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A MARKOV RENEWAL APPROACH TO THE ASYMPTOTIC DECAY OF THE TAIL PROBABILITIES IN RISK AND QUEUING PROCESSES

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It is well known that various characteristics in risk and queuing processes can be formulated as Markov renewal functions, which are determined by Markov renewal equations. However, those functions have not been utilized as they are expected. In this article, we show that they are useful for studying asymptotic decay in risk and queuing processes under a Markovian environment. In particular, a matrix version of the Cramér–Lundberg approximation is obtained for the risk process. The corresponding result for the $MAP/G/1$ queue is presented as well. Emphasis is placed on a straightforward derivation using the Markov renewal structure.

1. INTRODUCTION

In risk and queuing processes, it is interesting to consider them under Markovian random environments; namely, processes of primary interest are perturbed using continuous-time Markov chains with finite state spaces. They are frequently referred to as Markov-modulated processes. A Markovian arrival process, MAP in short, originally introduced by Neuts, is a typical example (see, e.g., [11]). In those processes, it is also frequently observed that characteristics of interests satisfy Markov renewal equations. For instance, those renewal equations are obtained for the ruin probability of a risk process with claims subject to MAP and the stationary workload process in the $MAP/G/1$ queue. However, those renewal structures seem to have not been fully utilized, because it seems hard to get closed-form results from them, as is often remarked.

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We are particularly interested in the asymptotic decay of the ruin probability of the risk processes with MAP claims when the loading factor $\rho < 1$ and of the tail probability of the stationary workload in the $MAP/G/1$ queue when the traffic intensity $\rho < 1$. For those probabilities, the decay rates have been obtained under certain modeling assumptions in addition to light tail conditions. Those asymptotic decays are referred to as *the Cramér-Lundberg approximations* in the risk process literature (see, e.g., [5,12]).

The aim of this article is to obtain the Cramér–Lundberg approximations and the asymptotics of the related queueing model under a Markovian environment, using the Markov renewal structure. For this, we use the general result on the asymptotic decay of the Markov renewal function due to Asmussen [2]. We present this result in a matrix form to make computations transparent.

In the literature, the Cramér–Lundberg approximation for a risk process under a Markovian environment has been studied using different approaches. Among them, the change of measure technique is most popular (see [5,12]). The technique not only verifies the Cramér–Lundberg approximation but also provides useful information on bounds. However, this technique demands some analytic arguments based on the martingale. Furthermore, the Cramér–Lundberg approximation has not been fully studied for a risk process with claims subject to MAP (see [5,6,13]). Either the Poisson rate modulations or the claim size distributions of the phase type have been assumed, although it would be routine work to remove these restrictions.

The advantage of the present derivation is to make computations straightforward. In this approach, a key step is to compute the moment-generating function of the Markov renewal kernel in a closed form. This not only addresses the decay rate in the risk process but also gives an informative expression to the coefficient of the exponential decay function. This approach also clarifies the role of the Markov renewal kernel for studying the decay rate problem under a Markovian environment, which seems to have not been well recognized.

This article is composed of four sections. In Section 2, the Markov renewal theory for the decay rate problem is briefly introduced. In Section 3, this result is applied to the risk process. In Section 4, the $MAP/G/1$ queue is discussed as a dual of the risk model.

2. ASYMPTOTICS OF A MARKOV RENEWAL FUNCTION

We basically follow Çinlar [7] for the notation on a Markov renewal process, but some changes are made to reformulate it in a matrix form. Let S be a finite set and let $P(x)$ be an $S \times S$ nonnegative matrix such that its ijth entry $P_{ij}(x)$ is a nondecreasing function of $x \geq 0$ and satisfies

$$\sum_{j \in S} P_{ij}(x) \leq 1, \qquad x \geq 0, i \in S.$$

The matrix $P(x)$ is said to be a Markov renewal kernel, which uniquely determine a Markov renewal process. It should be noted that $P(\infty)$ may not be a stochastic matrix

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(i.e., their row sums may not be unity) in this article, namely, we admit $P(x)$ to be a defective kernel, which means that the Markov renewal process may be terminated in a finite time. In this section, we assume the following:

1. $P(x)$ has a single irreducible recurrent class that can be reached from any state in S with probability one.
2. The return time to each state in the irreducible class has a nonarithmetic distribution.

Let $A(x)$ and $B(x)$ be an $S \times S$ nonnegative matrices such that $A(x)$ is nondecreasing in x and the ijth entry of $B(x)$ is bounded and measurable in x for all $i, j \in S$. Define the convolution $A * B(x)$ for those matrices by

$$[A * B(x)]_{ij} = \sum_{k \in S} \int_0^x A_{ik}(dy) B_{kj}(x - y), \qquad x \geq 0, i, j \in S.$$

Then, an $S \times S$ nonnegative matrix $U(x)$ is said to satisfy a Markov renewal equation in matrix form if

$$U(x) = A(x) + P * U(x), \qquad x \geq 0. \tag{1}$$

In [7], this equation is given for vector-valued functions.

As is well known, the matrix renewal equation has the minimal solution such that

$$U(x) = \sum_{n=0}^{\infty} (P^{(n*)} * A)(x), \qquad x \geq 0, \tag{2}$$

and U is a unique solution of (1) if $P(\infty)$ is strictly substochastic; that is, there is a row of $P(\infty)$ whose sum is less than unity, where $P^{(n*)}(x)$ is inductively defined by $P^{(n*)}(x) = (P^{((n-1)*)} * P)(x)$ with $P^{(0*)}(x) = I$.

Let $\boldsymbol{m}$ be a column vector for the mean sojourn times of the Markov renewal process (i.e., $\boldsymbol{m} = \int_0^\infty u P(du) \boldsymbol{e}$, where an integral is defined for a matrix componentwise, and $\boldsymbol{e}$ is the S-column vector, all of whose entries are unity). We assume that $\boldsymbol{m}$ is finite. If $P(\infty)$ is stochastic, $P(\infty)$ admits the stationary row vector $\boldsymbol{\pi}$ (i.e., $\boldsymbol{\pi} P(\infty) = \boldsymbol{\pi}$). We normalize $\boldsymbol{\pi}$ so as to satisfy $\boldsymbol{\pi e} = 1$. Throughout the article, we use *boldface greek* letters for row vectors, and *boldface Latin* letters are used for column vectors.

We refer to the following Markov renewal theorem under our terminology.

LEMMA 2.1 (Proposition 4.9 in [7]): *If $P(\infty)$ is stochastic and if each entry of $A(x)$ is directly Riemann integrable (see [7] or [8] for its definition), then*

$$\lim_{x \to \infty} U(x) = \frac{1}{\boldsymbol{\pi m}} \boldsymbol{e \pi} \int_0^\infty A(u)\, du. \tag{3}$$

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Let $\hat{P}(\theta)$ be the moment-generating function of $P(x)$ for real number θ; that is,

$$\hat{P}(\theta) = \int_0^\infty e^{\theta u} P(du).$$

Clearly, $\hat{P}(\theta)$ exists if θ is not positive. We assume that $P(x)$ has light tails; that is, there is a $\theta_0 > 0$ such that $\hat{P}(\theta)$ exists for $\theta < \theta_0$, where θ_0 may be infinity.

From now on, we assume that $P(\infty)$ is strictly substochastic. We briefly present the asymptotic decay of $U(x)$ due to Asmussen [2]. Since $\hat{P}(\theta)$ is nonnegative and irreducible for real θ, $\hat{P}(\theta)$ has unique positive eigenvalues, denoted by $\delta(\theta)$, as long as $\hat{P}(\theta)$ exists, and the associated left and right eigenvectors are positive by the Perron–Frobenius theorem (see, e.g., [14]). Denote these eigenvectors by $\boldsymbol{\nu}^{(\theta)}$ and $\boldsymbol{h}^{(\theta)}$, respectively.

We suppose that there exists an $\alpha > 0$ such that $\delta(\alpha) = 1$. For instance, this is the case when all of the row sums of $\hat{P}(\theta)$ go to infinity as θ goes to $\theta_0 > 0$, because $\delta(0) < 1$ and $\delta(\theta)$ is increasing in θ. We define a Markov renewal kernel by

$$P^\dagger(x) = \Delta_{h^{(\alpha)}}^{-1} \int_0^x e^{\alpha u} P(du) \Delta_{h^{(\alpha)}},$$

where Δ_a with vector $\boldsymbol{a}$ is the diagonal matrix whose ith entry is a_i (i.e., the ith entry of $\boldsymbol{a}$). It is easy to check that $P^\dagger(\infty)$ is stochastic and that $\boldsymbol{\nu}^{(\alpha)}\Delta_{h^{(\alpha)}}$ is the stationary probability vector of $P^\dagger(\infty)$. Obviously, (1) is obtained for $P^\dagger(x)$ with

$$U^\dagger(x) = \Delta_{h^{(\alpha)}}^{-1} e^{\alpha x} U(x), \qquad A^\dagger(x) = \Delta_{h^{(\alpha)}}^{-1} e^{\alpha x} A(x).$$

Since $\Delta_{h^{(\alpha)}} \boldsymbol{e}\boldsymbol{\nu}^{(\alpha)} = \boldsymbol{h}^{(\alpha)}\boldsymbol{\nu}^{(\alpha)}$, Lemma 2.1 yields the following result (see [2, pp. 230–231] for a complete proof).

LEMMA 2.2 (Theorem 2.6 of Chapter X in [2]): *Under the above existence conditions on $\alpha > 0$, if each entry of $e^{\alpha x}A(x)$ is directly Riemann integrable and if $\boldsymbol{\nu}^{(\alpha)}\hat{P}_{(1)}(\alpha)\boldsymbol{h}^{(\alpha)}$ is finite, then*

$$\lim_{x\to\infty} e^{\alpha x} U(x) = \frac{1}{\boldsymbol{\nu}^{(\alpha)}\hat{P}_{(1)}(\alpha)\boldsymbol{h}^{(\alpha)}} \boldsymbol{h}^{(\alpha)}\boldsymbol{\nu}^{(\alpha)} \int_0^\infty e^{\alpha u} A(u)\, du, \tag{4}$$

where $\hat{P}_{(1)}(\alpha) = (d/d\theta)\hat{P}(\theta)|_{\theta=\alpha}$.

Thus, to obtain the asymptotic decay, we must identify the matrix renewal kernel and the matrix renewal equation and find the root of $\delta(\theta) = 1$ with associated eigenvectors. In applications, these are not so obvious, because the matrix renewal kernel may be very complicated. Nevertheless, there are easier cases. These are the cases for Markov modulations in risk and queuing processes.

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3. CRAMÉR–LUNDBERG APPROXIMATION

In this section, we apply Lemma 2.2 to risk and queuing processes under Markovian environments. Both are generated by the following Markov additive jump process. Let C be an $S \times S$ matrix which has negative diagonal entries and nonnegative off-diagonal entries, and let $D(x)$ be an $S \times S$ matrix function whose entries are nonnegative and nondecreasing functions of $x \geq 0$. It is assumed that $C + D(\infty)$ is a rate matrix; that is,

$$(C + D(\infty))\boldsymbol{e} = \mathbf{0}.$$

Furthermore, we assume that $C + D(\infty)$ is irreducible. Let $M(t)$ be the Markov chain with rate matrix $C + D(\infty)$. Since the state space is finite, this Markov chain always has the stationary distribution, which is denoted by the vector $\boldsymbol{\pi}$; that is,

$$\boldsymbol{\pi}(C + D(\infty)) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\pi}\boldsymbol{e} = 1.$$

Transition instants of $M(t)$ include those that do not change the current state; that is, when the state is in $i \in S$, $M(t)$ has transitions with rate $-C_{ii}$, and the state changes to $j \in S$ with probability $(1(\neq j)C_{ij} + D_{ij}(\infty))/(-C_{ii})$. $M(t)$ is referred to as a background process. At each transition instant from state i to state j, jumps occur with probability $D_{ij}(\infty)/(-C_{ii})$, and their sizes are independent of everything else and are subject to the distribution function $D_{ij}(x)/D_{ij}(\infty)$. Let $Y(t)$ be the sum of those jumps up to time $t > 0$, starting from time 0; that is, $Y(t)$ is the additive process of the jumps. $M(t)$ and $Y(t)$ are assumed to be right continuous. Note that this type of Markov modulation, say MAP modulation, is more general than the rate modulation of the Poisson process depending on each background state.

A risk process $R(t)$ with unit premium rate is defined as

$$R(t) = R(0) + t - Y(t).$$

For $R(0) = x$, let

$$\tau_x = \inf\{u > 0; R(u) < 0\}.$$

τ_x is called the ruin time. We are interested in the ruin probability $P(\tau_x < \infty, M(\tau_x) = i | R(0) = x)$. In the risk process literature, the above risk process has been studied when $D(x)$ has nonzero entries only at diagonals. This means that the jumps arrive according to the Poisson process with rate $D_{ii}(\infty)$ while $M(t)$ is in state i, but no jump occurs at the transition instants. So, the above model extends those risk processes, but all of the arguments in the literature go through without essential changes.

According to Asmussen [5], we describe jumps in the risk process by the opposite sign, namely define

$$B(t) = Y(t) - t, \qquad t \geq 0.$$

Then, we have

$$\tau_x = \inf\{u > 0; B(u) > x\}, \qquad x \geq 0.$$

Thus, the ruin time is the hitting time of the Markov additive process $B(t)$ at level x. As in the risk process literature, we are only interested in the case when the loading factor ρ is less than 1; that is,

$$\rho \equiv \boldsymbol{\pi} \int_0^\infty u D(du) \boldsymbol{e} < 1, \tag{5}$$

where we assume that the integral is finite componentwise. This condition implies that $P(\tau_x < \infty | M(0) = i) < 1$ for any $i \in S$, so its decay rate is meaningful.

Define an $S \times S$ matrix function $U(x)$ by

$$U_{ij}(x) = P(M(\tau_x) = j | M(0) = i), \qquad x \geq 0, i, j, \in S,$$

where $\tau_x < \infty$ is included in the event $\{M(\tau_x) = j\}$. This convention will be used throughout the article. We also define $A(x)$ and $P(x)$ as

$$A_{ij}(x) = P(M(\tau_0) = j, Y(\tau_0) > x | M(0) = i),$$
$$P_{ij}(x) = P(M(\tau_0) = j, Y(\tau_0) \leq x | M(0) = i),$$

for $x \geq 0$, $i, j, \in S$. Since $(M(t), Y(t))$ constitutes a continuous-time Markov process and τ_0 is a stopping time with respect to this Markov process, conditioning on $(M(\tau_0), Y(\tau_0))$ yields (1). Thus, the ruin probabilities are indeed described by the matrix renewal equation.

A hard part of the above analysis is to find either $A(x)$ or $P(x)$, which simultaneously determine each other. We refer to Corollary 2.6 of [5] for this, and slightly extend it in the following way, where a square matrix is said to be Metzler–Leontief (ML) if it has nonnegative off-diagonal entries.

LEMMA 3.1: *There exists an $S \times S$ ML matrix K that is the minimal solution of the equation*

$$K = C + \int_0^\infty e^{uK} D(du) \tag{6}$$

and $A(x)$ is obtained as

$$A(x) = \int_0^\infty e^{yK} \int_{x+y}^\infty D(dz)\, dy. \tag{7}$$

Furthermore, $\boldsymbol{\pi} K = \mathbf{0}$; that is, the stationary vector $\boldsymbol{\pi}$ of $C + D(\infty)$ is the left eigenvector of K for eigenvalue 0, and the corresponding right eigenvector $\boldsymbol{k}$ of K is positive.

Remark 3.1:

(a) Corollary 2.6 of [5] assumes that $D_{ij}(x) = 0$ for $i \neq j$. As we mentioned earlier, this restriction is not essential.

(b) In the numerical evaluation, the matrix K is obtained as the limit of the following iteration. Let $K_0 = 0$ and

$$K_n = C + \int_0^\infty e^{uK_{n-1}} D(du), \qquad n = 1, 2, \ldots.$$

Note that K_n monotonically converges as n goes to infinity. It can be shown that the limit of this sequence is indeed the minimal solution of (6) (see, e.g., [3]).

(c) Let $Q^* = \Delta_\pi^{-1} K' \Delta_\pi$, where K' is the transpose of K. Then, Q^* satisfies

$$Q^* = C^* + \int_0^\infty D^*(du) e^{uQ^*}, \tag{8}$$

where $C^* = \Delta_\pi^{-1} C' \Delta_\pi$ and $D^*(x) = \Delta_\pi^{-1}[D(x)]' \Delta_\pi$. C^* and $D^*(x)$ generate the time-reversed processes of $M(t)$ and $-Y(t)$. It is known that Q^* is a rate matrix if $\rho < 1$. This implies that $\boldsymbol{\pi} K = \mathbf{0}$, and the corresponding right eigenvector $\boldsymbol{k}$ is nonnegative. Furthermore, for $\theta > 0$, $\theta I - K$ is nonsingular since $\theta I - Q^*$ is nonsingular.

From (7), it is easy to see that conditions 1 and 2 of Section 2 are satisfied. Since

$$P(x) = A(0) - A(x) = \int_0^\infty e^{yK} \int_0^x D(dz + y)\, dy \tag{9}$$

and $\theta I - K$ is nonsingular for $\theta > 0$ by Remark 3.1(c), (7) yields

$$\begin{aligned}
\hat{P}(\theta) &= \int_0^\infty e^{\theta x} \int_0^\infty e^{uK} D(dx + u)\, du \\
&= \int_0^\infty e^{\theta x} \int_0^x e^{u(K - \theta I)}\, du\, D(dx) \\
&= (\theta I - K)^{-1} \int_0^\infty (e^{\theta x} - e^{xK}) D(dx) \\
&= (\theta I - K)^{-1}(-K + C + \hat{D}(\theta)) \\
&= I + (\theta I - K)^{-1}(-\theta I + C + \hat{D}(\theta)),
\end{aligned}$$

where the fourth equality is obtained using (6). To apply Lemma 2.2, we need to find a θ such that $P(\theta)$ has a unit eigenvalue. From the above computation, $-\theta I + C + \hat{D}(\theta)$ must have a null eigenvalue for this. Note that $-\theta I + C + \hat{D}(\theta)$ is an ML matrix. Hence, by the Perron–Frobenius theory (e.g., see [14]), it has a real eigenvalue that dominates the real parts of all other eigenvalues, and the associated right and left eigenvectors are positive.

Denote this eigenvalue by $\kappa(\theta)$ and the associated right and left eigenvectors by $\boldsymbol{\mu}^{(\theta)}$ and $\boldsymbol{h}^{(\theta)}$, respectively. Let $\boldsymbol{k}$ be the right eigenvector of K for eigenvalue 0,

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where $\boldsymbol{k}$ is positive by Remark 3.1(c). We normalize $\boldsymbol{\mu}^{(\theta)}$ and $\boldsymbol{h}^{(\theta)}$ so that $\boldsymbol{\mu}^{(\theta)}\boldsymbol{h}^{(\theta)} = 1$ and $\boldsymbol{\mu}^{(\theta)}\boldsymbol{k} = 1$. Note that $\boldsymbol{\mu}^{(\theta)}$ and $\boldsymbol{h}^{(\theta)}$ are also obtained as the eigenvectors of $C + \hat{D}(\theta)$ for the eigenvalue that dominates the real parts of all other eigenvalues. In this case, the eigenvalue is $\kappa(\theta) + \theta$.

We now look for a positive solution of the equation such that $\kappa(\theta) = 0$. Note that $\kappa(0) = 0$ and $\kappa(\theta)$ is a convex function (see [9]). Thus, we need $\kappa'(0) < 0$ to have the positive solution. Assume that $\hat{D}(\theta)$ exists for some $\theta_0 > 0$, which is called a light tail condition. Then, this is, indeed, the case because $\kappa(\theta) = \boldsymbol{\mu}^{(\theta)}(-\theta I + C + \hat{D}(\theta))\boldsymbol{h}^{(\theta)}$ implies that

$$\begin{aligned}\kappa'(0) &= \boldsymbol{\mu}^{(0)}(-I + C + \hat{D}_{(1)}(0))\boldsymbol{h}^{(0)} \\ &= \boldsymbol{\pi}(-I + C + \hat{D}_{(1)}(0))\boldsymbol{e} \\ &= -1 + \rho < 0,\end{aligned}$$

where we have used the fact that $C + \hat{D}(0)$ is a rate matrix with the stationary vector $\boldsymbol{\pi}$. Hence, if $\kappa(\theta)$ becomes positive as θ is increased, $\kappa(\theta) = 0$ has a positive solution. For instance, this is the case if each row sum of $-\theta I + C + \hat{D}(\theta)$ diverges as θ is increased (see Corollary 1 of [14, Chap. 1]).

In what follows, we just assume that $\kappa(\theta) = 0$ has a positive solution and denote this solution by α. Then,

$$\hat{P}(\alpha)\boldsymbol{h}^{(\alpha)} = \boldsymbol{h}^{(\alpha)}.$$

Let $\boldsymbol{\nu}^{(\alpha)} = \boldsymbol{\mu}^{(\alpha)}(\alpha I - K)/\alpha$. Note that $\boldsymbol{\nu}^{(\alpha)}\boldsymbol{k} = 1$, since $\boldsymbol{\mu}^{(\alpha)}\boldsymbol{k} = 1$. It is easy to see that $\boldsymbol{\nu}^{(\alpha)}$ is the left eigenvector of $\hat{P}(\alpha)$ for eigenvalue 1. Define the †-system as in Section 3. Then, $\boldsymbol{\nu}^{(\alpha)}\Delta_{h^{(\alpha)}}$ is the stationary vector of $P^{\dagger}(\infty)$, so $\boldsymbol{\nu}^{(\alpha)}$ must be positive. It remains to check that $e^{\alpha u}A(u)$ is directly Riemann integrable. To this end, it is sufficient to show that $e^{\alpha u}A(u)$ is integrable, since $A(u)$ is decreasing function of u (see, e.g., Lemma 6.1.4 of [12]):

$$\begin{aligned}\int_0^\infty e^{\alpha u}A(u)\,du &= \int_0^\infty e^{\alpha u}\,du \int_0^\infty e^{yK}\int_{u+y}^\infty D(dz)\,dy \\ &= \int_0^\infty \left(\int_0^z \int_0^{z-y} e^{\alpha u}\,du\, e^{yK}\,dy\right) D(dz) \\ &= \int_0^\infty \frac{1}{\alpha}\left(\int_0^z (e^{\alpha(z-y)} - 1)e^{yK}\,dy\right) D(dz) \\ &= \frac{1}{\alpha}\int_0^\infty \left((K - \alpha I)^{-1}(e^{zK} - e^{\alpha z}) - \int_0^z e^{yK}\,dy\right) D(dz) \\ &= \frac{1}{\alpha}(K - \alpha I)^{-1}(K - C - \hat{D}(\alpha)) - \frac{1}{\alpha}\int_0^\infty \int_0^z e^{yK}\,dy\, D(dz),\end{aligned}$$

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where $\hat{D}(\alpha) = \int_0^\infty e^{\alpha u} D(du)$. We normalize the positive right eigenvector $\boldsymbol{k}$ of K for eigenvalue 0 in such a way that $\boldsymbol{\pi k} = 1$. Since

$$(\boldsymbol{k\pi} - K)\boldsymbol{k} = \boldsymbol{k},$$

we have $\boldsymbol{k} = (\boldsymbol{k\pi} - K)^{-1}\boldsymbol{k}$, where $\boldsymbol{k\pi} - K$ can be shown to be nonsingular. Hence, $\boldsymbol{\pi} K = \boldsymbol{0}$ implies that

$$\begin{aligned}
-\frac{1}{\alpha} & \int_0^\infty \int_0^z e^{yK}\,dy\,D(dz) \\
&= -\frac{1}{\alpha}(\boldsymbol{k\pi} - K)^{-1} \int_0^\infty \int_0^z (\boldsymbol{k\pi} - K) e^{yK}\,dy\,D(dz) \\
&= -\frac{1}{\alpha}(\boldsymbol{k\pi} - K)^{-1} \left(\int_0^\infty (1 - e^{zK}) D(dz) + \boldsymbol{k\pi} \int_0^\infty z D(dz) \right) \\
&= -\frac{1}{\alpha}(\boldsymbol{k\pi} - K)^{-1} (\boldsymbol{k\pi} - K - \boldsymbol{k\pi}(I - \hat{D}_{(1)}(0)) + C + D(\infty)) \\
&= \frac{1}{\alpha}(-I + \boldsymbol{k\pi}(I - \hat{D}_{(1)}(0)) - (\boldsymbol{k\pi} - K)^{-1}(C + D(\infty))).
\end{aligned}$$

Consequently, $e^{\alpha u} A(u)$ is integrable and we have

$$\boldsymbol{\nu}^{(\alpha)} \int_0^\infty e^{\alpha u} A(u)\,du = \frac{1}{\alpha} \boldsymbol{\nu}^{(\alpha)} (\boldsymbol{k\pi}(I - \hat{D}_{(1)}(0)) - (\boldsymbol{k\pi} - K)^{-1}(C + D(\infty))),$$

since $\boldsymbol{\mu}^{(\alpha)} = \alpha \boldsymbol{\nu}^{(\alpha)} (\alpha I - K)^{-1}$ is the left eigenvector of $(\alpha I - C - \hat{D}(\alpha))$ for eigenvalue 0. Thus, using Lemma 2.2, we arrive at the following asymptotics.

THEOREM 3.1: *For the risk process with claims subject to MAP, if* $\rho < 1$ *and if the maximal eigenvalue* $\kappa(\theta)$ *of* $\theta I - (C + \hat{D}(\theta))$ *admits* $\alpha > 0$ *such that* $\kappa(\alpha) = 0$, *then*

$$\begin{aligned}
\lim_{x \to \infty} & e^{\alpha x} P(M(\tau_x) = j \mid M(0) = i) \\
&= \frac{1}{\rho^\dagger - 1} [\boldsymbol{h}^{(\alpha)} \boldsymbol{\nu}^{(\alpha)} (\boldsymbol{k\pi}(I - \hat{D}_{(1)}(0)) - (\boldsymbol{k\pi} - K)^{-1}(C + D(\infty)))]_{ij}, \\
& \qquad i, j \in S, \qquad \textbf{(10)}
\end{aligned}$$

where $\rho^\dagger = \boldsymbol{\mu}^{(\alpha)} \hat{D}_{(1)}(\alpha) \boldsymbol{h}^{(\alpha)}$. *In particular,*

$$\lim_{x \to \infty} e^{\alpha x} P(\tau_x < \infty \mid M(0) = i) = \frac{(1 - \rho)\boldsymbol{\nu}^{(\alpha)} \boldsymbol{k}}{\rho^\dagger - 1} h_i^{(\alpha)}. \qquad \textbf{(11)}$$

PROOF: We only need to verify that

$$\alpha \boldsymbol{\nu}^{(\alpha)} \hat{P}_{(1)}(\alpha) \boldsymbol{h}^{(\alpha)} = \boldsymbol{\mu}^{(\alpha)} \hat{D}_{(1)}(\alpha) \boldsymbol{h}^{(\alpha)} - 1.$$

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This is indeed obtained from the following computations:

$$\begin{aligned}
\hat{P}_{(1)}(\alpha) &= \int_0^\infty ue^{\alpha u}\,du \int_0^\infty e^{uK} \int_0^\infty D(dy+u)\,du \\
&= \int_0^\infty e^{uK} \int_u^\infty (y-u)e^{\alpha(y-u)} D(dy)\,du \\
&= \int_0^\infty e^{\alpha y} \int_0^x (y-u)e^{u(K-\alpha I)}\,du\,D(dy) \\
&= (K-\alpha I)^{-1} \int_0^\infty e^{\alpha y} \left([(y-u)e^{u(K-\alpha I)}]_0^y + \int_0^x e^{u(K-\alpha I)}\,du \right) D(dy) \\
&= (K-\alpha I)^{-1} \int_0^\infty e^{\alpha y}(-y + (K-\alpha I)^{-1}(e^{y(K-\alpha I)} - I))D(dy) \\
&= (\alpha I - K)^{-1}(\hat{D}_{(1)}(\alpha) - I + (\alpha I - K)^{-1}(\alpha I - C - \hat{D}(\alpha))),
\end{aligned}$$

so we have

$$\alpha \boldsymbol{\nu}^{(\alpha)} \hat{P}_{(1)}(\alpha) \boldsymbol{h}^{(\alpha)} = \boldsymbol{\mu}^{(\alpha)} (\hat{D}_{(1)}(\alpha) - I)\boldsymbol{h}^{(\alpha)}. \qquad \blacksquare$$

Remark 3.2: The matrix version of Cramér–Lundberg approximation (10) seems to be new with respect to the coefficient. Similar results are obtained in Theorem 8.2 of [6], but its coefficient is more like those in Lemma 2.2 and less informative. Furthermore, the claim size distributions in [6] are limited to be of the phase type, although this is not essential. The asymptotic decay (11) extends Theorem 3.7 of [5] in the sense that the modeling assumption is relaxed to have jumps at the transition instants of $M(t)$.

4. THE *MAP*/*G*/1 QUEUE

We next consider the $MAP/G/1$ queue. In this case, $Y(t)$ is the total work arriving up to time t starting from time 0, so $B(t) = Y(t) - t$ is the net flow up to time t. We assume the stability condition $\rho < 1$. Under this condition, we are interested in the stationary distribution of the workload and its decay rate. Let V be the workload in the steady state and let M be the associated background state; then, it is well known that the time-reversed construction (see, e.g., Theorem 3.9 of [4] and [10]) yields

$$\begin{aligned}
P(V > x, M = i) &= P\left(\sup_{u \ge 0}(-B(-u)) > x, M(0) = i\right) \\
&= \pi_i P\left(\sup_{u\ge 0}(-Y(-u) - u) > x \,\middle|\, M(0) = i\right). \qquad \textbf{(12)}
\end{aligned}$$

The process $(-Y(-t), M(-t))$ is the time reversal of the process $(-Y(t), M(t))$. So, $(-Y(-t), M(-t))$ is generated by C^* and $D^*(x)$, which are introduced in Remark

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3.1(c). Hence, the computation of (12) is reduced to the hitting probability $P(\tau_x < \infty | M(0) = i)$ for the dual system with C^* and $D^*(x)$. Thus, converting all characteristics to those in the dual systems, we have the following result.

THEOREM 4.1: *For the MAP/G/1 queue, if $\rho < 1$ and if the maximal eigenvalue $\kappa(\theta)$ of $\theta I - (C + \hat{D}(\theta))$ admits $\alpha > 0$ such that $\kappa(\alpha) = 0$, then*

$$\lim_{x\to\infty} e^{\alpha x} P(V > x, M = i) = \frac{(1-\rho)\boldsymbol{\eta}\boldsymbol{h}^{(\alpha)}}{\boldsymbol{\mu}^{(\alpha)}\hat{D}_{(1)}(\alpha)\boldsymbol{h}^{(\alpha)} - 1}\,\mu_i^{(\alpha)}, \tag{13}$$

where $\boldsymbol{\eta}$ is the stationary probability vector of the rate matrix Q that is determined as the minimal solution of the following equation:

$$Q = C + \int_0^\infty D(du)e^{uQ}. \tag{14}$$

PROOF: If we replace C and $D(x)$ in Theorem 3.1 by C^* and $D^*(x)$, then we get

$$\lim_{x\to\infty} e^{\alpha x} P(V > x, M = i) = \frac{(1-\rho)\boldsymbol{\nu}^*\boldsymbol{k}^*}{\boldsymbol{\mu}^*\hat{D}^*_{(1)}(\alpha)\boldsymbol{h}^* - 1}\,\pi_i h_i^*, \tag{15}$$

where $\boldsymbol{\mu}^*$, $\boldsymbol{\nu}^*$, $\boldsymbol{h}^*$, and $\boldsymbol{k}^*$ correspond to $\boldsymbol{\mu}^{(\alpha)}$, $\boldsymbol{\nu}^{(\alpha)}$, $\boldsymbol{h}^{(\alpha)}$, and $\boldsymbol{k}$, respectively, in the dual system. Since $\boldsymbol{\mu}^*$ and $\boldsymbol{h}^*$ are the left and right eigenvectors of the following matrix for eigenvalue 0,

$$\alpha I - (C^* + D^*(\alpha)) = \Delta_\pi^{-1}(\alpha I - (C + D(\alpha)))'\Delta_\pi,$$

we have $\boldsymbol{\mu}^* = [\boldsymbol{h}^{(\alpha)}]'\Delta_\pi$ and $\boldsymbol{h}^* = \Delta_\pi^{-1}[\boldsymbol{\mu}^{(\alpha)}]'$. Since $Q = \Delta_\pi^{-1}(K^*)'\Delta_\pi$, $\boldsymbol{\eta}Q = 0$ and $K^*\boldsymbol{k}^* = 0$ imply that $\boldsymbol{k}^* = \Delta_\pi^{-1}\boldsymbol{\eta}'$. Finally, we see that

$$\alpha\boldsymbol{\nu}^*\boldsymbol{k}^* = \boldsymbol{\mu}^*(\alpha I - K^*)\boldsymbol{k}^* = \alpha\boldsymbol{\eta}\boldsymbol{h}^{(\alpha)}.$$

Substituting these expressions into (15) yields (13). ■

Remark 4.1: The asymptotics (13) is exactly the same as obtained in [16]. However, the current derivation in [16] needs to assume the existence of the limit in (13) since the Tauberian theorem of Feller [8] is used (see, e.g., [1]). We have not presented a direct application of Lemma 2.2 to the *MAP/G/*1 queue. Clearly, this can be done using the dual system of the risk process. For instance, from (9), the Markov renewal kernel is obtained as

$$\begin{aligned} P(x) &= \Delta_\pi^{-1}\left(\int_0^\infty e^{yK^*}\int_0^x D^*(dz+y)\,dy\right)'\Delta_\pi \\ &= \int_0^\infty \int_0^x D(dz+y)e^{yQ}\,dy. \end{aligned} \tag{16}$$

This formula is originally due to Asmussen [3], in which $D(x)$ is a diagonal matrix. Takine [15] differently derives it for the *MAP* arrivals using a last-come first-served

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preemptive resume (LCFS-PR) queue. In that literature, (16) is obtained as the ladder height distribution rather than the Markov transition kernel.

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