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Nagoya Math. J.
Vol. 50 (1973), 7–20

BOUNDARY ISOMORPHISM BETWEEN DIRICHLET FINITE SOLUTIONS OF $\Delta u = Pu$ AND HARMONIC FUNCTIONS

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Introduction

Consider an open Riemann surface R and a density $P(z)dxdy$ $(z = x + iy)$, well defined on R. As was shown by Myrberg in [3], if $P \not\equiv 0$ is a nonnegative α-Hölder continuous density on R $(0 < \alpha \leq 1)$ then there exists the Green's functions of the differential equation

$$(1) \qquad \Delta u(z) = P(z)u(z)$$

on R, where Δ means the Laplace operator. As a consequence, there always exists a nontrivial solution on R. In this paper we will be interested in such pairs (R, P) when the existence of a nontrivial Dirichlet finite solution is secured, i.e. when there will be such $u \not\equiv 0$, $\Delta u = Pu$ with $\int_R |\text{grad}\, u|^2 dxdy < \infty$. In such case the standard notation will be used. The real vector space of all Dirichlet finite solutions is called PD(R) and the space of all Dirichlet finite harmonic functions on R is denoted by HD(R). Studying related problems if $\int_R Pdxdy < \infty$, Royden proposed in [7] to use certain compactification of R which among other things reduces the study of class PD(R) into study of HD(R). In particular, he showed that then there exists an isomorphism between the subclasses PBD(R) and HBD(R), with those being subspaces of all bounded elements of PD(R) and HD(R). The isomorphism was meant in a sense that there is a one-one correspondence on the ideal boundary of R. Further investigation of the class PD(R) was done mostly by Nakai and Glasner-Katz, not mentioning Ozawa's originated paper (cf. references). This author worked out some conditions for the existence of such isomorphism in a general case of density P although he was not successful in establishing a necessary and sufficient condition for the existence of such

Received September 12, 1972.

mapping (cf. [9]). In this paper we proceed on a similar path as was suggested in [9]. We will try to answer the question what are the characteristics of Dirichlet finite harmonic functions in order for them to have a corresponding solution of (1) with the same behavior on the ideal boundary of R. Moreover in the final part we present an example of a density $\tilde{\mathrm{P}}$ defined on the unit disc, with $\tilde{\mathrm{P}}\mathrm{BD} \simeq \mathrm{HBD}$ but $\tilde{\mathrm{P}}\mathrm{D} \neq \mathrm{HD}$, with $\simeq$ meaning an isomorphism as it will be precisely described in the next paragraphs. This example leads us to suspect that even for finitely integrable densities the implication $(\mathrm{PBD} \simeq \mathrm{HBD}) \Rightarrow (\mathrm{PD} \simeq \mathrm{HD})$ may not be true in general, contrary to the conclusion of Royden in [7].

To treat the problem of isomorphism we will use the standard method of exhaustion of R by regular subregions since it proves to be very helpful method if one uses the Royden's compactification of R. Also we will exploit the so called P-unit as it was already introduced in [9]. Our results are of integral character on R although there are reasons to believe that they can be rewritten as near-boundary conditions.

Preliminaries

Let R be an open hyperbolic Riemann surface and P a nonzero density on R as was specified in the Introduction. We recall briefly some known facts and corresponding terminology.

A relatively compact open subregion Ω of R is called a regular subregion if its relative boundary $\partial\Omega$ consists of finitely many analytic Jordan curves in R. By an exhaustion of R we will always mean an exhaustion by an increasing sequence $\{\Omega_n\}$, $\bar{\Omega}_n \subset \Omega_{n+1}$ of regular subregions in R. We understand that the Dirichlet problem for the equation (1) with continuous boundary values on the relative boundary is always solvable for such domains. The maximum principle holds, i.e. a nonnegative solution of (1) on Ω does not attain its maximum in Ω, unless it is a constant solution. If $\{u_n\}$ is a monotone sequence of nonnegative solutions bounded at some point then there is a solution u to which $\{u_n\}$ converges uniformly on every compact set of R (Harnack's principle). We further assume the knowledge of the Dirichlet and energy principles (cf. [4]).

The convenient way to work with Dirichlet finite functions on R is to use Royden's compactification R^* of R (cf. [8]). The advantage of it is in the very simple formalism plus in the following distinctive characteristics:

a. Every Dirichlet finite Tonelli function on R is continuously ex-

https://doi.org/10.1017/S0027763000015555 Published online by Cambridge University Press

tendable to the whole R^*.

b. Let $\{f_n\}$ be a sequence of uniformly bounded Dirichlet finite Tonelli functions with compact supports in R. If there is Tonelli function f on R^* such that $f_n \to f$ on R pointwise and $\int_R |\mathrm{grad}\,(f - f_n)|^2 \to 0$, then $f = 0$ on the Royden's harmonic boundary $\Delta(R)$ of R.

c. Every Dirichlet finite Tonelli function f on R can be uniquely decomposed as $f = h + \Phi$, where h is also a Dirichlet finite and harmonic function and $f = h$ on $\Delta(R)$.

d. If $f \in \mathrm{HD}(R)$ (resp. $f \in \mathrm{PD}(R)$) then the maximum modulus principle holds. We have $\sup_R |f| = \sup_{\Delta(R)} |f|$ (confirm the above properties with [1], [8]).

From now on we will deal only with such pairs (R, P) which allow nontrivial space $\mathrm{PD}(R)$. Then $\mathrm{PD}(R)$ will become a real Hilbert space with scalar product given by Dirichlet integral; i.e. if $u, v \in \mathrm{PD}$ then $(u, v) = \int_R du \Lambda * dv$. Furthermore, space $\mathrm{PD}(R)$ is a vector lattice since there exists a Riesz' decomposition $u = u_+ - u_-$ for an arbitrary $u \in \mathrm{PD}(R)$ with both u_+, u_- being nonnegative and $\in \mathrm{PD}(R)$ (cf. Nakai [4]). We will understand that even if a Dirichlet finite Tonelli function is defined on R then according to (a) we can speak unambiguously about its values on the Royden boundary $R^* - R$ (cf. [8]).

Isomorphism of PD(R) into HD(R)

If $u \in \mathrm{PD}(R)$ is a Dirichlet finite solution of (1) on R, the orthogonal decomposition theorem says there exists a unique $h \in \mathrm{HD}(R)$, called the *harmonic projection* of u and a potential ϕ such that

$$u = h - \phi \tag{2}$$

on R and $u = h$ on the harmonic boundary $\Delta(R)$. We know (cf. [5]) that explicitly

$$u(z) = h(z) - \frac{1}{2\pi} \int_R G_R(z, \zeta) u(\zeta) P(\zeta) d\xi d\eta\,, \tag{3}$$

with $\zeta = \xi + i\eta$ and $G_R(z, \zeta)$ being the harmonic Green's function on R with the pole at z. Moreover, if $D_R[f]$ stays for the Dirichlet integral of f on R we can write

$$D_R[u] = D_R[h] + \frac{1}{2\pi}\iint_{R\times R} G_R(z,\zeta)u(z)u(\zeta)P(z)P(\zeta)dxdyd\xi d\eta \,. \tag{4}$$

Define now *the mapping* $T\colon \mathrm{PD}(R) \to \mathrm{HD}(R)$ as the correspondence given by (2). According to (2) and maximum modulus principle it is a well-defined linear mapping, obviously bijective and the range of T will be a linear subspace of $\mathrm{HD}(R)$. We denote this subspace by $\mathscr{H}_P(R)$ and certainly then $T\colon \mathrm{PD}(R) \to \mathscr{H}_P(R)$ is an algebraic isomorphism onto. We will *always assume that* $1 \in \mathscr{H}_P(R)$. Hence there exists a solution $e_R \in \mathrm{PD}(R)$ such that $e_R = 1$ on the harmonic boundary $\varDelta(R)$. The function e_R will be called *P-unit* on R. If $\{\Omega_n\}$ is an exhaustion of R and e_{Ω_n} are solutions on Ω_n with boundary values 1 on $\partial\Omega_n$ then the maximum principle and Harnack's principle yield

$$e_R = \lim_{\Omega_n \to R} e_{\Omega_n} \tag{5}$$

on every compact subset of R. On the other hand we must be careful not to take (5) as for the definition of e_R, since there is no reason for $\lim e_{\Omega_n}$ to be equal 1 on $\varDelta(R)$.

DEFINITION. A solution e_R of (1) on R is called a *P-unit* if it can be continuously extended to R^* with the values 1 on the harmonic boundary $\varDelta(R)$.

Using this definition of a P-unit we can reformulate a theorem, proved in [5].

THEOREM 1. *The necessary and sufficient condition for* $\mathrm{HBD}(R) \subset \mathscr{H}_P(R)$ *is the existence of a P-unit* e_R *on* R *with the finite Dirichlet integral* $D_R[e_R]$.

Proof. Necessity is obvious. Let $D_R[e_R] < \infty$ and $U^* = \{z \in R^*;\ e_R(z) > \frac{1}{2}\}$. Consequently, the relative boundary ∂U of $U = U^* \cap R$ is smooth and since $Te_R = 1$ the relation (4) gives

$$\iint_{U\times U} G_U(z,\zeta)P(z)P(\zeta)dxdyd\xi d\eta < \infty \,, \tag{6}$$

where $G_U(z,\zeta)$ is the harmonic Green's function on U. But (6) is a sufficient condition for $\mathrm{PBD} \simeq \mathrm{HBD}$ as was shown in [5]. This completes the proof.

At this point let's recall a lemma proved in [9] which was actually

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stated for e_R defined as by (5). Clearly the lemma will also hold with the present definition of P-unit.

LEMMA 1. *The necessary condition for an* $h \in \mathrm{HD}(R)$ *to be in* $\mathscr{H}_P(R)$ *is that*

$$D_R[he_R] < \infty . \tag{7}$$

Reduction operator *T*

As we have seen in the previous paragraph the mapping $T: \mathrm{PD}(R) \to \mathrm{HD}(R)$ is only a certain type of boundary correspondence between $\mathrm{PD}(R)$ and $\mathscr{H}_P(R)$ since the Dirichlet integral on $\mathscr{H}_P(R)$ is merely a seminorm. We would like to impose on $\mathscr{H}_P(R)$ a Hilbert-space structure which would distinguish $\mathscr{H}_P(R)$ as the only subspace of $\mathrm{HD}(R)$ to be isomorphic with $\mathrm{PD}(R)$ in a way that the inverse T^{-1} is a continuous well-defined operator. The Lemma 1 gives the way how to do it. Before we proceed to the main theorem let's formulate some simple lemmas for organizing purposes.

LEMMA 2. *If* $\{\alpha_n\}$ *is an infinite sequence of real numbers such that for all* $n, \alpha_n^2 \leq A\alpha_n + B$ *with some constants* A, B *then* $\limsup_{n\to\infty} \alpha_n < \infty$.

LEMMA 3. *Let* e_R *be the Dirichlet finite P-unit on* R. *If we put* $Q = |\mathrm{grad}\, e_R|^2$ *then there exists a Q-unit* q *on* R *such that* $q \geq e_R^2$ *on* R.

Proof. Take an exhaustion $\{\Omega_n\}$ of R and let for each n, q_n be the solution of $\Delta v = Qv$ on Ω_n with boundary values 1 on $\partial\Omega_n$. Then by (3), (4) and Fubini's theorem

$$D_{\Omega_n}[q_n] = \frac{1}{2\pi}\int_{\Omega_n} q_n(1 - q_n)Q . \tag{8}$$

For $0 < q_n \leq 1$ and $\int_R Q < \infty$, all integrals in (8) are uniformly bounded, hence by the Harnack's principle and Kawamura's lemma (cf. [8]) there exists $q = \lim q_n$ such that $q = 1$ on the harmonic boundary $\Delta(R)$ and $\Delta q = Qq$, where Δ means the Laplace operator; moreover $D_R[q] < \infty$. Because of $\Delta(q - e_R^2) = Qq - 2Pe_R^2 - 2Q \leq 0$ on R, the function $q - e_R^2$ is superharmonic and by maximum principle $q \geq e_R^2$ on the whole R (cf. [8]).

LEMMA 4. *Let* Q, q *be as in the Lemma* 3. *If* v *is a solution of* $\Delta v = Qv$ *with the finite energy then*

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$$\int_R h^2 q^2 Q < \infty \,, \tag{9}$$

with h being harmonic projection of v.

Proof. Let h_n be a harmonic function on Ω_n with boundary values v on $\partial\Omega_n$. Then by the Dirichlet principle $D_{\Omega_n}[h_n] \leq D_R[v]$ for all n and $h_n \to h$ uniformly on every compact subset of R. Using Green's formula for the energy $E[h_n q_n]$ we obtain

$$\begin{aligned} D_{\Omega_n}[h_n q_n] + \int_{\Omega_n} h_n^2 q_n^2 Q &= \int_{\partial\Omega_n} h_n q_n {*d}(h_n q_n) - 2\int_{\Omega_n} h_n q_n \operatorname{grad} h_n \operatorname{grad} q_n \\ &= \int_{\partial\Omega_n} v {*d}(h_n q_n) - 2\int_{\Omega_n} q_n \operatorname{grad} h_n \operatorname{grad}(h_n q_n) \\ &\quad + 2\int_{\Omega_n} q_n^2 |\operatorname{grad} h_n|^2 \end{aligned} \tag{10}$$

Furthermore

$$\begin{aligned} \int_{\partial\Omega_n} v{*d}(h_n q_n) &= \int_{\Omega_n} dv \Lambda {*d}(h_n q_n) + \int_{\Omega_n} v\Delta(h_n q_n) \\ &= \int_{\Omega_n} dv\Lambda{*d}(h_n q_n) + \int_{\Omega_n} v h_n q_n Q + 2\int_{\Omega_n} v \operatorname{grad} h_n \operatorname{grad} q_n \\ &< \sqrt{D_{\Omega_n}[v] D_{\Omega_n}[h_n q_n]} + \sqrt{\int_{\Omega_n} v^2 Q \cdot \int_{\Omega_n} h_n^2 q_n^2 Q} \\ &\quad + 2\int_{\Omega_n} \operatorname{grad} h_n \operatorname{grad}(v q_n) - 2\int_{\Omega_n} q_n \operatorname{grad} h_n \operatorname{grad} v \,. \end{aligned} \tag{11}$$

After substituting (11) into (10) and using Schwarz's inequality again with the estimate $D_{\Omega_n}[vq_n] \leq 25 D_{\Omega_n}[v]$ as was shown in [9] we obtain a positive constants A, B such that

$$E_{\Omega_n}[h_n q_n] \leq A\sqrt{E_{\Omega_n}[h_n q_n]} + B \tag{12}$$

for all n, with $E_{\Omega_n}[\cdot]$ meaning the energy integral

$$E_{\Omega_n}[\cdot] = D_{\Omega_n}[\cdot] + \int_{\Omega_n} (\cdot)^2 Q \,.$$

Thus finally, the Lemma 2 together with Fatou's lemma give

$$\int_R h^2 q^2 Q < \infty \,.$$

We are ready now to state and to prove the main theorem. After that we will be able to define a scalar product on $\mathscr{H}_P(R)$ which will turn this space into a Hilbert space.

MAIN THEOREM. *Let e_R be a Dirichlet finite P-unit on R. Then the necessary and sufficient condition for an $h \in \mathrm{HD}(R)$ to be in $\mathscr{H}_P(R)$ is $D_R[he_R] < \infty$.*

Proof. According to the Lemma 1 we have the necessity. Let's therefore assume that $D_R[he_R] < \infty$ and $h \not\equiv 0$, since the trivial case $h \equiv 0$ is evident. Because

$$D_R[he_R] = \int_R h^2 |\mathrm{grad}\, e_R|^2 + 2\int_R he_R dh \Lambda * de_R + \int_R e_R^2 |\mathrm{grad}\, h|^2$$

we can see after applying Schwarz's inequality and the Lemma 2 that $D_R[he_R] < \infty$ is equivalent to

$$\int_R h^2 |\mathrm{grad}\, e_R|^2 + D_R[h] < \infty \ . \tag{13}$$

Let's therefore assume that

$$\int_R h^2 |\mathrm{grad}\, e_R|^2 + D_R[h] \leq 1 \ . \tag{14}$$

If we put $g_+ = \max(h, 0)$ on R, g_+ will be continuous and subharmonic on R. Also $D_R[g_+] \leq D_R[h]$. Take an exhaustion $\{\Omega_n\}$ of R and consider the density $Q = |\mathrm{grad}\, e_R|^2$ on R. If $\{v_n\}$ are solutions of $\Delta v = Qv$ on $\{\Omega_n\}$ with continuous boundary values g_+ on $\{\partial\Omega_n\}$ then by the energy principle and (14) $E_{\Omega_n}[v_n] \leq 1$ for all n. Because $\int_R Q < \infty$ there exists a reproducing kernel in the Hilbert space of all energy finite solutions of $\Delta u = Qu$ (cf. [6]). Thus we conclude that there is a $v \in \mathrm{QD}(R)$, and a subsequence $\{v_{n_i}\} \subset \{v_n\}$ such that $v_{n_i} \to v$ on every compact subset of R and hence by Fatou's lemma $E_R[v] \leq 1$, i.e. $D_R[v] + \int_R v^2 Q \leq 1$. Furthermore by Kawamura's lemma $v = g_+$ on the harmonic boundary $\Delta(R)$. If orthogonally decomposing v on R we get a harmonic function $h_+ \in \mathrm{HD}(R)$ such that $h_+ = \max(h, 0)$ on $\Delta(R)$ and consequently by the Lemma 4

$$\int_R h_+^2 q^2 Q + D_R[h_+] < \infty \tag{15}$$

with q being the Q-unit on R, whose existence was given by the Lemma 3. Hence

$$\int_R h_+^2 q^2 |\mathrm{grad}\, e_R|^2 + D_R[h_+] < \infty \ . \tag{16}$$

Using Lemma 3 and (16) it follows

$$\int_U h_+^2 \,|\mathrm{grad}\, e_R|^2 + D_U[h_+] < \infty \,, \tag{17}$$

with $U = U^* \cap R$ and $U^* = \{z \in R^*, e_R(z) > \frac{1}{2}\}$. Set $4w = e_R - e_R^2$. Then $w = 0$ on $\varDelta(R)$ and $w = 1$ on the relative boundary ∂U of U which is sufficiently smooth. Thus (17) yields

$$\int_U h_+^2 \,|\mathrm{grad}\, w|^2 + D_U[h_+] < \infty \,. \tag{18}$$

Putting $f_n = \min(h_+, n)$ on U for all natural n we have by (18) a uniform bound, say K, such that

$$\int_U f_n^2 \,|\mathrm{grad}\, w|^2 + D_U[f_n] \leq K \tag{19}$$

for all n. Hence there is a finite constant $L > 0$ such that $D_U[f_n w] \leq L$, $n = 1, 2, \cdots$. If $f_n w = X_n + \varPhi_n$ is an orthogonal decomposition on U such that $X_n \in \mathrm{HD}(U)$ and $X_n = f_n \cdot w$ on $\varDelta(R)$ as well as on ∂U then obviously X_n is a nondecreasing sequence of nonnegative harmonics bounded by h_+, with the uniformly bounded Dirichlet integrals. Therefore there exists a Dirichlet finite harmonic function $X, X = \lim X_n$ such that $X = h_+$ on ∂U and $X = 0$ on $\varDelta(R)$. For $X \leq h_+$ on U and X is nonnegative we maintain that if $\lambda = h_+ - X$ then $\lambda = h_+$ on $\varDelta(R)$, $\lambda = 0$ on ∂U and by (17) we conclude

$$\int_U \lambda^2 \,|\mathrm{grad}\, e_R|^2 + D_U[\lambda] < \infty \,. \tag{20}$$

The function λ is a nonnegative harmonic function and the latter inequality implies $D_U[\lambda e_R] < \infty$. By using essentially the same methods as described in the proof of the Lemma 4 or in [9] we deduce

$$\int_U \lambda^2(\tilde{h} - e_R)e_R P + D_U[\lambda] < \infty \,, \tag{21}$$

where $\tilde{h} - e_R = 0$ on $\varDelta(R)$ & ∂U and $\tilde{h} \in \mathrm{HBD}(U)$ by the orthogonal decomposition of e_R on U. Denoting by e_U the P-unit on U (which exists), we observe that

$$\tilde{h} - e_R \geq \tfrac{1}{2}(1 - e_U) > 0 \tag{22}$$

on U since $e_R \geq \frac{1}{2} e_U$ on U and $(\tilde{h} - e_R) - \frac{1}{2}(1 - e_U)$ is superharmonic

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with zeros values on $\Delta(R)$ & ∂U. Then by (22) and (21)

$$\int_U \lambda^2(1-e_U)e_U P + D_U[\lambda] < \infty\,. \tag{23}$$

Because of $e_U \geq \frac{1}{2}$ on U (maximum principle) and the relation (3) applied to the function e_U we get from (23)

$$\iint_{U\times U} \lambda^2(z)G_U(z,\zeta)P(z)P(\zeta)dxdyd\xi d\eta < \infty\,. \tag{24}$$

If $\tilde{u}_+ \geq 0$ is a solution of (1) on U with $\tilde{u}_+ = \lambda$ on $\Delta(R)$ & ∂U then obviously $\tilde{u}_+ \leq \lambda$ on U and using the Schwarz's inequality with respect to the measure

$$d\mu = G_U(z,\zeta)P(z)P(\zeta)dxdyd\xi d\eta$$

on $U \times U$ we get from (24)

$$\iint_{U\times U} G_U(z,\zeta)\tilde{u}_+(z)\tilde{u}_+(\zeta)P(z)P(\zeta) < \infty\,. \tag{25}$$

Finally by (4) and $D_U[\lambda] < \infty$ we have $D_U[\tilde{u}_+] < \infty$. Using the canonical extension of $\tilde{u}_+$ into the whole R (cf. [5]) we thus obtain a solution u_+ of (1) on R such that $u_+ = \lambda = h_+ = \max(h, 0)$ on $\Delta(R)$ and $u_+ \in \mathrm{PD}(R)$. By the same steps as above we can show the existence of such $u_- \in \mathrm{PD}(R)$ that $u_- = -\inf(h, 0)$ on $\Delta(R)$. Putting $u = u_+ - u_-$ we have proved the theorem.

The Main Theorem provides us with the needed Hilbert-space structure in the space $\mathscr{H}_P(R)$. We state

THEOREM 2. *The vector space $\mathscr{H}_P(R)$ is a Hilbert space of all harmonic functions on R with finite*

$$\int_R h^2|\mathrm{grad}\, e_R|^2 + D_R[h]$$

and with scalar product defined as

$$\langle h, g\rangle = \int hg\,|\mathrm{grad}\, e_R|^2 + D_R[h,g] \tag{26}$$

on $\mathscr{H}_P(R)$.

Proof. We know that unless $P \equiv 0$, $\int_R |\mathrm{grad}\, e_R|^2 > 0$. Hence $\langle\cdot,\cdot\rangle$

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is a well-defined scalar product with $\sqrt{\langle h, h\rangle}$ being the norm of h. According to the Main Theorem we only need to show the completeness of $\mathscr{H}_P(R)$ with respect to the norm $\sqrt{\langle\cdot,\cdot\rangle}$. Assume that $\{h_n\}_1^\infty$ is a Cauchy sequence in $\mathscr{H}_P(R)$. Fix $z_0, z_0 \in R$. Then $\{h_n(z) - h_n(z_0)\}$ will be a Cauchy sequence in the HD(R) with respect to the Dirichlet norm. By Virtanen [10], there exists an $h_0 \in \mathrm{HD}(R)$, $h_0(z_0) = 0$ such that $D_R[h_0 - (h_n - h_n(z))]$ $\to 0$ and $(h_n - h_n(z_0)) \to h_0$ uniformly on each compact subset of R. On the other hand, from the integral part of the norm in $\mathscr{H}_P(R)$ we conclude that there is a function $s(z) \in L^2(R, |\mathrm{grad}\, e_R|^2)$ such that $h_n \to s$ in L^2-norm. Hence $\limsup_{n\to\infty} |h_n(z_0)| < \infty$ and consequently there exists a subsequence $\{h_{n_i}\} \subset \{h_n\}_1^\infty$ such that $h_{n_i} \to h_0 + \beta$ in $\mathscr{H}_P(R)$-norm where $\beta = \lim h_{n_i}(z_0)$. Obviously $h_0 + \beta \in \mathscr{H}_P(R)$, which was to be proved.

COROLLARY 1. *The Hilbert space $\mathscr{H}_P(R)$ poses a Riesz' decomposition.*

Proof. It can be observed from the proof of the Main Theorem.

COROLLARY 2. *Linear mapping T: PD$(R) \to \mathscr{H}_P(R)$ is a continuous operator onto $\mathscr{H}_P(R)$ with a continuous inverse T^{-1}.*

Proof. If $u_n \to 0$ in PD(R) then $D_R[h_n] \to 0$ where $h_n \in \mathrm{HD}(R)$, $h_n = u_n$ on $\Delta(R)$. From [9] it follows that also $D_R[h_n e_R] \to 0$, hence $\langle Tu_n, Tu_n\rangle$ $\to 0$. The existence of a continuous inverse T^{-1} is given by the open mapping theorem (cf. [11]).

We call the linear operator T the *reduction operator*. Its range in HD(R) is completely determined if we know the behavior of P-unit. Unfortunately we do not know too much about the range if taking (5) as a definition of P-unit except for the fact that in such cases $\mathscr{H}_P(R)$ is embedded into the real vector subspace of HD(R) of elements with finite norm $\sqrt{\langle\cdot,\cdot\rangle}$.

Application

In this last paragraph we will show that generally $\mathscr{H}_P(R) \subset \mathrm{HD}(R)$. We exhibit an example when the above inclusion is proper while maintaining the same assumptions as in the previous sections. We need to formulate two auxiliary lemmas, concerning the open unit disc $W = \{z\,;|z| < 1\}$.

LEMMA 5. *Let W be the open unit disc and $h \in \mathrm{HD}(W)$. Then* $\int_W h^2 dxdy < \infty$.

Proof. As shown in [8] there exists a constant C, $0 < C < \infty$ such that $\left|\int_{\partial\Omega_n} h^2(z)*dG_{\Omega_n}(z,0)\right| < C$, where $G_{\Omega_n}(z,0)$ is the harmonic Green's function on a domain Ω_n with the pole at $0 \in \Omega_n$. Choose $\{\Omega_n\}_1^\infty$ to be a sequence of concentric open discs with centers at 0 and with radii $\rho_n = 1 - 1/n$. Then $G_{\Omega_n}(z,0) = -ln(r/\rho_n)$ with $r^2 = z\cdot\bar{z}$. Hence

$$\rho_n^{-1}\int_{\partial\Omega_n} h^2 < C \quad \text{or} \quad \int_{\partial\Omega_n} h^2 < C \qquad \text{for all } n = 1, 2, \cdots,$$

when integrating in the positive direction. Take the function $e = r^2$ on W and apply the Green's formula to the Dirichlet integral $D_{\Omega_n}[he]$. We have then

$$\begin{aligned} D_{\Omega_n}[he] &= \int_{\partial\Omega_n} hr^2*d(hr^2) - \int_{\Omega_n} hr^2\Delta(hr^2) \\ &= \rho_n^4\int_{\partial\Omega_n} h*dh + \int_{\partial\Omega_n} 2r^3h^2*dr - 4\int_{\Omega_n} h^2r^2 \\ &\quad - 2\int_{\Omega_n} hr^2 \operatorname{grad} h \operatorname{grad} r^2 \leq 3D_W[h] \\ &\quad + 2C + 2\sqrt{D_W[h]D_{\Omega_n}[he]}\,, \end{aligned} \tag{27}$$

since

$$\int_{\Omega_n} hr^2 \operatorname{grad} h \operatorname{grad} r^2 = \int_{\Omega_n} r^2 \operatorname{grad} h \operatorname{grad}(hr^2) - \int_{\Omega_n} r^4 |\operatorname{grad} h|^2$$

and by using Schwarz's formula. With the Lemma 2 we conclude $D_W[he] < \infty$ and thus the first two equalities in (27) give $\int_W h^2dxdy < \infty$. This proves the lemma.

COROLLARY. *If P is such density on the open unit disc W that* $|\operatorname{grad} e_W|^2$ *is a bounded function on W then $\mathscr{H}_P(W) = \mathrm{HD}(W)$.*

LEMMA 6. *Let W be the open unit disc and $h \in \mathrm{HD}(W)$ be a Dirichlet finite harmonic function continuously extendable to the closure of W and finite there except possibly at the point 1 on the boundary of W. Then for every bounded and continuous Dirichlet finite harmonic function g on $\overline{W}$ such that g is zero on an open arc Γ on the boundary of W, $1 \in \Gamma$,* it follows $D_W[hg] < \infty$.

Proof. Considering that

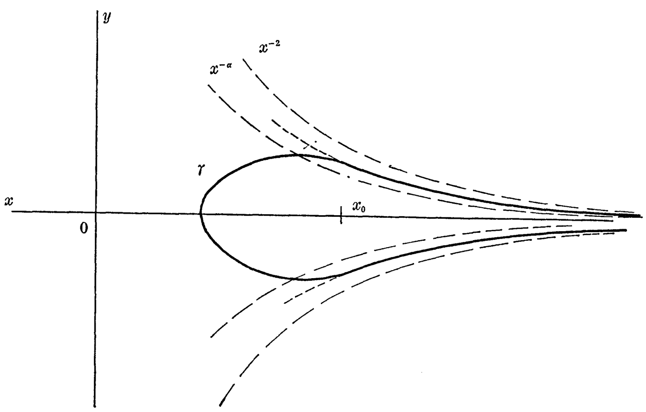
$$D_W[hg] = \int_W h^2|\operatorname{grad} g|^2 + 2\int_W hg \operatorname{grad} h \operatorname{grad} g + \int_W g^2|\operatorname{grad} h|^2\,,$$

we have to prove that $\int_W h^2|\mathrm{grad}\, g|^2 < \infty$. Because of the properties of g we can extend g "behind" the arc Γ in order to obtain a harmonic function $\tilde{g}$ such that $\tilde{g} = g$ on $\overline{W}$ and $\tilde{g}$ is harmonic also in some neighborhood of the point 1 in the complex plane (cf. [2]). Hence we can regard $|\mathrm{grad}\, g|^2$ as a bounded function in a certain neighborhood of 1 in $\overline{W}$. From the continuity of h on $\overline{W}$ and from the Lemma 5 we conclude easily that $\int_W h^2|\mathrm{grad}\, g|^2 < \infty$, what had to be proved.

Now we proceed to construct the example. We will not do all the computations as they are easy but quite cumbersome and require a large amount of space. Consider the complex plane $\boldsymbol{C}$ and the harmonic function $H = r^{-2}\cos 2\phi$ on $\boldsymbol{C} - \{0\}$, expressed in polar coordinates. Define the following function $\psi(z)$ on $\boldsymbol{C}$:

$$\psi(z) \begin{cases} x^4y^3 & (y \geq 0) \\ -x^4y^3 & (y \leq 0)\,. \end{cases} \tag{28}$$

Then the Laplacian $\Delta\psi \geq 0$ everywhere, hence $\psi(z)$ is subharmonic. Let's define in the right half-plane the curve γ given by $\gamma \equiv \{z \in \boldsymbol{C}; \mathrm{Re}\, z > 0$ and $H(z) = \psi(z)\}$. Explicitly, in polar coordinates $r^9 = \cos^{-4}\phi\cdot\sin^{-3}\phi\cdot\cos 2\phi$ for all $\phi, 0 < |\phi| < \pi/4$. The curve γ lies symmetrically around the x-axis and for an $\alpha > 2$ there exists a x_0 such that for $x > x_0$ the upper part of γ lies between the curves $y = x^{-2}$ and $y = x^{-\alpha}$. Choose an α such that $2 < \alpha < 11/5$. Then for an appropriate x_0, for all $x > x_0$, $x^{-\alpha} < \gamma_+(x) < x^{-2}$ if γ_+ means a part of γ in the first quadrant. Finally let's adjust the curve γ for $x \leq x_0$ in a way that it will look as on the picture 1.



Then call all the points "inside" the curve γ a Riemann surface R which in fact is a disc according to the Riemann mapping theorem and simple connectedness of that region. Take now the harmonic function $h = x$ on R. By direct computation we can show that

$$\int_R |\operatorname{grad} h|^2 < \infty\,, \tag{29}$$

$$\int_R h^2 |\operatorname{grad} \psi|^2 = \infty \tag{30}$$

and

$$\int_R h^2 |\operatorname{grad} H|^2 < \infty\,. \tag{31}$$

Put $g = H - \psi$ on γ. Then according to the construction of γ we have $g(z) = 0$ whenever $\operatorname{Re} z > x_0$. Moreover g is continuous on γ and bounded there. Let g be identified with the harmonic function on R with boundary values $H - \psi$ on γ. Then g is Dirichlet finite on R and if we map conformally R onto the unit disc W such that $\infty \to 1$, the corresponding image of g will satisfy the assumptions of the Lemma 6 and hence by (29)

$$\int_R h^2 |\operatorname{grad} g|^2 < \infty\,, \tag{32}$$

since the latter integral is invariant under conformal mappings of R. Put $K = \sup_R H(z)$. Then certainly

$$\tilde{e}_R(z) = \frac{K - H(z) + (g(z) + \psi(z))}{K}$$

is subharmonic on R with values 1 on γ. Moreover $\tilde{e}_R(z) > 0$ on R and we can define the C^1-density

$$\tilde{P}(z) = \frac{\Delta \tilde{e}_R}{\tilde{e}_R}\,. \tag{33}$$

Then according to (30), (31) and the way the function g was constructed, we maintain that $\tilde{e}_R$ is a $\tilde{P}$-unit on R, which is Dirichlet finite and $\int_R h^2 |\operatorname{grad} \tilde{e}_R|^2 = \infty$. But the latter follows from (30), (31) and (32). Now we conclude that although $\widetilde{\mathrm{P}}\mathrm{BD} \simeq \mathrm{HBD}$ on R, because of $D_R[\tilde{e}_R] < \infty$, we have found the function $h = x$, $h \in \mathrm{HD}(R)$ such that the necessary condition in the Main Theorem fails.

Still open question remains if the inclusion $\mathscr{H}_P(R) \subset \mathrm{HD}(R)$ can be proper for a finite integrable density P. In our case $\int_R \tilde{P} = \infty$ as can be checked easily by direct computation.

Remark. To get the estimates (29), (30), (31) and $\int_R \tilde{P} > \int_R \Delta\tilde{e}_R = \infty$ we have used the fact that the harmonic boundary of R for $x > x_0$ lies between curves $y = x^{-2}$ & $y = x^{-\alpha}$ and $y = -x^{-2}$ & $y = -x^{-\alpha}$.

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