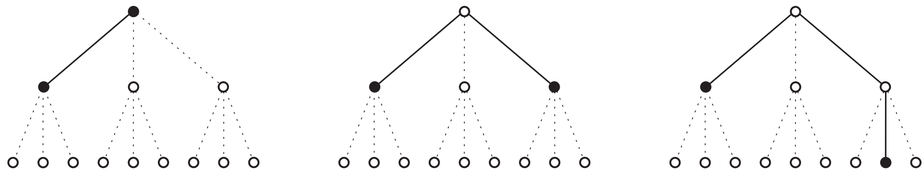

Figure 2. Doubletons in $3^{<\mathbb{N}}$ with parameters (0), (0,2) and (0,2,1).

Fact 6.4. *Every doubleton of a homogeneous tree T is either of type* I*, or of type* II*, or of type* III.

Proof. Let $s,t \in T$ with $s \neq t$ be arbitrary. We may assume that $\ell_T(s) \leqslant \ell_T(t)$. We set $w = s \wedge t$ and we consider the following cases.

Case 1: $\ell_T(w) = \ell_T(s)$. In this case we see that $s = s \wedge t$. Since $s \neq t$, there exists $p \in \{0,\dots,b_T - 1\}$ such that $t \in \mathrm{Succ}_T(s^\frown{}_T p)$. Therefore, it is possible to select $F \in \mathrm{Str}_3(T)$ such that $F(0) = s$ and $F(0)^\frown{}_F p = t$. So in this case the doubleton $\{s,t\}$ is of type I with parameter (p).

Case 2: $\ell_T(w) < \ell_T(s)$ *and* $\ell_T(s) = \ell_T(t)$. There exist $p,q \in \{0,\dots,b_T - 1\}$ such that $s \in \mathrm{Succ}_T(w^\frown{}_T p)$ and $t \in \mathrm{Succ}_T(w^\frown{}_T q)$. Observe that $p \neq q$. It is then possible to select $F \in \mathrm{Str}_3(T)$ such that $F(0) = w$, $F(0)^\frown{}_F p = s$ and $F(0)^\frown{}_F q = t$. Therefore, in this case the doubleton $\{s,t\}$ is of type II with parameters (p,q).

Case 3: $\ell_T(w) < \ell_T(s) < \ell_T(t)$. Notice first that there exist $p,q \in \{0,\dots,b_T - 1\}$ with $p \neq q$ such that $s \in \mathrm{Succ}_T(w^\frown{}_T p)$ and $t \in \mathrm{Succ}_T(w^\frown{}_T q)$. Since $\ell_T(s) < \ell_T(t)$, there exist $t' \in \mathrm{Succ}_T(w^\frown{}_T q)$ and $r \in \{0,\dots,b_T - 1\}$ such that $\ell_T(t') = \ell_T(s)$ and $t \in \mathrm{Succ}_T(t'^\frown{}_T r)$. Hence, we may select $F \in \mathrm{Str}_3(T)$ such that $F(0) = w$, $F(0)^\frown{}_F p = s$, $F(0)^\frown{}_F q = t'$ and $(F(0)^\frown{}_F q)^\frown{}_F r = t$. It follows that the doubleton $\{s,t\}$ is of type III with parameters (p,q,r). The proof is complete. □

We are now ready to proceed to the proof of Proposition 1.6.

Proof of Proposition 1.6. We fix $0 < \theta < \varepsilon$. Let $p,q,r \in \{0,\dots,b_T - 1\}$ with $p \neq q$ be arbitrary. We set

$$\mathcal{F}_{\mathrm{I}} = \Big\{F \in \mathrm{Str}_3(T) : \mu\Big(\bigcap_{t\in F[p]} A_t\Big) \geqslant \theta^2\Big\}, \tag{6.8}$$

$$\mathcal{F}_{\mathrm{II}} = \Big\{F \in \mathrm{Str}_3(T) : \mu\Big(\bigcap_{t\in F[p,q]} A_t\Big) \geqslant \theta^2\Big\} \tag{6.9}$$

and

$$\mathcal{F}_{\mathrm{III}} = \Big\{F \in \mathrm{Str}_3(T) : \mu\Big(\bigcap_{t\in F[p,q,r]} A_t\Big) \geqslant \theta^2\Big\}. \tag{6.10}$$

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By Theorem 2.1, there exists $S \in \mathrm{Str}_\infty(T)$ such that, for every $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, we have that either $\mathrm{Str}_3(S) \subseteq \mathcal{F}_i$ or $\mathrm{Str}_3(S) \cap \mathcal{F}_i = \varnothing$. Therefore, by Fact 6.4, the proof will be complete once we show that $\mathrm{Str}_3(S) \cap \mathcal{F}_i \neq \varnothing$ for every $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. The argument below is not uniform and depends on the type of doubletons we are dealing with. We set $N = \lceil (\varepsilon^2 - \theta^2)^{-1} \rceil$. Notice that we may (and we will) assume that S is the tree $b_T^{<\mathbb{N}}$.

Case 1: *type* I *doubletons.* We set $t_k = p^k \in b_T^{<\mathbb{N}}$ for every $k \in \{0, \dots, N-1\}$. By our assumptions, Lemma 2.4 can be applied to the family $(A_{t_k})_{k=0}^{N-1}$ and the fixed constant θ. Hence, there exist $0 \leqslant k_0 < k_1 < N$ such that $\mu(A_{t_{k_0}} \cap A_{t_{k_1}}) \geqslant \theta^2$. We select $F \in \mathrm{Str}_3(b_T^{<\mathbb{N}})$ such that

$$F \upharpoonright 1 = \{p^{k_0}\} \cup \{p^{k_0} j^{k_1 - k_0} : j \in \{0, \dots, b_T - 1\}\}.$$

Since $F[p] = \{p^{k_0}, p^{k_1}\} = \{t_{k_0}, t_{k_1}\}$, we conclude that $F \in \mathrm{Str}_3(b_T^{<\mathbb{N}}) \cap \mathcal{F}_{\mathrm{I}}$.

Case 2: *type* II *doubletons.* In this case we set $s_k = q^k p^{N-1-k} \in b_T^{<\mathbb{N}}$ for every $k \in \{0, \dots, N-1\}$. By Lemma 2.4, there exist $0 \leqslant k_0 < k_1 < N$ such that $\mu(A_{s_{k_0}} \cap A_{s_{k_1}}) \geqslant \theta^2$. We select $G \in \mathrm{Str}_3(b_T^{<\mathbb{N}})$ such that

$$G \upharpoonright 1 = \{q^{k_0}\} \cup \{q^{k_0} j^{k_1 - k_0} p^{N-1-k_1} : j \in \{0, \dots, b_T - 1\}\}.$$

Observe that $G[p, q] = \{q^{k_0} p^{N-1-k_0}, q^{k_1} p^{N-1-k_1}\} = \{s_{k_0}, s_{k_1}\}$. It follows that

$$G \in \mathrm{Str}_3(b_T^{<\mathbb{N}}) \cap \mathcal{F}_{\mathrm{II}}.$$

Case 3: *type* III *doubletons.* We set $w_k = (qr)^k p \in b_T^{<\mathbb{N}}$ for every $k \in \{0, \dots, N-1\}$, where $(qr)^k$ stands for the k-times concatenation of (qr) if $k \geqslant 1$ and $(qr)^0 = \varnothing$. Arguing as above, we find $0 \leqslant k_0 < k_1 < N$ such that $\mu(A_{w_{k_0}} \cap A_{w_{k_1}}) \geqslant \theta^2$. Let

$$H = \{(qr)^{k_0}\} \cup \{(qr)^{k_0} j : j \in \{0, \dots, b_T - 1\}\} \cup \\ \{(qr)^{k_0} j v (qr)^{k_1 - k_0 - 1} p : j, v \in \{0, \dots, b_T - 1\}\}$$

Notice that $H \in \mathrm{Str}_3(b_T^{<\mathbb{N}})$ and $H[p, q, r] = \{(qr)^{k_0} p, (qr)^{k_1} p\} = \{w_{k_0}, w_{k_1}\}$. Hence, $H \in \mathrm{Str}_3(b_T^{<\mathbb{N}}) \cap \mathcal{F}_{\mathrm{III}}$. The proof is complete. □

We close this subsection with the following consequence of Proposition 1.6. It is the analogue of Corollary 3.9 and it will be used in the proof of Theorem 1.3.

Corollary 6.5. *Let T be a homogeneous tree. Also, let $\{A_t : t \in T\}$ be a family of measurable events in a probability space (Ω, Σ, μ) and $Y \in \Sigma$ with $\mu(Y) > 0$ such that $\mu(A_t \mid Y) \geqslant \varepsilon > 0$ for every $t \in T$. Then, for every $0 < \theta < \varepsilon$ there exists $S \in \mathrm{Str}_\infty(T)$ such that for every $s, t \in S$ we have $\mu(A_t \mid Y \cap A_s) \geqslant \theta$.*

Proof. We fix $0 < \theta < \varepsilon$ and we set

$$\lambda = \left(\varepsilon \cdot \theta^{-1}\right)^{\frac{1}{3}}. \tag{6.11}$$

Notice that $\lambda > 1$. Also, let

$$r = \left\lceil \frac{\ln \varepsilon^{-1}}{\ln \lambda} \right\rceil. \tag{6.12}$$

By Theorem 2.1 and the choice of r, there exist $R \in \mathrm{Str}_\infty(T)$ and $i_0 \in \{1,\dots,r\}$ such that for every $t \in R$ we have

$$\varepsilon\lambda^{i_0-1} \leqslant \mu_Y(A_t) \leqslant \varepsilon\lambda^{i_0}. \tag{6.13}$$

By Proposition 1.6 applied for '$\theta = \varepsilon\lambda^{i_0-2}$', '$\varepsilon = \varepsilon\lambda^{i_0-1}$', the family '$\{A_t : t \in R\}$' and the probability space '(Ω, Σ, μ_Y)', there exists $S \in \mathrm{Str}_\infty(R)$ such that

$$\mu_Y(A_t \cap A_s) \geqslant \varepsilon^2\lambda^{2i_0-4}. \tag{6.14}$$

for every $s,t \in S$. By (6.13) and (6.14) and taking into account that $\lambda > 1$ and $i_0 \geqslant 1$, we conclude that

$$\begin{aligned}\mu(A_t \mid Y \cap A_s) &= \frac{\mu(A_t \cap Y \cap A_s)}{\mu(Y \cap A_s)} = \frac{\mu_Y(A_t \cap A_s)}{\mu_Y(A_s)} \geqslant \frac{\varepsilon^2\lambda^{2i_0-4}}{\varepsilon\lambda^{i_0}} \\ &= \varepsilon\lambda^{i_0-4} \geqslant \varepsilon\lambda^{-3} \overset{(6.11)}{=} \theta\end{aligned}$$

for every $s,t \in S$. The proof is complete. □

6.3. Proof of Theorem 1.3

Throughout the proof we will use the following notation. For every tree U and every finite subset F of U, we set

$$\mathrm{depth}_U(F) = \begin{cases} \min\{n \in \mathbb{N} : F \subseteq U \restriction n\} & \text{if } F \text{ is non-empty,} \\ -1 & \text{otherwise.}\end{cases}$$

The quantity $\mathrm{depth}_U(F)$ is called the *depth* of F in U (see, *e.g.*, [15]).

Now, fix $0 < \theta < \varepsilon \leqslant 1$. We select a sequence (δ_n) in $(0,1)$ such that

$$\prod_{n \in \mathbb{N}} (1-\delta_n) \geqslant \frac{\theta}{\varepsilon}. \tag{6.15}$$

Let (ε_n) be the sequence of positive reals defined recursively by the rule

$$\begin{aligned}\varepsilon_0 &= \varepsilon, \\ \varepsilon_{n+1} &= \varepsilon_n(1-\delta_n).\end{aligned} \tag{6.16}$$

Notice that the sequence (ε_n) is strictly decreasing. Moreover, it is easy to see that for every integer $n \geqslant 1$ we have

$$\prod_{i=0}^{n} \varepsilon_i = \varepsilon^{n+1} \cdot \left(\prod_{i=0}^{n-1} (1-\delta_i)^{n-i}\right). \tag{6.17}$$

Recursively, we will select a sequence (R_n) of strong subtrees of T of infinite height such that for every $n \in \mathbb{N}$ the following conditions are satisfied.

(C1) The tree R_{n+1} is a strong subtree of R_n.
(C2) We have $R_{n+1} \restriction n = R_n \restriction n$.

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(C3) For every finite subset F of R_n with $\mathrm{depth}_{R_n}(F) \leqslant n-1$ and every $t \in R_n$ with $n \leqslant \ell_{R_n}(t)$, if $F \cup \{t\} \in \mathrm{Fr}(R_n)$ then

$$\mu\Big(\bigcap_{w\in F\cup\{t\}} A_w\Big) \geqslant \prod_{i=0}^{|F|} \varepsilon_i. \tag{6.18}$$

(C4) For every finite subset F of R_n with $\mathrm{depth}_{R_n}(F) \leqslant n-1$ and every $s,t \in R_n$ with $s \neq t$ and $n \leqslant \min\{\ell_{R_n}(s), \ell_{R_n}(t)\}$, if $F \cup \{s,t\} \in \mathrm{Fr}(R_n)$ then

$$\mu\Big(A_t \mid \bigcap_{w\in F\cup\{s\}} A_w\Big) \geqslant \varepsilon_{|F|+1}. \tag{6.19}$$

As the reader might have already guessed, the above recursive selection is the main step of the proof of Theorem 1.3. We are mainly interested in conditions (C3) and (C4). The analytical information guaranteed by estimates (6.18) and (6.19) will be used later on to complete the proof of Theorem 1.3.

We proceed to the details. For $n = 0$ we apply Corollary 6.5 for '$Y = \Omega$' and '$\theta = \varepsilon_1$' and we get a strong subtree S of T of infinite height such that for every $s,t \in S$ we have $\mu(A_t \mid A_s) \geqslant \varepsilon_1$. We set '$R_0 = S$' and we observe that with this choice conditions (C3) and (C4) are satisfied. Since (C1) and (C2) are meaningless for $n = 0$, the first step of the recursive selection is complete.

Let $n \in \mathbb{N}$ and assume that we have selected the trees $R_0, \ldots, R_n$ so that conditions (C1)–(C4) are satisfied. We need to find the tree R_{n+1}. We start with the following fact.

Fact 6.6. *Let F be a non-empty finite subset of R_n with $\mathrm{depth}_{R_n}(F) \leqslant n$ and $t \in R_n$ with $n+1 \leqslant \ell_{R_n}(t)$. If $F \cup \{t\} \in \mathrm{Fr}(R_n)$, then the following hold.*

(i) *There exist $k \in \{0, \ldots, n\}$, a (possibly empty) subset G of R_k satisfying $\mathrm{depth}_{R_k}(G) \leqslant k-1$ and a node $s \in R_k$ with $k = \ell_{R_k}(s) < \ell_{R_k}(t)$ such that $F \cup \{t\} = G \cup \{s,t\}$.*

(ii) *We have*

$$\mu\Big(A_t \mid \bigcap_{w\in F} A_w\Big) \geqslant \varepsilon_{|F|}. \tag{6.20}$$

Proof. Part (i) follows by the definition of free sets and conditions (C1) and (C2) of the recursive selection. To see that part (ii) is also satisfied let k, G and s be as in part (i). By property ($\mathcal{P}3$) in Section 6.1 and our inductive assumptions, we have that $G \cup \{s,t\} \in \mathrm{Fr}(R_k)$. Therefore, by condition (C4) for the tree R_k applied for the set G and the doubleton $\{s,t\}$, we see that

$$\mu\Big(A_t \mid \bigcap_{w\in F} A_w\Big) = \mu\Big(A_t \mid \bigcap_{w\in G\cup\{s\}} A_w\Big) \geqslant \varepsilon_{|G|+1} = \varepsilon_{|F|}$$

and the proof is complete. □

The following consequence of Fact 6.6 shows that for the selection of the tree R_{n+1} we only have to worry about conditions (C1), (C2) and (C4).

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Corollary 6.7. *Let $W \in \mathrm{Str}_\infty(R_n)$ be such that $W \upharpoonright n = R_n \upharpoonright n$. Then condition* (C3) *is satisfied if we set $R_{n+1} = W$.*

Proof. Let F be a finite subset of W satisfying $\mathrm{depth}_W(F) \leqslant n$ and $t \in W$ with $n+1 \leqslant \ell_W(t)$, and assume that $F \cup \{t\} \in \mathrm{Fr}(W)$. If F is the empty set, then the estimate in (6.18) is straightforward. So we may assume that F is non-empty. Since $W \in \mathrm{Str}_\infty(R_n)$ and $W \upharpoonright n = R_n \upharpoonright n$, we see that

(a) F is a non-empty finite subset of R_n with $\mathrm{depth}_{R_n}(F) \leqslant n$,
(b) $n+1 \leqslant \ell_{R_n}(t)$, and
(c) $F \cup \{t\} \in \mathrm{Fr}(R_n)$.

By (a), (b) and (c) above and part (ii) of Fact 6.6, we have the estimate

$$\mu\Big(A_t \mid \bigcap_{w\in F} A_w\Big) \geqslant \varepsilon_{|F|}.$$

Also, let k, G and s be as in part (i) of Fact 6.6. Since $R_n \in \mathrm{Str}_\infty(R_k)$, by properties ($\mathcal{P}2$) and ($\mathcal{P}3$) in Section 6.1, we have $G \cup \{s\} \in \mathrm{Fr}(R_k)$. Hence, by condition (C3) for the tree R_k applied for the set G and the node s, we see that

$$\mu\Big(\bigcap_{w\in F} A_w\Big) = \mu\Big(\bigcap_{w\in G\cup\{s\}} A_w\Big) \geqslant \prod_{i=0}^{|G|} \varepsilon_i = \prod_{i=0}^{|F|-1} \varepsilon_i.$$

Noticing that

$$\mu\Big(\bigcap_{w\in F\cup\{t\}} A_w\Big) = \mu\Big(A_t \mid \bigcap_{w\in F} A_w\Big) \cdot \mu\Big(\bigcap_{w\in F} A_w\Big),$$

and combining the previous estimates we conclude that condition (C3) is satisfied if we set $R_{n+1} = W$. The proof of Corollary 6.7 is complete. □

We need one more preparatory step for the selection of the tree R_{n+1}.

Claim 6.8. *Let F be a finite subset of R_n such that $\mathrm{depth}_{R_n}(F) \leqslant n$. Also, let $U \in \mathrm{Str}_\infty(R_n)$ with $U \upharpoonright n = R_n \upharpoonright n$. Then there exists $W \in \mathrm{Str}_\infty(U)$ with the following properties.*

(P1) *We have $W \upharpoonright n = U \upharpoonright n$.*
(P2) *For every $s, t \in W$ with $s \neq t$ and such that $n+1 \leqslant \min\{\ell_W(s), \ell_W(t)\}$, if $F \cup \{s,t\} \in \mathrm{Fr}(W)$ then*

$$\mu\Big(A_t \mid \bigcap_{w\in F\cup\{s\}} A_w\Big) \geqslant \varepsilon_{|F|+1}. \tag{6.21}$$

Proof. Notice that we may assume that F is non-empty; indeed, for the empty set the result follows by condition (C3) for the tree R_0. Let $\{u_1 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} u_d\}$ be the lexicographical increasing enumeration of the $(n+1)$-level $U(n+1)$ of U (notice that $d = b_T^{n+1}$). Recursively, we will select a family $\{Z_j : j \in \{1, \dots, d\}\}$ of strong subtrees of T such that the following are satisfied.

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(A1) For every $j \in \{1,\dots,d\}$ we have $Z_j \in \mathrm{Str}_\infty\big(\mathrm{Succ}_U(u_j)\big)$.
(A2) For every $j \in \{1,\dots,d-1\}$ we have $L_T\big(Z_{j+1}\big) \subseteq L_T\big(Z_j\big)$.
(A3) If $j \in \{1,\dots,d\}$ is such that $F \cup \{u_j\} \in \mathrm{Fr}(U)$, then for every $s,t \in Z_j$ with $s \neq t$ we have

$$\mu\Bigg(A_t \mid \bigcap_{w \in F \cup \{s\}} A_w\Bigg) \geqslant \varepsilon_{|F|+1}.$$

As the first step is identical to the general one, let us assume that the selection has been carried out up to some $j \in \{1,\dots,d-1\}$ so that properties (A1)–(A3) are satisfied. Let Z be a strong subtree of $\mathrm{Succ}_U(u_{j+1})$ such that $L_T(Z) = L_T(Z_j)$; for the first step we simply set $Z = \mathrm{Succ}_U(u_1)$. We consider the following cases.

Case 1: *the set $F \cup \{u_{j+1}\}$ is not a free subset of U.* We set '$Z_{j+1} = Z$' and we observe that with this choice properties (A1)–(A3) are satisfied.

Case 2: *the set $F \cup \{u_{j+1}\}$ is a free subset of U.* In this case we see that for every $t \in \mathrm{Succ}_U(u_{j+1})$ the set $F \cup \{t\}$ is also a free subset of U. Since $U \in \mathrm{Str}_\infty(R_n)$, by part (ii) of Fact 6.6, for every $t \in \mathrm{Succ}_U(u_{j+1})$ we have

$$\mu\Bigg(A_t \mid \bigcap_{w \in F} A_w\Bigg) \geqslant \varepsilon_{|F|}.$$

We apply Corollary 6.5 for '$T = Z$', '$Y = \bigcap_{w\in F} A_w$', '$\varepsilon = \varepsilon_{|F|}$' and '$\theta = \varepsilon_{|F|+1}$', and we get $S \in \mathrm{Str}_\infty(Z)$ such that, for every $s,t \in S$, we have

$$\mu\Bigg(A_t \mid \bigcap_{w \in F \cup \{s\}} A_w\Bigg) = \mu(A_t \mid Y \cap A_s) \geqslant \varepsilon_{|F|+1}. \tag{6.22}$$

We set '$Z_{j+1} = S$', and we notice that with this choice properties (A1)–(A3) are satisfied. The recursive selection is thus complete.

Now, for every $j \in \{1,\dots,d-1\}$ we select a strong subtree W_j of Z_j with $L_T(W_j) = L_T(Z_d)$. We set $W_d = Z_d$ and we define

$$W = (U \upharpoonright n) \cup \bigcup_{j=1}^{d} W_j.$$

It is clear that $W \in \mathrm{Str}_\infty(U)$ and $W \upharpoonright n = U \upharpoonright n$. What remains is to show that property (P2) is satisfied for the tree W. To this end, let $s,t \in W$ with $s \neq t$ and $n+1 \leqslant \min\{\ell_W(s), \ell_W(t)\}$, and assume that $F \cup \{s,t\} \in \mathrm{Fr}(W)$. Since W is a strong subtree of U, we see that

$$\min\{\ell_U(s), \ell_U(t)\} \geqslant \min\{\ell_W(s), \ell_W(t)\} \geqslant n+1.$$

Therefore, by Fact 6.2, there exists $j_0 \in \{1,\dots,d\}$ such that $F \cup \{u_{j_0}\} \in \mathrm{Fr}(U)$ and $s,t \in \mathrm{Succ}_U(u_{j_0}) \cap W = W_{j_0} \subseteq Z_{j_0}$. Hence, by (A3) above, we conclude that property (P2) is satisfied. The proof of Claim 6.8 is complete. □

After this preliminary discussion we are ready to start the process for selecting the tree R_{n+1}. Let $\{F_1,\dots,F_m\}$ be an enumeration of the set of all subsets F of R_n with

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$\mathrm{depth}_{R_n}(F) \leqslant n$. By repeated applications of Claim 6.8, it is possible to construct a family $\{W_j : j \in \{1,\dots,m\}\}$ of strong subtrees of R_n with the following properties.

(a) For every $j \in \{1,\dots,m\}$ we have $W_j \upharpoonright n = R_n \upharpoonright n$.
(b) For every $j \in \{1,\dots,m-1\}$ the tree W_{j+1} is a strong subtree of W_j.
(c) For every $j \in \{1,\dots,m\}$ and every $s,t \in W_j$ with $s \neq t$ and such that $n+1 \leqslant \min\{\ell_{W_j}(s), \ell_{W_j}(t)\}$, if $F_j \cup \{s,t\} \in \mathrm{Fr}(W_j)$ then

$$\mu\Big(A_t \mid \bigcap_{w\in F_j\cup\{s\}} A_w\Big) \geqslant \varepsilon_{|F_j|+1}.$$

The construction is fairly standard and the details are left to the reader. We set '$R_{n+1} = W_m$'. By (a), (b) and (c) above, it is clear that with this choice conditions (C1), (C2) and (C4) are satisfied. On the other hand, by Corollary 6.7, condition (C3) is also satisfied. Therefore, the recursive selection is complete.

We are finally in a position to complete the proof of Theorem 1.3. We set

$$S = \bigcup_{n\in\mathbb{N}} R_n(n) \tag{6.23}$$

and we observe that $S \in \mathrm{Str}_\infty(T)$. We will show that S is the desired strong subtree. So let $G \in \mathrm{Fr}(S)$ be arbitrary. We need to prove that

$$\mu\Big(\bigcap_{t\in G} A_t\Big) \geqslant \theta^{|G|}.$$

To this end, clearly we may assume that $|G| \geqslant 2$. We will show, first, that

$$\mu\Big(\bigcap_{t\in G} A_t\Big) \geqslant \prod_{i=0}^{|G|-1} \varepsilon_i. \tag{6.24}$$

Indeed, by conditions (C1) and (C2) of the recursive selection and the choice of the tree S in (6.23), there exist $n \in \mathbb{N}$, a (possibly empty) subset F of R_n satisfying $\mathrm{depth}_{R_n}(F) \leqslant n-1$ and $s,t \in R_n$ with $s \neq t$ and $n = \ell_{R_n}(s) \leqslant \ell_{R_n}(t)$ such that $G = F \cup \{s,t\}$. Since $S \in \mathrm{Str}_\infty(R_n)$, by property ($\mathcal{P}3$) in Section 6.1, we see that $F \cup \{s,t\} \in \mathrm{Fr}(R_n)$. Therefore, by condition (C4) for the tree R_n applied for the set F and the doubleton $\{s,t\}$, we have

$$\mu\Big(A_t \mid \bigcap_{w\in F\cup\{s\}} A_w\Big) \geqslant \varepsilon_{|F|+1}. \tag{6.25}$$

By property ($\mathcal{P}2$) in Section 6.1, we have $F \cup \{s\} \in \mathrm{Fr}(R_n)$. Thus, by condition (C3),

$$\mu\Big(\bigcap_{w\in F\cup\{s\}} A_w\Big) \geqslant \prod_{i=0}^{|F|} \varepsilon_i. \tag{6.26}$$

Combining (6.25) and (6.26), we conclude that the estimate in (6.24) is satisfied. Therefore,

$$\begin{aligned}\mu\Big(\bigcap_{t\in G} A_t\Big) &\overset{(6.24)}{\geqslant} \prod_{i=0}^{|G|-1} \varepsilon_i \overset{(6.17)}{=} \varepsilon^{|G|}\cdot\Big(\prod_{i=0}^{|G|-2}(1-\delta_i)^{|G|-1-i}\Big)\\ &\geqslant \varepsilon^{|G|}\cdot\Big(\prod_{i=0}^{|G|-2}(1-\delta_i)\Big)^{|G|-1}\\ &\geqslant \varepsilon^{|G|}\cdot\Big(\prod_{i\in\mathbb{N}}(1-\delta_i)\Big)^{|G|-1}\\ &\overset{(6.15)}{\geqslant} \varepsilon^{|G|}\cdot\Big(\frac{\theta}{\varepsilon}\Big)^{|G|-1} \geqslant \varepsilon\cdot\theta^{|G|-1}\geqslant\theta^{|G|}.\end{aligned}$$

The proof of Theorem 1.3 is thus complete. □

Appendix A

We start by introducing some pieces of notation and terminology. For every finitely branching tree T and every $t\in T$, the *branching number* of t in T, denoted by $b_T(t)$, is defined to be cardinality of the set of all immediate successors of t in T. Next we introduce the following class of trees.

Definition A.1. Let (b_n) be a strictly increasing sequence of positive integers. A tree T will be called (b_n)-*large* if it is uniquely rooted, finitely branching and $b_T(t)\geqslant b_n$ for every $n\in\mathbb{N}$ and every $t\in T(n)$.

A tree T will be called *large* if it is (b_n)-large for some strictly increasing sequence (b_n) of positive integers.

Below we gather some elementary properties of large trees.

Fact A.2. *Let (b_n) be a strictly increasing sequence of positive integers and let T be a (b_n)-large tree. Then the following hold.*

(i) *If $S\in \mathrm{Str}_\infty(T)$, then S is (b_n)-large.*
(ii) *For every strictly increasing sequence (c_n) of positive integers there exists $S\in\mathrm{Str}_\infty(T)$ such that S is (c_n)-large.*

We have the following trichotomy.

Proposition A.3. *For every uniquely rooted, pruned and finitely branching tree T there exists a strong subtree S of T of infinite height such that either*

(i) *S is a chain, or*
(ii) *S is homogeneous, or*
(iii) *S is large.*

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Proof. Assume that neither (i) nor (ii) are satisfied. Recursively, and using the Halpern–Läuchli Theorem [7], we may select a sequence (R_n) of strong subtrees of T of infinite height such that, for every $n \in \mathbb{N}$, the following hold.

(C1) The tree R_{n+1} is a strong subtree of R_n.
(C2) We have $R_{n+1} \upharpoonright n = R \upharpoonright n$.
(C3) For every $t \in \bigcup_{m=n}^{\infty} R_n(m)$ we have $b_T(t) \geqslant n+1$.

The recursive selection is fairly standard and the details are left to the reader. Let

$$S = \bigcup_{n \in \mathbb{N}} R_n(n). \tag{A.1}$$

By conditions (C1) and (C2), we have that $S \in \mathrm{Str}_\infty(T)$. On the other hand, by condition (C3), we see that S is a (b_n)-large tree where $b_n = n+1$ for every $n \in \mathbb{N}$. The proof is complete. □

We remark that, by Proposition A.3, Theorem 1.1 still holds if the events are indexed by a uniquely rooted, pruned and boundedly branching tree.

On the other hand, if T is a uniquely rooted, pruned and finitely branching tree not containing a strong subtree of infinite height which is either a chain or homogeneous then, by Proposition A.3 and Fact A.2, for every strictly increasing sequence (b_n) of positive integers there exists a strong subtree of T which is (b_n)-large. Concerning this class of trees we have the following.

Proposition A.4. *Let* $0 < \delta < 1$. *Also, let* (b_n) *be a strictly increasing sequence of positive integers such that*

$$\sum_{n \in \mathbb{N}} \frac{1}{b_n} \leqslant \delta. \tag{A.2}$$

Then, for every (b_n)*-large tree* T *there exists a family* $\{A_t : t \in T\}$ *of Borel subsets of the interval* $[0,1]$ *satisfying* $\lambda(A_t) \geqslant 1-\delta$ *for every* $t \in T$, *and such that*

$$\bigcap_{t \in F} A_t = \varnothing \tag{A.3}$$

for every $F \in \mathrm{Str}_2(T)$.

Proof. We fix a (b_n)-large tree T. The family $\{A_t : t \in T\}$ will be defined by recursion on the length of nodes in T. For $n = 0$ we set $A_{T(0)} = [0,1]$. Let $n \in \mathbb{N}$ and assume that we have defined the family $\{A_t : t \in T(n)\}$. Let $t \in T(n)$ be arbitrary. We partition the set A_t into a family $\{\Delta_s : s \in \mathrm{ImmSucc}_T(t)\}$ of Borel sets of equal measure, and for every $s \in \mathrm{ImmSucc}_T(t)$ we set

$$A_s = A_t \setminus \Delta_s. \tag{A.4}$$

We notice two properties guaranteed by the above construction.

(P1) For every $t \in T$ and every $w \in \mathrm{Succ}_T(t)$ we have $A_w \subseteq A_t$.

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(P2) For every $n \in \mathbb{N}$, every $t \in T(n)$ and every $s \in \mathrm{ImmSucc}_T(t)$, we have $\lambda(A_s) = \lambda(A_t) \cdot \big(1 - b_T(t)^{-1}\big) \geqslant \lambda(A_t) - 1/b_n$.

Therefore, for every $t \in T$ we have

$$\lambda(A_t) \geqslant \lambda(A_{T(0)}) - \sum_{n\in\mathbb{N}} \frac{1}{b_n} \overset{\text{(A.2)}}{\geqslant} 1 - \delta.$$

Finally if $F \in \mathrm{Str}_2(T)$, then

$$\bigcap_{t\in F} A_t \subseteq \bigcap_{t\in F(1)} A_t \subseteq \bigcap_{s\in \mathrm{ImmSucc}_T(F(0))} A_s = \varnothing.$$

The proof is complete. □

Appendix B

Our goal is this appendix is to give the proof of the following result.

Proposition B.1. *Let $k \in \mathbb{N}$ with $k \geqslant 1$. Then, for every uniquely rooted and balanced tree T of height k and every non-empty finite subset F of T there exists a strong subtree S of T with $h(S) \leqslant \min\{k, 2|F| - 1\}$ such that $F \subseteq S$.* □

Since every homogeneous tree is uniquely rooted and balanced, by Proposition B.1 we get the following.

Corollary B.2. *Let T be a homogeneous tree and $n \in \mathbb{N}$ with $n \geqslant 1$. Then every subset F of T of cardinality n is contained in a strong subtree of T of height $2n - 1$.*

Before we give the proof of Proposition B.1 let us remark that the estimate on the height of the strong subtree obtained by Corollary B.2 is sharp.

Example 1. *For every integer $i \geqslant 1$ let $t_i = 0^{2i}1 \in 2^{<\mathbb{N}}$. Observe that for every pair of integers $1 \leqslant i < j$ we have $t_i \wedge t_j = 0^{2i}$. Now, fix an integer $n \geqslant 2$ and set $A_n = \{t_i : i \in \{1, \ldots, n\}\}$. Let S be an arbitrary strong subtree of $2^{<\mathbb{N}}$ with $A_n \subseteq S$. Notice, first, that the level set of S must contain the set $\{2i + 1 : i \in \{1, \ldots, n\}\}$. Since strong subtrees preserve infima, we see that $\{t_i \wedge t_{i+1} : i \in \{1, \ldots, n - 1\}\} \subseteq S$, and so, the level set of S must also contain the set $\{2i : i \in \{1, \ldots, n - 1\}\}$. Therefore, the height of S is at least $2n - 1$.*

We proceed to the proof of Proposition B.1.

Proof of Proposition B.1. The result will be proved by induction on k. The case $k = 1$ is straightforward. Let $k \in \mathbb{N}$ with $k \geqslant 1$ and assume that the result has been proved for every uniquely rooted and balanced tree of height at most k. Let T be a uniquely rooted and balanced tree of height $k + 1$ and let F be a non-empty finite subset of T. We need to find a strong subtree S of T with $h(S) \leqslant \min\{k + 1, 2|F| - 1\}$ such that $F \subseteq S$. Clearly we may assume that $|F| \geqslant 2$.

Let $w_0 = \wedge_T F$ be the infimum of F in T, and set

$$I(F) = \{t \in \mathrm{ImmSucc}_T(w_0) : F \cap \mathrm{Succ}_T(t) \neq \varnothing\}.$$

Notice that

$$\bigcup_{t\in I(F)} \big(F \cap \mathrm{Succ}_T(t)\big) \subseteq F \subseteq \{w_0\} \cup \bigcup_{t\in I(F)} \big(F \cap \mathrm{Succ}_T(t)\big), \tag{B.1}$$

and so

$$\sum_{t\in I(F)} |F \cap \mathrm{Succ}_T(t)| \leqslant |F| \leqslant 1 + \sum_{t\in I(F)} |F \cap \mathrm{Succ}_T(t)|. \tag{B.2}$$

Observe that $I(F)$ is non-empty (for if not, by (B.1), we would have that $F = \{w_0\}$).

Let $t \in I(F)$ be arbitrary. Since T is a balanced tree of height $k+1$, we see that $\mathrm{Succ}_T(t)$ is a uniquely rooted and balanced tree of height at most k. By our inductive assumptions, there exists a strong subtree W_t of $\mathrm{Succ}_T(t)$ such that

$$F \cap \mathrm{Succ}_T(t) \subseteq W_t \tag{B.3}$$

and

$$h(W_t) \leqslant 2|F \cap \mathrm{Succ}_T(t)| - 1. \tag{B.4}$$

Observe that $|F \cap \mathrm{Succ}_T(t)| < |F|$ (for if not, we would have that $F \subseteq \mathrm{Succ}_T(t)$, which yields that $w_0 \in \mathrm{Succ}_T(t)$, a contradiction). Therefore,

$$h(W_t) \leqslant 2|F| - 2. \tag{B.5}$$

We set

$$L = \bigcup_{t\in I(F)} L_T(W_t) \tag{B.6}$$

and we select a family $\{S_t : t \in \mathrm{ImmSucc}_T(w_0)\}$ of strong subtrees of T such that

(P1) $S_t \subseteq \mathrm{Succ}_T(t)$ and $L_T(S_t) = L$ for every $t \in \mathrm{ImmSucc}_T(w_0)$, and

(P2) $W_t \subseteq S_t$ for every $t \in I(F)$.

Such a selection is possible since the tree T is balanced. Finally, let

$$S = \{w_0\} \cup \{S_t : t \in \mathrm{ImmSucc}_T(w_0)\}. \tag{B.7}$$

By (B.1) and properties (P1) and (P2), we see that S is a strong subtree of T and that $F \subseteq S$. The proof will be complete once we show that $h(S) \leqslant 2|F| - 1$. Indeed notice that, by (B.7) and property (P1), we have

$$h(S) = |L| + 1. \tag{B.8}$$

We consider the following cases.

Case 1: $|I(F)| = 1$. Let $t_0 \in \mathrm{ImmSucc}_T(w_0)$ be the unique element of $I(F)$. By (B.6), we have $L = L_T(W_{t_0})$. Hence,

$$h(S) \overset{\text{(B.8)}}{=} |L| + 1 = |L_T(W_{t_0})| + 1 = h(W_{t_0}) + 1 \overset{\text{(B.5)}}{\leqslant} 2|F| - 1.$$

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Case 2: $|I(F)| \geqslant 2$. Notice that

$$|L| \overset{(B.6)}{\leqslant} \sum_{t\in I(F)} |L_T(W_t)| \overset{(B.4)}{\leqslant} 2\sum_{t\in I(F)} |F\cap \mathrm{Succ}_T(t)| - |I(F)|$$
$$\overset{(B.2)}{\leqslant} 2|F| - |I(F)| \leqslant 2|F| - 2.$$

Combining (B.8) and the above estimate we conclude that $h(S) \leqslant 2|F| - 1$. The above case are exhaustive, so the proof is complete. □

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