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Abstract

Given a height at most two Landweber exact $\mathbb{E}_\infty$-ring E whose homotopy is concentrated in even degrees, we show that any complex orientation of E which satisfies the Ando criterion admits a unique lift to an $\mathbb{E}_\infty$-complex orientation $\mathrm{MU} \to E$. As a consequence, we give a short proof that the level n elliptic genus lifts uniquely to an $\mathbb{E}_\infty$-complex orientation $\mathrm{MU} \to \mathrm{tmf}_1(n)$ for all $n \geq 2$.

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1. Introduction

Complex-oriented ring spectra play a central role in the chromatic approach to homotopy theory. Given a homotopy associative ring spectrum E, recall that a complex orientation is a choice of class $u \in \widetilde{E}^2(\mathbb{CP}^\infty)$ with the property that its restriction along $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is the unit $1 \in E^0(*) \cong \widetilde{E}^2(S^2)$. A complex orientation determines an isomorphism of graded rings

$$E^*(\mathbb{CP}^\infty) \cong E^*[\![u]\!].$$

Complex orientations may also be described in terms of the complex cobordism spectrum MU: complex orientations of E are in natural bijection with maps of homotopy associative ring spectra $\mathrm{MU} \to E$. For more background on complex orientations, we refer the reader to [Ada74, Part II].

The complex cobordism spectrum MU admits much more structure than that of a homotopy associative ring spectrum: it is an $\mathbb{E}_\infty$-ring spectrum. When E also admits the structure of an $\mathbb{E}_\infty$-ring spectrum, it is natural to ask whether a given complex orientation is induced by a map of $\mathbb{E}_\infty$-ring spectra

$$\mathrm{MU} \to E.$$

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We will call such maps $\mathbb{E}_\infty$-complex orientations. The $\mathbb{E}_\infty$-complex orientations of an $\mathbb{E}_\infty$-ring may be viewed as particularly canonical complex orientations.

Indeed, many of the most familiar (not necessarily complex) orientations admit lifts to $\mathbb{E}_\infty$-orientations. For example, the Atiyah–Bott–Shapiro orientations [ABS64]

$$\mathrm{MSpin} \to \mathrm{ko} \quad \text{and} \quad \mathrm{MSpin}^{\mathbb{C}} \to \mathrm{ku}$$

were refined to $\mathbb{E}_\infty$-orientations by Joachim [Joa04], who gave an explicit geometric construction of such an $\mathbb{E}_\infty$-orientation. Indeed, one expects that any orientation of geometric origin may, with enough care, be refined to an $\mathbb{E}_\infty$-orientation. A more sophisticated example is the Ando–Hopkins–Rezk $\mathbb{E}_\infty$-String orientation of the connective $\mathbb{E}_\infty$-ring of topological modular forms tmf (see [AHR10])

$$\mathrm{MString} \to \mathrm{tmf},$$

which refines the Witten genus [Wit87, Wit88]. While it is expected that this $\mathbb{E}_\infty$-orientation has a geometric origin, and much work has gone into developing such a viewpoint (for a small sampling, see [ST04, ST11, Cos10, Cos11, DH11, BE21]), such a description has so far remained elusive. In Theorem 1.7, we will prove that the Hirzebruch level n elliptic genera [Hir88], [Wit88, § 5] for $n \geq 2$ may be lifted to $\mathbb{E}_\infty$-complex orientations

$$\mathrm{MU} \to \mathrm{tmf}_1(n).$$

1.1 The Ando criterion

An algebraic approximation of what it means for a complex orientation $\mathrm{MU} \to E$ to be $\mathbb{E}_\infty$ is given by the *Ando criterion*. It asks that the complex orientation be compatible in a suitable sense with the power operations in E. In many cases, the Ando criterion is equivalent to the property that the complex orientation $\mathrm{MU} \to E$ be a map of $\mathbb{H}_\infty$-ring spectra (see, e.g., [AHS04, Proposition 6.1]).

Now let E denote an $\mathbb{E}_\infty$, or more generally an $\mathbb{H}_\infty$-ring spectrum, with a fixed complex orientation u.

Notation 1.1. Given a complex vector bundle $V \to X$ of dimension d, we let $t_u(V) \in E^{2d}(\mathrm{Th}(V))$ denote the Thom class of V.

Fix a prime p, let ρ denote the vector bundle over BC_p corresponding to the complex regular representation of C_p, and let γ_1 denote the tautological bundle over $\mathbb{CP}^\infty$. Let $I_{\mathrm{tr}} \subset E^*(\mathrm{BC}_p)$ denote the transfer ideal. Recall from [HL18, § 7] that there are additive power operations

$$\Psi_u : E^{2*}(\mathrm{Th}(\gamma_1)) \to E^{2p*}(\mathrm{Th}(\rho \boxtimes \gamma_1))/I_{\mathrm{tr}}.$$

Definition 1.2. We say that a complex orientation of E satisfies the Ando criterion at the prime p if

$$\Psi_u(t_u(\gamma_1)) = t_u(\rho \boxtimes \gamma_1)$$

in $E^{2p}(\mathrm{Th}(\rho \boxtimes \gamma_1))/I_{\mathrm{tr}}$. We say that a complex orientation of E satisfies the Ando criterion if it satisfies the Ando criterion for all primes p.

Remark 1.3. If E is p-local, then $E^*(\mathrm{BC}_\ell)/I_{\mathrm{tr}} = 0$ for all primes $\ell \neq p$. Therefore, a complex orientation of a p-local $\mathbb{E}_\infty$-ring satisfies the Ando criterion if and only if it satisfies the Ando criterion at p.

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The complex cobordism spectrum MU, equipped with the canonical complex orientation, satisfies the Ando criterion.[1] It follows that any $\mathbb{E}_\infty$-complex orientation, or more generally any $\mathbb{H}_\infty$-complex orientation, satisfies the Ando criterion.

1.2 Results

The first main theorem of this paper states that for many $\mathbb{E}_\infty$-rings of height at most two, this condition is sufficient, and that the resulting $\mathbb{E}_\infty$-complex orientations are determined up to homotopy by their underlying complex orientations.

Theorem 1.4. *Let E denote a height at most two Landweber exact $\mathbb{E}_\infty$-ring whose homotopy is concentrated in even degrees. Then any complex orientation $\mathrm{MU} \to E$ which satisfies the Ando criterion lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-ring map $\mathrm{MU} \to E$.*

In the above theorem, we say that a Landweber exact ring spectrum E is of height at most n if $v_n \in \pi_*(E)/(p, v_1, \ldots, v_{n-1})$ is a unit for all primes p.[2] As a corollary, we obtain the following result for height at most two Lubin–Tate theories.[3]

Corollary 1.5. *Let $k \subseteq \overline{\mathbb{F}}_p$ denote a field of characteristic $p > 0$ which is algebraic over the prime field $\mathbb{F}_p$, and let $\mathbb{G}$ denote a formal group of height at most two over k. Then any complex orientation of the associated 2-periodic Morava K-theory $K(k, \mathbb{G})$ lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-complex orientation of $E(k, \mathbb{G})$*

$$\mathrm{MU} \to E(k, \mathbb{G}).$$

Proof. This follows immediately from Theorem 1.4 and [Zhu20, Theorem 1.2], which implies that every complex orientation of $K(k, \mathbb{G})$ admits a unique lift to a complex orientation of $E(k, \mathbb{G})$ which satisfies the Ando criterion whenever k is algebraic over $\mathbb{F}_p$. □

Remark 1.6. Theorem 1.4 was inspired by recent work of Balderrama [Bal21, Theorem 6.5.3]. Balderrama showed that every periodic complex orientation of a Lubin–Tate theory of height at most two satisfying an analogue of the Ando criterion lifts to an $\mathbb{E}_\infty$-orientation. He also showed that $\mathbb{E}_\infty$-refinements of periodic complex orientations of even periodic $K(1)$-local $\mathbb{E}_\infty$-rings exist whenever the Ando criterion is satisfied and are unique up to homotopy.

The key observation he made is the presence of evenness in the Goerss–Hopkins obstruction theory for (periodic) $\mathbb{E}_\infty$-complex orientations, which implies that the obstructions to existence and uniqueness appearing in his theorem vanish for formal reasons. Our results about existence of $\mathbb{E}_\infty$-complex orientations will be obtained by observing a similar evenness in the Hopkins–Lawson obstruction theory [HL18].

In contrast to Theorem 1.4, Balderrama did not obtain any uniqueness results at height 2 (see [Bal21, Remark 6.5.4]). In Remark 3.6, we will prove that $\mathbb{E}_\infty$-refinements of periodic complex orientations of height 2 Lubin–Tate theories are unique.

Our second main theorem uses Theorem 1.4 to give a simple proof of the following theorem.

[1] This follows from [Qui71] as explained in [Law18, Proposition 4.1.2]. Alternatively, it follows directly from (the proof of) [HL18, Theorem 32].
[2] See [Rav86, Appendix A2.2] for the v_i.
[3] See Recollection 4.14 for a refresher on the Lubin–Tate theories $E(k, \mathbb{G})$. The associated 2-periodic Morava K-theories $K(k, \mathbb{G})$ are obtained by cofibering out by the sequence $(p, u_1, \ldots, u_{n-1})$, or equivalently by the sequence $(p, v_1, \ldots, v_{n-1})$.

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Theorem 1.7. *For $n \geq 2$, the Hirzebruch level n elliptic genus lifts uniquely up to homotopy to a map of $\mathbb{E}_\infty$-rings*

$$\mathrm{MU} \to \mathrm{tmf}_1(n).$$

While Theorem 1.4 does not apply directly to $\mathrm{tmf}_1(n)$, it does apply to $\mathrm{TMF}_1(n)$, and it is not hard to upgrade the resulting $\mathbb{E}_\infty$-complex orientation to one of $\mathrm{tmf}_1(n)$. This reduces us to verifying the Ando criterion, which may be done following the strategy of Ando, Hopkins, and Strickland [AHS04].

Remark 1.8. During the writing process, [Abs21] has also appeared, which follows the strategy of Ando, Hopkins, and Rezk [AHR10] to prove a similar result to Theorem 1.7 for $\mathrm{Tmf}_1(n)$. Our method is rather different from that of Absmeier and completely avoids the consideration of p-adic Eisenstein measures.

1.3 Further questions

One of the key inputs in our proof of Theorem 1.4 is Theorem 2.11, which states that the Morava K-theory of certain finite groups is concentrated in even degrees. This is closely related to these groups being *good* in the sense of Hopkins, Kuhn, and Ravenel [HKR00, § 7].

These groups come in a family, and to see that the Hopkins–Lawson obstruction theory is concentrated in even degrees one would like to show that the entire family has Morava K-theory concentrated in even degrees; see Remark 2.13.

Question 1.9. Is the Morava K-theory of the groups $\Gamma_k^{(n)}$ of Definition 2.7 concentrated in even degrees for $k > 2$, at least for $n \gg 0$? The groups $\Gamma_k^{(1)}$ are the extraspecial p-groups of type p_+^{1+2k}.

One could also ask about C_2-equivariant refinement of our results.

Question 1.10. The complex cobordism spectrum may be refined to a C_2-equivariant $\mathbb{E}_\infty$-ring $\mathrm{MU}_\mathbb{R}$, and $\mathrm{tmf}_1(n)$ admits the natural structure of a C_2-equivariant $\mathbb{E}_\infty$-ring [Mei21, Theorem 2.20]. Moreover, the Hirzebruch level n elliptic genus admits a refinement to a map of homotopy C_2-ring spectra $\mathrm{MU}_\mathbb{R} \to \mathrm{tmf}_1(n)$ (see [Mei21, Theorem 3.5]). Is there a suitable C_2-equivariant analogue of Theorem 1.4 which may be used to prove a C_2-equivariant refinement of Theorem 1.7? See [HL18, Remark 13] for a comment on a C_2-equivariant version of the Hopkins–Lawson obstruction theory.

On the other hand, it would be very interesting to study $\mathbb{E}_\infty$-complex orientations at heights 3 and above. A natural choice of spectra to study would be Lubin–Tate spectra. Since the obstruction groups do not vanish for formal reasons at these heights, it seems likely that this will require an explicit analysis of the Goerss–Hopkins obstruction theory for $\mathbb{E}_\infty$-maps $\mathrm{MU} \to E(k, \mathbb{G})$. In particular, one would have to compute the E_2-page.

Problem 1.11. Compute the Goerss–Hopkins obstruction groups for $\mathbb{E}_\infty$-maps $\mathrm{MU} \to E(k, \mathbb{G})$ for height 3 and above Lubin–Tate theories $E(k, \mathbb{G})$.

2. Existence of $\mathbb{E}_\infty$-orientations

In this section, we will prove the half of Theorem 1.4 concerning the existence of $\mathbb{E}_\infty$-complex orientations. The main tool that we will utilize is an obstruction theory for $\mathbb{E}_\infty$-complex orientations studied by Hopkins and Lawson [HL18]. Given this obstruction theory, the existence half of Theorem 1.4 reduces to the statement that certain obstruction groups vanish. Using work of

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Arone and Lesh [AL07], this can be further reduced to the evenness of the Morava K-theory of certain extraspecial p-groups and related groups, which we are able to extract from the literature.

2.1 Hopkins–Lawson obstruction theory

We begin by summarizing the main results of the Hopkins–Lawson obstruction theory [HL18]. First, a definition.

Definition 2.1. Let E denote a homotopy commutative ring spectrum. We let $\mathrm{Or}(E)$ denote the space of complex orientations of E, i.e. the fiber

$$\mathrm{Or}(E) \to \mathrm{Map}(\Sigma^{\infty-2}\,\mathbb{CP}^\infty, E) \to \mathrm{Map}(\Sigma^{\infty-2}\,\mathbb{CP}^2, E) \simeq \mathrm{Map}(\mathbb{S}, E)$$

above the unit map $\mathbb{S} \to E$.

Theorem 2.2 [HL18, Theorems 1 and 32]. *There exists a diagram of $\mathbb{E}_\infty$-ring spectra*

$$\mathbb{S} \to \mathrm{MX}_1 \to \mathrm{MX}_2 \to \mathrm{MX}_3 \to \cdots \to \mathrm{MU}$$

such that the following hold:

(i) *the natural map $\varinjlim \mathrm{MX}_n \to \mathrm{MU}$ is an equivalence;*
(ii) *the $\mathbb{E}_\infty$-ring MX_1 is equipped with a natural complex orientation inducing an equivalence $\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_1, E) \xrightarrow{\sim} \mathrm{Or}(E)$ for each $\mathbb{E}_\infty$-ring E;*
(iii) *given $m > 0$ and an $\mathbb{E}_\infty$-ring E, there is a pullback square*

$$\begin{array}{ccc} \mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_m, E) & \longrightarrow & \mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_{m-1}, E) \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{Map}_*(F_m, \mathrm{Pic}(E)), \end{array}$$

where F_m is a pointed space described in Recollection 2.6;
(iv) *the map $\mathrm{MX}_{m-1} \to \mathrm{MX}_m$ is a rational equivalence if $m > 1$, a p-local equivalence if m is not a power of p, and a $K(n)$-local equivalence if $m > p^n$;*
(v) *let E denote an $\mathbb{E}_\infty$-ring such that π_*E is p-local and p-torsion free. Then an $\mathbb{E}_\infty$-map $\mathrm{MX}_1 \to E$ extends to an $\mathbb{E}_\infty$-map $\mathrm{MX}_p \to E$ if and only if the corresponding complex orientation of E satisfies the Ando criterion.*

Using this theorem, we will reduce the proof of Theorem 1.4 to the following.

Lemma 2.3. *Let E denote a p-complete Landweber exact ring spectrum with homotopy concentrated in even degrees. Then $E^{2n}(F_p) \cong E^{2n+1}(F_{p^2}) \cong 0$ for all $n \in \mathbb{Z}$.*

Remark 2.4. In fact, we only need that $E^{2n+1}(F_{p^2}) \cong 0$. However, we include the statement $E^{2n}(F_p) \cong 0$ since it is no harder for us to prove. This extra evenness implies uniqueness up to homotopy for $\mathbb{E}_\infty$-refinements of complex orientations of height at most one. However, we will prove uniqueness in a different way in § 3.

Question 2.5. Given a p-complete Landweber exact ring spectrum E with homotopy concentrated in even degrees, is $E^{2*+k-1}(F_{p^k}) \cong 0$ for $k \geq 3$?

Proof of existence in Theorem 1.4 assuming Lemma 2.3. We begin by reducing to the case where E is p-complete for some prime p. We make use of the following fracture square (see

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[Bou79, Proposition 2.9]).

$$\begin{array}{ccc} E & \longrightarrow & \prod_p E_p^\wedge \\ \downarrow & & \downarrow \\ E_{\mathbb{Q}} & \longrightarrow & \left(\prod_p E_p^\wedge\right)_{\mathbb{Q}} \end{array}$$

Note that $E_p^\wedge$ is again even and Landweber exact; see Appendix A.

By Theorem 2.2, we have $\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MU}, R) \simeq \mathrm{Or}(R)$ for a rational $\mathbb{E}_\infty$-ring R; it follows further that $\pi_1\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MU}, R) \cong \pi_1\mathrm{Or}(R) \cong 0$ if R has homotopy concentrated in even degrees. As a consequence, there are pullback squares of sets:

$$\begin{array}{ccc} \pi_0\,\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MU}, E) & \longrightarrow & \pi_0\,\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MU}, \prod_p E_p^\wedge) \\ \downarrow & & \downarrow \\ \pi_0\mathrm{Or}(E_{\mathbb{Q}}) & \longrightarrow & \pi_0\mathrm{Or}((\prod_p E_p^\wedge)_{\mathbb{Q}}) \end{array} \tag{1}$$

and

$$\begin{array}{ccc} \pi_0\mathrm{Or}(E) & \longrightarrow & \pi_0\mathrm{Or}(\prod_p E_p^\wedge) \\ \downarrow & & \downarrow \\ \pi_0\mathrm{Or}(E_{\mathbb{Q}}) & \longrightarrow & \pi_0\mathrm{Or}((\prod_p E_p^\wedge)_{\mathbb{Q}}). \end{array}$$

To lift a complex orientation of E to an $\mathbb{E}_\infty$-complex orientation, it therefore suffices to lift the induced complex orientation of $E_p^\wedge$. We may therefore assume that E is p-complete.

Let E now denote an p-complete Landweber exact $\mathbb{E}_\infty$-ring with homotopy concentrated in even degrees. Using Theorem 2.2, we see that it suffices to show that

$$\pi_0\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_{p^2}, E) \to \pi_0\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_p, E)$$

is surjective.

By Theorem 2.2, there is a fiber sequence

$$\mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_{p^2}, E) \to \mathrm{Map}_{\mathbb{E}_\infty}(\mathrm{MX}_p, E) \to \mathrm{Map}_*(F_{p^2}, \mathrm{Pic}(E)).$$

Now, there are equivalences

$$\mathrm{Map}_*(F_m, \mathrm{Pic}(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \mathrm{pic}(E)) \simeq \mathrm{Hom}(\Sigma^\infty F_m, \Sigma E),$$

where in the second equivalence we have used the fact that $\Sigma^\infty F_m$ is $(2m-1)$-connected by [HL18, Corollary 4(5)]. It therefore follows from the above fiber sequence that it suffices to show that

$$E^1(\Sigma^\infty F_{p^2}) \cong 0.$$

Since E is p-complete, this follows from Lemma 2.3. □

2.2 Proof of Lemma 2.3

In the remainder of this section, we will prove Lemma 2.3. First, we must recall the definition of the spaces F_m.

Recollection 2.6. Let L_m denote the nerve of the topologized poset of proper direct-sum decompositions of $\mathbb{C}^m$, and let $(L_m)^\diamond$ denote its unreduced suspension. The natural action of $U(m)$ on $\mathbb{C}^m$ endows L_m and $(L_m)^\diamond$ with the structure of $U(m)$-spaces.

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Furthermore, view S^{2m} as a $U(m)$-space via its identification with the one-point compactification of $\mathbb{C}^m$, viewed as the fundamental representation of $U(m)$. Then F_m is given by

$$F_m \simeq ((L_m)^\diamond \wedge S^{2m})_{hU(m)}.$$

To prove Lemma 2.3, we will use a result of Arone–Lesh to reduce the study of the E-cohomology of F_m to the E-cohomology of certain groups Γ_k, whose definition we now recall.

Definition 2.7 [Oli94, Definition 1]. Let $\sigma_0, \dots, \sigma_{k-1} \in \Sigma_{p^k}$ denote the permutations

$$\sigma_r(i) = \begin{cases} i + p^r & \text{if } i \equiv 1, \dots, (p-1)p^r \mod p^{r+1}, \\ i - (p-1)p^r & \text{if } i \equiv (p-1)p^r + 1, \dots, p^{r+1} \mod p^{r+1}. \end{cases}$$

We let $\Gamma_k \subset U(p^k)$ denote the subgroup generated by the permutation matrices corresponding to $\sigma_0, \dots, \sigma_{k-1}$, the central S^1, and the diagonal matrices $A_0, \dots, A_{k-1}$ given by

$$(A_r)_{ii} = \zeta_p^{\lfloor (i-1)/p^r \rfloor},$$

where ζ_p is a primitive pth root of unity. Then Γ_k lies in a central extension

$$1 \to S^1 \to \Gamma_k \to \mathbb{F}_p^{2k} \to 1.$$

For each $n \geq 1$, there is a normal subgroup $\Gamma_k^{(n)} \subset \Gamma_k$ which only contains the central p^nth roots of unity instead of the full S^1. Then there are central extensions

$$1 \to C_{p^n} \to \Gamma_k^{(n)} \to \mathbb{F}_p^{2k} \to 1$$

and exact sequences

$$1 \to \Gamma_k^{(n)} \to \Gamma_k \to S^1 \to 1.$$

Remark 2.8. The groups $\Gamma_k^{(1)}$ are examples of *extraspecial p-groups*, and in this language are commonly denoted by p_+^{1+2k}.

Proposition 2.9 [AL07, Propositions 9.6 and 10.3]. *The p-completion of the spectrum*

$$\Sigma^\infty F_m \simeq \Sigma^\infty ((L_m)^\diamond \wedge S^{2m})_{hU(m)}$$

is null unless $m = p^k$, in which case it is a summand of the p-completion of

$$\Sigma^k (S^{2p^k})_{h\Gamma_k},$$

where Γ_k acts on S^{2p^k} via the inclusion $\Gamma_k \subset U(p^k)$.

Now, $(S^{2p^k})_{h\Gamma_k}$ may also be described as the Thom spectrum associated to the composition

$$B\Gamma_k \to \mathrm{BU}(p^k) \to \mathbb{Z} \times \mathrm{BU},$$

from which it follows that

$$E^*((S^{2p^k})_{h\Gamma_k}) \cong \widetilde{E}^{*-2p^k}(B\Gamma_k).$$

Lemma 2.3 therefore reduces to the following lemma.

Lemma 2.10. *Let E denote a p-local Landweber exact ring spectrum whose homotopy is concentrated in even degrees. Then, for $k \leq 2$, $E^*(B\Gamma_k)$ is concentrated in even degrees.*

We will deduce Lemma 2.10 from the following Morava K-theory computations.

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THEOREM 2.11. *For all $n \geq 0$, the following groups are concentrated in even degrees:*

(i) *(Tezuka and Yagita [TY89, Theorem 4.2]) $K(n)^*(B\Gamma_1^{(1)})$ at all primes p;*
(ii) *(Schuster and Yagita [SY04, Theorem 5.4]) $K(n)^*(B\Gamma_2^{(1)})$ at the prime 2;*
(iii) *(Yagita [Yag05, Theorem 1.2]) $K(n)^*(B\Gamma_2^{(2)})$ at odd primes p.*

A lemma of Strickland which builds on the work of Ravenel, Wilson, and Yagita [RWY98] allows us to transport this evenness from Morava K-theory to E-cohomology.

COROLLARY 2.12. *Given a p-local Landweber exact ring spectrum E whose homotopy is concentrated in even degrees, the following groups are concentrated in even degrees:*

(i) *$E^*(B\Gamma_1^{(1)})$ at all primes p;*
(ii) *$E^*(B\Gamma_2^{(1)})$ at the prime 2;*
(iii) *$E^*(B\Gamma_2^{(2)})$ at odd primes p.*

Proof. Combine Theorem 2.11 with [Str99, Lemma 8.25].[4,5] □

Proof of Lemma 2.10. The short exact sequence

$$\Gamma_k^{(n)} \to \Gamma_k \to S^1$$

induces a fiber sequence

$$B\Gamma_k^{(n)} \to B\Gamma_k \to BS^1.$$

Using the associated Atiyah–Hirzebruch spectral sequence

$$\mathrm{H}^*(BS^1; E^*(B\Gamma_k^{(n)})) \Rightarrow E^*(B\Gamma_k),$$

we find that if $E^*(B\Gamma_k^{(n)})$ is even for some n, then $E^*(B\Gamma_k)$ must be as well. Therefore, Lemma 2.10 follows from Corollary 2.12. □

Remark 2.13. By the same arguments, to give a positive answer to Question 2.5 it suffices to show that $K(n)^*(B\Gamma_k^{(i_k)})$ is concentrated in even degrees for a fixed i_k not depending on n. This is closely related to the question of whether $\Gamma_k^{(i_k)}$ is a *good group* in the sense of Hopkins, Kuhn, and Ravenel [HKR00, § 7].

3. Uniqueness of $\mathbb{E}_\infty$-orientations

Our goal in this section is to prove the following result.

THEOREM 3.1. *Let E denote an L_2-local[6] complex orientable $\mathbb{E}_\infty$-ring with the property that $K(1)_*E$ and $K(1)_*L_{K(2)}E$ are concentrated in even degrees at all primes p. Then each complex orientation of E admits at most one refinement to an $\mathbb{E}_\infty$-complex orientation up to homotopy.*

Example 3.2. Any Landweber exact ring spectrum E of height at most two whose homotopy is concentrated in even degrees satisfies the hypotheses of Theorem 3.1. By Lemma A.1, $L_{K(2)}E$

[4] Note that we cannot apply [Str99, Lemma 8.25] directly to $B\Gamma_k$, since it only applies to spaces with bounded above $\mathbb{Q}$-cohomology.

[5] While it is assumed in [Str99] that E is even-periodic, this is not used in the proof of [Str99, Lemma 8.25]. We also note that E may be viewed as a summand of an even-periodic ring spectrum $E[x_2^{\pm 1}]$, so that we may conclude from [Str99, Lemma 8.25] as stated.

[6] Recall that L_2-localization refers to Bousfield localization with respect to height two Johnson–Wilson theory $E(2)$, or equivalently with respect to a height two Lubin–Tate theory: see [Rav84, Definition 5.1].

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is again Landweber exact and has homotopy concentrated in even degrees. It therefore suffices to show that $K(1)_*E$ is concentrated in even degrees. This is true because $K(1)_*E \cong K(1)_*\mathrm{MU} \otimes_{\pi_*\mathrm{MU}} \pi_*E$, and $K(1)_*\mathrm{MU}$ is concentrated in even degrees.

Combining Theorem 3.1 with Example 3.2, we obtain the uniqueness half of Theorem 1.4.

Example 3.3. The ring spectra $\mathrm{Tmf}_1(n)$ satisfy the hypotheses of Theorem 3.1. This follows from [Wil15, Propositions 2.4 and 2.6], which imply that the p-complete complex K-theory of these spectra and their $K(2)$-localizations is p-torsionfree and concentrated in even degrees. The proof is exactly the same as the case without level structure, which is treated in [Beh14]. The key point is that the Igusa tower becomes formally affine at a finite stage.

Our proof of Theorem 3.1 will be based on the orientation theory of [ABGHR14] and the following lemma.

Lemma 3.4. *Let E denote an* MU*-module with the property that $K(1)_*E$ is concentrated in even degrees. Then $[\mathrm{KU}_p, L_{K(1)}E]$ is torsionfree and $[\Sigma\mathrm{KU}_p, L_{K(1)}E] = 0$.*

Definition 3.5. We say that a $(\mathrm{KU}_p)_*$-module is *pro-free* if it is the p-completion of a free module. Moreover, given a spectrum X, we write $\mathrm{KU}^\vee_*(X)$ for $\pi_*L_{K(1)}(\mathrm{KU} \otimes X)$.

Proof. Since $K(1)_*(\mathrm{KU}_p)$ is even, $\mathrm{KU}^\vee_*(\mathrm{KU}_p)$ is pro-free by [HSt99, Proposition 8.4(f)]. Therefore, by [BH16, Proposition 1.14], there is an isomorphism

$$\pi_*\mathrm{Hom}(\mathrm{KU}_p, L_{K(1)}(\mathrm{KU}_p \otimes E) \cong \mathrm{Hom}_{(\mathrm{KU}_p)_*}(\mathrm{KU}^\vee_*(\mathrm{KU}_p), \mathrm{KU}^\vee_*(E)).$$

By assumption, $K(1)_*(E)$ is even and, hence, $\mathrm{KU}^\vee_*(E)$ is even and pro-free by [HSt99, Proposition 8.4(f)]. In particular, it is torsionfree, so that $\mathrm{Hom}_{(\mathrm{KU}_p)_*}(\mathrm{KU}^\vee_*(\mathrm{KU}_p), \mathrm{KU}^\vee_*(E))$ is even and torsionfree.

The result then follows from the following facts.

(i) Since E is an MU-module, the unit map $L_{K(1)}E \to L_{K(1)}(\mathrm{MU} \otimes E)$ admits a splitting given by the module structure map.
(ii) There is a splitting of the map $L_{K(1)}\mathrm{MU} \to L_{K(1)}E(1)$ which is compatible with the unit map [HSa99, Theorem 4.1].
(iii) There is an equivalence of spectra $\mathrm{KU}_p \simeq \bigoplus_{i=0}^{p-2} \Sigma^{2i} L_{K(1)}E(1)$. □

Proof of Theorem 3.1. Using the pullback square of sets (1), we may assume that E is p-complete.

Recall the map $\Sigma^\infty\mathbb{CP}^\infty \to \mathrm{bu}$ which is adjoint to the canonical map $\mathbb{CP}^\infty \to \mathrm{BU}$. By orientation theory [ABGHR14], we must show that

$$[\mathrm{bu}, \mathrm{gl}_1(E)] \to [\Sigma^\infty\mathbb{CP}^\infty, \mathrm{gl}_1(E)]$$

is injective. Since E is p-complete, $\mathrm{gl}_1(E)$ agrees with $\mathrm{gl}_1(E)^\wedge_p$ in degrees ≥ 2, so we may as well replace the former by the latter. Letting F denote the fiber of the map $\mathrm{gl}_1(E) \to L_2\mathrm{gl}_1(E)$, we find that there is a fiber sequence

$$F^\wedge_p \to \mathrm{gl}_1(E)^\wedge_p \to L_{K(1)\oplus K(2)}\mathrm{gl}_1(E).$$

It therefore suffices to show that

$$[\mathrm{bu}, F^\wedge_p] \to [\Sigma^\infty\mathbb{CP}^\infty, F^\wedge_p]$$

and

$$[\mathrm{bu}, L_{K(1)\oplus K(2)}\mathrm{gl}_1(E)] \to [\Sigma^\infty\mathbb{CP}^\infty, L_{K(1)\oplus K(2)}\mathrm{gl}_1(E)]$$

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are injective. The first is injective because $F_p^\wedge$ is 3-coconnective by [AHR10, Theorem 4.11] and the cofiber of $\Sigma^\infty\mathbb{CP}^\infty \to \mathrm{bu}$ is 4-connective.

To prove that the second is injective, we first note that the Bousfield–Kuhn functor [Bou87, Kuh89] and the chromatic fracture square for $L_{K(1)\oplus K(2)}\mathrm{gl}_1(E)$ imply that there is an exact sequence

$$[\Sigma\mathrm{KU}_p, L_{K(1)}L_{K(2)}E] \to [\mathrm{bu}, L_{K(1)\oplus K(2)}E] \to [\mathrm{KU}_p, L_{K(1)}E].$$

Applying Lemma 3.4, we learn that $[\mathrm{bu}, L_{K(2)\oplus K(1)}E]$ is torsionfree. It therefore injects into its rationalization, so that the result follows from the fact that $\Sigma^\infty\mathbb{CP}^\infty \to \mathrm{bu}$ is a rational equivalence. □

Remark 3.6. Balderrama has shown that *periodic* complex orientations of height 2 Lubin–Tate theories $E(k,\mathbb{G})$ which satisfy a version of the Ando criterion admit lifts to periodic $\mathbb{E}_\infty$-complex orientations $\mathrm{MUP} \to E(k,\mathbb{G})$ (see [Bal21, Theorem 6.5.3(3)]). In this remark, we prove that such lifts are unique up to homotopy.

We begin with a result which has been proven by Rezk at height 2 (which is the case that we use) [Rez13] and in general is an unpublished theorem of Hopkins and Lurie that has now appeared in work of Burklund, Schlank, and Yuan [BSY22, Theorem H]. Let $\overline{k}$ denote the algebraic closure of k. Then we have

$$\pi_*\mathbb{G}_m(E(\overline{k},\mathbb{G})) := \pi_*\mathrm{Map}(\mathbb{Z}, \mathrm{gl}_1(E(\overline{k},\mathbb{G}))) \cong \begin{cases} \overline{k}^\times & * = 0,\\ \mathbb{Z}_p & * = 3,\\ 0 & \text{otherwise.}\end{cases}$$

On π_0, the map $\mathbb{Z} \to \mathrm{gl}_1(E(\overline{k},\mathbb{G}))$ corresponding to $a \in \overline{k}^\times$ picks out the Teichmuller lift $[a]$.

Since $\mathrm{Gal}(k)$ has p-cohomological dimension at most 1 (see [GS06, Proposition 6.1.9]), it follows that

$$\pi_0\mathbb{G}_m(E(k,\mathbb{G})) \cong k^\times.$$

In particular, the map $\pi_0\mathbb{G}_m(E(k,\mathbb{G})) \to \pi_0\mathrm{gl}_1(E(k,\mathbb{G}))$ is injective. Now, by orientation theory it suffices to show that

$$[\mathrm{ku}, \mathrm{gl}_1(E(k,\mathbb{G}))] \to [\Sigma^\infty_+\mathbb{CP}^\infty, \mathrm{gl}_1(E(k,\mathbb{G}))]$$

is injective. By what we have proven above about uniqueness of $\mathbb{E}_\infty$-complex orientations, it suffices to show that

$$[\mathbb{Z}, \mathrm{gl}_1(E(k,\mathbb{G}))] \to [\mathbb{S}^0, \mathrm{gl}_1(E(k,\mathbb{G}))]$$

is injective, which is what we showed above.

As noted in [Bal21, Remark 6.5.4], this implies that the Goerss–Hopkins obstruction group $\mathrm{Ext}^2_\Delta(\hat{Q}(E(k,\mathbb{G})^\wedge_0\mathrm{MUP}), \omega)$ is equal to 0.

4. The level n elliptic genus

Convention 4.1. In this section n will denote an integer greater than or equal to 2.

In this section, we will prove Theorem 1.7, which states that the Hirzebruch level n elliptic genus lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-complex orientation $\mathrm{MU} \to \mathrm{tmf}_1(n)$.

We will begin by recalling some background material about $\mathbb{E}_\infty$-rings of topological modular forms with level-$\Gamma_1(n)$ structures in § 4.1. In § 4.2, we recall from [AHS01, AHS04] how complex orientations may be described in terms of Θ^1-structures. We then describe the level n elliptic

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genus in this language, following Meier [Mei21, § 3]. In § 4.3, we show that a complex orientation for $\mathrm{TMF}_1(n)$ satisfies the Ando criterion at p if and only if its composition along a map $\mathrm{TMF}_1(n) \to E(k, \mathbb{G})$ to a Lubin–Tate theory does.

In § 4.4, we recall from [AHS04] how the Ando criterion for Lubin–Tate theories may be rephrased in terms of Θ^1-structures. We then prove that the Hirzebruch level n genus satisfies the Ando criterion. As a consequence of Theorem 1.4, it lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-ring map $\mathrm{MU} \to \mathrm{TMF}_1(n)$. Finally, in § 4.5, we prove that this $\mathbb{E}_\infty$-ring map admits a unique up to homotopy lift to an $\mathbb{E}_\infty$-ring map $\mathrm{MU} \to \mathrm{tmf}_1(n)$, completing the proof of Theorem 1.7.

4.1 Topological modular forms with level structures

In this section, we will recall some basic facts about the $\mathbb{E}_\infty$-rings $\mathrm{tmf}_1(n)$, $\mathrm{Tmf}_1(n)$ and $\mathrm{TMF}_1(n)$ of topological modular forms with level-$\Gamma_1(n)$ structure. We begin by recalling the algebraic background.

Definition 4.2. We let $\mathcal{M}_1(n)$ denote the Deligne–Mumford moduli stack of elliptic curves with level $\Gamma_1(n)$ structure over $\mathbb{Z}[1/n]$. Concretely, given a scheme S on which n is invertible, we have

$$\mathcal{M}_1(n)(S) = \text{elliptic curves } E \text{ over } S \text{ with a point } P \in E[n](S) \text{ of exact order } n.$$

Moreover, we write $\overline{\mathcal{M}_1}(n)$ for the Deligne–Rapoport moduli stack of generalized elliptic curves with level $\Gamma_1(n)$ structure [DR73]. This is again a Deligne–Mumford stack over $\mathbb{Z}[1/n]$, and $\mathcal{M}_1(n) \subset \overline{\mathcal{M}_1}(n)$ is an open substack.

We denote the universal family of curves by $\pi : \mathcal{C} \to \overline{\mathcal{M}_1}(n)$, and write $\pi^{\mathrm{sm}} : \mathcal{C}^{\mathrm{sm}} \to \overline{\mathcal{M}_1}(n)$ for the smooth locus. Then $\mathcal{C}^{\mathrm{sm}}$ admits a natural structure of a group scheme, and we write ω for the line bundle of invariant differentials.

Finally, we write $\mathcal{M} = \mathcal{M}_1(1)$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}_1}(1)$ for the moduli stacks of (generalized) elliptic curves without level structure.

Recollection 4.3 (Goerss, Hopkins, and Miller [DFHH14, Chapter 12], Lurie [Lur18b], and Hill and Lawson [HL16]). There is a sheaf $\mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}}$ of $\mathbb{E}_\infty$-ring spectra on the Kummer log-étale site of $\overline{\mathcal{M}}$ with the following properties.

- (i) There is a natural isomorphism of sheaves $\pi_0 \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}} \cong \mathcal{O}_{\overline{\mathcal{M}}}$.
- (ii) There are natural isomorphisms of quasicoherent sheaves $\pi_{2i} \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}} \cong \omega^i$ and $\pi_{2i+1} \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}} \cong 0$.
- (iii) Write $\widehat{\pi^{\mathrm{sm}}} : \widehat{\mathcal{C}^{\mathrm{sm}}} \to \overline{\mathcal{M}}$ for the completion of $\pi^{\mathrm{sm}} : \mathcal{C}^{\mathrm{sm}} \to \overline{\mathcal{M}}$ along the zero section. There is a natural isomorphism of sheaves of rings $\pi_0 \mathrm{Hom}(\Sigma^\infty_+ \mathbb{CP}^\infty, \mathcal{O}^{\mathrm{top}}) \cong (\widehat{\pi^{\mathrm{sm}}})_* \mathcal{O}_{\widehat{\mathcal{C}^{\mathrm{sm}}}}$.

Since the natural morphisms $\overline{\mathcal{M}_1}(n) \to \overline{\mathcal{M}}$ are Kummer log-étale (we refer the reader to [HL16] for more details), we may define $\mathbb{E}_\infty$-rings:

$$\mathrm{TMF}_1(n) \coloneqq \Gamma(\mathcal{M}_1(n), \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}})$$

and

$$\mathrm{Tmf}_1(n) \coloneqq \Gamma(\overline{\mathcal{M}_1(n)}, \mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}}).$$

By definition, there are spectral sequences

$$\mathrm{H}^s(\mathcal{M}_1(n), \omega^i) \Rightarrow \pi_{2i-s} \mathrm{TMF}_1(n)$$

and

$$\mathrm{H}^s(\overline{\mathcal{M}_1}(n), \omega^i) \Rightarrow \pi_{2i-s} \mathrm{Tmf}_1(n).$$

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Remark 4.4. Let $n \geq 2$. It follows from [Mei22a, Proposition 2.4(4)] that $\mathrm{H}^s(\mathcal{M}_1(n), \omega^i) = 0$ for all $s > 0$, so that $\mathrm{TMF}_1(n)$ has homotopy groups concentrated in even degrees and that there are natural isomorphisms

$$\pi_{2i}(\mathrm{TMF}_1(n)) \cong \Gamma(\mathcal{M}_1(n), \omega^i).$$

Moreover, $\mathrm{TMF}_1(n)$ is Landweber exact. Indeed, this is a consequence of flatness of $\mathcal{M}_1(n)$ over $\mathbb{Z}[1/n]$, the integrality of $\mathcal{M}_1(n)_{\mathbb{F}_p}$ and the fact that the formal group of an elliptic curve is of height at most two.

However, because the groups $\mathrm{H}^1(\overline{\mathcal{M}_1}, \omega^i)$ do not in general vanish, we do not have a similar theorem for $\mathrm{Tmf}_1(n)$. Instead, we have the $\mathbb{E}_\infty$-ring spectrum $\mathrm{tmf}_1(n)$ from [Mei21].

Recollection 4.5 [Mei21, Theorem 1.1]. There is an essentially unique connective $\mathbb{E}_\infty$-ring spectrum $\mathrm{tmf}_1(n)$ whose homotopy groups are concentrated in even degrees and which is equipped with an $\mathbb{E}_\infty$-ring map

$$\mathrm{tmf}_1(n) \to \mathrm{Tmf}_1(n)$$

such that the induced maps

$$\pi_{2i}\mathrm{tmf}_1(n) \to \pi_{2i}\mathrm{Tmf}_1(n) \to \Gamma(\overline{\mathcal{M}_1}(n), \omega^i)$$

are isomorphisms.

Remark 4.6. There is a sequence of natural maps

$$\mathrm{tmf}_1(n) \to \mathrm{Tmf}_1(n) \to \mathrm{TMF}_1(n).$$

4.2 Θ^1-structures

In this section, we describe complex orientations in terms of Θ^1-structures and give a description of the Hirzebruch level n genus in this language.

Suppose that we are given a base Deligne–Mumford stack S and a formal group or generalized elliptic curve G over S. We denote the structure map by $p : G \to S$ and the zero section by $0 : S \to G$. Given a line bundle $\mathcal{L}$ on G, we set

$$\Theta^1(\mathcal{L}) := p^*0^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

There is a canonical trivialization $0^*\Theta^1(\mathcal{L}) \cong \mathcal{O}_S$.

Definition 4.7. A Θ^1-structure on a line bundle $\mathcal{L}$ over G is a trivialization of $\Theta^1(\mathcal{L})$ which pulls back to the canonical trivialization of $0^*\Theta^1(\mathcal{L})$.

Equivalently, a Θ^1-structure on $\mathcal{L}$ is an isomorphism $p^*0^*\mathcal{L} \cong \mathcal{L}$ which pulls back to the canonical isomorphism $0^*p^*0^*\mathcal{L} \cong 0^*\mathcal{L}$.

Remark 4.8. It is clear from the definition that there is a natural bijection between Θ^1-structures on $\mathcal{L}$ and $\mathcal{L}^{-1}$.

Remark 4.9. Note that a line bundle $\mathcal{L}$ on G admits a Θ^1-structure if and only if it is pulled back from a line bundle on S via the structure map p.

When G is a generalized elliptic curve, Θ^1-structures are unique when they exist.

Lemma 4.10. *Suppose that G is a generalized elliptic curve. Then the set of Θ^1-structures on a line bundle $\mathcal{L}$ over G is either empty or consists of a single element.*

Proof. The set of isomorphisms $p^*0^*\mathcal{L} \cong \mathcal{L}$ is a torsor for $\mathcal{O}_G(G)^\times$, whereas the set of isomorphisms $0^*p^*0^*\mathcal{L} \cong 0^*\mathcal{L}$ is a torsor for $\mathcal{O}_S(S)^\times$. It therefore suffices to show that the map

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$\mathcal{O}_G(G) \to \mathcal{O}_S(S)$ induced by pulling back along the zero section is an isomorphism. This follows from [Sta23, Tag 0E0L]. □

Now let E denote an even weakly periodic homotopy commutative ring spectrum, and let $\mathbb{G}_E = \mathrm{Spf}\, E^0(\mathbb{CP}^\infty)$ denote the associated formal group over Spec $\pi_0 E$. We denote by γ_1 the canonical line bundle over $\mathbb{CP}^\infty$ and let $\mathcal{O}_{\mathbb{G}_E}(1)$ denote the line bundle over $\mathbb{G}_E$ corresponding to

$$E^0(\mathrm{Th}(\gamma_1)) \cong \widetilde{E^0}(\mathbb{CP}^\infty) \cong \ker(\mathcal{O}_{\mathbb{G}_E} \xrightarrow{0^*} \pi_0 E).$$

Then we have the following theorem.

Proposition 4.11 [AHS01, Theorem 2.48]. *There is a natural bijection between complex orientations of E and Θ^1-structures on $\mathcal{O}_{\mathbb{G}_E}(1)$.*

Proof. Recall that a complex orientation of E consists of an element of $\widetilde{E^2}(\mathbb{CP}^\infty)$ which restricts to the unit along the map $\widetilde{E^2}(\mathbb{CP}^\infty) \to \widetilde{E^2}(\mathbb{CP}^1) \cong \pi_0 E$. Since the map $E^0(\mathbb{CP}^\infty) \to \pi_0 E$ is an infinitesimal thickening, complex orientations of E may be identified with $E^0(\mathbb{CP}^\infty)$-module isomorphisms $E^0(\mathbb{CP}^\infty) \cong \widetilde{E^2}(\mathbb{CP}^\infty)$ that become equal to the canonical isomorphism $\widetilde{E^2}(\mathbb{CP}^1) \cong \pi_0 E$ after tensoring down along $E^0(\mathbb{CP}^\infty) \to \pi_0 E$.

By Remark 4.8, we may replace $\mathcal{O}_{\mathbb{G}_E}(1)$ with $\mathcal{O}_{\mathbb{G}_E}(1)^{-1}$ in the statement of the proposition. To prove the proposition, it therefore suffices to identify the global sections of $\Theta^1(\mathcal{O}_{\mathbb{G}_E}(1)^{-1})$ with $\widetilde{E^2}(\mathbb{CP}^\infty)$ and pullback along the zero section with $\widetilde{E^2}(\mathbb{CP}^\infty) \to \widetilde{E^2}(\mathbb{CP}^1) \cong \pi_0 E$. This is an immediate consequence of the definitions. □

Even though $\mathrm{tmf}_1(n)$ is not weakly even periodic, Meier has shown that its complex orientations may still be described in terms of Θ^1-structures.

Proposition 4.12 [Mei21, Lemma 3.2]. *There is a natural bijection between complex orientations of $\mathrm{tmf}_1(n)$ and Θ^1-structures on $\mathcal{O}_{\widehat{\mathcal{C}}}(1)$ over $\widehat{C} \to \overline{\mathcal{M}_1}(n)$.*

We now recall the treatment of Hirzebruch's level n elliptic genus from [Mei21, § 3]. The following proposition is a mild rephrasing of [Mei21, Lemma 3.3].

Proposition 4.13. *Let $P : \overline{\mathcal{M}_1}(n) \to \mathcal{C}^{\mathrm{sm}}$ denote the universal level $\Gamma_1(n)$ structure.*

(i) *The pullback of the line bundle $\mathcal{O}_{\mathcal{C}}([0]-[P])$ on $\mathcal{C}$ to $\widehat{\mathcal{C}}$ is naturally isomorphic to $\mathcal{O}_{\widehat{\mathcal{C}}}(1)$.*
(ii) *There is a degree n étale cover $q : \mathcal{C}' \to \mathcal{C}$ of generalized elliptic curves so that $q^*\mathcal{O}_{\mathcal{C}}([0]-[P])$ admits a (necessarily unique) Θ^1-structure.*

Since the induced map $\widehat{q} : \widehat{\mathcal{C}'} \xrightarrow{\sim} \widehat{\mathcal{C}}$ is an isomorphism, we obtain a Θ^1-structure on $\mathcal{O}_{\widehat{\mathcal{C}}}(1)$ and, hence, by Proposition 4.12 a complex orientation of $\mathrm{tmf}_1(n)$. This is the complex orientation corresponding to the *Hirzebruch level n elliptic genus*.

4.3 Reduction to Lubin–Tate theory

Our goal in this section is to prove Proposition 4.16. This proposition implies that to verify the Ando criterion for the Hirzebruch level n elliptic genus, it suffices to verify the Ando criterion after composition with a map $\mathrm{TMF}_1(n) \to E(k, \mathbb{G})$ to a Lubin–Tate theory. This will be useful to us because work of Ando, Hopkins, and Strickland [AHS04] rephrases the Ando criterion for Lubin–Tate theories in terms of Θ^1-structures.

We begin by recalling some basic facts about Lubin–Tate theories.

Recollection 4.14 (Goerss and Hopkins [GH04] and Lurie [Lur18b]). Let k denote a perfect field of characteristic $p > 0$ and let $\mathbb{G}$ denote a formal group of finite height h over k. To the pair

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$(k, \mathbb{G})$ one may associate an $\mathbb{E}_\infty$-ring spectrum $E(k, \mathbb{G})$, known as the *Lubin–Tate spectrum* of $(k, \mathbb{G})$. The ring $\pi_0 E(k, \mathbb{G})$ is naturally isomorphic to the universal deformation ring of $(k, \mathbb{G})$, which is non-canonically isomorphic to $\mathbb{W}(k)[\![u_1, \ldots, u_{h-1}]\!]$. We let $\mathbb{G}^{\mathrm{univ}}$ denote the universal deformation of $\mathbb{G}$ over $\pi_0 E(k, \mathbb{G})$, and denote by $\omega_{\mathbb{G}^{\mathrm{univ}}}$ its module of invariant differentials. There are natural isomorphisms of $\pi_{2i} E(k, \mathbb{G})$ with $\omega^i_{\mathbb{G}^{\mathrm{univ}}}$.

Construction 4.15. Let k denote a perfect field of characteristic $p > 0$. Associated to a supersingular k-point $(E, \alpha \in E[n](k))$ of $\mathcal{M}_1(n)$, there is a map of $\mathbb{E}_\infty$-ring spectra $\mathrm{TMF}_1(n) \to E(k, \widehat{E})$. Indeed, this follows from the description of $E(k, \widehat{E})$ as an oriented deformation ring [Lur18b, § 6], Lurie's Serre–Tate theorem for strict abelian varieties [Lur18a, § 7], and the universal property of $\mathrm{TMF}_1(n)$.

Proposition 4.16. *Let k denote a perfect field of characteristic p and let $(E, \alpha \in E[n](k))$ denote an object of $\mathcal{M}_1(n)(k)$ with E a supersingular elliptic curve. Then a complex orientation of $\mathrm{TMF}_1(n)$ satisfies the Ando criterion at p if and only if its composite with the canonical map $\mathrm{TMF}_1(n) \to E(k, \widehat{E})$ does.*

Our proof of Proposition 4.16 rests on the following lemma.

Lemma 4.17. *Let $E \to F$ denote a map of complex orientable homotopy commutative ring spectra. Suppose that $\pi_* E$ and $\pi_* F$ are p-torsionfree and that the induced map $\pi_* E/p \to \pi_* F/p$ is an injection. Then the map $E^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to F^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$ is an injection.*

Proof. Since $\pi_* E$ is p-torsionfree, a choice of complex orientation gives rise to an isomorphism (see [Qui71, Proposition 4.2]):

$$E^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \cong E^*[[t]]/\langle p \rangle(t)$$

where $|t| = 2$, $\langle p \rangle(t) = [p](t)/t$, and $[p](t)$ is the p-series of the formal group on E^*. Moreover, since $\pi_* E$ is p-torsionfree, we may identify $t^{n-1} E^*[[t]]/(\langle p \rangle(t), t^n)$ with a shift of E^*/p. The analogous statements for F also hold.

Taking the induced complex orientation of F, we identify $E^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to F^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$ with the natural map $E^*[[t]]/\langle p \rangle(t) \to F^*[[t]]/\langle p \rangle(t)$. To show that this map is injective, it suffices to show that it is injection on the associated graded for the t-adic filtration. This identifies with a shift of the natural map $E^*/p \to F^*/p$ in each degree, which is an injection by hypothesis. ☐

Proof of Proposition 4.16. It is clear that if a complex orientation of $\mathrm{TMF}_1(n)$ satisfies the Ando criterion at p, so does the induced complex orientation of $E(k, \widehat{E})$. To prove the converse, it suffices to show that the induced map

$$\mathrm{TMF}_1(n)^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to E(k, \widehat{E})^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$$

is an injection. We begin by reducing to the case $n \geq 5$. Any map $\mathrm{Spec}\, k \to \mathcal{M}_1(n)$ fits into a diagram

$$\begin{array}{ccc} \mathrm{Spec}\, k' & \longrightarrow & \mathcal{M}_1(n^2) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}\, k & \longrightarrow & \mathcal{M}_1(n) \end{array}$$

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for k' a finite separable extension of k. It follows that there is a diagram

$$\begin{array}{ccc} \mathrm{TMF}_1(n) & \longrightarrow & E(k, \widehat{E}) \\ \downarrow & & \downarrow \\ \mathrm{TMF}_1(n^2) & \longrightarrow & E(k', \widehat{E}_{k'}), \end{array}$$

so that it suffices to show that

$$\mathrm{TMF}_1(n)^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to \mathrm{TMF}_1(n^2)^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$$

and

$$\mathrm{TMF}_1(n^2)^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to E(k', \widehat{E}_{k'})^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$$

are injective.

We may therefore assume that $n \geq 5$ if we can show that the first map is an injection. Both $\pi_*\mathrm{TMF}_1(n)$ and $\pi_*\mathrm{TMF}_1(n^2)$ are p-torsionfree, and

$$\pi_{2i}\mathrm{TMF}_1(n)/p \cong \Gamma(M_1(n)_{\mathbb{F}_p}, \omega^i) \to \Gamma(M_1(n^2)_{\mathbb{F}_p}, \omega^i) \cong \pi_{2i}\mathrm{TMF}_1(n)/p$$

is an injection since $\mathcal{M}_1(n^2)_{\mathbb{F}_p} \to \mathcal{M}_1(n)_{\mathbb{F}_p}$ is a finite étale cover. Therefore, we may apply Lemma 4.17 to show that the first map above is an injection.

We may now assume that $n \geq 5$, so that $\mathcal{M}_1(n)$ is represented by an affine scheme Spec R_n (see [Mei22a, Proposition 2.4(2)]). Let $\mathfrak{m} \subset R_n$ denote the kernel of the map $R_n \to k$. Then $R_n/\mathfrak{m}$ is a finite field (hence, perfect), and the pair $(E, P \in E[n](k))$ descends to $R_n/\mathfrak{m}$. As a consequence, there is a factorization $\mathrm{TMF}_1(n) \to E(R_n/\mathfrak{m}, \widehat{E}) \to E(k, \widehat{E})$. It follows immediately from Lemma 4.17 that the induced map

$$E(R_n/\mathfrak{m}, \widehat{E})^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to E(k, \widehat{E})^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$$

is an injection, so that it suffices to show that

$$\mathrm{TMF}_1(n)^*(\mathrm{BC}_p)/I_{\mathrm{tr}} \to E(R_n/\mathfrak{m}, \widehat{E})^*(\mathrm{BC}_p)/I_{\mathrm{tr}}$$

is an injection. To apply Lemma 4.17, we need to show that $\pi_*\mathrm{TMF}_1(n)/p \to \pi_*E(R_n/\mathfrak{m}, \widehat{E})/p$ is an injection. By abuse of notation, we let ω denote the invertible R_n-module corresponding to the line bundle ω on $\mathcal{M}_1(n)$. Then the above map can be identified with the $\mathfrak{m}$-adic completion map $(\omega^{*/2}/p) \to (\omega^{*/2}/p)^{\wedge}_{\mathfrak{m}}$. This is an injection by the Krull intersection theorem, since $\mathcal{M}_1(n)_{\mathbb{F}_p}$ is an integral scheme. □

4.4 The Ando criterion and Θ^1-structures

In this section, we recall from the work of Ando, Hopkins, and Strickland how the Ando criterion for Lubin–Tate theories may be rephrased in terms of Θ^1-structures. We refer the reader to [AHS04] for proofs and further details. We then combine this rephrasing with Proposition 4.16 to prove that the Hirzebruch level n elliptic genus satisfies the Ando criterion. Finally, we deduce from Theorem 1.4 that the Hirzebruch level n elliptic genus lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-ring map

$$\mathrm{MU} \to \mathrm{TMF}_1(n).$$

We begin by recalling from [AHS04, § 14] how Θ^1-structures may be normed along isogenies.

Recollection 4.18 [AHS04, § 14]. Suppose we are given an isogeny $G \to G'$ of formal groups or elliptic curves. Given a line bundle $\mathcal{L}$ over G, there is a line bundle $N(\mathcal{L})$ over G', called the

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norm of $\mathcal{L}$. Moreover, given a Θ^1-structure s on $\mathcal{L}$, there is an associated Θ^1-structure $N(s)$ on $N(\mathcal{L})$, called the *norm* of s.

Remark 4.19. Given an isogeny of formal groups $\mathbb{G} \to \widetilde{\mathbb{G}}$, there is a natural isomorphism of line bundles $N(\mathcal{O}_{\mathbb{G}}(1)) \cong \mathcal{O}_{\widetilde{\mathbb{G}}}(1)$.

Remark 4.20. Suppose that we are given an elliptic curve E with a point P of exact order n, i.e. a level $\Gamma_1(n)$ structure, and an isogeny $E \to \widetilde{E}$ of degree p coprime to n. Then the image $\widetilde{P}$ of P in $\widetilde{E}$ is again a point of exact order n, and there is a natural isomorphism of line bundles $N(\mathcal{O}_E([0]-[P])) \cong \mathcal{O}_{\widetilde{E}}([0]-[\widetilde{P}])$.

Following [AHS04], we may now use this language to rephrase the Ando criterion for Lubin–Tate theory. We begin with some setup.

Recollection 4.21. Let k denote a perfect field of characteristic $p > 0$ and let $\mathbb{G}$ denote a formal group of finite height over k, so that we have an associated Lubin–Tate theory $E(k, \mathbb{G})$. Then there are two ring maps $i, \psi : \pi_0 E(k, \mathbb{G}) \to E(k, \mathbb{G})^0(\mathrm{BC}_p)/I_{\mathrm{tr}}$. The first, i, is induced by the projection $\mathrm{BC}_p \to *$. The second ψ, is the total power operation. Over the ring $E(k, \mathbb{G})^0(\mathrm{BC}_p)/I_{\mathrm{tr}}$, there is a degree p isogeny

$$i^*\mathbb{G}^{\mathrm{univ}} \to \psi^*\mathbb{G}^{\mathrm{univ}}$$

induced by the total power operation on $E(k, \mathbb{G})^0(\mathbb{CP}^\infty)$.

Given a Θ^1-structure s on $\mathcal{O}_{\mathbb{G}^{\mathrm{univ}}}(1)$, there are therefore two naturally induced Θ^1-structures on $\mathcal{O}_{\psi^*\mathbb{G}^{\mathrm{univ}}}(1) \cong N(\mathcal{O}_{i^*\mathbb{G}^{\mathrm{univ}}}(1))$: the pullback $\psi^*(s)$ and the norm $N(i^*(s))$.

DEFINITION 4.22. We say that a Θ^1-structure s on $\mathbb{G}^{\mathrm{univ}}$ satisfies the Ando criterion if $\psi^*(s) = N(i^*(s))$.

It follows from [AHS04, § 5] that this is compatible with our previous definition of the Ando criterion.

PROPOSITION 4.23. *A complex orientation of $E(k, \mathbb{G})$ satisfies the Ando criterion if and only if the associated Θ^1-structure on $\mathcal{O}_{\widetilde{\mathbb{G}}}(1)$ satisfies the Ando criterion.*

We are now able to prove the main theorem of this section.

THEOREM 4.24. *The Hirzebruch level n elliptic genus* $\mathrm{MU} \to \mathrm{tmf}_1(n) \to \mathrm{TMF}_1(n)$ *satisfies the Ando criterion. As a consequence of Theorem 1.4 and Remark 4.4, it lifts uniquely up to homotopy to an $\mathbb{E}_\infty$-complex orientation*

$$\mathrm{MU} \to \mathrm{TMF}_1(n).$$

Proof. Choose, for each p not dividing n, $(E, P \in E[n](k)) \in \mathcal{M}_1(n)(k)$ with E supersingular and k a perfect field of characteristic p. By Proposition 4.16, it suffices to show that the induced complex orientation of $E(k, \widehat{E})$ satisfies the Ando criterion.

By the Serre–Tate theorem [Kat81, § 1], $\pi_0 E(k, \widehat{E})$ is the universal deformation ring of E. We let E^{univ} denote the universal deformation of E over $\pi_0 E(k, \widehat{E})$, and let $P^{\mathrm{univ}} \in E^{\mathrm{univ}}[n](\pi_0 E(k, \mathbb{G}))$ denote the unique lift of P. The associated formal group $\widehat{E^{\mathrm{univ}}}$ is a universal deformation of $\widehat{E}$.

Applying Proposition 4.23, we must show that the Θ^1-structure s on $\mathcal{O}_{\widehat{E^{\mathrm{univ}}}}(1)$ corresponding to the level n elliptic genus satisfies the Ando criterion, i.e. that $\psi^*(s) = N(i^*(s))$. By the definition of the level n elliptic genus, there is a degree n étale isogeny $q : (E^{\mathrm{univ}})' \to E^{\mathrm{univ}}$ and

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a Θ^1-structure $\overline{s}$ on $q^*\mathcal{O}_{E^{\mathrm{univ}}}([0]-[P^{\mathrm{univ}}])$ which induces s. By the Serre–Tate theorem, the isogeny of formal groups

$$i^*\widehat{E^{\mathrm{univ}}} \to \psi^*\widehat{E^{\mathrm{univ}}}$$

over $E(k,\widehat{E})^0(\mathrm{BC}_p)/I_{\mathrm{tr}}$ lifts to the following diagram of isogenies of elliptic curves.

$$\begin{array}{ccc} i^*(E^{\mathrm{univ}})' & \longrightarrow & \psi^*(E^{\mathrm{univ}})' \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_\psi} \\ i^*E^{\mathrm{univ}} & \longrightarrow & \psi^*E^{\mathrm{univ}}. \end{array}$$

Let $\overline{P^{\mathrm{univ}}}$ denote the image of $i^*(P^{\mathrm{univ}})$ in ψ^*E^{univ}. From the above diagram, we obtain Θ^1-structures $\psi^*(\overline{s})$ and $N(i^*(\overline{s}))$ on

$$(q_\psi)^*\mathcal{O}_{\psi^*E^{\mathrm{univ}}}([0]-[\overline{P^{\mathrm{univ}}}]) \cong N((q_i)^*\mathcal{O}_{i^*E^{\mathrm{univ}}}([0]-[i^*(P^{\mathrm{univ}})])).$$

As these induce $\psi^*(s)$ and $N(i^*(s))$, it suffices to show that $\psi^*(\overline{s}) = N(i^*(\overline{s}))$. But this follows immediately from the uniqueness of Θ^1-structures over elliptic curves proven in Lemma 4.10. □

4.5 Lift to $\mathbf{tmf}_1(n)$

In this section, we will complete the proof of Theorem 1.7 by proving the following two lemmas.

LEMMA 4.25. *Suppose that we are given a complex orientation* $\mathrm{MU} \to \mathrm{Tmf}_1(n)$ *with the property that the composite*

$$\mathrm{MU} \to \mathrm{Tmf}_1(n) \to \mathrm{TMF}_1(n)$$

lifts to an $\mathbb{E}_\infty$*-ring map. Then this complex orientation lifts uniquely to an* $\mathbb{E}_\infty$*-complex orientation* $\mathrm{MU} \to \mathrm{Tmf}_1(n)$.

LEMMA 4.26. *Any* $\mathbb{E}_\infty$*-complex orientation of* $\mathrm{Tmf}_1(n)$ *lifts uniquely to an* $\mathbb{E}_\infty$*-complex orientation of* $\mathrm{tmf}_1(n)$.

Proof of Lemma 4.25. The uniqueness will follow from Theorem 3.1 and Example 3.3 once we know that a complex orientation of $\mathrm{Tmf}_1(n)$ is determined by the induced complex orientation of $\mathrm{TMF}_1(n)$. This follows from the fact that the map $\pi_*\mathrm{Tmf}_1(n) \to \pi_*\mathrm{TMF}_1(n)$ is injective in even degrees by [Mei22b, Proposition 2.5] and the descent spectral sequence.

It therefore suffices to show that the $\mathbb{E}_\infty$-map $\mathrm{MU} \to \mathrm{TMF}_1(n)$ lifts to an $\mathbb{E}_\infty$-map $\mathrm{MU} \to \mathrm{Tmf}_1(n)$. We begin with the pullback square of $\mathbb{E}_\infty$-rings coming from [HL16]:

$$\begin{array}{ccc} \mathrm{Tmf}_1(n) & \longrightarrow & \mathrm{TMF}_1(n) \\ \downarrow & & \downarrow \\ K_1^{\mathrm{cusp}}(n) & \longrightarrow & \Delta^{-1}K_1^{\mathrm{cusp}}(n) \end{array}$$

This square satisfies the following properties.

(i) The $\mathbb{E}_\infty$-rings $K_1^{\mathrm{cusp}}(n)$ and $\Delta^{-1}K_1^{\mathrm{cusp}}(n)$ are Landweber exact and of height at most one.
(ii) The induced map $\pi_*K_1^{\mathrm{cusp}}(n)/p \to \pi_*\Delta^{-1}K_1^{\mathrm{cusp}}(n)/p$ is injective for all p.

It follows that to construct an $\mathbb{E}_\infty$-lifting $\mathrm{MU} \to \mathrm{Tmf}_1(n)$, it suffices to lift the composite

$$\mathrm{MU} \to \mathrm{TMF}_1(n) \to \Delta^{-1}K_1^{\mathrm{cusp}}(n)$$

to an $\mathbb{E}_\infty$-map

$$\mathrm{MU} \to K_1^{\mathrm{cusp}}(n).$$

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By the uniqueness in Theorem 1.4 and (i) above, it suffices to lift the complex orientation and verify that it satisfies the Ando criterion. However, we are given a lift of the complex orientation by assumption, and it follows from Lemma 4.17 and (ii) above that it satisfies the Ando criterion. □

Proof of Lemma 4.26. For this, we use orientation theory [ABGHR14]. We have the sequence of maps

$$\mathrm{bu} \xrightarrow{J} \mathrm{bgl}_1(\mathbb{S}) \to \mathrm{bgl}_1(\mathrm{tmf}_1(n)) \to \mathrm{bgl}_1(\mathrm{Tmf}_1(n)),$$

and $\mathbb{E}_\infty$-complex orientations of tmf_1 and $\mathrm{Tmf}_1(n)$ correspond to nullhomotopies of the respective composites. Since bu is 2-connective, we may as well replace all occurrences of bgl_1 with $\mathrm{bsl}_1 := \tau_{\geq 2}\mathrm{bgl}_1$.

Now, it follows from the definition of $\mathrm{tmf}_1(n)$ (see [Mei21]) that the map $\mathrm{tmf}_1(n) \to \tau_{\geq 0}\mathrm{Tmf}_1(n)$ fits into the following pullback square.

$$\begin{array}{ccc} \mathrm{tmf}_1(n) & \longrightarrow & \tau_{\geqslant 0}\mathrm{Tmf}_1(n) \\ \downarrow & & \downarrow \\ \pi_0\mathrm{Tmf}_1(n) & \longrightarrow & \tau_{[0,1]}\mathrm{Tmf}_1(n). \end{array}$$

As a consequence, $\mathrm{tmf}_1(n) \to \mathrm{Tmf}_1(n)$ induces an isomorphism on π_k for $k \geq 2$. Since $\pi_1\mathrm{tmf}_1(n) = 0$, it follows that the composite

$$\mathrm{bsl}_1(\mathrm{tmf}_1(n)) \to \mathrm{bsl}_1(\mathrm{Tmf}_1(n)) \to \tau_{\geq 3}\mathrm{bsl}_1(\mathrm{Tmf}_1(n))$$

is an equivalence, so that there is a splitting

$$\mathrm{bsl}_1(\mathrm{Tmf}_1(n)) \simeq \mathrm{bsl}_1(\mathrm{tmf}_1(n)) \oplus \Sigma^2\pi_1\mathrm{Tmf}_1(n),$$

from which the lemma follows. □

Proof of Theorem 1.7. Combine Theorem 4.24 with Lemmas 4.25 and 4.26. □

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CONFLICTS OF INTEREST
None.

Appendix A. $K(n)$-localizations of Landweber exact ring spectra

In this appendix, we prove the following lemma, which the author was unable to find a reference for in the literature.

LEMMA A.1. *Let E denote a Landweber exact ring spectrum whose homotopy is concentrated in even degrees. Then $L_{K(n)}E$ is also a Landweber exact ring spectrum whose homotopy is concentrated in even degrees.*

Before we proceed to the proof of Lemma A.1, we need a lemma from commutative algebra.

LEMMA A.2. *Let R_* denote a graded commutative ring and let $x_1, \ldots, x_n \in R_*$ denote a regular sequence of homogeneous elements. Then the sequence $x_1, \ldots, x_n$ remains regular in the completion $(R_*)^\wedge_{(x_1,\ldots x_n)}$.*

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Proof. Let $I = (x_1, \dots x_n)$. We claim that there are short exact sequences

$$0 \to R_*/I^{k-1} \xrightarrow{x_1} R_*/I^k \to R_*/(I^k + (x_1)) \to 0 \tag{A.1}$$

for all $k \geq 1$. Supposing this for the moment, we find by taking the limit that

$$0 \to (R_*)^\wedge_I \xrightarrow{x_1} (R_*)^\wedge_I \to (R_*/x_1)^\wedge_I \to 0$$

is short exact. In particular, we find that x_1 is a regular element in $(R_*)^\wedge_I$ and that $(R_*)^\wedge_I/x_1 \cong (R_*/x_1)^\wedge_I$. Taking R_*/x_1 as our new ring and the image of $x_2, \dots, x_n$ in R_*/x_1 as our exact sequence, we may conclude by induction on the length of our regular sequence.

It remains to establish (A.1). It is sufficient to prove that (A.1) is exact on the associated graded of the I-adic filtration where x_1 is considered as a map of I-adic filtration degree 1. Since $x_1, \dots, x_n$ is regular, this associated graded may be identified with the sequence

$$0 \longrightarrow (R_*/I)[\overline{x}_1, \dots \overline{x}_n]/(\overline{x}_1, \dots, \overline{x}_n)^{k-1} \xrightarrow{\overline{x}_1} (R_*/I)[\overline{x}_1, \dots \overline{x}_n]/(\overline{x}_1, \dots, \overline{x}_n)^k \longrightarrow (R_*/I)[\overline{x}_2, \dots \overline{x}_n]/(\overline{x}_2, \dots, \overline{x}_n)^k \longrightarrow 0,$$

which is easily verified to be short exact. □

Proof of Lemma A.1. By abuse of notation, we let $v_i \in \pi_* E$ inductively denote an arbitrary lift of the class $v_i \in \pi_* E/(p, \dots, v_{i-1})$. Given a positive integer k, let $I_k = (p, v_1, \dots, v_{k-1})$ and let $I_0 = (0)$. It follows from [HSt99, Proposition 7.10] that $\pi_* L_{K(n)} E \cong (v_n^{-1}\pi_* E)^\wedge_{I_n}$. In particular, $\pi_{2*-1} L_{K(n)} E = 0$. It is clear that the operation of inverting v_n^{-1} preserves Landweber exactness, so that it suffices to prove that completion with respect to I_n does as well. However, this follows immediately from Lemma A.2. □

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