

Hyperbolic actions of higher rank lattices come from rank-one factors

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Abstract. We study actions of higher rank lattices $\Gamma < G$ on hyperbolic spaces and we show that all such actions satisfying mild properties come from the rank-one factors of G . In particular, all non-elementary isometric actions on an unbounded hyperbolic space are of this type.

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1. Introduction

How can a given group act by isometries on a hyperbolic space? The aim of this paper is to study this question for irreducible lattices in a semisimple Lie group G of rank ≥ 2 . Haettel [13] addressed the case where all simple factors of the ambient product G have rank ≥ 2 and showed, in that case, that the isometric actions of the lattice on hyperbolic spaces are all degenerate (see below for a more precise formulation). In this paper, we allow G to have simple factors of rank one. Since rank-one simple groups have a natural geometric action on a proper hyperbolic space (namely, a rank-one symmetric space), the lattices in G do

admit non-degenerate actions on hyperbolic spaces via their projections on the rank-one simple factors of G . We show that, up to a natural equivalence, those are the only actions of lattices in G on hyperbolic spaces.

1.1. *Generalities on actions on hyperbolic spaces.* Before stating our main theorem, we now explain some general facts about actions on hyperbolic spaces. First of all, any group has isometric actions on hyperbolic spaces that fix a bounded set, as well as actions that fix a point at infinity. Such actions can therefore not be used to deduce anything about the group: from our viewpoint, they are degenerate and we will disregard them. Moreover, given an action on a hyperbolic space, one could make a larger hyperbolic space containing the first one as a quasi-convex subspace, maintaining the group action. This can be done, for example, by attaching equivariantly geodesic rays. To take this possibility into account, it is natural to also rule out actions that admit a quasi-convex invariant set that is not coarsely dense. In view of all this, we define *coarsely minimal* actions (Definition 4.1) by, essentially, ruling out the pathological behaviours discussed above. Arguably, those are the most general actions that one might want to classify. Moreover, actions on hyperbolic spaces that admit an equivariant quasi-isometry should be considered equivalent, and we capture this in Definition 4.2, where there is a subtlety to deal with actions where given subgroups fix a bounded set rather than single points.

1.2. *Rigidity of hyperbolic actions.* We say that a lattice Γ in a product group $G = G_1 \times \cdots \times G_N$ is *irreducible* if the closure of the projection of Γ to any proper sub-product has finite index. It follows from results of Margulis [17, Theorem II.6.7] that if each G_i is a simple algebraic group over a local field, then Γ is irreducible if and only if Γ has no finite index subgroup that splits as a direct product in a non-trivial way. Here is the main result of this paper.

THEOREM 1.1. *Let $N \geq n \geq 0$ be integers. Let $G = \prod_{i=1}^N G_i$ be a product of N centreless simple Lie groups, where for all $i \in \{1, \dots, n\}$, the factor G_i has real rank-one, and for all $j \in \{n+1, \dots, N\}$, the factor G_j has real rank at least two. Let $\Gamma < G$ be a lattice. Assume that $n \geq 2$ or that $N > n$. If $N > 1$, assume in addition that Γ is irreducible.*

Then any coarsely minimal isometric action of Γ on a geodesic hyperbolic space is equivalent to one of the actions

$$\Gamma \longrightarrow G \xrightarrow{\text{Pr}_i} G_i \longrightarrow \text{Isom}(X_i, d_i) \quad (1 \leq i \leq n),$$

where each X_i is the rank-one symmetric space of the factor G_i .

As mentioned above, we refer to Definition 4.2 for the precise notion of *equivalence* appearing in the theorem.

In the special case where G consists only of a single higher rank factor, that is the case where $n = 0$ and $N = 1$, our considerations recover for Lie groups the main theorem of [13].

COROLLARY 1.2. *Let G be a simple Lie group of real rank ≥ 2 , and $\Gamma < G$ a lattice. Then Γ does not admit any coarsely minimal isometric action on a geodesic hyperbolic space.*

One can also view Theorem 1.1 as a generalization of the main result from Margulis' paper [16], where he studied possible amalgam decompositions of lattices in higher rank.

1.3. Hyperbolic structures. The setup adopted here is inspired by the notion of *hyperbolic structures*, defined in [1] to capture *cobounded* actions on hyperbolic spaces. Coarsely minimal actions provide a similar but broader setup (see [1, Proposition 3.12] for a comparison). In what follows, we regard a hyperbolic structure as an equivalence class (in the sense of Definition 4.2) of cobounded actions on hyperbolic spaces. It is a general fact that any such action is either coarsely minimal or the hyperbolic space being acted on is bounded (giving rise to what is called the trivial structure). While a coarsely minimal action may fail to be cobounded in general, it is worth noting that Theorem 1.1 yields another rigidity feature of higher rank lattices, namely the following corollary.

COROLLARY 1.3. *Let $\Gamma < G$ be as in Theorem 1.1. Then any coarsely minimal isometric Γ -action on a geodesic hyperbolic space is cobounded.*

Indeed, since the projection $\Gamma \rightarrow G_i$ has dense image, the result follows from Theorem 1.1 because the G_i -action on the model space X_i is cocompact.

By Corollary 1.3, the number of hyperbolic structures up to equivalence on Γ is the number of coarsely minimal Γ -actions up to equivalence plus one. Therefore, in the language of [1], Theorem 1.1 implies that the lattices under consideration have exactly $n + 1$ inequivalent hyperbolic structures. Note that in [1], for every integer $n \geq 1$, the authors construct a finitely generated group Γ admitting precisely n distinct hyperbolic structures; irreducible lattices in higher rank semisimple Lie groups provide naturally occurring examples of that same phenomenon. Note that any lattice in a higher rank simple Lie group has only the trivial hyperbolic structure by [13] or Corollary 1.2.

Example 1.4. (Groups with $n \geq 2$ non-trivial hyperbolic structures) Let $n \geq 2$ be an integer. Let also K be a totally real number field of degree n . Such a field can be constructed as a subextension of degree n in $\mathbb{Q}(\cos(2\pi/p))$, where p is any prime congruent to 1 modulo $2n$ (in the special case of $n = 2$, one can of course simply take $K = \mathbb{Q}(\sqrt{2})$). Let \mathcal{O} be the ring of integers of K . Then K has exactly n inequivalent embeddings as subfields of \mathbb{R} , whose restrictions to \mathcal{O} have dense image. The corresponding diagonal embedding of \mathcal{O} in \mathbb{R}^n has discrete image. Consequently, the group

$$\Gamma = \mathrm{SL}_2(\mathcal{O})$$

embeds as an irreducible lattice in the product of n copies of $\mathrm{SL}_2(\mathbb{R})$. Theorem 4.4 implies that it has precisely n non-trivial hyperbolic structures that arise from its actions on the hyperbolic plane via the projections $\mathrm{pr}_1, \dots, \mathrm{pr}_n$.

Example 1.5. (A group with a single non-trivial hyperbolic structure) Consider the quadratic form $q(x_1, \dots, x_5) = x_1^2 + x_2^2 + x_3^2 + \sqrt{2}x_4^2 - x_5^2$, its orthogonal group $\mathrm{SO}(q) = \{g \in \mathrm{SL}_5 \mid q \circ g = q\}$ and let $\Gamma = \mathrm{SO}(q)_{\mathbb{Z}[\sqrt{2}]}$ be the group of its integer points. This group has only one non-trivial hyperbolic structure, because Γ is an irreducible lattice in the semisimple real Lie group $\mathrm{SO}(4, 1) \times \mathrm{SO}(3, 2)$ that has a single rank-one factor $\mathrm{SO}(4, 1) \simeq \mathrm{Isom}(\mathbb{H}^4)$ and another simple $\mathrm{SO}(3, 2)$ factor of rank two.

1.4. *Outline of proofs.* In the proofs, we will use boundary theory as outlined in [3]. Roughly, given a group Γ , one can associate to it a Lebesgue space B , called Γ -boundary, which on one hand has very strong ergodic properties, and on the other hand has the property that whenever Γ acts on a compact space Z , there is a Γ -equivariant map $B \rightarrow \text{Prob}(Z)$, called a *boundary map*. In the case of a group Γ acting nicely on a hyperbolic space X , as in Theorem 1.1, a key step consists in constructing an equivariant measurable map $\Phi : B \rightarrow \partial X$ into the Gromov boundary of X . To do so, one starts from a boundary map into $\text{Prob}(\bar{X})$ —the space of probability measures on a certain compactification of X —and uses the ergodic properties of the Γ -action on B in conjunction with the geometric properties for the Γ -action on X , to show that the map Φ takes values in Dirac measures $\Phi(x) = \delta_{\phi(x)}$, where $\phi(x) \in \partial X \subset \bar{X}$. The case where X is a proper Gromov-hyperbolic space had been considered already in [3], and indeed the main result of §3 is a direct generalization of [3, Theorem 3.2].

The fact that X is potentially a non-proper space causes several issues. One of them requires to study the relation between the horofunction compactification \bar{X} and the Gromov boundary ∂X . Similar considerations appear in the work by Duchesne [9] and by Maher and Tiozzo [15]. Related ideas appeared already in the much earlier work [16] of Margulis, where he studied actions of higher rank lattices on trees of possibly infinite valency. Another issue is that the space $\text{Bdd}_C(X)$ of C -bounded subsets of X , which is naturally quasi-isometric to X , need no longer be separable. This causes critical separability issues when dealing with boundary maps. To circumvent those difficulties, we introduce in §2 a new type of ergodicity of boundary Γ -spaces, called *coarse metric ergodicity*. We show in Theorem 2.7 that it is always satisfied by Furstenberg–Poisson boundaries. In the context of Theorem 1.1, we also show that the standard Furstenberg–Poisson boundary of G also enjoys the coarse metric ergodicity as a Γ -space (Proposition 2.9). In §3, we construct boundary maps for any non-degenerate isometric Γ -action on a hyperbolic space X , under the sole hypothesis of existence of an amenable coarsely metrically ergodic Γ -space (Theorem 3.1).

In §4, we focus again on the case where Γ is a higher rank lattice, as in our main theorem. In this case, the Γ -boundary of Γ splits as a product, with factors corresponding to the factors of the ambient locally compact group G . Due to the ergodicity properties of Γ -boundaries, we see that when Γ acts nicely on a hyperbolic space X , the boundary map from the Γ -boundary to ∂X factors through one of the algebraic factors of the ambient group G . At this point, there are two cases to analyse. The first case is when the said factor is of rank ≥ 2 : we have to show that this cannot occur. This is done in §4.4 by adapting the Weyl group method of Bader and Furman [2]. The second case is that the factor as above corresponds to a rank-one factor G_i of G . In that case, we have to show that X is equivalent to the model symmetric space X_i for G_i . This is done in §§4.5 and 4.6. By hypothesis, the group G has at least two factors in this case, and the projection of Γ to G_i has dense image. To show the equivalence between X_i and X , metric properties are transferred from X_i to X via the boundary map. A key ingredient is that bounded subsets of X_i correspond to precompact subsets of G_i , and the latter property can be rephrased in terms of the boundedness of Radon–Nikodym derivatives for the action on the G_i -boundary.

Remark 1.6. In a draft version of this paper that was circulated as a preprint, we had adopted a more general setup, encompassing irreducible lattices in products whose factors were allowed to be simple algebraic groups over local fields or standard rank-one groups in the sense of [8]. However, the strategy used in that version was hampered by the measurability issues alluded to above, related to the inseparability of the space $\text{Bdd}_C(X)$. This is resolved here using coarse metric ergodicity. It is likely that Theorem 1.1 nevertheless extends to the general setup we had originally envisaged. A possible approach to establish this would be to generalize Proposition 2.9 beyond the Lie group case.

2. Coarse metric ergodicity

2.1. *Definitions.* The goal of this section is to introduce a coarse version of metric ergodicity. Let us recall some definitions before recalling metric ergodicity and discussing why we need a different version.

Let G be a second countable locally compact group (abbreviated *lcsc group*). This includes the case of a countable discrete group Γ . A *Lebesgue G -space* is a Lebesgue space (Ω, μ) with a measurable, measure class preserving action map $G \times \Omega \rightarrow \Omega$. A *Borel G -space* V is a standard Borel space V with a Borel action map $G \times V \rightarrow V$. By a *Lebesgue map* $\Omega \rightarrow V$, we mean an equivalence class of functions from Ω to V , defined almost everywhere and up to an almost everywhere identification, which has a Borel measurable representative.

The Lebesgue G -space Ω is *metrically ergodic* if, given any separable metric space S and continuous homomorphism $\pi : G \rightarrow \text{Isom}(S)$, the only G -equivariant measurable maps $\Omega \rightarrow S$ are constant. (Ergodicity in the usual sense corresponds to taking $S = \{0, 1\}$ and π trivial.)

Roughly, the relevance of metric ergodicity in our setting is that, given a higher rank lattice Γ acting on a hyperbolic space X , we will need to rule out the existence of certain Γ -equivariant maps $B \rightarrow \partial X^{(3)}$, where B is a Γ -boundary and $\partial X^{(3)}$ is the space of distinct triples of boundary points of X . We are not aware of a natural way of endowing $\partial X^{(3)}$ with a *separable* metric. Instead, we can endow it with a ‘coarsely separable’ metric and we will apply coarse metric ergodicity to this metric. Since it does not cost any additional work, we will actually deal with coarse metrics in the following sense.

Definition 2.1. A *coarse metric* on a set U is a function $d : U \times U \rightarrow [0, \infty)$ such that there exists $C > 0$ such that for all $x, y, z \in U$,

$$d(x, x) \leq C, \quad d(x, y) \leq d(y, x) + C, \quad \text{and} \quad d(x, z) \leq d(x, y) + d(y, z) + C.$$

The function d is a *C -coarse metric* on U .

Two coarse metrics are coarsely equivalent if their difference is uniformly bounded.

Remark 2.2. Every C -coarse metric d lies within bounded distance of a genuine metric d' . Indeed, one can start with d , change it so that $d(x, x) = 0$ and then set $d_1(x, y) = \max\{d(x, y), d(y, x)\}$. At this point, it is readily checked that d_1 is again a C -coarse metric and to get d' , one can add to it C times the discrete $\{0, 1\}$ -metric.

Definition 2.3. Let G be an lcsc group. A Lebesgue G -space Ω is said to be *coarsely metrically ergodic* if every Borel coarse metric d , defined on some conull subset $\Omega_0 \subseteq \Omega$ and G -invariant as a Lebesgue map $\Omega \times \Omega \rightarrow [0, \infty)$, is essentially bounded, that is, there exists $R > 0$ such for almost every (a.e.) pair $(x, x') \in \Omega_0 \times \Omega_0$, we have $d(x, x') \leq R$. Equivalently, by Remark 2.2, if every G -invariant Borel metric on any Ω_0 as above is essentially bounded.

Example 2.4. If B is a *doubly ergodic* G -space, that is, the diagonal G -action on $B \times B$ is ergodic, then B is coarsely metrically ergodic. This follows easily from the fact that given a G -invariant Borel coarse metric d on B , the subsets $\Delta_R = \{(b, b') : d(b, b') \leq R\} \subseteq B \times B$ are G -invariant and their union is the whole $B \times B$. Hence, one of them needs to have positive measure, and hence full measure by ergodicity.

Remark 2.5. Coarse metric ergodicity does not imply ergodicity, as any finite space with a trivial action is clearly coarsely metrically ergodic. However, it is easy to see that any coarsely metrically ergodic space has a finite number of ergodic components.

Note that in the definition of coarse metric ergodicity, we do not assume separability of d , merely that it is Borel. An example to keep in mind is the discrete $\{0, 1\}$ -metric, which is Borel, but not separable unless Ω is countable. A coarse metric space is *coarsely separable* if it is covered by a countable collection of balls of a fixed radius. By the following remark, essentially all coarse metrics of concern to us are coarsely separable.

Remark 2.6. Given an lcsc group G , every G -invariant coarse metric on an ergodic space Ω is coarsely separable by the following argument. The space is an exhaustion of countably many concentric balls of growing radii, so there exists a radius r and a non-null r -ball. Translating this ball by a dense countable subgroup of G (which acts ergodically due to the weak* continuity of the G -representation on $L^\infty(\Omega)$), we find a countable collection of non-null r -balls whose union is of full measure.

2.2. Examples. A first source of coarsely metrically ergodic actions of a group Γ is given by doubly ergodic Γ -spaces, as pointed out in Example 2.4. For our purposes, we need amenable Γ -spaces B_- and B_+ such that the Γ -action on $B_- \times B_+$ is coarsely metrically ergodic, see Theorem 3.1 below. Our main source for such Γ -spaces is provided by the following theorem.

THEOREM 2.7. *Let G be an lcsc group. Let μ be a spread-out generating probability measure on G , and let B_- and B_+ be the corresponding past and future Furstenberg–Poisson boundaries, that is, the Furstenberg–Poisson boundaries associated with the measures $\check{\mu}$ and μ correspondingly. Then, B_- and B_+ are amenable Lebesgue G -spaces such that the diagonal G -action on $B_- \times B_+$ is metrically ergodic and coarsely metrically ergodic.*

In the case where μ is a symmetric measure, we obtain a single amenable G -space B , equal to B_- and B_+ . We emphasize that the coarse metric ergodicity of the G -action on $B \times B$ should not be expected to hold for the same reason as in Example 2.4, since the G -action on $B \times B \times B \times B$ need not be ergodic: indeed, in the case where G is a rank-one simple Lie group, the space B can be identified with the visual boundary of the

symmetric space of G , and $B \times B \times B \times B$ is the space of 4-tuple of distinct points. The cross-ratio is a G -invariant function on that space.

Instead, the proof of Theorem 2.7 relies on the following lemma.

LEMMA 2.8. (Cf. [3, Lemma 2.8]) *Let G be an lcsc group and let μ be a spread-out generating probability measure on G . Let (B_-, ν_-) and (B_+, ν_+) be the corresponding past and future Furstenberg–Poisson boundaries. Given a positive measure subset $E \subset B_- \times B_+$ and $\epsilon > 0$, there is $g \in G$ and a positive measure subset $E_1^- \subset E^- \cap g^{-1}E^-$, where E^- is the projection of E to B_- , so that for every $b_- \in E_1^-$, $\nu_+(gE_{b_-}) > 1 - \epsilon$, where $E_{b_-} = \{b_+ \in B_+ \mid (b_-, b_+) \in E\}$.*

Proof of Theorem 2.7. The fact that B_- and B_+ are amenable Lebesgue G -spaces such that the diagonal G -action on $B_- \times B_+$ is metrically ergodic is proven in [3, Theorem 2.7]. We will show that diagonal G -action on $B_- \times B_+$ is coarsely metrically ergodic.

Fix a G -invariant Borel metric $d : (B_- \times B_+)^2 \rightarrow [0, \infty)$. By Remark 2.6 and upon normalization, we assume, as we may, that d is 1-separable. We will show that for a.e. $b_- \in B_-$, the essential diameter of $\{b_-\} \times B_+$ is bounded by 3. By symmetry, for a.e. $b_+ \in B_+$, the essential diameter of $B_- \times \{b_+\}$ is bounded by 3, and thus the essential diameter of $B_- \times B_+$ is bounded by 6.

We find a countable collection A_i of non-null 1-balls whose union is of full measure. We consider the projection $B_- \times B_+ \rightarrow B_-$ and for every i , we let A_i^- be the image of A_i . For every natural n , we let $A_{i,n}^-$ be the set of those $b_- \in A_i^-$ such that

$$\nu_+(\{b_+ \in B_+ \mid (b_-, b_+) \in A_i\}) \geq 1/n.$$

Note that there exists N_i such that the sets $A_{i,n}^-$ are non-null for every $n \geq N_i$. Using the symbol $\stackrel{\circ}{=}$ for equation up to null-sets, we get $A_i^- \stackrel{\circ}{=} \bigcup_{n \geq N_i} A_{i,n}^-$. We let $A_{i,n}$ be the preimage of $A_{i,n}^-$ in A_i and get

$$A_i \stackrel{\circ}{=} \bigcup_{n \geq N_i} A_{i,n} \quad \text{and} \quad B_+ \times B_- \stackrel{\circ}{=} \bigcup_i A_i \stackrel{\circ}{=} \bigcup_{i,n \geq N_i} A_{i,n}.$$

By modifying the sets $A_{i,n}$, we assume as we may that for every $i', n', i'', n'', A_{i',n'}^- \cap A_{i'',n''}^-$ is either non-null or empty, upon throwing away the preimages of the intersection in case it is null. By Fubini, for a.e. $b_- \in B_-$, for a.e. $b_+ \in B_+$, $(b_-, b_+) \in \bigcup_{i,n \geq N_i} A_{i,n}$. We fix a generic $b_- \in B_-$ and consider the full measure subset of the fibre over b_- ,

$$D = (\{b_-\} \times B_+) \cap \left(\bigcup_{i,n \geq N_i} A_{i,n} \right).$$

We claim that the set D is bounded by 3. To see this, let us fix $b'_+, b''_+ \in B_+$ such that $(b_-, b'_+) \in D$. Fix i', n', i'', n'' such that $(b_-, b'_+) \in A_{i',n'}$, $(b_-, b''_+) \in A_{i'',n''}$ and $n' > N_{i'}, n'' > N_{i''}$. Set

$$\epsilon = \min\{1/n', 1/n''\} \quad \text{and} \quad E^- = A_{i',n'}^- \cap A_{i'',n''}^-.$$

Let E be the preimage of E^- in $A_{i',n'}$ and note that it is non-null. By Lemma 2.8, there exist $g \in G$ and a non-null subset E_1^- in $E^- \cap g^{-1}E^-$ such that for every $b'_- \in E_1^-$,

$$\nu_+(gE_{b'_-}) > 1 - \epsilon,$$

where $E_{b'_-} = \{b_+ \mid (b'_-, b_+) \in E\}$. Fixing such b'_- , we conclude that the set $gE_{b'_-}$ is bounded by 1 and it intersects both sets $A_{i',n'}$ and $A_{i'',n''}$. We conclude that $d((b_-, b'_+), (b_-, b''_+)) < 3$. This finishes the proof. \square

For lattices in a semisimple Lie group G , one can realize the unique G -invariant measure class on G/P as a Furstenberg–Poisson boundary, leading to the following proposition.

PROPOSITION 2.9. *Let G be a semisimple Lie group and $\Gamma < G$ a lattice. Let $P < G$ be a minimal parabolic and endow the coset space G/P with the unique G -invariant measure class and consider it as a Lebesgue Γ -space. Then G/P is an amenable Lebesgue Γ -space such that the diagonal Γ -action on $G/P \times G/P$ is metrically ergodic and coarsely metrically ergodic.*

Proof. We fix a maximal compact subgroup $K < G$. In [10], Furstenberg proved that all bounded harmonic functions on the symmetric space G/K are represented via a Poisson formula as bounded measurable functions on G/P with respect to a certain G -quasi-invariant measure $\nu \in \text{Prob}(G/P)$, namely the unique K -invariant measure. Following the discretization process applied by Furstenberg in the special case of the hyperbolic plane, Lyons and Sullivan obtained in [14, Theorem 5] a discretization scheme which applies to any $*$ -recurrent discrete subset of a Riemannian manifold of bounded geometry (see [14, §7] for the definition of $*$ -recurrent), an example of such a setting being the orbit of Γ on G/K . Their discretization procedure for the pair $(G/K, \Gamma.K)$ commutes with the symmetries of the pair, and thus provides a probability measure μ on Γ for which the bounded μ -harmonic functions are exactly the restriction to Γ of the bounded harmonic functions on G/K . That is, the future Furstenberg–Poisson boundary B_+ of (Γ, μ) is $(G/P, \nu)$. The main theorem of [4] shows further that the measure μ could be chosen to be symmetric, that is, $\check{\mu} = \mu$. This shows that the past Furstenberg–Poisson boundary B_- of (Γ, μ) is $(G/P, \nu)$ as well. We are thus done by Theorem 2.7. \square

2.3. Coarse invariance. One can also make a further coarsification.

Definition 2.10. A coarse metric on a G -space U is said to be coarsely invariant if there exists $C > 0$ such that for every $x, y \in U$ and $g \in G$, $|d(gx, gy) - d(x, y)| < C$.

This coarsification turns out to not correspond to a more restrictive type of ergodicity.

LEMMA 2.11. *Assume Γ is a countable group and Ω is a coarsely metrically ergodic Lebesgue Γ -space. Then every coarsely invariant Borel coarse metric on Ω is bounded. In fact, every coarsely invariant Borel function $\Omega \times \Omega \rightarrow [0, \infty)$ which is coarsely symmetric and satisfies the coarse triangle inequality is a bounded coarse metric.*

Proof. Given a coarsely invariant coarse metric d on Ω , observe that $d'(x, y) = \sup\{d(gx, gy) \mid g \in \Gamma\}$ is a Γ -invariant measurable coarse metric. Its boundedness implies the boundedness of d . The last part of the lemma follows from the fact that the function $x \mapsto d'(x, x)$ is automatically bounded, by Remark 2.5. \square

Remark 2.12. We could say that Ω is coarsely coarse metrically ergodic, but we honestly think this is too much.

2.4. *Countable quotients.* The following lemma shows that up to coarse equivalence, every coarsely separable coarse metric is pulled back from a countable quotient.

LEMMA 2.13. *Let d be a coarsely separable coarse metric on U . Then there exists a countable measurable quotient $\pi : U \rightarrow V$ and a coarse metric d' on V such that d is coarsely equivalent to the metric π^*d' , defined by $d(x, y) = d'(\pi(x), \pi(y))$.*

Proof. We choose a countable C -dense subset V in U , identify it with \mathbb{N} and define π inductively, sending points at the C -ball around $v \in V$ to v , unless their π value was already determined. We set $d' = d|_V$. □

2.5. *Spaces of probabilities.* Given a Borel space U , we endow the space $\text{Prob}(U)$ with the Borel structure generated by the evaluation maps associated with all bounded Borel functions on U . We will need Proposition 2.15 below to rule out the existence of maps from the (double) boundary of the lattices we will be considering to various probability spaces $\text{Prob}(U)$, including for U the space of distinct boundary triples of a hyperbolic space being acted on.

If U is a standard Borel space, then $\text{Prob}(U)$ is also a standard Borel space. If $\pi : U \rightarrow V$ is a Borel map, then the induced map $\pi_* : \text{Prob}(U) \rightarrow \text{Prob}(V)$ is a Borel map.

We assume henceforth that the Borel space U is endowed with a coarse metric d . In that case, we define the function

$$\bar{d} : \text{Prob}(U) \times \text{Prob}(U) \rightarrow [0, \infty)$$

by setting

$$\bar{d}(\mu_1, \mu_2) = \inf\{r > 0 \mid \text{there exists a Borel set } A \text{ in } U, \text{diam}(A) < r, \mu_1(A), \mu_2(A) > 1/2\}.$$

Note that $\bar{d}(\mu, \mu)$ might be arbitrarily large, so this is not necessarily a coarse metric. However, we have the following.

LEMMA 2.14. *The function \bar{d} is symmetric and it satisfies the coarse triangle inequality: the difference $\bar{d}(\mu_1, \mu_2) - \bar{d}(\mu_1, \mu_0) - \bar{d}(\mu_0, \mu_2)$ is uniformly bounded from above. The restriction of \bar{d} to U via the identification $x \mapsto \delta_x$ gives the coarse metric $(x, y) \mapsto \max\{d(x, y), d(y, x)\}$ which is at a bounded distance away from d .*

Moreover, if U is a countable set and d is a Borel coarse metric on U , then \bar{d} is a Borel function.

Proof. The first assertions of the lemma are straightforward. For the last assertion under the extra hypothesis that U is countable, we observe that the infimum in the definition of \bar{d} could be taken over the countable collection of the finite subsets of U . This ensures that \bar{d} is indeed Borel. □

PROPOSITION 2.15. *Let Γ be a countable group and Ω be a coarsely metrically ergodic Lebesgue Γ -space. Let U be a Borel space and consider the Borel structure on $\text{Prob}(U)$ generated by evaluation at bounded Borel functions on U . If there exists a Borel Γ -map from Ω to $\text{Prob}(U)$, then every Γ -invariant Borel, coarsely separable coarse metric on U has bounded Γ -orbits.*

Proof. We assume $f : \Omega \rightarrow \text{Prob}(U)$ is a Borel Γ -map and d is a coarsely separable Γ -invariant Borel coarse metric on U , and we argue to show that the Γ -orbits in U are d -bounded. Since the restriction of \bar{d} to U is at a bounded distance away from d (see Lemma 2.14), it is enough to show that $\text{Prob}(U)$ contains a \bar{d} -bounded subset.

We let $\pi : U \rightarrow V$ and d' be as in Lemma 2.13, so that the pull back π^*d' is coarsely equivalent to d . It follows $\overline{\pi^*d'}$ is coarsely equivalent to \bar{d} . By Lemma 2.14, \bar{d} is Borel, symmetric and it satisfies the coarse triangle inequality. Hence, the same applies also to $\overline{\pi^*d'} = (\pi_*)^*\bar{d}'$.

By Lemma 2.11, $f^*\overline{\pi^*d'}$ is a bounded coarse metric on Ω . We deduce that $\text{Prob}(U)$ contains a $\overline{\pi^*d'}$ -bounded subset. The proof is now complete, as this subset is \bar{d} -bounded. □

2.6. *Barycentre maps.* The last result in this section is not about coarse metric ergodicity, but it fits here since we are discussing the spaces $\text{Prob}(U)$. Let U be a Borel space and d a measurable metric on U . Roughly, we will construct ‘barycentres’ for probability measures, and the property that we will need is roughly that measures with bounded Radon–Nykodim derivatives with respect to each other have approximately the same barycentre.

We denote by $\text{Bdd}(U)$ the space of non-empty d -bounded subsets of U . Given a measure $m \in \text{Prob}(U)$ and $\epsilon \in (0, 1/2)$, we define

$$r_\epsilon(m) = \inf\{\text{diam}(A) \mid A \text{ is a Borel set in } U, m(A) \geq 1 - \epsilon\},$$

and we let $\beta_\epsilon(m)$ be the union of all Borel sets A in U of diameter less than $r_\epsilon(m) + 1$ and $m(A) \geq 1 - \epsilon$. Note that since $1 - \epsilon < 1$, the infimum in the definition of r_ϵ is taken over a non-empty set and $\beta_\epsilon(m)$ is a non-empty subset of U .

LEMMA 2.16. *Let U be a Borel space and d a measurable metric on U . For every $\epsilon \in (0, 1/2)$ and for every $m \in \text{Prob}(U)$, $\beta_\epsilon(m)$ is a non-empty d -bounded subset of U , and the map $\beta_\epsilon : \text{Prob}(U) \rightarrow \text{Bdd}(U)$ is equivariant under all isometries of U . Moreover, given $C \geq 1$, if $m_1, m_2 \in \text{Prob}(U)$ are such that for every measurable $E \subseteq U$, we have*

$$m_2(E) \leq C m_1(E),$$

then

$$\beta_\epsilon(m_2) \subseteq N_R(\beta_{\epsilon/C}(m_1)),$$

where $R = r_{\epsilon/C}(m_1) + 1$.

Proof. By the fact that $1 - \epsilon > 1/2$, all the Borel subsets A with $m(A) \geq 1 - \epsilon$ intersect in pairs, so the diameter of $\beta_\epsilon(m)$ is bounded by $2r_\epsilon(m) + 2$. Equivariance is clear.

Let now C, m_1, m_2 as in the last statement. Note that every Borel subset A of U with $m_1(A) \geq 1 - \epsilon/C$ satisfies $m_2(U \setminus A) \leq C m_1(U \setminus A) \leq \epsilon$, and hence $m_2(A) \geq 1 - \epsilon$. It follows that $r_\epsilon(m_2) \leq r_{\epsilon/C}(m_1)$.

If $u \in \beta_\epsilon(m_2)$, then by definition, u belongs to a Borel set A with $m_2(A) \geq 1 - \epsilon$ and diameter less than $r_\epsilon(m_2) + 1 \leq r_{\epsilon/C}(m_1) + 1$. Fix any Borel set B of diameter less than $r_{\epsilon/C}(m_1) + 1$ and $m_1(B) \geq 1 - \epsilon/C$. Then, $m_2(B) \geq 1 - \epsilon$ and we get $A \cap B \neq \emptyset$. In particular, u lies within $r_{\epsilon/C}(m_1) + 1$ of B , whence of $\beta_{\epsilon/C}(m_1)$. \square

3. Boundary maps

In this section, we fix a countable group Γ and discuss *boundary maps* associated to various hyperbolic structures on Γ . In particular, we prove Theorem 3.1 below. The novel aspect of this theorem is that the Gromov hyperbolic spaces it deals with are not assumed to be proper. Recall that the Gromov boundary of a proper Gromov hyperbolic space is compact and the associated action is a *convergence group* action. Boundary maps associated with such actions were considered in [3, Theorem 3.2].

Given a Lebesgue Γ -space Ω and a standard Borel Γ -space V , we denote by $\text{Map}_\Gamma(\Omega, V)$ the space of equivalence classes of measurable Γ -equivariant maps $f : \Omega \rightarrow V$, that is, those measurable maps that satisfy $f(g.\omega) = g.f(\omega)$ for a.e. $g \in \Gamma$ and a.e. $\omega \in \Omega$. Two such maps $f, f' : \Omega \rightarrow V$ are identified if $f(\omega) = f'(\omega)$ for a.e. $\omega \in \Omega$. Any such map f is equivalent to $f_0 : \Omega \rightarrow V$ such that for every $g \in \Gamma$, we have $f_0(g.\omega) = g.f_0(\omega)$ a.e. $\omega \in \Omega$ [18, Proposition B.5].

In the following theorem, (X, d) is a separable Gromov hyperbolic space, Γ a countable group and $\Gamma \rightarrow \text{Isom}(X, d)$ a homomorphism. We denote by ∂X the associated Gromov boundary, which is a Polish space on which Γ acts by homeomorphisms, let ∂X^2 be its square and

$$\partial X^{(2)} = \{(\xi, \eta) \mid \xi \neq \eta \in \partial X\}$$

the subset of pairs of distinct boundary points. Since X is not assumed to be proper, ∂X is not necessarily compact. Yet, ∂X is a standard Borel space and so are $\partial X^{(2)} \subset \partial X^2$. The action of Γ on all these spaces is Borel.

THEOREM 3.1. (Compare [3, Theorem 3.2]) *Let Γ be a countable group. Assume B_- and B_+ are amenable Lebesgue Γ -spaces such that the diagonal Γ -action on $B_- \times B_+$ is ergodic and coarsely metrically ergodic. Let (X, d) be a separable, Gromov hyperbolic (possibly non-proper), geodesic metric space and let Γ act on X by isometries. Assume that Γ does not fix a bounded set in X and does not fix a point or a pair of points in ∂X .*

Then there exist $\phi_- \in \text{Map}_\Gamma(B_-, \partial X)$, $\phi_+ \in \text{Map}_\Gamma(B_+, \partial X)$ such that the image of the map $\phi_{\bowtie} \in \text{Map}_\Gamma(B_- \times B_+, \partial X^2)$ given by

$$\phi_{\bowtie}(x, y) = (\phi_-(x), \phi_+(y))$$

is contained, modulo a null set, in the set of distinct pairs $\partial X^{(2)} \subset \partial X^2$. Moreover, each of the following assertions holds:

- (i) $\text{Map}_\Gamma(B_-, \text{Prob}(\partial X)) = \{\delta \circ \phi_-\}$ and $\text{Map}_\Gamma(B_+, \text{Prob}(\partial X)) = \{\delta \circ \phi_+\}$;
- (ii) $\text{Map}_\Gamma(B_- \times B_+, \partial X) = \{\phi_- \circ \text{pr}_-, \phi_+ \circ \text{pr}_+\}$;
- (iii) $\text{Map}_\Gamma(B_- \times B_+, \partial X^{(2)}) = \{\phi_{\triangleright\triangleleft}, \tau \circ \phi_{\triangleright\triangleleft}\}$, where $\tau(\xi, \xi') = (\xi', \xi)$.

The rest of this section is devoted to the proof of this theorem.

3.1. *The horoclosure of a separable metric space.* Let (X, d) be a separable metric space. We consider the vector space of functions from X to \mathbb{R} endowed with the topology of pointwise convergence, that is, the product space \mathbb{R}^X , and the constant function $\mathbf{1} \in \mathbb{R}^X$. We endow $\mathbb{R}^X / \mathbb{R} \cdot \mathbf{1}$ with the quotient topological vector space structure. We map X to \mathbb{R}^X by $x \mapsto d(\cdot, x)$ and consider its image in $\mathbb{R}^X / \mathbb{R} \cdot \mathbf{1}$. We denote the closure of the image of X in $\mathbb{R}^X / \mathbb{R} \cdot \mathbf{1}$ by \bar{X} and call it the *horoclosure* of X . We denote the obvious map $X \rightarrow \bar{X}$ by i and the preimage of \bar{X} in \mathbb{R}^X by \tilde{X} . Elements of \bar{X} (and, by abuse of notation, also elements of \tilde{X}) are called *horofunctions*. It is a common practice to fix a base point $x \in X$ and to consider the subspace

$$\tilde{X} \supset \tilde{X}_x = \{h \in \tilde{X} \mid h(x) = 0\}.$$

It is well known that the horoclosure of X is a compactification of it.

LEMMA 3.2. *\bar{X} is a compact metrizable space and the map $i : X \rightarrow \bar{X}$ is an injective continuous map. For a fixed $x \in X$, the map $\tilde{X}_x \rightarrow \bar{X}$ is a homeomorphism.*

Proof. The fact that i is continuous is obvious. For $x \neq y$ in X , note that the difference function $d(x, \cdot) - d(y, \cdot)$ is not constant, as it attains different values at x and y . Thus, i is injective. We now fix $x \in X$. First, we note that \tilde{X}_x is closed subset of

$$\prod_{y \in X} [-d(x, y), d(x, y)] \subset \prod_{y \in X} \mathbb{R} = \mathbb{R}^X,$$

and thus it is compact. Fixing a countable dense subset X_0 in X , the obvious map $\tilde{X}_x \rightarrow \mathbb{R}^X \rightarrow \mathbb{R}^{X_0}$ is a continuous injection (as \tilde{X}_x consists of continuous functions), and hence a homeomorphism onto its image. The image is a Frechet space, and thus metrizable. It follows that \tilde{X}_x is metrizable. Since the natural map $\tilde{X}_x \rightarrow \bar{X}$ is also a continuous bijection, we conclude that it is a homeomorphism, and deduce that \bar{X} is compact and metrizable. □

Loosely speaking, we often identify X with $i(X) \subset \bar{X}$. Note, however, that the image of X is in general not open in \bar{X} , and the map i is not a homeomorphism onto its image.

We decompose \tilde{X} as follows:

$$\begin{aligned} \tilde{X}^b &= \{h \in \tilde{X} \mid h \text{ is bounded from below}\}, \\ \tilde{X}^u &= \{h \in \tilde{X} \mid h \text{ is unbounded from below}\}. \end{aligned}$$

This decomposition is constant on the fibres of $\tilde{X} \rightarrow \bar{X}$, and thus gives a corresponding decomposition $\bar{X} = \bar{X}^b \cup \bar{X}^u$. Clearly, we have $i(X) \subseteq \bar{X}^b$ so that \bar{X}^b is dense in \bar{X} .

LEMMA 3.3. *The decompositions $\tilde{X} = \tilde{X}^b \cup \tilde{X}^u$ and $\bar{X} = \bar{X}^b \cup \bar{X}^u$ are measurable and $\text{Isom}(X)$ -equivariant.*

Proof. The equivariance of the decompositions is obvious. Fix a dense countable subset X_0 in X and use the fact that \tilde{X} consists of continuous functions to note that

$$\tilde{X}^u = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in X_0} \{h \in \tilde{X} \mid h(x) \leq -n\},$$

and thus $\tilde{X}^u \subset \tilde{X}$ is measurable. Fixing $x \in X$, using the measurability of \tilde{X}_x^u , we observe that $\tilde{X}^u \subset \tilde{X}$ is measurable. □

3.2. *The horoclosure of a hyperbolic metric space.* We now assume, in addition, that the separable metric space (X, d) is geodesic and Gromov hyperbolic (as before, it is possibly non-proper), and that there is a Γ -action on it for some group Γ .

The following lemma will allow us to use coarse metric ergodicity in the setting of Theorem 4.4 to exclude the existence of Γ -maps to \tilde{X}^b .

LEMMA 3.4. *There is a Borel coarsely separable pseudo-metric on \tilde{X}^b on which Γ acts by isometries. Moreover, if the Γ -action on X is unbounded, then the Γ -action on \tilde{X}^b is unbounded.*

Proof. We will show the analogous statement for \tilde{X}^b rather than \tilde{X}^b , and we note that the pseudo-metric we construct below passes to \tilde{X}^b (since the sets $\tilde{I}(h)$ below do not change when adding a constant to h).

We first show that the function

$$\inf : \tilde{X}^b \rightarrow \mathbb{R}, \quad h \mapsto \inf\{h(x) \mid x \in X\}$$

is measurable and $\text{Isom}(X)$ -invariant, and that for every $h \in \tilde{X}^b$, the set

$$\tilde{I}(h) = \{x \in X \mid h(x) < \inf(h) + 1\}$$

is bounded in X .

To see that \inf is measurable, we fix a dense countable subset X_0 in X and we use the continuity of the functions in \tilde{X}^b to observe that

$$\inf(h) = \inf\{h(x) \mid x \in X_0\}.$$

The invariance of this function under $\text{Isom}(X)$ is clear.

Fix $h \in \tilde{X}^b$. We now argue that $\tilde{I}(h)$ is of diameter bounded by $C = 8 + 4\delta$, where δ is the hyperbolicity constant associated with the thin triangle property of X . Without loss of generality, we assume that $\inf(h) = 0$. Assuming the negation, we fix two points x, x' satisfying $d(x, x') > C$ and $h(x), h(x') < 1$. We consider a finite sequence of points x_0, x_1, \dots, x_n on a geodesic segment from x to x' such that $x_0 = x, x_n = x'$ and $d(x_i, x_{i+1}) < 1$. We consider the image of h in \tilde{X} along with its neighbourhood given by

$$U = \{f + \mathbb{R} \cdot \mathbf{1} \mid f \in \mathbb{R}^X \text{ for all } 0 \leq i, j \leq n, |(f(x_i) - f(x_j)) - (h(x_i) - h(x_j))| < 1\}.$$

We fix a point $y \in X$ whose image in \tilde{X} is in U . We thus have

$$\text{for all } 0 \leq i, j \leq n, \quad |d(y, x_i) - h(x_i) - d(y, x_j) + h(x_j)| < 1. \tag{3.1}$$

We consider geodesic segments from y to x and from y to x' and, using that x, x' and y are the vertices of a thin triangle, we fix i such that x_i lies at distance at most $1 + \delta$ from these segments. Thus,

$$\begin{aligned} d(y, x_i) + d(x_i, x) &\leq d(y, x) + 2 + 2\delta, \\ d(y, x_i) + d(x_i, x') &\leq d(y, x') + 2 + 2\delta. \end{aligned}$$

Note that $d(x, x_i) + d(x_i, x') = d(x, x')$. Upon possibly interchanging the roles of x and x' , we will assume that $d(x, x_i) \geq d(x, x')/2$. In particular, $d(x, x_i) \geq C/2$. Taking $j = 0$ in equation (3.1), we now have

$$\begin{aligned} 0 &= \inf(h) \\ &\leq h(x_i) \\ &< 1 + d(y, x_i) + h(x) - d(y, x) \\ &\leq 1 + (d(y, x) + 2 + 2\delta - d(x_i, x)) + h(x) - d(y, x) \\ &< (3 + h(x)) + 2\delta - d(x_i, x) \\ &\leq 4 + 2\delta - C/2 = 0. \end{aligned}$$

This is a contradiction, and thus indeed the diameter of $\tilde{I}(h)$ is bounded by C . We can now define the required pseudo-metric ρ by setting

$$\rho(h, h') = d_{\text{Haus}}(\tilde{I}(h), \tilde{I}(h')).$$

This is the pull-back of a metric, so it is a pseudo-metric, and Γ clearly acts on it by isometries. The fact that the $\tilde{I}(\cdot)$ are bounded implies the claim on unboundedness of the action. Moreover, coarse separability follows from separability of X and (uniform) boundedness of the $\tilde{I}(\cdot)$. What is left to show is that ρ is Borel, and to do so, we have to show that for all $r \in \mathbb{R}$, we have that $\Delta_r = \{(h, h') : \rho(h, h') > r\} \subseteq \tilde{X}^b \times \tilde{X}^b$ is Borel. Expanding the definitions, Δ_r is the set of pairs (h, h') such that $\tilde{I}(h)$ and $\tilde{I}(h')$ lie at Hausdorff distance more than r . In turn, this means that there exists $x \in \tilde{I}(h) \cap X_0$ such that all $y \in \tilde{I}(h') \cap X_0$ lie at distance more than r from x , or that the same holds switching the roles of h and h' (to justify the ‘ $\cap X_0$ ’, note that $\tilde{I}(h) \cap X_0$ is dense in $\tilde{I}(h)$ in view of the continuity of h , and similarly for h'). Denoting $\tilde{J}(z) = \{k \in \tilde{X}^b : z \in \tilde{I}(k)\}$ for $z \in Z$, the first case is further equivalent to h' not lying in $\bigcup_{y \in X_0: d(x,y) \leq r} \tilde{J}(y)$, which is the subset of \tilde{X}^b of all k such that $\tilde{I}(k)$ contains some $y \in X_0$ within distance r of x (the other case is similar). Hence, we have

$$\begin{aligned} \Delta_r &= \bigcup_{x \in X_0} \left(\tilde{J}(x) \times \left(\tilde{X}^b - \bigcup_{y \in X_0: d(x,y) \leq r} \tilde{J}(y) \right) \right) \\ &\cup \bigcup_{y \in X_0} \left(\left(\tilde{X}^b - \bigcup_{x \in X_0: d(x,y) \leq r} \tilde{J}(x) \right) \times \tilde{J}(y) \right). \end{aligned}$$

Hence, Δ_r is Borel provided that all $\tilde{J}(z)$ are Borel. However, $\tilde{J}(z) = \{k \in \tilde{X}^b : k(z) < \inf(k) + 1\}$, so it is measurable in view of the measurability of \inf , as required. \square

3.3. *The Gromov boundary.* While the construction of the Gromov boundary ∂X is fairly standard, it is commonly taken under a properness assumption on X . In preparation for our more general discussion, we review this construction (for X possibly non-proper) below, and we also relate it to horoclosures. As is common, we fix from now on a base point $o \in X$. For $x \in X$, we use the shorthand notation $|x| = d(o, x)$. Gromov products will be taken, unless otherwise stated, with respect to o . That is, for $x, y \in X$, we set

$$(x, y) = \frac{1}{2}(|x| + |y| - d(x, y)).$$

In our discussion below, we fix $\delta > 0$ such that for every $x, y, z \in X$, we have

$$(x, z) \geq \min\{(x, y), (y, z)\} - \delta.$$

We recall that a sequence of points (x_n) in X is said to *converge to infinity* if the real numbers (x_n, x_m) approach infinity when both indices m and n tend to infinity.

LEMMA 3.5. *Assume (x_n) is a sequence of points in X which converges in \bar{X} and denote $\bar{h} = \lim x_n$. Then, (x_n) converges to infinity if and only if $\bar{h} \in \bar{X}^u$. In that case, if (x'_n) is another sequence in X satisfying $\lim x'_n = \bar{h}$, then $(x_n, x'_n) \rightarrow \infty$.*

Proof. We will denote the lift of \bar{h} in \tilde{X}_o by h and show that (x_n) converges to infinity if and only if $h \in \tilde{X}_o^u$. Note that for every $x \in X$, we have $d(x_n, x) - |x| \rightarrow h(x)$.

Assuming first (x_n) converges to infinity, we will show that $h \in \tilde{X}_o^u$. Fix $r > 0$. Fix N such that for $n, m > N$, $(x_n, x_m) > r$. Fix $m > N$, note that $|x_m| \geq r$ and let x be a point on a geodesic segment from o to x_m with $|x| = r$. Then, by hyperbolicity,

$$(x_n, x) \geq \min\{(x_n, x_m), (x_m, x)\} - \delta = r - \delta.$$

Thus,

$$h(x) = \lim_{n \rightarrow \infty} (d(x_n, x) - |x_n|) = \lim_{n \rightarrow \infty} (|x| - 2(x_n, x)) \leq 2\delta - r.$$

As r was arbitrary, we get indeed that $h \in \tilde{X}_o^u$.

Assuming now $h \in \tilde{X}_o^u$, we will show that (x_n) converges to infinity. Fix $r > 0$. Fix x such that $h(x) < -r$. Fix N such that for every $n > N$, $d(x_n, x) - |x_n| < -r$ and observe that for such n ,

$$(x_n, x) = \frac{1}{2}(|x_n| + |x| - d(x_n, x)) \geq -\frac{1}{2}(d(x_n, x) - |x_n|) > \frac{1}{2}r.$$

Then, by hyperbolicity, for $n, m > N$,

$$(x_n, x_m) \geq \min\{(x_n, x), (x, x_m)\} - \delta > \frac{1}{2}r - \delta.$$

As r was arbitrary, we deduce indeed that the sequence (x_n) converges to infinity.

In the setting of the former paragraph, if (x'_n) is another sequences in X satisfying $x'_n \rightarrow \bar{h}$, fixing $N' \geq N$ such that for every $n > N'$, $d(x'_n, x) - |x'_n| < -r$, the same computation shows that $(x_n, x'_n) > r/2 - \delta$. Thus indeed, we obtain $(x_n, x'_n) \rightarrow \infty$. \square

Two sequences which converge to infinity, (x_n) and (y_n) , are said to be *equivalent* if $(x_n, y_n) \rightarrow \infty$. We conclude that if (x_n) and (x'_n) are two sequences in X satisfying

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x'_n \in \bar{X}^u,$$

then (x_n) is equivalent to (x'_n) .

A point in ∂X is, by definition, an equivalence class of sequences which converge to infinity. We denote by π the unique map

$$\pi : \bar{X}^u \rightarrow \partial X$$

satisfying

$$\lim_{n \rightarrow \infty} x_n \in \bar{X}^u \implies \pi \left(\lim_{n \rightarrow \infty} x_n \right) = [x_n].$$

For a point $\xi \in \partial X$ and $r > 0$, we set

$$U(\xi, r) = \left\{ \eta \in \partial X \mid \sup \left\{ \liminf_{n \rightarrow \infty} (x_n, x'_n) \mid (x_n) \in \xi, (x'_n) \in \eta \right\} \geq r \right\}.$$

We note that the collection of sets $U(\xi, r)$ forms a basis for a topology and endow ∂X with this topology. In fact, one can extend the sets $U(\xi, r)$ to $X \cup \partial X$, and define a topology there using the extended sets as bases.

LEMMA 3.6. *The map π is continuous and $\text{Isom}(X)$ -equivariant.*

Proof. The equivariance of π is obvious. To show continuity, we fix a point $\bar{h} \in \bar{X}^u$ and show the continuity of π at \bar{h} . Thus, we fix $r > 0$ and argue to show that there exists a neighbourhood V of \bar{h} in \bar{X}^u such that $\pi(V) \subset U(\pi(\bar{h}), r)$. We will denote the lift of \bar{h} in \bar{X}_o^u by h , set $t = 2(r + \delta)$ and fix a point $x \in X$ such that $h(x) < -t$. We let $V \subset \bar{X}_o^u$ be the open neighbourhood of \bar{h} corresponding to the set $\{h' \in \bar{X}_o^u \mid h'(x) < -t\}$. Fix $\bar{h}' \in V$ and denote its lift in \bar{X}_o^u by h' . Let (x_n) and (x'_n) be sequences in X converging to \bar{h} and \bar{h}' , respectively. In particular, we have

$$h(x) = \lim_{n \rightarrow \infty} (d(x_n, x) - |x_n|) \quad \text{and} \quad h'(x) = \lim_{n \rightarrow \infty} (d(x'_n, x) - |x'_n|).$$

Fix N such that for every $n > N$, both $(d(x_n, x) - |x_n|) < -t$ and $(d(x'_n, x) - |x'_n|) < -t$. Note that for $n > N$,

$$(x_n, x) = \frac{1}{2}(|x_n| + |x| - d(x_n, x)) \geq -\frac{1}{2}(d(x_n, x) - |x_n|) > \frac{1}{2}t$$

and similarly $(x'_n, x) > t/2$. Thus, we have

$$(x_n, x'_n) \geq \min\{(x_n, x), (x, x'_n)\} - \delta > \frac{1}{2}t - \delta = r.$$

It follows that $\liminf_{n \rightarrow \infty} (x_n, x'_n) \geq r$ and, in particular,

$$\sup \left\{ \liminf_{n \rightarrow \infty} (x_n, x'_n) \mid (x_n) \in \pi(\bar{h}), (x'_n) \in \pi(\bar{h}') \right\} \geq r.$$

Therefore, $\pi(\bar{h}') \in U(\pi(\bar{h}), r)$ and we conclude that indeed $\pi(V) \subset U(\pi(\bar{h}), r)$. □

Here and in the following, we use the term ‘ Γ -map’ to mean a measurable Γ -equivariant map between spaces equipped with a Γ -action.

The following lemma will be used, similarly to Lemma 3.4, in conjunction with coarse metric ergodicity to exclude the existence of Γ -maps to $\partial X^{(3)}$.

LEMMA 3.7. *Let Γ be a countable group acting by isometries on the separable hyperbolic space X . Then there is a Borel coarsely separable pseudo-metric ρ on $\partial X^{(3)}$ such that the natural Γ -action on $\partial X^{(3)}$ is an action by isometries of ρ . Moreover, if $\mathcal{U} \subseteq \Gamma$ acts on X with unbounded orbits, then the \mathcal{U} -orbits in $(\partial X^{(3)}, \rho)$ are also unbounded.*

Proof. Denote $\mathcal{T} = \partial X^{(3)}$ for convenience. Fix a Γ -invariant dense subset A of X .

We claim that there exists a Γ -equivariant map $\tau : \mathcal{T} \rightarrow \text{Bdd}(A)$ with the property that for any $x \in X$, we have that $\mathcal{T}_x = \tau^{-1}(\{B \in \text{Bdd}(X) : x \in B\}) \subseteq \mathcal{T}$ is closed.

To show the claim, we will in fact first define a closed subset \mathcal{T}_x for each $x \in A$ and then set $\tau(t) = \{x \in A : t \in \mathcal{T}_x\}$. Let $\delta > 0$ be a hyperbolicity constant for X . For $x \in A$, let $T_x \subseteq X^3$ be the set of all triples (x_1, x_2, x_3) such that $d(x_i, x_j) > d(x_i, x) + d(x, x_j) - 10\delta$ for all distinct $i, j \in \{1, 2, 3\}$. We then define \mathcal{T}_x to be the intersection of \mathcal{T} with closure of T_x (taken in $(X \cup \partial X)^3$). Notice that $\tau(t)$ defined as above is then a bounded, non-empty subset of A , and that τ is Γ -equivariant.

We can now pull-back the Hausdorff distance d_{Haus} on $\text{Bdd}(A)$ to \mathcal{T} , that is, we set $\rho(t, t') = d_{\text{Haus}}(\tau(t), \tau(t'))$. Clearly, ρ is a pseudo-metric and Γ acts on \mathcal{T} by isometries of ρ . Moreover, orbits are unbounded since (\mathcal{T}, ρ) is Γ -equivariantly isometrically embedded in $\text{Bdd}(A)$, and on the latter, the Γ -orbits are unbounded since orbits are unbounded in X (a similar argument applies to subsets of Γ). Coarse separability also arises from this isometric embedding together with separability of X , which gives coarse separability of $\text{Bdd}(A)$. It remains to show that ρ is Borel. To do so, it suffices to show that for all $r \in \mathbb{R}$, we have that $\Delta_r = \{(t, t') : \rho(t, t') > r\} \subseteq \mathcal{T} \times \mathcal{T}$ is Borel. Expanding the definitions, Δ_r is the set of pairs (t, t') such that $\tau(t)$ and $\tau(t')$ lie at Hausdorff distance more than r . In turn, this means that there exists $x \in \tau(t)$ such that all $y \in \tau(t')$ lie at distance more than r from x , or that the same holds switching the roles of t and t' . The first case is further equivalent to t' not lying in $\bigcup_{y \in A: d(x,y) \leq r} \mathcal{T}_y$, so that $\tau(t)$ does not contain any y with $d(x, y) \leq r$ (and the other case is similar). Hence, we have

$$\Delta_r = \bigcup_{x \in A} \left(\mathcal{T}_x \times \left(\mathcal{T} - \bigcup_{y \in A: d(x,y) \leq r} \mathcal{T}_y \right) \right) \cup \bigcup_{y \in A} \left(\left(\mathcal{T} - \bigcup_{x \in A: d(x,y) \leq r} \mathcal{T}_x \right) \times \mathcal{T}_y \right).$$

Since all \mathcal{T}_x are closed, Δ_r is Borel, as required. □

3.4. *Atom-less measures.* We now show a result needed in the next section, but not needed for the proof of Theorem 3.1; we include it here since we established the setup for its proof.

The result provides a Γ -map from atom-less probability measures on ∂X to probability measures on $\partial X^{(3)}$.

Given a hyperbolic space X , denote by $\text{Prob}_c(\partial X)$ the set of all atom-less probability measures on the standard Borel space ∂X .

LEMMA 3.8. *Let Γ be a countable group acting by isometries on the hyperbolic space X . Then there is a Γ -map*

$$\Psi : \text{Prob}_c(\partial X) \longrightarrow \text{Prob}(\partial X^{(3)}).$$

Moreover, for all $\mu, \nu \in \text{Prob}_c(\partial X)$ with μ absolutely continuous with respect to ν , we have

$$\left\| \frac{d\Psi(\mu)}{d\Psi(\nu)} \right\|_\infty \leq \left(\left\| \frac{d\mu}{d\nu} \right\|_\infty \right)^3.$$

Proof. We have a Γ -map

$$\Phi : \text{Prob}_c(\partial X) \longrightarrow \text{Prob}(\partial X^3), \quad \mu \mapsto \mu \times \mu \times \mu.$$

In fact, the assumption that μ has no atoms on a space ∂X implies that $\mu \times \mu \times \mu$ gives zero mass to the diagonals in $\partial X \times \partial X \times \partial X$, and so is fully supported on $\partial X^{(3)}$. We thus get a well-defined map

$$\text{Prob}_c(\partial X) \longrightarrow \text{Prob}(\partial X^{(3)}), \quad \mu \mapsto \mu \times \mu \times \mu,$$

as required.

To prove the moreover part, let $\mu, \nu \in \text{Prob}_c(\partial X)$ and let $f \in L^\infty(\partial X)$ be the Radon–Nikodym derivative of μ with respect to ν . Then $(x, y, z) \mapsto f(x)f(y)f(z)$ is the Radon–Nikodym derivative of $\Phi(\mu)$ with respect to $\Phi(\nu)$, so that $\|d\Phi(\mu)/d\Phi(\nu)\|_\infty = (\|d\mu/d\nu\|_\infty)^3$. The measures $\Psi(\mu)$ and $\Psi(\nu)$ are push-forward measures of $\Phi(\mu)$ and $\Phi(\nu)$, and push-forwards do not increase the L^∞ norm of Radon–Nikodym derivatives, so we obtain the required bound. □

3.5. *Proof of Theorem 3.1.* We are now ready to prove Theorem 3.1. By hypothesis, the group Γ is a countable group acting by isometries on the hyperbolic space X , without fixing any bounded subset of X , any point of ∂X and any pair of points of ∂X .

We start with some preliminary claims that we will use a few times.

First of all, we exclude the existence of Γ -maps with various targets.

CLAIM 3.9. *We have*

$$\begin{aligned} \text{Map}_\Gamma(B_-, \text{Prob}(\bar{X}^b)) &= \text{Map}_\Gamma(B_+, \text{Prob}(\bar{X}^b)) \\ &= \text{Map}_\Gamma(B_- \times B_+, \text{Prob}(\partial X^{(3)})) \\ &= \emptyset. \end{aligned}$$

Proof. Recall Lemmas 3.4 and 3.7 about the existence of Borel coarsely separable pseudo-metrics on \bar{X}^b and on $\partial X^{(3)}$, respectively, on which Γ acts by isometries with unbounded orbits. Then, in view of Proposition 2.15, all sets of Γ -maps in the statement need to be empty (note that any pseudo-metric is a coarse metric). □

We shall make use of the following general observation with Ω being B_- , B_+ or $B_- \times B_+$. Note that these Lebesgue Γ -spaces are ergodic.

CLAIM 3.10. *Let $\Gamma \curvearrowright \Omega$ be an ergodic action and $\Gamma \curvearrowright V$ be a Borel action on a Polish space. Suppose $V = V_0 \cup V_1$ with $V_0 \cap V_1 = \emptyset$ is a decomposition into Γ -invariant Borel sets and suppose $\text{Map}_\Gamma(\Omega, \text{Prob}(V_0)) = \emptyset$. Then the inclusion $\text{Prob}(V_1) \subset \text{Prob}(V)$ gives an identification*

$$\text{Map}_\Gamma(\Omega, \text{Prob}(V)) = \text{Map}_\Gamma(\Omega, \text{Prob}(V_1)).$$

Proof. Consider a measurable Γ -equivariant map $\phi: \Omega \rightarrow \text{Prob}(V)$. We also define $f_0, f_1: \Omega \rightarrow [0, 1]$ by setting $f_i(\omega) = \phi(\omega)(V_i)$. These are Γ -invariant measurable functions. By ergodicity, $f_i(\omega) = c_i$ constants and $c_i \geq 0$ with $c_0 + c_1 = 1$. Assuming $c_0 > 0$, one obtains a contradiction, because the normalized restriction $\phi_0(\omega) = c_0^{-1} \cdot \phi(\omega)|_{V_0}$ is a Γ -map to $\text{Prob}(V_0)$, which are ruled out by assumption. Hence, $c_0 = 0$ and $\phi(\omega)$ gives full mass to V_1 , and hence ϕ can be considered as a Γ -map $\Omega \rightarrow \text{Prob}(V_1)$. \square

CLAIM 3.11. *For all $\psi_- \in \text{Map}(B_-, \text{Prob}(\partial X))$ and $\psi_+ \in \text{Map}(B_+, \text{Prob}(\partial X))$, the diagonal $\Delta = \{(t, t) \mid t \in \partial X\} \subset \partial X^2$ satisfies*

$$\psi_-(x) \times \psi_+(y)(\Delta) = 0$$

for a.e. $(x, y) \in B_- \times B_+$.

Proof. Given a probability measure $\eta \in \text{Prob}(\partial X)$, denote by $\text{Atom}(\eta)$ the set of atoms of η ; and for $\epsilon > 0$, by $\text{Atom}_\epsilon(\eta) = \{a \mid \eta(\{a\}) \geq \epsilon\}$ the subset of atoms with weight $\geq \epsilon$. The cardinality of $\text{Atom}_\epsilon(\eta)$ is bounded by $\lceil 1/\epsilon \rceil$ and $\text{Atom}(\eta)$ is at most countable. Let $\alpha, \beta \in \text{Prob}(\partial X)$ be two probability measures. A standard application of Fubini's theorem shows that

$$\alpha \times \beta(\Delta) = \sum_{t \in \partial X} \alpha(\{t\}) \cdot \beta(\{t\}),$$

where the non-zero terms correspond to $t \in \text{Atom}(\alpha) \cap \text{Atom}(\beta)$.

For $(x, y) \in B_- \times B_+$, observe that the sets $\text{Atom}(\psi_-(x))$, $\text{Atom}(\psi_+(y))$ and their intersection vary measurably and Γ -equivariantly.

Assume, towards contradiction, that $\psi_-(x) \times \psi_+(y)(\Delta) > 0$. This is equivalent to asserting that for some $\epsilon > 0$, the set

$$\{(x, y) \in B_- \times B_+ \mid \text{Atom}_\epsilon(\psi_-(x)) \cap \text{Atom}_\epsilon(\psi_+(y)) \neq \emptyset\}$$

has positive measure. Since the Γ -action on $B_- \times B_+$ is ergodic, we deduce that this set is conull, and the cardinality of the set $A(x, y) = \text{Atom}_\epsilon(\psi_-(x)) \cap \text{Atom}_\epsilon(\psi_+(y))$ is essentially constant.

By Fubini's theorem for a.e. $x \in B_-$, we have $A(x, y) = A(x, y')$ for a.e. $y, y' \in B_+$. Thus, $A(x, -)$ is essentially independent of the second variable. For the same reason, it is essentially independent of the first variable; hence, $A(x, y) = A$ for a fixed finite subset $A \subset \partial X$. Since the map $(x, y) \mapsto A(x, y) = A$ is equivariant, it follows that Γ has a finite orbit O on ∂X . By hypothesis, Γ does not fix a point or a pair of points in ∂X , so that $|O| \geq 3$. By considering the restriction of the pseudo-metric on $\partial X^{(3)}$ from Lemma 3.7

to the finite set of triples of distinct points in O , we obtain a Γ -invariant orbit in $\partial X^{(3)}$, contradicting that the action on $\partial X^{(3)}$ has unbounded orbits.

Hence, $\psi_-(x) \times \psi_+(y)(\Delta) = 0$ almost everywhere on $B_- \times B_+$, as claimed. □

End of proof of Theorem 3.1. Since the actions $\Gamma \curvearrowright B_-$, $\Gamma \curvearrowright B_+$ are amenable, the spaces $\text{Map}_\Gamma(B_\pm, \text{Prob}(\bar{X}))$ are not empty. Consider the Γ -invariant Borel decomposition $\bar{X} = \bar{X}^b \cup \bar{X}^u$. In view of Claims 3.9 and 3.10, we have

$$\text{Map}_\Gamma(B_\pm, \text{Prob}(\bar{X})) = \text{Map}_\Gamma(B_\pm, \text{Prob}(\bar{X}^u))$$

and post-composing with the Γ -map $\pi_* : \text{Prob}(\bar{X}^u) \rightarrow \text{Prob}(\partial X)$ (see §3.3), we obtain

$$\text{Map}_\Gamma(B_\pm, \text{Prob}(\partial X)) \neq \emptyset.$$

Let us choose some measurable Γ -equivariant maps

$$\psi_- : B_- \longrightarrow \text{Prob}(\partial X), \quad \psi_+ : B_+ \longrightarrow \text{Prob}(\partial X).$$

Consider the partition of $(\partial X)^3$ into Γ -invariant Borel sets

$$(\partial X)^3 = \partial X^{(3)} \cup \Delta_{12} \cup \Delta'_{13} \cup \Delta'_{23},$$

where $\Delta_{ij} = \{(\xi_1, \xi_2, \xi_3) \mid \xi_i = \xi_j\}$, $\Delta_{123} = \{(\xi, \xi, \xi) \mid \xi \in \partial X\}$, $\Delta'_{ij} = \Delta_{ij} \setminus \Delta_{123}$.

Note that $\text{Map}_\Gamma(B_- \times B_+, \text{Prob}(\partial X^{(3)})) = \emptyset$ by Claim 3.9. Therefore, using Claim 3.10, every measurable Γ -map $\Psi : B_- \times B_+ \longrightarrow \text{Prob}((\partial X)^3)$ gives full measure to $\Delta_{12} \cup \Delta'_{13} \cup \Delta'_{23}$. We shall apply this to the map

$$\Psi(x, y) = \psi_-(x) \times \psi_+(y) \times \frac{1}{2}(\psi_-(x) + \psi_+(y)).$$

Note that Claim 3.11 implies that almost everywhere, $\Psi(x, y)(\Delta_{12}) = 0$.

Let Δ denote the diagonal in $\partial X \times \partial X$. The projection $\Delta'_{13} \rightarrow \Delta$, $(\xi, \eta, \xi) \mapsto (\xi, \xi)$ shows that

$$\Psi(x, y)(\Delta'_{13}) \leq \Psi(x, y)(\Delta_{13}) = \frac{1}{2}\psi_-(x) \times \psi_+(y)(\Delta) + \frac{1}{2}\psi_-(x) \times \psi_-(x)(\Delta).$$

Moreover, the first term of the right-hand side vanishes by Claim 3.11, so that $\Psi(x, y)(\Delta'_{13}) \leq \frac{1}{2}\psi_-(x) \times \psi_-(x)(\Delta)$. Similarly, using the projection $\Delta'_{23} \rightarrow \Delta$, $(\eta, \xi, \xi) \mapsto (\xi, \xi)$ gives

$$\Psi(x, y)(\Delta'_{23}) \leq \Psi(x, y)(\Delta_{23}) = \frac{1}{2}\psi_+(y) \times \psi_+(y)(\Delta).$$

We conclude that

$$\begin{aligned} 1 &= \Psi(x, y)(\Delta'_{13}) + \Psi(x, y)(\Delta'_{23}) \\ &\leq \frac{1}{2}\psi_-(x) \times \psi_-(x)(\Delta) + \frac{1}{2}\psi_+(y) \times \psi_+(y)(\Delta) \\ &\leq 1, \end{aligned}$$

so that $\psi_-(x) \times \psi_-(y)(\Delta) = \psi_+(y) \times \psi_+(y)(\Delta) = 1$. Recall that, by Fubini's theorem, for any probability measure ν , one has $\nu \times \nu(\Delta) = 1$ if and only if ν is a Dirac measure $\nu = \delta_z$ for some z . Hence, there exist maps $\phi_\pm : B_\pm \longrightarrow \partial X$ so that

$$\psi_-(x) = \delta_{\phi_-(x)}, \quad \psi_+(y) = \delta_{\phi_+(y)}.$$

The maps $\phi_{\pm} : B_{\pm} \rightarrow \partial X$ are measurable and Γ -equivariant because the maps $\psi_{\pm} : B_{\pm} \rightarrow \text{Prob}(\partial X)$ are.

We also note that, since $\Psi(\Delta_{12}) = 0$ almost everywhere, we have $\phi_{-}(x) \neq \phi_{+}(y)$ for a.e. $(x, y) \in B_{-} \times B_{+}$. Therefore, the map $\phi_{\bowtie} = \phi_{-} \times \phi_{+}$ can be viewed as a measurable Γ -equivariant map $B_{-} \times B_{+} \rightarrow \partial X^{(2)}$ into distinct pairs.

Next, we show that ϕ_{\pm} are essentially unique. Indeed, we showed that every $\psi_{\pm} \in \text{Map}_{\Gamma}(B_{\pm}, \text{Prob}(\partial X))$ takes values in Dirac measures. Assume we have two pairs of Γ -maps $\psi_{\pm}^i : B_{\pm} \rightarrow \text{Prob}(\partial X)$ with $i = 1, 2$. Consider the Γ -maps $B_{\pm} \rightarrow \text{Prob}(\partial X)$ given by $z \mapsto \frac{1}{2}(\psi_{\pm}^1(z) + \psi_{\pm}^2(z))$. Such a map has the form $z \mapsto \delta_{\phi_{\pm}(z)}$ only if $\psi_{\pm}^1(z) = \psi_{\pm}^2(z) = \delta_{\phi_{\pm}(z)}$.

Next, consider an arbitrary map $\theta \in \text{Map}_{\Gamma}(B_{-} \times B_{+}, \partial X)$, consider the measurable Γ -equivariant map to be defined by

$$\Theta : B_{-} \times B_{+} \rightarrow (\partial X)^3, \quad \Theta(x, y) = (\theta(x, y), \phi_{-}(x), \phi_{+}(y)).$$

Using the Γ -equivariant Borel decomposition $(\partial X)^3 = \partial X^{(3)} \cup \Delta'_{12} \cup \Delta'_{13} \cup \Delta_{23}$ and ergodicity of the Γ -action on $B_{-} \times B_{+}$, we deduce that $\Theta(x, y)$ lies in one of these sets for a.e. (x, y) . Lemma 3.7 combined with coarse metric ergodicity of $B_{-} \times B_{+}$ rules out the set of distinct triples $\partial X^{(3)}$. Since the 23-components of Θ is just ϕ_{\bowtie} , and the latter take values in the space of distinct pairs $\partial X^{(2)}$, rule out Δ_{23} . It follows that, up to a null set, either $\Theta(x, y) \in \Delta'_{12}$ or $\Theta(x, y) \in \Delta'_{13}$. This corresponds to $\theta = \psi_{-} \circ \text{pr}_{-}$ or $\theta = \psi_{+} \circ \text{pr}_{+}$ as claimed in assertion (ii).

Finally, notice that assertion (iii) can be deduced from assertion (ii) by looking at the coordinates in $(\partial X)^{(2)}$.

4. Classification of actions on hyperbolic spaces

In this section, we prove Theorem 1.1.

4.1. *Lattices acting on hyperbolic spaces.* Recall that a subset A of a metric space is *coarsely dense* if there exists a constant R such that X is the R -neighbourhood of A . Recall moreover that given a group Γ acting on a hyperbolic space X , its limit set is the set of boundary points that are equivalence classes of sequence of points γx for some fixed $x \in X$.

Definition 4.1. We say that an action $\Gamma \curvearrowright X$ on a hyperbolic space is *coarsely minimal* if X is unbounded, the limit set of Γ in ∂X is not a single point and every quasi-convex Γ -invariant subset of X is coarsely dense.

Note that coarse minimality is a stronger requirement than asking that the orbits of Γ have full limit set. Indeed, consider for example, the free group F_2 acting on its Cayley tree T , which is the universal cover of a graph with 1 vertex and two loops. Glueing, for each n , an interval of length n to that vertex, one obtains a locally infinite graph whose universal cover is a tree \tilde{T} on which F_2 acts with full limit set. The tree T embeds as a convex F_2 -invariant subtree of \tilde{T} , but T is not coarsely dense in \tilde{T} , so that the F_2 -action on \tilde{T} is not coarsely minimal.

Notice that if H is an infinite normal subgroup of infinite index of the hyperbolic G , then the action of H on a Cayley graph of G is coarsely minimal, but not cobounded.

Given a metric space X and $C \geq 0$, denote by $\text{Bdd}_C(X)$ the set of all closed subsets of diameter at most C , endowed with the Hausdorff metric. Notice that $\text{Bdd}_C(X)$ is quasi-isometric to X .

Definition 4.2. Two actions $\Gamma \curvearrowright X_1, X_2$ on metric spaces X_1, X_2 are *equivalent* if there exists an equivariant quasi-isometry $X_1 \rightarrow \text{Bdd}_C(X_2)$ for some $C \geq 0$.

The reason for having $\text{Bdd}_C(X_2)$ instead of X_2 is that we want to allow the situation where some group element has a fixed point in X_1 but merely a bounded orbit in X_2 ; for example, we want to declare all actions on bounded metric spaces to be equivalent.

We note that the definition above is very similar to [1, Definition 3.5] and, in fact, the two definitions are easily seen to be equivalent for cobounded actions.

Remark 4.3. Consider an action $\Gamma \curvearrowright X$ on a geodesic hyperbolic space.

(1) If the action is cobounded, then it is coarsely minimal.

The following two items follow from a construction well known to experts, namely taking the coarse convex hull of an orbit and approximating it with a graph; this is explained, for example, in [12, Remark 4].

(2) If the limit set of Γ is not a single point, then there is a coarsely minimal action $\Gamma \curvearrowright Y$ on a geodesic hyperbolic space Y and an equivariant quasi-isometric embedding $Y \rightarrow X$.

(3) If Γ is countable and $\Gamma \curvearrowright X$ is coarsely minimal, then $\Gamma \curvearrowright X$ is equivalent to an action on a separable geodesic hyperbolic space (in fact, a graph).

Consider a semisimple Lie group $G = G_1 \times \cdots \times G_N$ without compact factors. Assume that either $N \geq 2$ or $N = 1$ and $G = G_1$ has rank at least 2.

Re-order the factors in such a way that G_i is rank-one if and only if $1 \leq i \leq n$ for some $n \leq N$.

We now re-state our main theorem.

THEOREM 4.4. *Let Γ be an irreducible lattice in $G = G_1 \times \cdots \times G_N$ as above. Then every coarsely minimal action of Γ on a geodesic hyperbolic space is equivalent to the action*

$$\Gamma \longrightarrow G \xrightarrow{\text{pr}_i} G_i \longrightarrow \text{Isom}(X_i, d_i)$$

for some $i \in \{1, \dots, n\}$, where X_i is the symmetric space for G_i .

Let us extend the notation of the theorem by denoting X_j , for $j > n$, the symmetric space for G_j .

Let $\Gamma \curvearrowright (X, d)$ be a coarsely minimal action of Γ on the geodesic hyperbolic space X . Denote by d_i the pseudo-metric on Γ corresponding to $\Gamma \curvearrowright X_i$ (with respect to some basepoint x_i , for $i \leq n$), and d the pseudo-metric corresponding to $\Gamma \curvearrowright (X, d)$.

4.2. *Ruling out elementary actions.* To be able to apply Theorem 3.1, we have to rule out that Γ fixes a pair of points in ∂X (the case that it fixes one point being ruled out by hypothesis). If that were the case, the subgroup Γ' of index at most 2 of Γ that fixes a boundary point would admit the quasimorphism described in [8, Proposition 3.7]. According to [6, 7], Γ' does not admit unbounded quasimorphisms, so that according to [8, Lemma 3.8], the action $\Gamma \curvearrowright X$ has a single limit point, contradicting the coarse minimality of the action. From now on, we will assume that Γ does not fix a point or a pair of points in X .

Finally, we can assume that X is separable by Remark 4.3.

4.3. *Boundary map from one factor.* For every $1 \leq i \leq N$, we let $B_i = G_i/P_i$ and we let ν_i be a measure in the Haar class on B_i , and similarly we denote $B = G/P = B_1 \times \cdots \times B_N$. Note that we are in the setting of Proposition 2.9 about coarse metric ergodicity.

Theorem 3.1 affords now two Γ -maps $B \rightarrow \partial X$ satisfying various properties. The first of these properties implies that these two maps must coincide almost everywhere. We assume henceforth that both maps are identical, and we denote it by $\phi: B \rightarrow \partial X$.

CLAIM 4.5. *The map ϕ factors through one of B_i . In other words, there is an index $i \in \{1, \dots, N\}$ and a Γ -map $B_i \xrightarrow{\phi_i} \partial X$ such that*

$$\phi: B \xrightarrow{\text{pr}_i} B_i \xrightarrow{\phi_i} \partial X.$$

Proof. The equivariant map $\phi: B \rightarrow \partial X$ cannot be essentially constant, because Γ does not have a fixed point in ∂X . Fix an $i \in \{1, \dots, N\}$ so that ϕ is not essentially constant over the B_i -factor of $B = B_1 \times \cdots \times B_N$. We let B' be the product of the other factors, and identify $B \simeq B_i \times B'$. Using this identification, we consider the map

$$\Phi: B_i \times B' \times B_i \times B' \rightarrow \partial X^2, \quad (x, y, x', y') \mapsto (\phi(x, y), \phi(x, y')).$$

By Theorem 3.1(iii), we have three cases: $\Phi(B \times B)$ is contained in the diagonal $\Delta \subset \partial X^2$, $\Phi = \phi_{\bowtie}$ or $\Phi = \tau \circ \phi_{\bowtie}$, where $\phi_{\bowtie} = \phi \times \phi$ and $\tau(\xi, \eta) = (\eta, \xi)$. In the first case, we see that $\phi(x, y)$ is independent of $y \in B'$ and therefore descends to a Γ -map $B_i \rightarrow \partial X$ as required. In the second and third cases, ϕ is independent of $x \in B_i$, contradicting our choice of i . □

From now on, we fix the index $i \in \{1, \dots, N\}$ afforded by Claim 4.5.

4.4. *The factor B_i is associated with a rank-one factor.* We now explain that the factor B_i afforded by Claim 4.5 is associated with a rank-one factor, that is, $i \leq n$. We argue by contradiction, assuming $i > n$, that is, G_i is of higher rank.

As before, we let $P_i < G_i$ be a minimal parabolic. In the following, we will omit the index i and denote $P = P_i$. Let $A < P$ be a maximal split torus. We let $W = N_{G_i}(A)/Z_{G_i}(A)$ be the corresponding Weyl group and let $S \subset W$ be the standard Coxeter generators associated with the positivity defined by P . Letting $Z = Z_{G_i}(A)$ be the centralizer of A , we note that W acts naturally on G_i/Z by G_i -automorphisms.

As usual, we identify the set S with the set of simple roots of G_i associated with the pair (A, P) . Any subset T of S generates a subgroup $W_T < W$ and it corresponds to a standard parabolic $P_T < G_i$ containing $P = P_\emptyset$. All the subgroups of G_i containing P are of this form. Denoting by $\pi_T : G_i/Z \rightarrow G_i/P_T$ the standard map coming from the inclusion $Z < P < P_T$, we note that

$$W_T = \{w \in W \mid \pi_T \circ w = \pi_T\}.$$

We let $\pi = \pi_\emptyset : G_i/Z \rightarrow G_i/P$ be the standard map and let $w_0 \in W$ be the longest element (with respect to the word distance induced by S). It is a standard fact that the map $\pi \times \pi \circ w_0 : G_i/Z \rightarrow G_i/P \times G_i/P$ is injective and its image is Zariski open (this image is the big cell in the Bruhat decomposition of $G_i/P \times G_i/P$).

We also endow G_i/Z with the Haar measure class. By the preceding paragraph, we may identify it, as measured G_i -spaces, with $B_i \times B_i = G_i/P \times G_i/P$ via the map $\pi \times \pi \circ w_0$.

We set $\phi_i : B_i \rightarrow \partial X$ to be the map given in Claim 4.5. Note that ϕ_i is not essentially constant, as ∂X has no Γ -fixed points. We thus may find a bounded measurable function $f_0 : \partial X \rightarrow \mathbb{C}$ such that $f_0 \circ \phi_i$ is not essentially constant. We fix such a function f_0 .

We consider the map $\psi = \phi_i \circ p_1 : B_i \times B_i \rightarrow \partial X$, where $p_1 : B_i \times B_i \rightarrow B_i$ is the projection on the first factor. Under our identifications $B_i = G_i/P$ and $B_i \times B_i = G_i/Z$, the projection p_1 is identified with the map $\pi = \pi_\emptyset : G_i/Z \rightarrow G_i/P$. We set

$$U = \{w \in W \mid \psi \circ w \text{ agrees almost everywhere with } \psi\} < W.$$

By Theorem 3.1(ii), we conclude that $U < W$ is of index at most 2.

We shall now show that U is contained in a proper parabolic subgroup W_T of W . Consider now the algebra $L^\infty(G_i/Z)$ and its subalgebra $\pi^*(L^\infty(G_i/P))$ consisting of functions pulled back from $L^\infty(G_i/P)$ under $\pi : G_i/Z \rightarrow G_i/P$ (which we identify with $p_1 : B_i \times B_i \rightarrow B_i$). Consider the subalgebra

$$\{f \in L^\infty(G_i/Z) \mid f \in L^\infty(G_i/P) \text{ and for every } u \in U, f \circ u \text{ agrees almost everywhere with } f\}.$$

This is a weak*-closed G_i -invariant subalgebra of $L^\infty(G_i/Z)$. By Mackey’s point realization theorem, this algebra coincides with the subalgebra of functions pulled back from a G_i -factor of G_i/Z . As all functions in it are pulled back from G_i/P , this factor is of the form G_i/P_T for some $T \subset S$. As the algebra includes the non-constant function $f_0 \circ \psi$, we conclude that $P_T \neq G_i$, and thus $T \neq S$ and $W_T \neq W$. We have that $\pi_T \circ u = \pi_T$ for every $u \in U$, and thus $U < W_T$. It follows that W_T is of index 2 in W and, in particular, it is a normal subgroup.

To obtain a contradiction and conclude this step of the proof, we use that W is an irreducible finite Coxeter group with $|W| > 2$, since G_i is simple of rank ≥ 2 . Therefore, there is $t \in T$ and $s \in S \setminus T$ such that s and t do not commute. In particular, sts is a reduced word in W . Since W_T has index 2, we have $sts \in W_T$. By [5, Corollary 1 in Ch. IV, §1, no. 1.8], we deduce that $s \in T$, which is a contradiction.

4.5. *Bounded in G_i is d -bounded.* Next, we show that d is ‘smaller’ than d_i for i as in Claim 4.5.

CLAIM 4.6. *There exist L, C so that for all γ, γ' , we have*

$$d(\gamma, \gamma') \leq L \cdot d_i(\gamma, \gamma') + C.$$

Proof. As G is of higher rank, we deduce from §4.4 that $N > 1$ and, in particular, we get that $\text{pr}_i(\Gamma)$ is dense in G_i . Hence, in the metric d_i any pair of points is connected by a $(1, 1)$ -quasi-geodesic, and to prove the claim, it suffices to show that sequences that are bounded in G_i are bounded in (Γ, d) . In other words, it suffices to show that any sequence $\{\gamma_j\}$ in Γ for which $\{\text{pr}_i(\gamma_j)\}$ is precompact in G_i , one has

$$\sup_j d(\gamma_j, 1) < +\infty.$$

Let $\mu = \phi_* v_i \in \text{Prob}(\partial X)$ be the push-forward of the measure v_i on B_i . By the metric ergodicity of B_i , μ has no atomic part. Indeed, if it had, we would get a countable invariant subset of ∂X and upon endowing it with the discrete metric, in view of the metric ergodicity of B_i , we will conclude that this set contains a single point which is Γ -invariant, which contradicts the hypothesis.

By Lemma 3.8, we have a Γ -map

$$\Psi : \text{Prob}_c(\partial X) \longrightarrow \text{Prob}(\partial X^{(3)}),$$

where, as before, $\text{Prob}_c(\partial X)$ is the set of all atom-less probability measures on ∂X .

We assume that $\{\gamma_j\}$ is such that $\{\text{pr}_i(\gamma_j)\}$ is precompact in G_i . We now need the following lemma.

LEMMA 4.7. *Given any precompact subset $\{\gamma_j\} \subseteq G_i$, the corresponding Radon–Nikodym derivatives are uniformly bounded, that is, we have*

$$\sup_{j \geq 1} \left\| \frac{d\gamma_j v_i}{dv_i} \right\|_\infty < +\infty.$$

Proof. Since v_i is the Patterson–Sullivan measure for G_i , we have for any $g \in G_i$ and a.e. $\xi \in \partial X_i$,

$$\frac{dg v_i}{dv_i}(\xi) = e^{-\delta_i \beta(g, \xi)},$$

where δ_i is a constant associated with the symmetric space X_i for G_i and $\beta(g, \xi)$ is the Busemann function. Since the Busemann function is bounded by the distance in X_i , $|\beta(g, \xi)| \leq d_i(g o, o)$, the lemma is proved. □

Hence, by Lemma 4.7, we have

$$\sup_{j \geq 1} \left\| \frac{d\gamma_j v_i}{dv_i} \right\|_\infty = C < +\infty.$$

We thus have the same bound on the Radon–Nikodym derivatives of μ

$$\sup_{j \geq 1} \left\| \frac{d\gamma_j \mu}{d\mu} \right\|_\infty \leq C$$

and also, by the ‘moreover’ part of Lemma 3.8,

$$\sup_{j \geq 1} \left\| \frac{d\Psi(\gamma_j \mu)}{d\Psi(\mu)} \right\|_\infty \leq C^3.$$

Setting $U = \text{Prob}(\partial X^{(3)})$, recall the maps $\beta_\epsilon : \text{Prob}(\partial X^{(3)}) \rightarrow \text{Bdd}(\partial X^{(3)})$ from §2.6, which are Γ -equivariant (as they are equivariant with respect to all isometries). Fix $\epsilon \in (0, 1/2)$ and set $R = r_{\epsilon/C}(\Psi(\mu)) + 1$. Fixing j and applying Lemma 2.16 with $m_1 = \Psi(\mu)$ and $m_2 = \Psi(\gamma_j \mu)$, we get

$$\gamma_j(\beta_\epsilon \circ \Psi(\mu)) = \beta_\epsilon \circ \Psi(\gamma_j \mu) \subset N_R(\beta_{\epsilon/C}(\Psi(\mu))).$$

Since $N_R(\beta_{\epsilon/C}(\Psi(\mu)))$ is a bounded subset of $\partial X^{(3)}$, we get that the sequence $\{\gamma_j\}$ has bounded orbits in $\partial X^{(3)}$, whence it is bounded in (Γ, d) by the ‘moreover’ part of Lemma 3.7. □

4.6. *Unbounded in G_i is d -unbounded.*

CLAIM 4.8. *There exist L, C so that for all γ, γ' , we have*

$$d_i(\gamma, \gamma') \leq L \cdot d(\gamma, \gamma') + C.$$

Proof. Let $\gamma_0 \in \Gamma$ be a loxodromic element for $\Gamma \curvearrowright X$, which exists by Gromov’s classification of actions on hyperbolic spaces [11, §8]. Since the identity map $(\Gamma, d_i) \rightarrow (\Gamma, d)$ is coarsely Lipschitz, γ_0 is loxodromic for $\Gamma \curvearrowright X_i$ as well. Let us choose a constants A so that for each $j \geq 0$, we have $j/A \leq d_i(1, \gamma_0^j) \leq Aj$ and $j/A \leq d(1, \gamma_0^j) \leq Aj$. Note that, up to increasing A , since G_i is rank-one and γ_0 is loxodromic, the set

$$\{\kappa \gamma_0^j \mid \kappa \in K_i, j \geq 0\}$$

is A -dense in X_i , where we denote by K_i a maximal compact subgroup of G_i . This is because, since K_i acts transitively on ∂X , there exists a constant C such that given any two points on a sphere around x_i , there is an element of K_i that moves the first point C -close to the second one.

Since the rank of G_i equals one, we know by hypothesis that $N > 1$ and, in particular, we get that $\text{pr}_i(\Gamma)$ is dense in G_i . Approximating elements of K_i by elements of Γ , we get that the set

$$\{\kappa \gamma_0^j \mid d_i(1, \kappa) \leq 1, j \geq 0\}$$

is $A + 1$ -dense in Γ for the metric d_i . Enlarging A if necessary, let us further assume that $d(1, \gamma) \leq Ad_i(1, \gamma) + A$ for all $\gamma \in \Gamma$.

Consider an arbitrary $\gamma \in \Gamma$. Using the above, we find $\kappa \in \Gamma$ such that $d_i(1, \kappa) \leq 1$ and $d_i(\kappa \gamma, \gamma_0^j) \leq A + 1$ for some $j \geq 0$. Therefore, we obtain

$$\begin{aligned}
d_i(1, \gamma) &= d_i(\kappa, \kappa\gamma) \\
&\leq d_i(1, \gamma_0^j) + A + 2 \\
&\leq Aj + A + 2 \\
&\leq A^2 d(1, \gamma_0^j) + A + 2 \\
&\leq A^2(d(1, \kappa) + d(\kappa, \kappa\gamma) + d(\kappa\gamma, \gamma_0^j)) + A + 2 \\
&\leq A^2 d(1, \gamma) + A^3 + 2A^2 + 3A + 2,
\end{aligned}$$

as required. \square

4.7. Conclusion. We have seen in §4.4 that the rank of G_i is 1. Claims 4.6 and 4.8 imply that there is a Γ -equivariant quasi-isometric embedding $f : X_i \rightarrow \text{Bdd}_C(X)$ for some $C \geq 0$ (since (Γ, d_i) is Γ -equivariantly quasi-isometric to X_i , and there is a Γ -equivariant quasi-isometric embedding $(\Gamma, d_i) \rightarrow X$). Since X_i is hyperbolic, the image of that quasi-isometric embedding is quasi-convex. Since the Γ -action on X is coarsely minimal, it follows that the Γ -action on X is cobounded, so that the quasi-isometric embedding $(\Gamma, d_i) \rightarrow X$ is in fact a quasi-isometry. This proves that $\Gamma \curvearrowright X_i$ is indeed equivalent to $\Gamma \curvearrowright X$, as required.

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