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EXIT PROBLEMS FOR GENERAL DRAW-DOWN TIMES OF SPECTRALLY NEGATIVE LÉVY PROCESSES

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Abstract

For spectrally negative Lévy processes, we prove several fluctuation results involving a general draw-down time, which is a downward exit time from a dynamic level that depends on the running maximum of the process. In particular, we find expressions of the Laplace transforms for the two-sided exit problems involving the draw-down time. We also find the Laplace transforms for the hitting time and creeping time over the running-maximum related draw-down level, respectively, and obtain an expression for a draw-down associated potential measure. The results are expressed in terms of scale functions for the spectrally negative Lévy processes.

Keywords: Spectrally negative Lévy process; draw-down time; exit problem; potential measure; creeping time; hitting time

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1. Introduction

As part of fluctuation theory, exit problems for spectrally negative Lévy processes and the associated reflected processes have been studied extensively over the past ten years. Such problems often concern the joint distributions of the process when it first leaves either a finite or a semi-finite interval, or when its draw-down (from the running maximum) or its draw-up (from the running minimum) first exceeds a fixed level. These results are often expressed in terms of the scale functions. We refer to [8] and references therein for a collection of such results; see also [1], [10], [13], and [16] for research on the draw-down times and draw-up times for spectrally negative Lévy processes reflected at the running maximum and running minimum processes, respectively. We refer to [15] for recent work on draw-down(up) times of regular diffusions.

Exit problems involving more general first passage times had been considered earlier for time homogeneous diffusions. In [11] an exit problem with exit level depending on the running maximum of the diffusion was studied and a joint Laplace transform was found for such a general draw-down time. The general draw-down times find interesting applications in [3], defining the Azéma–Yor martingales to solve the Skorokhod embedding problem. The draw-down problem for renewal processes was studied in [9]. More recent work on applications of draw-down times can be found in [5], [9], and references therein.

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Pistorius [14] studied the general draw-down times of spectrally negative Lévy processes with an excursion theory approach, to obtain the Skorokhod embedding for the spectrally negative Lévy process and for the associated reflected process from its maximum. Following a similar approach, Mijatović and Pistorius [12] then considered a sextuple law related to the draw-down time. In [2] we considered a perturbed spectrally negative Lévy risk process, the so-called Lévy tax process, with a draw-down exit level that is a linear function of the running maximum process, and found expressions on the expected present values of amount of tax for this process. To this end, we applied both the excursion theory and an approximation approach using solutions to the exit problems with fixed boundaries.

In this paper we continue to investigate the exit problems for a spectrally negative Lévy process with a dynamic draw-down exit level that depends on the running maximum in a general way. Applying the excursion theory, which comes in handy for analysing the draw-down fluctuation behaviours for the spectrally negative Lévy processes, we first find the expressions for a joint Laplace transform of the process at the draw-down time. We also find the Laplace transforms for the hitting time and creeping time of the draw-down level, respectively. In addition, we obtain an expression for a potential measure associated with the draw-down time.

This paper is structured as follows. After the Introduction, in Section 2 we review basic facts on spectrally negative Lévy processes to prepare for the main proofs. The main results are presented in Section 3. Section 4 focuses on results with linear draw-down functions for which the expressions can be simplified and some previous results can be easily recovered. Proofs of the main results are deferred to Section 5.

2. Spectrally negative Lévy processes

Throughout this paper, let $X=\{X_t, t\geq 0\}$ be a spectrally negative Lévy process (SNLP for short), i.e. a one-dimensional stochastic process with stationary independent increments and with no positive jumps, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We also assume that X is not the negative of a subordinator. Denote by $\mathbb{P}_x$ the probability law of X given $X_0=x$, and the corresponding expectation by $\mathbb{E}_x$. Write $\mathbb{P}$ and $\mathbb{E}$ when $x=0$. Its Laplace transform always exists, with the Laplace exponent given by

$$\psi(\lambda) := \frac{1}{t}\log \mathbb{E}(\mathrm{e}^{\lambda X_t}) \quad \text{for } \lambda\geq 0$$

where

$$\psi(\lambda)=\mu\lambda+\frac{1}{2}\sigma^2\lambda^2+\int_{(-\infty,0)}(\mathrm{e}^{\lambda x}-1-\lambda x\mathbf{1}_{\{x>-1\}})\Pi(\mathrm{d}x)$$

for $\mu\in\mathbb{R}$, $\sigma\geq 0$ and the σ-finite Lévy measure Π on $(-\infty, 0)$ satisfying

$$\int_{(-\infty,0)}(1\wedge x^2)\Pi(\mathrm{d}x)<\infty.$$

Further, there exists a function $\Phi:[0,\infty)\to[0,\infty)$ defined by

$$\Phi(q) := \sup\{\lambda\geq 0\colon \psi(\lambda)=q\} \quad \text{for } q\geq 0.$$

Scale functions play a central role in the fluctuation theory for SNLPs. For $q\geq 0$, the q-scale function $W^{(q)}$ for process X is defined as a continuous function on $[0,\infty)$ satisfying

$$\int_0^\infty \mathrm{e}^{-\lambda y}W^{(q)}(y)\,\mathrm{d}y=\frac{1}{\psi(\lambda)-q} \quad \text{for } \lambda>\Phi(q).$$

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For convenience, we extend the domain of $W^{(q)}$ to the whole real line by setting $W^{(q)}(x)=0$ for all $x<0$. Given $W^{(q)}$, the second scale function is defined by

$$Z^{(q)}(x):=1+q\int_0^x W^{(q)}(y)\,\mathrm{d}y.$$

Write $W=W^{(0)}$ and $Z=Z^{(0)}$ whenever $q=0$.

It is well known that $W^{(q)}(x)$ is strictly increasing on $\mathbb{R}^+$. $W^{(q)}(0)=W(0)>0$ if and only if the process X has paths of bounded variation, and if and only if $\sigma=0$ and $\int_{-1}^0 |x|\Pi(\mathrm{d}x)<\infty$. The scale function $W^{(q)}$ is continuously differentiable on $(0,\infty)$ if the process X has paths of unbounded variation (and, in particular, if it has a nontrivial Gaussian component) or if the process X has paths of bounded variation and the Lévy measure has no atoms. Moreover, if $\sigma>0$, $W^{(q)}$ has a continuous derivative of order two on $(0,\infty)$ and $W'(0+)=2/\sigma^2$. We refer the readers to [6] for more detailed discussions on the smoothness of scale functions.

For $c\geq 0$, process $\{\mathrm{e}^{cX_t-\psi(c)t},\ t\geq 0\}$ is a martingale under $\mathbb{P}$. Introduce a new probability measure satisfying

$$\frac{\mathrm{d}\mathbb{P}^{(c)}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t}=\mathrm{e}^{cX_t-\psi(c)t}\quad \text{for every } t\geq 0.$$

It is well known that X is still an SNLP under $\mathbb{P}^{(c)}$. Denoting the associated Laplace exponent and scale functions with a subscript c under $\mathbb{P}^{(c)}$, a straightforward calculation shows that, for $c\geq 0$, $q+\psi(c)\geq 0$,

$$\psi_c(s)=\psi(c+s)-\psi(c)\quad\text{and}\quad \Phi_c(s)=\Phi(s+\psi(c))-c\quad \text{for } s\geq 0;$$

in addition,

$$W_c^{(q)}(x)=\mathrm{e}^{-cx}W^{(q+\psi(c))}(x)\quad\text{and}\quad Z_c^{(q)}(x)=1+q\int_0^x W_c^{(q)}(y)\,\mathrm{d}y.$$

Note that, for $x\geq 0$, $W^{(q)}(x)$ and $Z^{(q)}(x)$ are analytically extendable to all $q\in\mathbb{C}$ ([8, Lemma 8.3, Corollary 8.5]), and identity (2.2) in Lemma 2.1 below holds for all $u,v\geq 0$.

For any c, $b\in\mathbb{R}$, defining the first passage times

$$\tau_b^+:=\inf\{t\geq 0\colon X_t>b\}\quad\text{and}\quad \tau_c^-:=\inf\{t\geq 0\colon X_t<c\}$$

with the convention that $\inf\varnothing=\infty$, we have the following result.

Lemma 2.1. *For $c\leq x\leq b$, q, u, $v\geq 0$ and for $p:=u-\psi(v)$, we have*

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_b^+}\colon \tau_b^+<\tau_c^-)=\frac{W^{(q)}(x-c)}{W^{(q)}(b-c)}, \tag{2.1}$$

$$\mathbb{E}_x(\mathrm{e}^{-u\tau_c^-+vX(\tau_c^-)}\colon \tau_c^-<\tau_b^+)=\mathrm{e}^{vx}\bigg(Z_v^{(p)}(x-c)-\frac{W_v^{(p)}(x-c)}{W_v^{(p)}(b-c)}Z_v^{(p)}(b-c)\bigg), \tag{2.2}$$

and, for $x\in(c,b)$,

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_c^-}\colon X(\tau_c^-)=c,\ \tau_c^-<\tau_b^+)=\frac{\sigma^2}{2}\bigg(W^{(q)\prime}(x-c)-W^{(q)}(x-c)\frac{W^{(q)\prime}(b-c)}{W^{(q)}(b-c)}\bigg). \tag{2.3}$$

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In addition, the resolvent of process X killed at the first exit time of interval $[c, b]$ is specified by

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}_x(f(X_t),\, t < \tau_b^+ \wedge \tau_c^-)\,\mathrm{d}t = \int_c^b f(y)\left(\frac{W^{(q)}(x-c)}{W^{(q)}(b-c)}W^{(q)}(b-y) - W^{(q)}(x-y)\right)\mathrm{d}y.$$

Identity (2.1) can be found in [8, Theorem 8.1]. To obtain the joint Laplace transform in (2.2), we apply [8, Theorem 8.1] under the new measure $\mathbb{P}_x^{(v)}$. Identity (2.3) can be found in [12, equation (2.6)], and $\{X(\tau_c^-) = c\}$ is known as the creeping event, which happens for an SNLP when the first downward passage over a level occurs by hitting the level with a positive probability. The result shows that an SNLP creeps downwards if and only if it has a Gaussian component.

Let $\tau^{\{a\}} := \inf\{t > 0, X_t = a\}$ be the first hitting time of level a. We could not find the following result in the literature, and provide a proof for the readers' convenience.

Lemma 2.2. *For $x, a \in (c, b)$, we have*

$$\mathbb{E}_x(\mathrm{e}^{-q\tau^{\{a\}}} : \tau^{\{a\}} < \tau_b^+ \wedge \tau_c^-) = \frac{W^{(q)}(x-c)}{W^{(q)}(a-c)} - \frac{W^{(q)}(x-a)W^{(q)}(b-c)}{W^{(q)}(b-a)W^{(q)}(a-c)}. \tag{2.4}$$

Proof of Lemma 2.2. As observed in [7, Lemma 11] that $\{\tau_b^+ < \tau^{\{a\}}\} = \{\tau_b^+ < \tau_a^-\}$ for $a < b$, applying the strong Markov property of X at $\tau^{\{a\}}$, we have

$$\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_c^-)\\
&\quad = \mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau^{\{a\}} < \tau_b^+ < \tau_c^-) + \mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_c^- \wedge \tau^{\{a\}})\\
&\quad = \mathbb{E}_x(\mathrm{e}^{-q\tau^{\{a\}}} : \tau^{\{a\}} < \tau_b^+ \wedge \tau_c^-)\mathbb{E}_a(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_c^-) + \mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_a^-).
\end{aligned}$$

The Laplace transform (2.4) for the hitting time follows by applying (2.1). □

3. Main results

Write $\bar{X}_t := \sup_{0\le s\le t} X_s$ for the running maximum process for X. The process X reflected at its running maximum is defined by $Y_t := \bar{X}_t - X_t$. Let $\xi(\cdot)$ be a measurable function on $\mathbb{R}$. Define the draw-down time for X with respect to the draw-down function ξ as

$$\tau_\xi := \inf\{t > 0, X_t < \xi(\bar{X}_t)\} = \inf\{t > 0, Y_t > \bar{\xi}(\bar{X}_t)\},$$

where $\bar{\xi}(z) := z - \xi(z)$ and $\{\xi(\bar{X}_t), t \ge 0\}$ is the associated draw-down level process. For the case of constant function $\bar{\xi}$, a sextuple law was found in [12], where the concerned quantities include the time of reaching the last maximum and minimum value of X before τ_ξ together with the undershoot at τ_ξ. The process Y is referred to as the draw-down process in [12]. In this paper, we focus on an arbitrary measurable and strictly positive function $\bar{\xi}$ on $\mathbb{R}$ and we assume that $W^{(q)\prime}(x)$ exists on $(0, \infty)$ for simplicity.

In this section we first present expressions for the solutions to the two-sided exit problems involving τ_ξ.

Proposition 3.1. *For any $q > 0$ and $x < b$, we have*

$$\mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_\xi) = \exp\left(-\int_x^b \frac{W^{(q)\prime}(\bar{\xi}(y))}{W^{(q)}(\bar{\xi}(y))}\,\mathrm{d}y\right). \tag{3.1}$$

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For any $u, v > 0$, $k \in \mathbb{R}$ and $x < b$, with $p := u - \psi(v)$, we have

$$\mathbb{E}_x(\mathrm{e}^{-u\tau_\xi + vX(\tau_\xi) + k\bar{X}(\tau_\xi)} : \tau_\xi < \tau_b^+)$$
$$= \mathrm{e}^{vx} \int_x^b \mathrm{e}^{ky - \int_x^y \frac{W_v^{(p)\prime}(\bar{\xi}(z))}{W_v^{(p)}(\bar{\xi}(z))}\,\mathrm{d}z} \left(\frac{W_v^{(p)\prime}(\bar{\xi}(y))}{W_v^{(p)}(\bar{\xi}(y))} Z_v^{(p)}(\bar{\xi}(y)) - pW_v^{(p)}(\bar{\xi}(y)) \right) \mathrm{d}y. \tag{3.2}$$

Remark 3.1. The assumption $\bar{\xi} > 0$ on $[x, b]$ is necessary. In fact, if $\bar{\xi}(a) = 0$ for some $a \in (x, b)$, one can find that $\tau_\xi \le \tau_a^+$ $\mathbb{P}_x$-a.s. by definition, and then (3.1) and (3.2) fail to hold. It is often the case that the process X is bounded from above by a constant b for the event of interest. When this happens, under $\mathbb{P}_x$ the effective domain of $\bar{\xi}$ is $[x, b]$ instead of $\mathbb{R}$, that is, only the values of $\bar{\xi}(y)$ for $y \in [x, b]$ really matter, and the conditions on $\bar{\xi}(y)$ only need to be imposed for $y \in [x, b]$.

For the potential measure up to time $\tau_b^+ \wedge \tau_\xi$, we also have the next result.

Proposition 3.2. *For any $x < b$, we have*

$$\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(X_t \in \mathrm{d}y : t < \tau_b^+ \wedge \tau_\xi)\,\mathrm{d}t$$
$$= \left(\int_x^b \mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\bar{\xi}(s))}{W^{(q)}(\bar{\xi}(s))}\,\mathrm{d}s} \left(W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(\bar{\xi}(z))}{W^{(q)}(\bar{\xi}(z))} W^{(q)}(z-y) \right) \mathbf{1}_{\{y \in (\xi(z), z)\}}\,\mathrm{d}z \right) \mathrm{d}y$$
$$+ W(0) \left(\mathrm{e}^{-\int_x^y \frac{W^{(q)\prime}(\bar{\xi}(z))}{W^{(q)}(\bar{\xi}(z))}\,\mathrm{d}z} \mathbf{1}_{\{y \in (x,b)\}} \right) \mathrm{d}y. \tag{3.3}$$

Remark 3.2. The resolvent density in (3.3) consists of two parts, where the second term degenerates if process X has sample paths of unbounded variation. By further analysis, one can find that it is contributed by the total amount of time in

$$\mathcal{L} := \{t > 0 : X_t = \bar{X}_t\}$$

until time $\tau_b^+ \wedge \tau_\xi$, that is,

$$\int_0^\infty \mathrm{e}^{-qt} \mathbf{1}_{\{t \in \mathcal{L}\}} \mathbf{1}_{\{t < \tau_b^+ \wedge \tau_\xi\}} \mathbf{1}_{\{X_t \in \mathrm{d}y\}}\,\mathrm{d}t;$$

and for the case of $W(0) = 0$, we see that $\mathcal{L}$ is a Lebesgue null set $\mathbb{P}_x$-a.s.

Remark 3.3. If $\xi \equiv c$ for some $c < x$, then Proposition 3.2 implies that the support of the resolvent is $[c, b]$. For any $y \in (c, b)$, since $y > \xi(z) = c$ in the following integral, we have

$$\int_x^b \mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\bar{\xi}(u))}{W^{(q)}(\bar{\xi}(u))}\,\mathrm{d}u} \left(W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(\bar{\xi}(z))}{W^{(q)}(\bar{\xi}(z))} W^{(q)}(z-y) \right) \mathbf{1}_{\{y \in (\xi(z), z)\}}\,\mathrm{d}z$$
$$= \int_{x \vee y}^b \mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(u-c)}{W^{(q)}(u-c)}\,\mathrm{d}u} \left(W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(z-c)}{W^{(q)}(z-c)} W^{(q)}(z-y) \right) \mathrm{d}z$$
$$= \left(\mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(u-c)}{W^{(q)}(u-c)}\,\mathrm{d}u} W^{(q)}(z-y) \right) \Big|_{x \vee y}^b = W^{(q)}(x-c) \left(\frac{W^{(q)}(z-y)}{W^{(q)}(z-c)} \right) \Big|_{x \vee y}^b$$
$$= \frac{W^{(q)}(x-c)}{W^{(q)}(b-c)} W^{(q)}(b-y) - W^{(q)}(x-y) \mathbf{1}_{\{x \ge y\}} - W(0) \frac{W^{(q)}(x-c)}{W^{(q)}(y-c)} \mathbf{1}_{\{y > x\}}$$

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and

$$W(0)\,\mathrm{e}^{-\int_x^y \frac{W^{(q)\prime}(\bar{\xi}(z))}{W^{(q)}(\bar{\xi}(z))}\,\mathrm{d}z}\mathbf{1}_{\{y>x\}} = W(0)\,\mathrm{e}^{-\int_x^y \frac{W^{(q)\prime}(z-c)}{W^{(q)}(z-c)}\,\mathrm{d}z}\mathbf{1}_{\{y>x\}} = W(0)\frac{W^{(q)}(x-c)}{W^{(q)}(y-c)}\mathbf{1}_{\{y>x\}}.$$

Making use of the fact that $W^{(q)}(z)=0$ for $z<0$ again, we recover the expression of the classical potential density.

Notice that $\tau_\xi \wedge \tau_c^- = \tau_{\xi\vee c}$ for any $c<x$ and initial value x. Applying Proposition 3.2, in the next result we can also obtain a joint distribution involving the running minimum and maximum before τ_ξ together with $X_{\tau_\xi -}$ and X_{τ_ξ} when there is an overshoot at the draw-down time τ_ξ.

Let $\underline{X}_t := \inf_{s\in[0,t]} X_s$ be the running minimum process of X.

Corollary 3.1. *For any nonnegative measurable function f on $\mathbb{R}^2$ satisfying $f(z,z)=0$ for all $z\in\mathbb{R}$ and for any $c<x<b$, we have*

$$\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi} f(X_{\tau_\xi -}, X_{\tau_\xi})\colon \underline{X}_{\tau_\xi -} > c, \bar{X}_{\tau_\xi} \le b)\\
&\quad = W(0)\int_c^b \Big(\mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\overline{\xi\vee c}(s))}{W^{(q)}(\overline{\xi\vee c}(s))}\,\mathrm{d}s}\Big)\,\mathrm{d}z \int_{-\infty}^{-\bar{\xi}(z)} f(z, z+u)\Pi(\mathrm{d}u)\\
&\qquad + \int_c^b \mathrm{d}z \int_{\xi(z)\vee c}^{z} \mathrm{d}y \int_{-\infty}^{\xi(z)-y} f(y, y+u)\Pi(\mathrm{d}u)\\
&\qquad \times \bigg(\mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\overline{\xi\vee c}(s))}{W^{(q)}(\overline{\xi\vee c}(s))}\,\mathrm{d}s} W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(\overline{\xi\vee c}(z))}{W^{(q)}(\overline{\xi\vee c}(z))}W^{(q)}(z-y)\bigg).
\end{aligned}$$

We now consider the hitting problem of a draw-down level. Let

$$\tau^{\{\xi\}} := \inf\{t>0, X_t = \xi(\bar{X}_t)\} = \inf\{t>0, Y_t = \bar{\xi}(\bar{X}_t)\}$$

denote the first time for X to hit the draw-down level $\xi(\bar{X})$. A particularly interesting case of hitting is the event of creeping, $\{\tau^{\{\xi\}} = \tau_\xi\}$, which happens for an SNLP when the first downward passage over a level occurs by hitting the level with a positive probability. It is well known that the classical creeping of a fixed level happens only if $\sigma>0$, i.e. only if process X has a nontrivial Brownian motion component. If the downward passage time is replaced with a draw-down time, observe that within the duration of each downward sample path of excursion away from the running maximum, the draw-down level $\xi(\bar{X})$ remains constant. Therefore, one would expect that the draw-down creeping occurs if and only if process X has a nontrivial Brownian motion component.

In the following proposition, another draw-down level with draw-down function $\theta(z)$ is introduced with $\bar{\theta}(z) = z - \theta(z) > 0$ for all $z\in\mathbb{R}$.

Proposition 3.3. *For any $x<b$ and $I := \{z\in\mathbb{R}\colon \theta(z) < \xi(z)\}$, we have*

$$\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi}\colon \tau^{\{\xi\}} = \tau_\xi < \tau_b^+ \wedge \tau_\theta)\\
&\quad = \frac{\sigma^2}{2}\int_{[x,b]\cap I} \mathrm{e}^{-\int_x^y \frac{W^{(q)\prime}(\overline{\xi\vee\theta}(z))}{W^{(q)}(\overline{\xi\vee\theta}(z))}\,\mathrm{d}z}\bigg(\frac{(W^{(q)\prime}(\bar{\xi}(y)))^2}{W^{(q)}(\bar{\xi}(y))} - W^{(q)\prime\prime}(\bar{\xi}(y))\bigg)\,\mathrm{d}y. \qquad (3.4)
\end{aligned}$$

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Moreover, for any $x < b$ we have

$$\mathbb{E}_x(\mathrm{e}^{-q\tau^{\{\xi\}}} : \tau^{\{\xi\}} < \tau_b^+ \wedge \tau_\theta)$$
$$= \int_{[x,b]\cap I} \mathrm{e}^{-\int_x^y \frac{W^{(q)\prime}(\overline{\xi\vee\theta}(z))}{W^{(q)}(\overline{\xi\vee\theta}(z))}\,\mathrm{d}z} \frac{W^{(q)}(\bar{\theta}(y))}{W^{(q)}(\bar{\theta}(y)-\bar{\xi}(y))}\left(\frac{W^{(q)\prime}(\bar{\xi}(y))}{W^{(q)}(\bar{\xi}(y))} - \frac{W^{(q)\prime}(\bar{\theta}(y))}{W^{(q)}(\bar{\theta}(y))}\right)\mathrm{d}y. \quad (3.5)$$

Remark 3.4. If $\theta < \xi = c$ for some $c < x < b$, Proposition 3.3 reduces to the classical result (2.3). In this case, $I = \mathbb{R}$ and $\overline{\xi \vee \theta}(y) = y - c = \bar{\xi}(y)$. As in Remark 3.3, we have

$$\int_x^y \frac{W^{(q)\prime}(\overline{\xi\vee\theta}(z))}{W^{(q)}(\overline{\xi\vee\theta}(z))}\,\mathrm{d}z = \int_x^y \frac{W^{(q)\prime}(z-c)}{W^{(q)}(z-c)}\,\mathrm{d}z = \log\left(\frac{W^{(q)}(y-c)}{W^{(q)}(x-c)}\right) \quad \text{for } y \in (x, b).$$

Then the right-hand side of (3.4) is equal to

$$\frac{\sigma^2}{2}\int_x^b \frac{W^{(q)}(x-c)}{W^{(q)}(y-c)}\left(\frac{(W^{(q)\prime}(y-c))^2}{W^{(q)}(y-c)} - W^{(q)\prime\prime}(y-c)\right)\mathrm{d}y$$
$$= \frac{\sigma^2}{2} W^{(q)}(x-c)\cdot\left(-\frac{W^{(q)\prime}(y-c)}{W^{(q)}(y-c)}\right)\bigg|_x^b$$
$$= \frac{\sigma^2}{2}\left(W^{(q)\prime}(x-c) - W^{(q)}(x-c)\frac{W^{(q)\prime}(b-c)}{W^{(q)}(b-c)}\right),$$

which recovers (2.3).

Similarly, one can recover Lemma 2.2 from (3.5) by taking $\xi = a$, $\theta = c$ with $c < a < b$ and $x \in (c, b)$.

It is also interesting to study similar problems associated with the draw-up times (from the running minimum) with a general draw-up function for a spectrally negative Lévy process. However, it seems challenging to express the desired results in terms of the scale functions.

4. Applications

In this section we present two applications of the results from Section 3.

4.1. Selling a stock at a draw-down time

Example 4.1. The decision to sell a stock is a combination of art and science. In general, it is ideal to sell a stock at a price as high as possible or just before it starts to decline. However, very few investors can buy at the absolute bottom and sell at the absolute high. If one does not sell at the right time, the profit disappears. There are a number of considerations to determine the best time. In this example, we assume that the price process of an underlying security is given by $S = \{S_t = \mathrm{e}^{X_t}, t \geq 0\}$. The investor sells a stock either when it hits a price target in order to lock in gains or before the ratio $S/\overline{S}$ leaves too far below 1 to stop the loss, where $\overline{S}$ is the historical high process for S. Using the Cobb–Douglas function, the investor sells out of a stock when the utility process

$$\{(S_t/S_0)^\gamma (S_t/\overline{S}_t)^{1-\gamma}, \ \ t \geq 0\}$$

leaves a predetermined interval $[a, b]$ for some $\gamma, a \in (0, 1)$ and $b > 1$. It can be checked directly that $T_{a,b} = T_{a,b}^+ \wedge T_{a,b}^-$, where

$$T_{a,b}^+ := \inf\{t > 0 : S_t > b^{1/\gamma} S_0\} \quad \text{and} \quad T_{a,b}^- := \inf\{t > 0 : S_t/S_0 < a\cdot(\overline{S}_t/S_0)^{1-\gamma}\}.$$

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Without loss of generality, we take $S_0 = 1$ and then $X_0 = 0$. Formulated in terms of the first passage time for X under $\mathbb{P}$, we have $T^+_{a,b} = \tau^+_{(\log b)/\gamma}$ and $T^-_{a,b} = \tau_\xi$ for $\xi(z) = (1-\gamma)z + \log a$. Let $q > 0$ be the risk-free interest rate and $p = q - \psi(1)$. Then we have

$$\mathbb{E}_1(\mathrm{e}^{-qT_{a,b}} S_{T_{a,b}} : T_{a,b} = T^+_{a,b}) = b^{1/\gamma} \left(\frac{W^{(q)}(\log 1/a)}{W^{(q)}(\log b/a)} \right)^{1/\gamma} \tag{4.1}$$

and

$$\begin{aligned}\mathbb{E}_1(\mathrm{e}^{-qT_{a,b}} S_{T_{a,b}} : T_{a,b} = T^-_{a,b}) &= Z_1^{(p)}(\log 1/a) - Z_1^{(p)}(\log b/a) \left(\frac{W_1^{(p)}(\log 1/a)}{W_1^{(p)}(\log b/a)} \right)^{1/\gamma} \\ &\quad - p \frac{1-\gamma}{\gamma} \int_{\log 1/a}^{\log b/a} (W_1^{(p)}(\log 1/a))^{1/\gamma} (W_1^{(p)}(y))^{(\gamma-1)/\gamma} \, \mathrm{d}y.\end{aligned} \tag{4.2}$$

Proof of (4.1) *and* (4.2). In this case, for $z \in [0, (\log b)/\gamma]$, $\bar{\xi}(z) = \gamma z - \log a > 0$ by definition. Then $\bar{\xi}'(z) = \gamma$ and, for $x \in [0, (\log b)/\gamma]$, we have

$$\int_0^x \frac{W^{(q)\prime}(\bar{\xi}(y))}{W^{(q)}(\bar{\xi}(y))} \, \mathrm{d}y = \frac{1}{\gamma} (\log (W^{(q)}(\bar{\xi}(y))))\Big|_0^x = \frac{1}{\gamma} \log \left(\frac{W^{(q)}(\bar{\xi}(x))}{W^{(q)}(\bar{\xi}(0))} \right).$$

Applying the formula (3.1) and the fact that $S_{T^+_{a,b}} = b^{1/\gamma}$ on the event $\{T^+_{a,b} < \infty\}$ gives (4.1).

Similarly, we have

$$\begin{aligned}&\int_0^{(\log b)/\gamma} \exp\left(-\int_0^y \frac{W_1^{(p)\prime}(\bar{\xi}(z))}{W_1^{(p)}(\bar{\xi}(z))} \, \mathrm{d}z \right) \left(\frac{W_1^{(p)\prime}(\bar{\xi}(y))}{W_1^{(p)}(\bar{\xi}(y))} Z_1^{(p)}(\bar{\xi}(y)) - p\gamma W_1^{(p)}(\bar{\xi}(y)) \right) \mathrm{d}y \\ &= \left(-Z_1^{(p)}(\bar{\xi}(y))\, \mathrm{e}^{-\int_0^y \frac{W_1^{(p)\prime}(\bar{\xi}(z))}{W_1^{(p)}(\bar{\xi}(z))} \, \mathrm{d}z} \right)\Bigg|_0^{(\log b)/\gamma} \\ &= Z_1^{(p)}\left(\log \frac{1}{a} \right) - Z_1^{(p)}\left(\log \frac{b}{a} \right) \left(\frac{W_1^{(p)}(\log 1/a)}{W_1^{(p)}(\log b/a)} \right)^{1/\gamma}.\end{aligned}$$

The identity (4.2) can be proved by applying (3.2) and a change of variable argument. □

4.2. First passage times of the reflected SNLP

Example 4.2. For function $\xi(z) = z - d$ for some $d > 0$, the draw-down time τ_ξ reduces to the first passage time of SNLP reflected at its running maximum, which was investigated in [1] and [13], that is, $\tau_\xi = \kappa_d^+$ for

$$\kappa_d^+ := \inf\{t > 0,\ Y_t > d\}.$$

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For b, q, u, $v>0$ and $k\in\mathbb{R}$, we have

$$\mathbb{E}(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \kappa_d^+) = \exp\left(-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)}b\right), \tag{4.3}$$

$$\mathbb{E}(\mathrm{e}^{-u\kappa_d^+ + vX(\kappa_d^+) + k\bar{X}(\kappa_d^+)} : \kappa_d^+ < \tau_b^+) = \left(\frac{W_v^{(p)\prime}(d)}{W_v^{(p)}(d)} Z_v^{(p)}(d) - pW_v^{(p)}(d)\right) \times \left(1 - \mathrm{e}^{\left(k - \frac{W_v^{(p)\prime}(d)}{W_v^{(p)}(d)}\right)b}\right) \Big/ \left(\frac{W_v^{(p)\prime}(d)}{W_v^{(p)}(d)} - k\right), \tag{4.4}$$

and

$$\mathbb{E}(\mathrm{e}^{-q\kappa_d^+} : Y(\kappa_d^+) = d, \kappa_d^+ < \tau_b^+) = \frac{\sigma^2}{2}\left(W^{(q)\prime}(d) - \frac{W^{(q)}(d)W^{(q)\prime\prime}(d)}{W^{(q)\prime}(d)}\right)\left(1 - \mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)}b}\right), \tag{4.5}$$

where $p = u - \psi(v)$. In addition, for $y \in (-d, b)$ we have

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(X_t \in \mathrm{d}y, t < \kappa_d^+ \wedge \tau_b^+)\,\mathrm{d}t = \left(W^{(q)}(d \wedge (b-y))\,\mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)}(y+d)\wedge b} - W^{(q)}(-y)\right)\mathrm{d}y. \tag{4.6}$$

Note that $\bar{\xi}(t) \equiv d$ in this case, so the above results follow directly from Propositions 3.1, 3.3, and 3.2, respectively, where an argument similar to Remark 3.3 is applied in obtaining (4.6).

Remark 4.1. Denote by e_q an exponential random variable with parameter q and independent of X. Since

$$\mathbb{E}(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \kappa_d^+) = \mathbb{P}(\tau_b^+ < e_q \wedge \kappa_d^+) = \mathbb{P}(\bar{X}_{e_q\wedge\kappa_d^+} > b),$$

it follows from (4.3) that $\bar{X}_{e_q\wedge\kappa_d^+}$ is exponentially distributed with parameter $W^{(q)\prime}(d)/W^{(q)}(d)$. Taking $k = -v < 0$ in (4.4) and letting $b \to \infty$, we have

$$\mathbb{E}(\mathrm{e}^{-u\kappa_d^+ - vY(\kappa_d^+)} : \kappa_d^+ < \infty) = Z_v^{(p)}(d) - W_v^{(p)}(d)\frac{pW_v^{(p)}(d) + vZ_v^{(p)}(d)}{W_v^{(p)\prime}(d) + vW_v^{(p)}(d)},$$

which coincides with [1, Theorem 1].

Remark 4.2. For $0 < b - y < d$ and $b > 0$, identity (4.6) can be rewritten as

$$\mathbb{P}(X_{e_q} \in \mathrm{d}y, e_q < \kappa_d^+, \bar{X}_{e_q} < b) = q(W^{(q)}(b-y)\,\mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)}b} - W^{(q)}(-y))\,\mathrm{d}y.$$

Then

$$\mathbb{P}(X_{e_q} \in \mathrm{d}y, e_q < \kappa_d^+, \bar{X}_{e_q} \in \mathrm{d}b) = q\left(W^{(q)\prime}(b-y) - \frac{W^{(q)\prime}(d)}{W^{(q)}(d)}W^{(q)}(b-y)\right)\mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)}b}\,\mathrm{d}y\,\mathrm{d}b.$$

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Therefore, for $z \in (0, d)$ we have

$$\mathbb{P}(Y_{e_q} \in \mathrm{d}z,\, e_q < \kappa_d^+) = q\left(W^{(q)\prime}(z) - \frac{W^{(q)\prime}(d)}{W^{(q)}(d)} W^{(q)}(z)\right) \int_0^\infty \mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)} b}\, \mathrm{d}b$$
$$= q\left(W^{(q)}(d) \frac{W^{(q)\prime}(z)}{W^{(q)\prime}(d)} - W^{(q)}(z)\right) \mathrm{d}z,$$

which coincides with the resolvent given in [13, Theorem 1(ii)].

By Remark 3.2, for the case $W(0) > 0$, we have for $y \in (0, b)$

$$\mathbb{P}(e_q < \kappa_d^+ \wedge \tau_b^+,\, \bar{X}(e_q) \in \mathrm{d}y,\, Y(e_q) = 0) = qW(0)\, \mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)} y}\, \mathrm{d}y.$$

Therefore,

$$\mathbb{P}(e_q < \kappa_d^+ \wedge \tau_b^+,\, Y(e_q) = 0) = q \frac{W(0) W^{(q)}(d)}{W^{(q)\prime}(d)} \left(1 - \mathrm{e}^{-\frac{W^{(q)\prime}(d)}{W^{(q)}(d)} b}\right).$$

It follows that

$$\mathbb{P}(Y(e_q) = 0,\, e_q < \kappa_d^+) = qW(0) \frac{W^{(q)}(d)}{W^{(q)\prime}(d)},$$

which gives the time Y spent at 0 before κ_d^+ and coincides with [13, Theorem 1(ii)].

5. Proofs of the main results

This section is dedicated to the proofs for our main results, where we make use of the excursion theory for Markov processes, and appeal to the compensation formula and the exponential formula for Poisson point processes; see for example [4, O.5]. To this end, we first restate the formula concerned in terms of excursions, and then apply the compensation formula. For this we use the following notations from [1], [4, IV], and [14], and refer the readers to the book for a detailed discussion of the related excursion theory. Moreover, by the spatial homogeneity of X, we mainly focus on the cases under $\mathbb{P}$. More general results for $\mathbb{P}_x$ can be derived by a shifting argument, as shown in the proof of Proposition 3.1.

Recall that $\bar{X}$ is the running maximum process of X and let $Y := \bar{X} - X$ be the reflected process. It is known that Y is a 'nice' Markov process with 0 being instantaneous whenever $W(0) = 0$. Let $\mathcal{L} := \{t > 0,\, Y_t = 0\}$ be the zero set of Y and let $\overline{\mathcal{L}}$ be its closure. A local time process L of Y at 0 is a continuous process that increases only on $\overline{\mathcal{L}}$ and is unique up to a multiplicative factor. Thus, there exists $\nu \geq 0$ such that

$$\int_0^t \mathbf{1}_{\{s \in \mathcal{L}\}}\, \mathrm{d}s = \nu L(t) \quad \text{for all } t \geq 0 \tag{5.1}$$

(see [4, Corollary IV.6]). The right inverse of L is defined by

$$L_t^{-1} := \inf\{s > 0 \colon L(s) > t\}, \qquad t \geq 0.$$

Under the new time scale, the excursion process of Y away from zero, associated with L and denoted by $\epsilon \equiv \{\epsilon_r,\, r \geq 0\}$, takes values in the so-called excursion space of paths away from 0 with an additional isolated point γ, $\mathcal{E} \cup \{\gamma\}$, and is defined by

$$\epsilon_r := \{Y_t,\, L_{r-}^{-1} \leq t < L_r^{-1}\} \quad \text{if } L_{r-}^{-1} < L_r^{-1},$$

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and $\epsilon_r := \gamma$ otherwise. The excursion process ϵ is a Poisson point process, possibly stopped at time $L(\infty)$ with an excursion of infinite lifetime, characterized by a σ-finite measure $n(\cdot)$ on $\mathcal{E}$. Set $(\overline{\mathcal{L}})^c$ consists of countable excursion intervals, and $\mathcal{L}$ differs from $\overline{\mathcal{L}}$ by at most countable points. In particular,

$$\int_0^{L_r^{-1}} \mathbf{1}_{\{s\in\overline{\mathcal{L}}\}}\,\mathrm{d}s = \nu r$$

on $\{L_r^{-1} < \infty\}$ under $\mathbb{P}$ (see [4, Lemma VI.8]). For a Borel function $f \geq 0$ on $\mathcal{E} \cup \{\gamma\}$ with $f(\gamma) = 0$, we write

$$n(f) := \int_{\mathcal{E}} f(\varepsilon) n(\mathrm{d}\varepsilon).$$

For any $c \geq 0$, $n^{(c)}$ denotes the associated excursion measure for X under $\mathbb{P}^{(c)}$.

For an SNLP X under $\mathbb{P}$, its running maximum process $\bar{X}$ fulfils the condition of a local time and is chosen to be the local time of Y at 0 (see [4, VII]). For this choice of local time L,

$$L(t) = \sup_{s\in[0,t]} X_s,$$

and $L_s^{-1} = \tau_s^+$ is a subordinator with Laplace exponent Φ. Since ν is the drift parameter of L^{-1} (see [4, Theorem IV.8]), we have

$$\nu = \lim_{s\to\infty} \frac{\Phi(s)}{s} = \lim_{s\to\infty} \frac{s}{\psi(s)} = W(0).$$

For an excursion $\varepsilon \in \mathcal{E}$, its lifetime is denoted by ζ and its excursion height is denoted by $\bar{\varepsilon}$. The first passage and the first hitting time of ε are defined by

$$\rho_c^+ := \inf\{s \in (0, \zeta) \colon \varepsilon(s) > c\} \quad \text{and} \quad \rho^{\{c\}} := \inf\{s \in (0, \zeta) \colon \varepsilon(s) = c\}$$

respectively, with the convention that $\inf \varnothing = \infty$. We write $\rho_c^+(r)$, $\rho^{\{c\}}(r)$ and $\zeta(r)$ for the first passage time, the first hitting time and the lifetime, respectively, of the excursion at local time r, that is, $\epsilon_r = \varepsilon \in \mathcal{E}$. As before, we let e_q denote an exponential variable with parameter $q > 0$ and independent of X.

Proof of Proposition 3.1. Observe from their definitions that under $\mathbb{P}$,

- on the event $\{\tau_b^+ < \infty\}$,

$$\tau_b^+ = L_b^{-1} = \int_0^{L_b^{-1}} \mathbf{1}_{\{t\in\overline{\mathcal{L}}\}}\,\mathrm{d}t + \int_0^{L_b^{-1}} \mathbf{1}_{\{t\notin\overline{\mathcal{L}}\}}\,\mathrm{d}t = \nu b + \sum_{r\in[0,b]} \zeta(r);$$

- on the event $\{\tau_\xi < \infty\}$, $\tau_\xi = L_{r-}^{-1} + \rho_{\bar{\xi}(r)}^+(r)$ and $\bar{X}(\tau_\xi) = r$ for $r = L(\tau_\xi)$;
- on the event $\{\tau^{\{\xi\}} < \infty\}$, $\tau^{\{\xi\}} = L_{r-}^{-1} + \rho^{\{\bar{\xi}(r)\}}(r)$ and $\bar{X}(\tau_\xi) = r$ for $r = L(\tau^{\{\xi\}})$.

From the idea in the proof of [4, Theorem VII.8], it holds that

$$\{\tau_b^+ < \tau_\xi\} = \{\bar{\epsilon}_r \leq \bar{\xi}(r) \text{ for all } r \in [0, b]\}.$$

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Therefore, using the exponential compensation formula (see [4, O.5]), we obtain

$$\mathbb{E}(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_\xi) = \mathrm{e}^{-q\nu b} \times \mathbb{E}\Big(\mathrm{e}^{-q\sum_{r\in[0,b]}\zeta(r)} \prod_{r\in[0,b]} \mathbf{1}_{\{\bar{\epsilon}_r \le \bar{\xi}(r)\}}\Big)$$

$$= \mathrm{e}^{-q\nu b} \times \mathbb{E}\Big(\exp\Big(-\sum_{r\in[0,b]} (q\zeta(r) + \infty\cdot \mathbf{1}_{\{\bar{\epsilon}_r > \bar{\xi}(r)\}})\Big)\Big)$$

$$= \exp\bigg(-\int_0^b \bigg(q\nu + \int_{\mathcal{E}} (1 - \mathrm{e}^{-q\zeta}\mathbf{1}_{\{\bar{\varepsilon}\le\bar{\xi}(r)\}})n(\mathrm{d}\varepsilon)\bigg)\,\mathrm{d}r\bigg), \tag{5.2}$$

with the understanding that $\mathrm{e}^{-\infty} = 0$ and $\infty \times 0 = 0$. To solve the problem, we recall the classical case of constant $\xi \equiv c < 0$ in Lemma 2.1:

$$\frac{W^{(q)}(-c)}{W^{(q)}(b-c)} = \mathbb{E}(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_c^-)$$

$$= \exp\bigg(-\int_0^b \bigg(q\nu + \int_{\mathcal{E}} (1 - \mathrm{e}^{-q\zeta}\mathbf{1}_{\{\bar{\varepsilon}\le(r-c)\}})n(\mathrm{d}\varepsilon)\bigg)\,\mathrm{d}r\bigg).$$

Differentiating in b on both sides of the equation above gives

$$q\nu + \int_{\mathcal{E}} (1 - \mathrm{e}^{-q\zeta}\mathbf{1}_{\{\bar{\varepsilon}\le z\}})n(\mathrm{d}\varepsilon) = \frac{W^{(q)\prime}(z)}{W^{(q)}(z)} \quad \text{for } z > 0. \tag{5.3}$$

Applying the formula (5.3) to the previous equation (5.2) gives identity (3.1) under $\mathbb{P}$.

For the event $\{\tau_\xi < \tau_b^+\}$ and for $p = u - \psi(v) > 0$, taking a change of measure we have

$$\mathbb{E}(\mathrm{e}^{-u\tau_\xi + vX(\tau_\xi) + k\bar{X}(\tau_\xi)} : \tau_\xi < \tau_b^+)$$

$$= \mathbb{E}^{(v)}(\mathrm{e}^{-p\tau_\xi + k\bar{X}(\tau_\xi)} : \tau_\xi < \tau_b^+)$$

$$= \mathbb{E}^{(v)}\bigg(\sum_{r\in[0,b]} \Big(\mathrm{e}^{kr - pL_{r-}^{-1}} \prod_{s<r} \mathbf{1}_{\{\bar{\epsilon}_s \le \bar{\xi}(s)\}}\Big) \times \mathrm{e}^{-p\rho^+_{\bar{\xi}(r)}(r)}\mathbf{1}_{\{\bar{\epsilon}_r > \bar{\xi}(r)\}}\bigg)$$

$$= \int_0^b \mathrm{e}^{kr}\mathbb{E}^{(v)}\Big(\mathrm{e}^{-pL_{r-}^{-1}} \prod_{s<r} \mathbf{1}_{\{\bar{\epsilon}_s \le \bar{\xi}(s)\}}\Big) \times n^{(v)}(\mathrm{e}^{-p\rho^+_{\bar{\xi}(r)}} : \bar{\varepsilon} > \bar{\xi}(r))\,\mathrm{d}r$$

$$= \int_0^b \mathrm{e}^{kr}\mathbb{E}^{(v)}(\mathrm{e}^{-pL_r^{-1}} : L_r^{-1} < \tau_\xi) \times n^{(v)}(\mathrm{e}^{-p\rho^+_{\bar{\xi}(r)}} : \bar{\varepsilon} > \bar{\xi}(r))\,\mathrm{d}r,$$

where a compensation formula is applied for the third equality (see e.g. [4, O.5]) and where we used the fact that $L_t^{-1} \ne L_{t-}^{-1}$ for at most countably many values of t.

Considering now the case where $\xi \equiv c < 0$, $u = q > 0$ and $v = k = 0$, we have

$$Z^{(q)}(-c) - \frac{W^{(q)}(-c)}{W^{(q)}(b-c)}Z^{(q)}(b-c)$$

$$= \mathbb{E}(\mathrm{e}^{-q\tau_c^-} : \tau_c^- < \tau_b^+)$$

$$= \int_0^b \mathbb{E}(\mathrm{e}^{-q\tau_r^+} : \tau_r^+ < \tau_c^-) \times n(\mathrm{e}^{-q\rho^+_{r-c}} : \bar{\varepsilon} > (r-c))\,\mathrm{d}r$$

$$= \int_0^b \frac{W^{(q)}(-c)}{W^{(q)}(r-c)} \times n(\mathrm{e}^{-q\rho^+_{r-c}} : \bar{\varepsilon} > (r-c))\,\mathrm{d}r.$$

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Differentiating in b on both sides of the above equation gives

$$n(\mathrm{e}^{-q\rho_z^+} : \bar{\varepsilon} > z) = \frac{W^{(q)\prime}(z)}{W^{(q)}(z)} Z^{(q)}(z) - qW^{(q)}(z) \quad \text{for } z > 0. \tag{5.4}$$

Plugging (3.1) and (5.4) into the equation gives the formula (3.2) under $\mathbb{P}$. The general result for $u, v > 0$ and $k \in \mathbb{R}$ follows by an analytic extension.

To consider the general case of $X(0) = x$ with $x < b$, we introduce a function $\varsigma(y) := \xi(y + x) - x$. Then

$$\bar{\varsigma}(y) = y + x - \xi(y + x) = \bar{\xi}(y + x).$$

Since X is spatially homogeneous, we have

$$(X, \bar{X}, \tau_\xi)|_{\mathbb{P}_x} = (x + X, x + \bar{X}, \tau_\varsigma)|_{\mathbb{P}}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}_x(\mathrm{e}^{-q\tau_b^+} : \tau_b^+ < \tau_\xi) &= \mathbb{E}(\mathrm{e}^{-q\tau_{b-x}^+} : \tau_{b-x}^+ < \tau_\varsigma) \\
&= \exp\bigg(-\int_0^{b-x} \frac{W^{(q)\prime}(\bar{\varsigma}(y))}{W^{(q)}(\bar{\varsigma}(y))} \,\mathrm{d}y\bigg) \\
&= \exp\bigg(-\int_x^{b} \frac{W^{(q)\prime}(\bar{\xi}(y))}{W^{(q)}(\bar{\xi}(y))} \,\mathrm{d}y\bigg),
\end{aligned}$$

which gives (3.1). Similarly, we have from the spatial homogeneity of X that

$$\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-u\tau_\xi + vX(\tau_\xi) + k\bar{X}(\tau_\xi)} : \tau_\xi < \tau_b^+) \\
&\quad = \mathrm{e}^{vx+kx}\mathbb{E}(\mathrm{e}^{-u\tau_\varsigma + vX(\tau_\varsigma) + k\bar{X}(\tau_\varsigma)} : \tau_\varsigma < \tau_{b-x}^+) \\
&\quad = \mathrm{e}^{vx+kx}\int_0^{b-x} \mathrm{e}^{ky - \int_0^y \frac{W_v^{(p)\prime}(\bar{\varsigma}(z))}{W_v^{(p)}(\bar{\varsigma}(z))}\,\mathrm{d}z}\bigg(\frac{W_v^{(p)\prime}(\bar{\varsigma}(y))}{W_v^{(p)}(\bar{\varsigma}(y))} Z_v^{(p)}(\bar{\varsigma}(y)) - pW_v^{(p)}(\bar{\varsigma}(y))\bigg)\,\mathrm{d}y \\
&\quad = \mathrm{e}^{vx}\int_x^{b} \mathrm{e}^{ky - \int_x^y \frac{W_v^{(p)\prime}(\bar{\xi}(z))}{W_v^{(p)}(\bar{\xi}(z))}\,\mathrm{d}z}\bigg(\frac{W_v^{(p)\prime}(\bar{\xi}(y))}{W_v^{(p)}(\bar{\xi}(y))} Z_v^{(p)}(\bar{\xi}(y)) - pW_v^{(p)}(\bar{\xi}(y))\bigg)\,\mathrm{d}y.
\end{aligned}$$

This concludes the proof of Proposition 3.1. □

In the following proofs, we use an idea from [4, Lemma VI.8] and the compensation formula, and we only focus on the case under $\mathbb{P}$ and for $b > 0$.

Proof of Proposition 3.2. Let $f \geq 0$ be a bounded and continuous function on $\mathbb{R}$. For the resolvent of X killed at $\tau_b^+ \wedge \tau_\xi$, which is defined in (3.3), we have

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}(f(X_t) : t < \tau_b^+ \wedge \tau_\xi)\,\mathrm{d}t \\
&\quad = \int_0^\infty \mathrm{e}^{-qt}\mathbb{E}(f(X_t), t \in \overline{\mathcal{L}}, t < \tau_b^+ \wedge \tau_\xi)\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{-qt}\mathbb{E}(f(X_t), t \notin \overline{\mathcal{L}}, t < \tau_b^+ \wedge \tau_\xi)\,\mathrm{d}t \\
&\quad =: I_1 + I_2.
\end{aligned}$$

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Recalling equation (5.1), applying Fubini's theorem and a change of variable, we have

$$I_1 = \nu\mathbb{E}\bigg(\int_0^\infty \mathrm{e}^{-qt}f(X_t)\mathbf{1}_{\{t<\tau_b^+\wedge\tau_\xi\}}\,\mathrm{d}L_t\bigg)$$
$$= \nu\mathbb{E}\bigg(\int_0^\infty \mathrm{e}^{-qL_r^{-1}}f(X(L_r^{-1}))\mathbf{1}_{\{L_r^{-1}<\tau_b^+\wedge\tau_\xi\}}\,\mathrm{d}r\bigg)$$
$$= \nu\int_0^b f(r)\mathbb{E}(\mathrm{e}^{-qL_r^{-1}} : L_r^{-1} < \tau_\xi)\,\mathrm{d}r,$$

where $L_r^{-1} = \tau_r^+$ and $X(L_r^{-1}) = r$ on event $\{L_r^{-1} < \infty\}$. On the other hand, we have

$$qI_2 = \mathbb{E}(f(X_{e_q})\colon e_q \notin \overline{\mathcal{L}}, e_q < \tau_b^+ \wedge \tau_\xi)$$
$$= \mathbb{E}\bigg(\sum_{r\in[0,b]} f(r-\epsilon_r(e_q - L_{r-}^{-1})) \times \prod_{s<r}\mathbf{1}_{\{\overline{\epsilon}_s\le\bar{\xi}(s)\}}\mathbf{1}_{\{e_q<L_{r-}^{-1}+\rho^+_{\bar{\xi}(r)}(r)\}} : L_{r-}^{-1} < e_q < L_r^{-1}\bigg),$$

where $\bar{X}_{e_q} = L(e_q) = r$ on the event $\{L_{r-}^{-1} < e_q < L_r^{-1}\}$. By the memoryless property of e_q and the compensation formula, we further have

$$qI_2 = \mathbb{E}\bigg(\sum_{r\in[0,b]} \mathrm{e}^{-qL_{r-}^{-1}}\prod_{s<r}\mathbf{1}_{\{\overline{\epsilon}_s\le\bar{\xi}(s)\}} \times f(r-\epsilon_r(e_q))\mathbf{1}_{\{e_q<\rho^+_{\bar{\xi}(r)}(r)\wedge\zeta(r)\}}\bigg)$$
$$= \int_0^b \mathbb{E}(\mathrm{e}^{-qL_r^{-1}} : L_r^{-1} < \tau_\xi) \times q\int_0^\infty n(\mathrm{e}^{-qs}f(r-\varepsilon(s))\colon s<\rho^+_{\bar{\xi}(r)}\wedge\zeta)\,\mathrm{d}s\,\mathrm{d}r.$$

Putting these together gives

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}(f(X_t)\colon t<\tau_b^+\wedge\tau_\xi)\,\mathrm{d}t$$
$$= \int_0^b \mathbb{E}(\mathrm{e}^{-q\tau_r^+} : \tau_r^+ < \tau_\xi) \times \bigg(\nu f(r) + \int_0^\infty \mathrm{e}^{-qs}n(f(r-\varepsilon(s))\colon s<\rho^+_{\bar{\xi}(r)}\wedge\zeta)\,\mathrm{d}s\bigg)\,\mathrm{d}r.$$

For the case of $\xi \equiv c < 0$, we have

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{E}(f(X_t)\colon t<\tau_b^+\wedge\tau_c^-)\,\mathrm{d}t$$
$$= \int_0^b \frac{W^{(q)}(-c)}{W^{(q)}(r-c)}\bigg(\nu f(r) + \int_0^\infty \mathrm{e}^{-qs}n(f(r-\varepsilon(s))\colon s<\rho^+_{r-c}\wedge\zeta)\,\mathrm{d}s\bigg)\,\mathrm{d}r$$
$$= \int_c^b f(y)\bigg(\frac{W^{(q)}(-c)}{W^{(q)}(b-c)}W^{(q)}(b-y) - W^{(q)}(-y)\bigg)\,\mathrm{d}y.$$

Further differentiating in b on the above equation, we eventually have

$$\nu f(b) + \int_0^\infty \mathrm{e}^{-qs}n(f(b-\varepsilon(s))\colon s<\rho^+_{b-c}\wedge\zeta)\,\mathrm{d}s$$
$$= W(0)f(b) + \int_c^b f(y)\bigg(W^{(q)\prime}(b-y) - \frac{W^{(q)\prime}(b-c)}{W^{(q)}(b-c)}W^{(q)}(b-y)\bigg)\,\mathrm{d}y. \qquad (5.5)$$

Since $\nu = W(0)$, formula (3.3) is thus proved for $x = 0$. □

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Proof of Proposition 3.3. As in the proofs of Propositions 3.1 and 3.2, for the event of creeping $\{\tau^{\{\xi\}}=\tau_\xi<\tau_b^+\wedge\tau_\theta\}$, i.e. the event that the draw-down event happens before X leaves interval $[\theta(\bar{X}), b]$ by hitting the draw-down level $\xi(\bar{X})$, we have

$$
\begin{aligned}
&\mathbb{E}(\mathrm{e}^{-q\tau_\xi}: \tau^{\{\xi\}}=\tau_\xi<\tau_b^+\wedge\tau_\theta)\\
&\quad=\mathbb{E}\bigg(\sum_{r\in[0,b]}\bigg(\mathrm{e}^{-qL_{r-}^{-1}}\prod_{s<r}\mathbf{1}_{\{\bar{\epsilon}_s\le\bar{\theta}(s)\}}\mathbf{1}_{\{\bar{\epsilon}_s\le\bar{\xi}(s)\}}\bigg)\times(\mathrm{e}^{-q\rho^+_{\bar{\xi}(r)}(r)}\mathbf{1}_{\{\epsilon_r(\rho^+_{\bar{\xi}(r)})=\bar{\xi}(r)\}}\mathbf{1}_{\{\bar{\xi}(r)<\bar{\theta}(r)\}})\bigg)\\
&\quad=\int_0^b\mathbb{E}(\mathrm{e}^{-qL_r^{-1}}: L_r^{-1}<\tau_{\xi\vee\theta})\times n(\mathrm{e}^{-q\rho^+_{\bar{\xi}(r)}}: \varepsilon(\rho^+_{\bar{\xi}(r)})=\bar{\xi}(r))\times\mathbf{1}_{\{r\in I\}}\,\mathrm{d}r.
\end{aligned}
$$

For the case of $\xi(z)=\theta(z)+1\equiv c$, we have $I=\mathbb{R}$ and $\{\tau_\xi\le\tau_b^+\wedge\tau_\theta\}=\{\tau_c^-\le\tau_b^+\}$. Since

$$
\begin{aligned}
\mathbb{E}(\mathrm{e}^{-q\tau_c^-}: X(\tau_c^-)=c,\,\tau_c^-<\tau_b^+)&=\frac{\sigma^2}{2}\bigg(W^{(q)\prime}(-c)-W^{(q)}(-c)\frac{W^{(q)\prime}(b-c)}{W^{(q)}(b-c)}\bigg)\\
&=\int_0^b\frac{W^{(q)}(-c)}{W^{(q)}(r-c)}\times n(\mathrm{e}^{-q\rho^+_{r-c}}: \varepsilon(\rho^+_{r-c})=r-c)\,\mathrm{d}r,
\end{aligned}
$$

differentiating on both sides of the equation above, we have

$$
n(\mathrm{e}^{-q\rho_z^+}: \varepsilon(\rho_z^+)=z)=\frac{\sigma^2}{2}\bigg(\frac{(W^{(q)\prime}(z))^2}{W^{(q)}(z)}-W^{(q)\prime\prime}(z)\bigg)\quad\text{for } z>0. \tag{5.6}
$$

Identity (3.4) is thus proved by applying (3.1) for $x=0$.

The hitting of a maximum dependent level cannot be derived by applying the strong Markov property of X as in the classical case in Lemma 2.2. However, due to the absence of positive jumps, a similar observation is that

$$
\{L_s^{-1}<\tau^{\{\xi\}}\wedge\tau_\theta\}=\{L_s^{-1}<\tau_\xi\wedge\tau_\theta\}\quad\text{on the event }\{L_s^{-1}<\infty\},
$$

that is, every excursion at time $s<r=L(\tau^{\{\xi\}})$ fails to go above level $\overline{\xi\vee\theta}(s)$. Therefore,

$$
\begin{aligned}
&\mathbb{E}(\mathrm{e}^{-q\tau^{\{\xi\}}}: \tau^{\{\xi\}}<\tau_b^+\wedge\tau_\theta)\\
&\quad=\mathbb{E}\bigg(\sum_{r\in[0,b]}\bigg(\mathrm{e}^{-qL_{r-}^{-1}}\prod_{s<r}\mathbf{1}_{\{\bar{\epsilon}_s\le\bar{\xi}(s)\}}\mathbf{1}_{\{\bar{\epsilon}_s\le\bar{\theta}(s)\}}\bigg)\times(\mathrm{e}^{-q\rho^{\{\bar{\xi}(r)\}}(r)}\mathbf{1}_{\{\rho^{\{\bar{\xi}(r)\}}(r)<\rho^+_{\bar{\theta}(r)}(r)\}})\bigg)\\
&\quad=\int_0^b\mathbb{E}(\mathrm{e}^{-qL_r^{-1}}: L_r^{-1}\le\tau_{\xi\vee\theta})\times n(\mathrm{e}^{-q\rho^{\{\bar{\xi}(r)\}}}: \rho^{\{\bar{\xi}(r)\}}<\rho^+_{\bar{\theta}(r)})\mathbf{1}_{\{\bar{\xi}(r)<\bar{\theta}(r)\}}\,\mathrm{d}r.
\end{aligned}
$$

For the case of $\xi\equiv a$ and $\theta\equiv c$ with $c<a<0<b$, we have

$$
\begin{aligned}
&\frac{W^{(q)}(-c)}{W^{(q)}(a-c)}-\frac{W^{(q)}(-a)W^{(q)}(b-c)}{W^{(q)}(b-a)W^{(q)}(a-c)}\\
&\quad=\mathbb{E}(\mathrm{e}^{-q\tau^{\{a\}}}: \tau^{\{a\}}<\tau_b^+\wedge\tau_c^-)\\
&\quad=\int_0^b\mathbb{E}(\mathrm{e}^{-q\tau_r^+}: \tau_r^+<\tau_a^-)\times n(\mathrm{e}^{-q\rho^{\{r-a\}}}: \rho^{\{r-a\}}<\rho^+_{r-c})\,\mathrm{d}r.
\end{aligned}
$$

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Differentiating in b on both sides of the above equation gives, for $b > a > c$,

$$n(\mathrm{e}^{-q\rho^{\{b-a\}}}:\ \rho^{\{b-a\}} < \rho^+_{b-c}) = \frac{W^{(q)}(b-c)}{W^{(q)}(a-c)}\left(\frac{W^{(q)\prime}(b-a)}{W^{(q)}(b-a)} - \frac{W^{(q)\prime}(b-c)}{W^{(q)}(b-c)}\right), \tag{5.7}$$

which leads to identity (3.5). □

Remark 5.1. We remark that the excursion theory for a (reflected) SNLP has been employed to solve several similar problems. The formulas (5.4), (5.5), (5.6), (5.3) for $q = 0$ and (5.7) for $b - c = \infty$ have also been found as well; see [14] for a collection of such results.

Proof of Corollary 3.1. Since $\{t < \tau_b^+\} = \{\bar{X}_t < b\}$ and $\{t < \tau_c^-\} = \{\underline{X}_t > c\}$ for every $t > 0$, we have by Proposition 3.2 that, for $z > x \ge c$ and $\xi(z) \vee c < y \le z$,

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, \bar{X}_t \in \mathrm{d}z, t < \tau_\xi, \underline{X}_t \ge c)\,\mathrm{d}t \\
&\quad = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y, \bar{X}_t \in \mathrm{d}z, t < \tau_{\xi\vee c})\,\mathrm{d}t \\
&\quad = \left(\mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\overline{\xi\vee c}(s))}{W^{(q)}(\overline{\xi\vee c}(s))}\,\mathrm{d}s} W^{(q)\prime}(z-y) - \frac{W^{(q)\prime}(\overline{\xi\vee c}(z))}{W^{(q)}(\overline{\xi\vee c}(z))}W^{(q)}(z-y)\right)\mathrm{d}z\,\mathrm{d}y \\
&\qquad + W(0)\left(\mathrm{e}^{-\int_x^z \frac{W^{(q)\prime}(\overline{\xi\vee c}(s))}{W^{(q)}(\overline{\xi\vee c}(s))}\,\mathrm{d}s}\right)\mathrm{d}z\cdot\delta_z(\mathrm{d}y),
\end{aligned}$$

where δ_z denotes the Dirac measure concentrated at z. Note that $\tau_\xi \in \{t > 0, X_{t-} \ne X_t\}$ on the event $\{X_{\tau_\xi} \ne X_{\tau_\xi-}\} \cap \{\tau_\xi < \infty\}$, and process $\{(X_t - X_{t-}), t \ge 0\}$ can be identified as a Poisson point process with characteristic measure $\Pi(\cdot)$. Therefore, for $f \ge 0$ satisfying $f(z, z) = 0$, we have by Fubini's theorem

$$\begin{aligned}
&\mathbb{E}_x(\mathrm{e}^{-q\tau_\xi} f(X_{\tau_\xi-}, X_{\tau_\xi}) \colon \underline{X}_{\tau_\xi-} \ge c, \bar{X}_{\tau_\xi} \le b) \\
&\quad = \mathbb{E}_x(\mathrm{e}^{-q\tau_\xi} f(X_{\tau_\xi-}, X_{\tau_\xi}) \colon X_{\tau_\xi} \ne X_{\tau_\xi-}, \underline{X}_{\tau_\xi-} \ge c, \bar{X}_{\tau_\xi} \le b) \\
&\quad = \mathbb{E}_x\Bigg(\sum_{\{t\colon X_{t-}\ne X_t\}} \mathrm{e}^{-qt} f(X_{t-}, X_{t-} + \Delta X_t)\mathbf{1}(\underline{X}_{t-} \ge c)\mathbf{1}(\bar{X}_t \le b) \\
&\qquad\quad \times \mathbf{1}(t \le \tau_\xi)\mathbf{1}(X_{t-} + \Delta X_t < \xi(\bar{X}_t))\Bigg) \\
&\quad = \int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}t \int_x^b \int_c^z \mathbb{P}_x(X_{t-} \in \mathrm{d}y, \bar{X}_{t-} \in \mathrm{d}z, t < \tau_\xi, \underline{X}_{t-} \ge c) \\
&\qquad \times \int_{-\infty}^{\xi(z)-y} f(y, y+u)\Pi(\mathrm{d}u),
\end{aligned}$$

where the compensation formula is applied for the last equation. The desired result then follows from the quasi-left-continuity for process X. □

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