*Proceedings of the Royal Society of Edinburgh*, **149**, 1603–1625, *2019*
DOI:10.1017/prm.2018.49

# Radial symmetry of non-maximal entire solutions of a bi-harmonic equation with exponential nonlinearity

**Hongxia Guo**
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, P.R. China (guohongxia@zzu.edu.cn)

**Zongming Guo**
Department of Mathematics, Henan Normal University, Xinxiang 453007, P.R. China (gzm@htu.cn)

**Fangshu Wan**
School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, P.R. China (fangshuwan@163.com)

(MS received 12 July 2016; accepted 7 April 2017)

We study radial symmetry of entire solutions of the equation

$$\Delta^2 u = 8(N-2)(N-4)e^u \quad \text{in } R^N \quad (N \geqslant 5). \tag{0.1}$$

It is known that (0.1) admits infinitely many radially symmetric entire solutions. These solutions may have either a (negative) logarithmic behaviour or a (negative) quadratic behaviour at infinity. Up to translations, we know that there is only one radial entire solution with the former behaviour, which is called 'maximal radial entire solution', and infinitely many radial entire solutions with the latter behaviour, which are called 'non-maximal radial entire solutions'. The necessary and sufficient conditions for an entire solution $u$ of (0.1) to be the maximal radial entire solution are presented in [7] recently. In this paper, we will give the necessary and sufficient conditions for an entire solution $u$ of (0.1) to be a non-maximal radial entire solution.

*Keywords:* Bi-harmonic equation; radial symmetry; entire solutions; exponential nonlinearity

2010 *Mathematics subject classification:* Primary 35B45; Secondary 35J40

## 1. Introduction

We consider radial symmetry of entire solutions of the equation

$$\Delta^2 u = \lambda_0 e^u \quad \text{in } \mathbb{R}^N, \;\; \lambda_0 = 8(N-2)(N-4), \;\; N \geqslant 5. \tag{1.1}$$

The necessary and sufficient conditions for an entire solution of (1.1) to be a radially symmetric entire solution are established.

We first consider the structure of radial entire solutions of (1.1). It is known from [1, 2] that for any $\alpha, \beta \in \mathbb{R}$, the initial value problem

$$\begin{cases} \Delta^2 u = \lambda_0 e^u, & r \in (0, R), \\ u(0) = \alpha, \ u'(0) = 0, \ \Delta u(0) = \beta, \ u'''(0) = 0 \end{cases} \tag{1.2}$$

admits a unique solution $u_{\alpha,\beta}(r)$, where $R$ is the maximal interval of existence of solutions and $r = |x|$. Moreover, there exists $\beta_0 = \beta_0(\alpha) < 0$ such that the radial solution is entire if and only if $\beta \leqslant \beta_0$. Moreover, (i) if $\beta < \beta_0$, then

$$\overline{\lim}_{r\to\infty} r^{-2} u_{\alpha,\beta}(r) < -c < 0 \tag{1.3}$$

for some $c > 0$ depending on $\beta$ and $\beta_0$; (ii) if $\beta = \beta_0$, then

$$\lim_{r\to\infty} [u_{\alpha,\beta}(r) + 4 \ln r] = 0. \tag{1.4}$$

Therefore, the radial entire solution of (1.1) exists for $\beta \leqslant \beta_0$, but their asymptotic behaviours at $\infty$ may be different. For each fixed $\alpha \in \mathbb{R}$, the comparison principle (lemma 3.2 in [9]) ensures that $u_{\alpha,\beta_0} > u_{\alpha,\beta}$ in $(0, \infty)$ when $-\infty < \beta < \beta_0$. Therefore, $u_{\alpha,\beta_0}$ is the maximal radial entire solution with respect to $u(0) = \alpha$.

The necessary and sufficient conditions for an entire solution $u$ of (1.1) to be the maximal radial entire solution of (1.1) are presented in [7] recently. In this paper, we will give the necessary and sufficient conditions for an entire solution $u$ of (1.1) to be a radial entire solution, but not the maximal radial entire solution. The main result is the following theorem.

Theorem 1.1. *Let $u \in C^4(\mathbb{R}^N)$ be an entire solution of (1.1) with $N \geqslant 5$. Then $u$ is a radially symmetric solution about some $x^0 \in \mathbb{R}^N$ (i.e. $u(x) = u(r)$ with $r = |x - x^0|$), but is not the maximal radial entire solution about $x^0$ of (1.1), if and only if there exists $D > 0$ such that*

$$|x|^{-2} u(x) \to -D \quad \text{as } |x| \to \infty. \tag{1.5}$$

*The constant $D$ then determines a particular non-maximal radial entire solution.*

In a recent paper [5], the authors constructed a non-radial entire solution of (1.1) with the asymptotic behaviour (see theorem 1 in [5]):

$$u(x) = -p(x) + O(|x|^{4-N}) \quad \text{as } |x| \to \infty,$$

where

$$p(x) = \Sigma_{i=1}^N \alpha_i (x_i - x_i^*)^2, \quad \alpha_1, \alpha_2, \ldots, \alpha_N > 1 + N/2.$$

It is easily seen that if the coefficients $\alpha_i$ are not all equal, then $u(x)$ is not radial about any point. Our theorem 1.1 implies that if $\alpha_i$ are all equal, then $u(x)$ is radial about $x = x^*$, where $x^* = (x_1^*, x_2^*, \ldots, x_N^*)$.

When $N = 4$, radial symmetry of an entire solution $u$ of the equation

$$(Q) \qquad \Delta^2 u = 24 e^u \quad \text{in } \mathbb{R}^4$$

satisfying $|u(x)| = o(|x|^2)$ at $|x| = \infty$ has been obtained by Lin [8]. Another interesting question is: can we show that an entire solution $u(x)$ of (1.1) is radially

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

symmetric in $\mathbb{R}^N$ for $N \geqslant 5$ provided that there exists $D > 0$ such that $u(x) = -D|x|^2 + o(|x|^2)$ at $|x| = \infty$? Our theorem 1.1 gives a positive answer to this question.

The structure and properties of entire solutions to (1.1) in the conformal dimension $N = 4$ and the 'supercritical dimension' $N \geqslant 5$ have been studied by many authors, see [**1**–**3**, **5**, **6**, **11**, **12**] and the references therein. Entire solutions of (1.1) with $N \geqslant 5$ have been classified via Morse index theory. In [**11**], the author determined the stability properties of the solutions of (1.1). In [**2**, **5**], the authors found that there exist both unstable and finite Morse index solutions of (1.1) in 'lower dimensions' $5 \leqslant N \leqslant 12$ and any radially symmetric solution to (1.1) is fully stable in 'high dimensions' $N \geqslant 13$. The second author of the present paper [**6**] classified all the maximal radial entire solutions of (1.1) in 'lower dimensions' and 'high dimensions'. He also obtained the asymptotic expansions of the maximal radial entire solutions at $\infty$.

We will use the moving-plane argument of system of equations to prove theorem 1.1. Note that the equation can be written to a system of equations:

$$\begin{cases} -\Delta u = w, & \text{in } \mathbb{R}^N, \\ -\Delta w = \lambda_0 e^u, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

which is cooperative. Unfortunately, the asymptotic behaviour (1.5) at $|x| = \infty$ of an entire solution of (1.1) is not enough to make the moving-plane procedure work. In order to use the moving-plane argument, we need to know more information on the asymptotic behaviour at $|x| = \infty$ of an entire solution $u \in C^4(\mathbb{R}^N)$ of (1.1) satisfying (1.5).

In § 2, we obtain the asymptotic behaviours of $u_{\alpha,\beta}(r)$ (given in (1.2)) at $\infty$ with $-\infty < \beta < \beta_0$. In § 3, we will give the exact asymptotic behaviour of an entire solution of (1.1) satisfying (1.5) and in the final section, we present the proof of theorem 1.1. In this paper, we use $C$ to denote a positive constant which may change from one line to another line.

## 2. Asymptotic behaviours of $u_{\alpha,\beta}(r)$ at $\infty$ with $-\infty < \beta < \beta_0$

In this section, we obtain more information for $u_{\alpha,\beta}(r)$ with $-\infty < \beta < \beta_0$, where $u_{\alpha,\beta}$ is given in (1.2).

PROPOSITION 2.1. *There exists $d > 0$ depending on $\alpha$ and $\beta$ such that*

$$\Delta u_{\alpha,\beta}(r) \to -d, \;\; r^{-2} u_{\alpha,\beta}(r) \to -\frac{d}{2N} \quad \text{as } r \to \infty. \tag{2.1}$$

*Proof.* It is easily seen from the equation in (1.2) that $\Delta u_{\alpha,\beta}(r)$ is increasing in $(0, \infty)$ with $\Delta u_{\alpha,\beta}(0) = \beta < 0$. Therefore, there are three cases for $\Delta u_{\alpha,\beta}(r)$:

(i) $\Delta u_{\alpha,\beta}(r) \to e > 0$ ($e$ may be $+\infty$) as $r \to \infty$,

(ii) $\Delta u_{\alpha,\beta}(r) \to 0$ as $r \to \infty$,

(iii) $\Delta u_{\alpha,\beta}(r) \to -d < 0$ as $r \to \infty$.

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

We show that the cases (i) and (ii) do not happen.

If (i) occurs, we see that for any small $\epsilon > 0$, there is an $R = R(\epsilon) > 1$ such that

$$\Delta u_{\alpha,\beta}(r) > e - \epsilon \ \text{ for } r > R. \tag{2.2}$$

(We may assume $0 < e < \infty$. If $e = \infty$, we can choose any $0 < e_1 < \infty$ such that (2.2) holds.) This implies

$$r^{N-1}u'_{\alpha,\beta}(r) - R^{N-1}u'_{\alpha,\beta}(R) \geqslant \frac{(e-\epsilon)}{N}(r^N - R^N)$$

and

$$u'_{\alpha,\beta}(r) \geqslant \frac{R^{N-1}}{r^{N-1}}u'_{\alpha,\beta}(R) + \frac{(e-\epsilon)}{N}(r - R^N r^{1-N}).$$

Therefore,

$$\begin{aligned} u_{\alpha,\beta}(r) \geqslant & u_{\alpha,\beta}(R) + \frac{R^{N-1}u'_{\alpha,\beta}(R)}{2-N}(r^{2-N} - R^{2-N}) \\ & + \frac{(e-\epsilon)}{2N}(r^2 - R^2) + \frac{(e-\epsilon)R^N}{N(N-2)}(r^{2-N} - R^{2-N}). \end{aligned}$$

This implies

$$\underline{\lim}_{r\to\infty} r^{-2}u_{\alpha,\beta}(r) \geqslant \frac{e}{2N} > 0 \tag{2.3}$$

by sending $\epsilon$ to 0. This contradicts to (1.3).

If (ii) occurs, arguments similar to those in the proof of case (i) imply that

$$\underline{\lim}_{r\to\infty} r^{-2}u_{\alpha,\beta}(r) \geqslant 0. \tag{2.4}$$

This also contradicts to (1.3).

Therefore, case (iii) occurs. Arguments similar to those in the proof of case (i) imply that

$$\overline{\lim}_{r\to\infty} r^{-2}u_{\alpha,\beta}(r) \leqslant -\frac{d}{2N} \tag{2.5}$$

and

$$\underline{\lim}_{r\to\infty} r^{-2}u_{\alpha,\beta}(r) \geqslant -\frac{d}{2N}. \tag{2.6}$$

Both (2.5) and (2.6) imply

$$\lim_{r\to\infty} r^{-2}u_{\alpha,\beta}(r) = -\frac{d}{2N}. \tag{2.7}$$

This completes the proof of this proposition. □

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

REMARK 2.2. We can easily see that for any $\alpha \in \mathbb{R}$, $d := d(\alpha, \beta) > 0$ for $\beta \in (-\infty, \beta_0)$ and $d$ is decreasing with respect to $\beta$,

$$\lim_{\beta \to \beta_0^-} d(\alpha, \beta) = 0$$

and

$$\lim_{\beta \to -\infty} d(\alpha, \beta) = \infty.$$

## 3. Exact asymptotic behaviour of an entire solution of (1.1) satisfying (1.5)

To prove the sufficiency of theorem 1.1, we need to know more information on the asymptotic behaviour of an entire solution $u \in C^4(\mathbb{R}^N)$ of (1.1) satisfying (1.5).

Let $u \in C^4(\mathbb{R}^N)$ be an entire solution of (1.1). We introduce the Kelvin-type transformation:

$$v(y) = |x|^{-2}u(x) + D, \quad y = \frac{x}{r^2}, \ \ r = |x| > 0, \ \ D > 0. \tag{3.1}$$

Then $v(y) = v(s, \theta)$ with $s = |y| = r^{-1}$ satisfies $v(s, \theta) \to 0$ uniformly for $\theta \in S^{N-1}$ as $s \to 0$ and the equation:

$$\begin{aligned} &v_s^{(4)} - 2(N-3)s^{-1}v_{sss} + (N-1)(N-3)s^{-2}v_{ss} - (N-1)(N-3)s^{-3}v_s \\ &\quad + 2Ns^{-4}\Delta_\theta v - 2(N-1)s^{-3}\Delta_\theta v_s + 2s^{-2}\Delta_\theta v_{ss} + s^{-4}\Delta_\theta^2 v \\ &\quad - \lambda_0 s^{-6} e^{s^{-2}(v-D)} = 0. \end{aligned} \tag{3.2}$$

LEMMA 3.1. *For any integer $\tau \geqslant 0$, there exist constants $M = M(u) > 0$, $0 < s_* = s_*(u) < 1$, such that*

$$\lim_{s \to 0} v(y) = 0, \ \ |\nabla^\tau v(y)| \leqslant \frac{M}{s^\tau} \ \ \text{for } s = |y| \leqslant s_*. \tag{3.3}$$

*Proof.* The estimates in (3.3) can be obtained from (1.5) and the standard elliptic theory. □

Define

$$w(s, \theta) = v(s, \theta) - \overline{v}(s),$$

where

$$\overline{v}(s) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} v(s, \theta) \mathrm{d}\theta.$$

Then $\overline{v}$ and $w$, respectively, satisfy

$$\begin{aligned} &\overline{v}_s^{(4)} - 2(N-3)s^{-1}\overline{v}_{sss} + (N-1)(N-3)s^{-2}\overline{v}_{ss} - (N-1)(N-3)s^{-3}\overline{v}_s \\ &\quad - \lambda_0 s^{-6}\overline{e^{s^{-2}(v-D)}} = 0 \end{aligned} \tag{3.4}$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

and

$$w_s^{(4)} - 2(N-3)s^{-1}w_{sss} + (N-1)(N-3)s^{-2}w_{ss} - (N-1)(N-3)s^{-3}w_s$$
$$+ 2Ns^{-4}\Delta_\theta w - 2(N-1)s^{-3}\Delta_\theta w_s + 2s^{-2}\Delta_\theta w_{ss} + s^{-4}\Delta_\theta^2 w$$
$$- s^{-4}g(s,\xi,w) = 0, \tag{3.5}$$

where by the Mean Value Theorem,

$$g(s,\xi,w) = \lambda_0 s^{-4}[e^{s^{-2}(\xi(s,\theta)-D)}w(s,\theta) - \overline{e^{s^{-2}(\xi(s,\theta)-D)}w(s,\theta)}]$$

and $\min\{v(s,\theta),\overline{v}(s)\} \leqslant \xi(s,\theta) \leqslant \max\{v(s,\theta),\overline{v}(s)\}$. Note that $\xi(s,\theta) \to 0$ uniformly for $\theta \in S^{N-1}$ as $s \to 0$, since $v(s,\theta) \to 0$ uniformly for $\theta \in S^{N-1}$ as $s \to 0$ and $\overline{v}(s) \to 0$ as $s \to 0$. If we define

$$\zeta(s) = \lambda_0 s^{-4} \max_{\theta \in S^{N-1}} e^{s^{-2}(\xi(s,\theta)-D)},$$

we see that

$$\zeta(s) = o(e^{-((D)/(2))s^{-2}}) \quad \text{for } s \text{ near } 0. \tag{3.6}$$

Since $\overline{w}(s) = 0$, we have the expansion:

$$w(s,\theta) = \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} w_j^i(s)Q_j^i(\theta),$$

where $\{Q_1^0(\theta), Q_1^1(\theta), \ldots, Q_{m_1}^1(\theta), Q_1^2(\theta), Q_2^2(\theta), \ldots, Q_{m_2}^2(\theta), Q_1^3(\theta), \ldots\}$ is the standard eigenfunction basis of $-\Delta_\theta$ or $\Delta_\theta^2$ in $H^2(S^{N-1}) \cap L^2(S^{N-1})$, i.e., $\int_{S^{N-1}} Q_l^i(\theta)Q_m^j(\theta)\mathrm{d}\theta = 0$ if $i \neq j$ or $l \neq m$, $\int_{S^{N-1}} (Q_i^j)^2(\theta)\mathrm{d}\theta = 1$ and

$$m_k = \frac{(N-3+k)!(N-2+2k)}{k!(N-2)!}.$$

It is known from [4] that the eigenvalues of the problem

$$-\Delta_\theta Q = \sigma Q, \quad \theta \in S^{N-1}$$

are

$$\sigma_k = k(N+k-2), \quad k \geqslant 0$$

with multiplicity $m_k$. Therefore, $\{Q_1^k, Q_2^k, \ldots Q_{m_k}^k\}$ is the standard eigenfunction basis of the eigenspace of $\sigma_k$. In particular, we have that

$$\sigma_0 = 0, \quad m_0 = 1, \quad Q_1^0 = \frac{1}{\sqrt{|S^{N-1}|}},$$

$$\sigma_1 = N-1, \quad m_1 = N, \quad Q_i^1(\theta) = \frac{x_i|_{S^{N-1}}}{\sqrt{\int_{S^{N-1}}(x_i|_{S^{N-1}})^2\mathrm{d}\theta}}, \quad 1 \leqslant i \leqslant N,$$

$$\sigma_2 = 2N.$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

Moreover, the eigenspace of $\sigma_k$ consists of homogeneous harmonic polynomials on $\mathbb{R}^N$, of degree $k$, restricted to $S^{N-1}$. The boot-strap argument implies that for $1 \leqslant j \leqslant m_k$,

$$\max_{\theta \in S^{N-1}} |Q_j^k(\theta)| \leqslant D_k, \quad \max_{\theta \in S^{N-1}} |(Q_j^k)_\theta(\theta)| \leqslant E_k, \tag{3.7}$$

where

$$D_k := C(1 + \sigma_k + \ldots + \sigma_k^\tau), \quad E_k := C(1 + \sigma_k + \ldots + \sigma_k^{\tau_1}) \tag{3.8}$$

with $C > 0$ being independent of $k$ and $\tau, \tau_1 \geqslant 1$ being positive integers such that $2\tau > N - 1$, $2\tau_1 > N$. Note that $(Q_j^k)_\theta(\theta)$ is in the eigenspace of $\sigma_k$. We also see that for each $i$ and $1 \leqslant j \leqslant m_i$, $w_j^i(s)$ satisfies the equation

$$\begin{aligned}
&(w_j^i)^{(4)} - 2(N-3)s^{-1}(w_j^i)_{sss} + [(N-1)(N-3) - 2\sigma_i]s^{-2}(w_j^i)_{ss} \\
&\quad - (N-1)[N-3-2\sigma_i]s^{-3}(w_j^i)_s - (2N\sigma_i - \sigma_i^2)s^{-4}w_j^i \\
&\quad = s^{-4}\tilde{g}_j^i(s),
\end{aligned} \tag{3.9}$$

where

$$\tilde{g}_j^i(s) = \int_{S^{N-1}} g(s, \xi, w)Q_j^i(\theta)\mathrm{d}\theta = \lambda_0 s^{-4} \int_{S^{N-1}} \mathrm{e}^{s^{-2}(\xi(s,\theta)-D)} w(s,\theta)Q_j^i(\theta)\mathrm{d}\theta. \tag{3.10}$$

Note that the Hölder inequality implies that

$$|\tilde{g}_j^i(s)| \leqslant \zeta(s)W(s), \tag{3.11}$$

where

$$W(s) = \left(\int_{S^{N-1}} w^2(s,\theta)\mathrm{d}\theta\right)^{1/2}.$$

This implies that for $s \in (0, s_*)$,

$$|\tilde{g}_j^i(s)| = o(e^{-((D)/(2s^2))}). \tag{3.12}$$

Note also that for any fixed $(i, j)$ and $0 < \kappa < 1$, if a sequence $\{s_k\}$ satisfies $s_* > s_1 > s_2 > \ldots > s_k > \ldots$ with $s_k \to 0$ as $k \to \infty$ and $w_j^i(s_k) = \int_{S^{N-1}} w(s_k,\theta)Q_j^i(\theta)\mathrm{d}\theta \neq 0$ for all $k$, we have that there exists $10 < C < \infty$ such that

$$\overline{\lim}_{k\to\infty} \left| \frac{\int_{S^{N-1}} (1 - \lambda_0 s_k^{-(\kappa+4)} \mathrm{e}^{s_k^{-2}(\xi(s_k,\theta)-D)}) w(s_k,\theta)Q_j^i(\theta)\mathrm{d}\theta}{\int_{S^{N-1}} w(s_k,\theta)Q_j^i(\theta)\mathrm{d}\theta} \right| \leqslant C, \tag{3.13}$$

since $s_k^{-(\kappa+4)} \max_{\theta \in S^{N-1}} e^{s_k^{-2}(\xi(s_k,\theta)-D)} \to 0$ as $k \to \infty$. Note that $\max_{\theta \in S^{N-1}} |\xi(s_k,\theta)| \to 0$ as $k \to \infty$ and, for $k$ sufficiently large,

$$\left|\int_{S^{N-1}} \lambda_0 s_k^{-(\kappa+4)} \mathrm{e}^{s_k^{-2}(\xi(s_k,\theta)-D)} w(s_k,\theta)Q_j^i(\theta)\mathrm{d}\theta\right| = o\left(\left|\int_{S^{N-1}} w(s_k,\theta)Q_j^i(\theta)\mathrm{d}\theta\right|\right).$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

This implies that there is $K > 0$ such that for all $k \geqslant K$,

$$|\tilde{g}_j^i(s_k)| \leqslant (2C+1)s_k^{\kappa}|w_j^i(s_k)|. \tag{3.14}$$

This also implies that there is $0 < s_{**} \leqslant s_*$ such that for $s \in (0, s_{**})$ and $w_j^i(s) \neq 0$,

$$|\tilde{g}_j^i(s)| = O(s^{\kappa})|w_j^i(s)|. \tag{3.15}$$

We have the following proposition.

Proposition 3.2. *For $N \geqslant 5$, there exist a sufficiently small $0 < s_0 < 1/10$ and $C > 0$ independent of $s$ such that for $s \in (0, s_0)$,*

$$W(s) \leqslant Cs. \tag{3.16}$$

*Proof.* Let $t = -\ln s$, $z_j^i(t) = w_j^i(s)$ and $Z(t) = W(s)$ (note that $Z(t) \to 0$ as $t \to \infty$). Then $z_j^i(t)$ satisfies the equation

$$\begin{aligned}(z_j^i)^{(4)} &+ 2N(z_j^i)_{ttt} + (N^2+2N-4-2\sigma_i)(z_j^i)_{tt} + 2N(N-2-\sigma_i)(z_j^i)_t \\ &- \sigma_i(2N-\sigma_i)z_j^i = f_j^i(t),\end{aligned} \tag{3.17}$$

where $f_j^i(t) = \tilde{g}_j^i(e^{-t})$. We also know from (3.11) and (3.6) that $|f_j^i(t)| \leqslant o(e^{-D/2e^{2t}})Z(t)$ as $t \to +\infty$. The corresponding polynomial of (3.17) is

$$\nu^4 + 2N\nu^3 + (N^2+2N-4-2\sigma_i)\nu^2 + 2N(N-2-\sigma_i)\nu - \sigma_i(2N-\sigma_i) = 0. \tag{3.18}$$

Using Matlab, we obtain four roots of (3.18):

$$\begin{aligned}\nu_1^{(i)} &= \frac{1}{2}\left(2-N+\sqrt{(N-2)^2+4\sigma_i}\right), \\ \nu_2^{(i)} &= \frac{1}{2}\left(2-N-\sqrt{(N-2)^2+4\sigma_i}\right), \\ \nu_3^{(i)} &= \frac{1}{2}\left(-2-N+\sqrt{(N-2)^2+4\sigma_i}\right), \\ \nu_4^{(i)} &= \frac{1}{2}\left(-2-N-\sqrt{(N-2)^2+4\sigma_i}\right).\end{aligned} \tag{3.19}$$

Therefore, we have

$$\nu_1^{(i)} = i, \quad \nu_2^{(i)} = 2-N-i, \quad \nu_3^{(i)} = i-2, \quad \nu_4^{(i)} = -N-i. \tag{3.20}$$

We easily see that

$$\nu_4^{(i)} < \nu_2^{(i)} < \nu_3^{(i)} < \nu_1^{(i)}.$$

Therefore, for $i = 1$,

$$\nu_1^{(1)} = 1, \quad \nu_2^{(1)} = 1-N, \quad \nu_3^{(1)} = -1, \quad \nu_4^{(1)} = -N-1$$

and

$$\nu_4^{(1)} < \nu_2^{(1)} < \nu_3^{(1)} = -1 < 0 < \nu_1^{(1)}.$$

For $i = 2$,

$$\nu_1^{(2)} = 2,\ \ \nu_2^{(2)} = -N,\ \ \nu_3^{(2)} = 0,\ \ \nu_4^{(2)} = -N-2$$

and

$$\nu_4^{(2)} < \nu_2^{(2)} < -1 < \nu_3^{(2)} = 0 < \nu_1^{(2)}.$$

For $i \geqslant 3$, we see that

$$\nu_4^{(i)} < \nu_2^{(i)} < -1 < 0 < \nu_3^{(i)} < \nu_1^{(i)}.$$

For $i \geqslant 2$ and $1 \leqslant j \leqslant m_i$, we see from (3.17) and ODE theory that for any $T \gg T_* := -\ln s_{**}$ ($s_{**}$ is given in (3.15)), there are constants $A^i_{j,k}$, $B^i_k$ $(k = 1, 2, 3, 4)$ such that, for $t > T$,

$$z^i_j(t) = \Sigma^4_{k=1}[A^i_{j,k} e^{\nu_k^{(i)} t} + B^i_k \int_T^t \mathrm{e}^{\nu_k^{(i)}(t-\tau)} f^i_j(\tau)\mathrm{d}\tau],$$

where each $A^i_{j,k}$ depends on $T$ and $\nu_k^{(i)}$, but each $B^i_k$ depends only on $\nu_k^{(i)}$. The detailed calculations show that

$$A^i_{j,1} = \frac{F^i_{j,1}(T)}{(\nu_1^{(i)} - \nu_2^{(i)})(\nu_1^{(i)} - \nu_3^{(i)})(\nu_1^{(i)} - \nu_4^{(i)})} e^{-\nu_1^{(i)} T},$$

$$A^i_{j,2} = \left[\frac{F^i_{j,1}(T)}{(\nu_2^{(i)} - \nu_1^{(i)})(\nu_2^{(i)} - \nu_3^{(i)})(\nu_2^{(i)} - \nu_4^{(i)})} + \frac{F^i_{j,2}(T)}{(\nu_2^{(i)} - \nu_3^{(i)})(\nu_2^{(i)} - \nu_4^{(i)})}\right] e^{-\nu_2^{(i)} T},$$

$$\begin{aligned} A^i_{j,3} = \Bigg[&\frac{F^i_{j,1}(T)}{(\nu_3^{(i)} - \nu_1^{(i)})(\nu_3^{(i)} - \nu_2^{(i)})(\nu_3^{(i)} - \nu_4^{(i)})} \\ &+ \frac{F^i_{j,2}(T)}{(\nu_3^{(i)} - \nu_2^{(i)})(\nu_3^{(i)} - \nu_4^{(i)})} + \frac{F^i_{j,3}(T)}{(\nu_3^{(i)} - \nu_4^{(i)})}\Bigg] e^{-\nu_3^{(i)} T}, \end{aligned}$$

$$\begin{aligned} A^i_{j,4} = \Bigg[&\frac{F^i_{j,1}(T)}{(\nu_4^{(i)} - \nu_1^{(i)})(\nu_4^{(i)} - \nu_2^{(i)})(\nu_4^{(i)} - \nu_3^{(i)})} \\ &+ \frac{F^i_{j,2}(T)}{(\nu_4^{(i)} - \nu_2^{(i)})(\nu_4^{(i)} - \nu_3^{(i)})} + \frac{F^i_{j,3}(T)}{(\nu_4^{(i)} - \nu_3^{(i)})} + z^i_j(T)\Bigg] e^{-\nu_4^{(i)} T}, \end{aligned}$$

where

$$F^i_{j,1}(T) = (\partial_t - \nu_2^{(i)})(\partial_t - \nu_3^{(i)})(\partial_t - \nu_4^{(i)}) z^i_j(T),$$

$$F^i_{j,2}(T) = (\partial_t - \nu_3^{(i)})(\partial_t - \nu_4^{(i)}) z^i_j(T),$$

$$F^i_{j,3}(T) = (\partial_t - \nu_4^{(i)}) z^i_j(T)$$

and

$$B_1^i = \frac{1}{(\nu_1^{(i)} - \nu_2^{(i)})(\nu_1^{(i)} - \nu_3^{(i)})(\nu_1^{(i)} - \nu_4^{(i)})},$$

$$B_2^i = \frac{1}{(\nu_2^{(i)} - \nu_1^{(i)})(\nu_2^{(i)} - \nu_3^{(i)})(\nu_2^{(i)} - \nu_4^{(i)})},$$

$$B_3^i = \frac{1}{(\nu_3^{(i)} - \nu_1^{(i)})(\nu_3^{(i)} - \nu_2^{(i)})(\nu_3^{(i)} - \nu_4^{(i)})},$$

$$B_4^i = \frac{1}{(\nu_4^{(i)} - \nu_1^{(i)})(\nu_4^{(i)} - \nu_2^{(i)})(\nu_4^{(i)} - \nu_3^{(i)})}.$$

Therefore,

$$\begin{aligned} z_j^i(t) = {} & M_{j,1}^i e^{\nu_1^{(i)} t} + M_{j,3}^i e^{\nu_3^{(i)} t} + A_{j,2}^i e^{\nu_2^{(i)} t} + A_{j,4}^i e^{\nu_4^{(i)} t} \\ & - B_1^i \int_t^\infty e^{\nu_1^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau - B_3^i \int_t^\infty \mathrm{e}^{\nu_3^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \\ & + B_2^i \int_T^t e^{\nu_2^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau + B_4^i \int_T^t e^{\nu_4^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \end{aligned} \tag{3.21}$$

by using that $\int_T^t = \int_T^\infty - \int_t^\infty$, where

$$M_{j,1}^i = A_{j,1}^i + B_1^i \int_T^\infty \mathrm{e}^{\nu_1^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau, \quad M_{j,3}^i = A_{j,3}^i + B_3^i \int_T^\infty \mathrm{e}^{\nu_3^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau.$$

Note that the limits

$$\int_t^\infty \mathrm{e}^{\nu_1^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \to 0, \quad \int_t^\infty \mathrm{e}^{\nu_3^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \to 0 \quad \text{as } t \to \infty \tag{3.22}$$

hold for $i \geqslant 3$ since $\nu_1^{(i)} > \nu_3^{(i)} > 0$ and $|f_j^i(t)| \leqslant o(e^{-((D)/(2))e^{2t}})Z(t)$. For $i = 2$, we see that $\nu_1^{(2)} > \nu_3^{(2)} = 0$. Then we also know that

$$\int_t^\infty \mathrm{e}^{\nu_3^{(2)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau = \int_t^\infty f_j^i(\tau)\mathrm{d}\tau \to 0 \quad \text{as } t \to \infty.$$

Since $z_j^i(t) \to 0$ as $t \to \infty$, we see from (3.21) that $M_{j,1}^i = M_{j,3}^i = 0$. Therefore,

$$\begin{aligned} z_j^i(t) = {} & A_{j,2}^i e^{\nu_2^{(i)} T} e^{\nu_2^{(i)}(t-T)} + A_{j,4}^i e^{\nu_4^{(i)} T} e^{\nu_4^{(i)}(t-T)} \\ & - B_1^i \int_t^\infty \mathrm{e}^{\nu_1^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau - B_3^i \int_t^\infty \mathrm{e}^{\nu_3^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \\ & + B_2^i \int_T^t \mathrm{e}^{\nu_2^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau + B_4^i \int_T^t \mathrm{e}^{\nu_4^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau. \end{aligned} \tag{3.23}$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

Without loss of generality, we assume that $z_j^i(t) \not\equiv 0$ for $t \in [T, \infty)$ in the following. We claim that

$$|z_j^i(t)| = O(ie^{\nu_2^{(i)}(t-T)}) \tag{3.24}$$

for $t > T$ and $i \geqslant 2$, $1 \leqslant j \leqslant m_i$. For any fixed $(i, j)$, we show that $z_j^i(t)$ can only admit isolated zeroes in $(T, \infty)$ provided that $z_j^i(t) \not\equiv 0$ for $t > T$. Suppose that there exist $t_1 < t_2 < t_3$ such that $z_j^i(t) \equiv 0$ for $t \in [t_1, t_2] \subset (T, \infty)$ and $z_j^i(t) \neq 0$ for $t \in (t_2, t_3)$. We see that $z_j^i(t_2) = (z_j^i)'(t_2) = (z_j^i)''(t_2) = (z_j^i)'''(t_2) = 0$. Note that it is known from (3.15) that

$$|f_j^i(t)| = o_t(1)|z_j^i(t)| \quad \text{for } t \in (t_2, t_3).$$

It follows from the initial value problem of the equation in (3.17) for $t \in (t_2, t_3)$ and the standard ODE theory that

$$z_j^i(t) \equiv 0 \quad \text{for } t \in (t_2, t_3).$$

This is a contradiction. This contradiction implies that $z_j^i(t)$ can only admit isolated zeroes in $(T, \infty)$ provided that $z_j^i(t) \not\equiv 0$ for $t > T$. Without loss of generality, we assume that $T < t_{i,j}^1 < t_{i,j}^2 < \ldots < t_{i,j}^k < \ldots$ are the zeroes of $z_j^i$ in $(T, \infty)$ and $t_{i,j}^k \to \infty$ as $k \to \infty$. (The case that $z_j^i$ does not admit any zero or only admits finite zeroes in $(T, \infty)$ can be studied similarly.) Then, we have from (3.15) that

$$|f_j^i(t)| = O(e^{-\kappa t})|z_j^i(t)| \quad \text{for } t \in (T, \infty)\backslash\{t_{i,j}^k\}_{k=1}^{\infty}. \tag{3.25}$$

It follows from lemma 3.1 and (3.23) that, for $t \in (T, \infty)\backslash\{t_{i,j}^k\}_{k=1}^{\infty}$ and $0 < \kappa < 1$ given in (3.15),

$$\begin{aligned} |z_j^i(t)| \leqslant & O(ie^{\nu_2^{(i)}(t-T)}) + C\int_T^t \mathrm{e}^{\nu_2^{(i)}(t-\tau)} O(\mathrm{e}^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau \\ & + C\int_t^{\infty} \mathrm{e}^{\nu_3^{(i)}(t-\tau)} O(\mathrm{e}^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau. \end{aligned} \tag{3.26}$$

Note that

$$e^{\nu_1^{(i)}(t-\tau)} \leqslant e^{\nu_3^{(i)}(t-\tau)} \quad \text{for } \tau \geqslant t, \quad e^{\nu_4^{(i)}(t-\tau)} \leqslant e^{\nu_2^{(i)}(t-\tau)} \quad \text{for } \tau \leqslant t.$$

Note also that for $\ell = 1, 3$ and any fixed $t > T$ with $t \notin \{t_{i,j}^k\}_{k=1}^{\infty}$ and $t < t_{i,j}^m < t_{i,j}^{m+1} < \ldots$,

$$\begin{aligned} & \left|\int_t^{\infty} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau\right| \\ & = \left|\int_t^{t_{i,j}^m} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau + \Sigma_{k=m}^{\infty} \int_{t_{i,j}^k}^{t_{i,j}^{k+1}} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau\right| \\ & \leqslant \int_t^{t_{i,j}^m} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau + \Sigma_{k=m}^{\infty} \int_{t_{i,j}^k}^{t_{i,j}^{k+1}} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau \\ & = \int_t^{\infty} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau. \end{aligned}$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

Similar arguments imply that for $\ell = 2, 4$,

$$\begin{aligned}
&\left|\int_T^t \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau\right| \\
&= \left|\int_T^{t_{i,j}^1} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau + \Sigma_{k=1}^{m-2} \int_{t_{i,j}^k}^{t_{i,j}^{k+1}} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau \right.\\
&\left. + \int_{t_{i,j}^{m-1}}^t \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} f_j^i(\tau)\mathrm{d}\tau\right| \\
&\leqslant \int_T^{t_{i,j}^1} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau + \Sigma_{k=1}^{m-2} \int_{t_{i,j}^k}^{t_{i,j}^{k+1}} \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau \\
&+ \int_{t_{i,j}^{m-1}}^t \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau \\
&= \int_T^t \mathrm{e}^{\nu_\ell^{(i)}(t-\tau)} O(e^{-\kappa\tau})|z_j^i(\tau)|\mathrm{d}\tau.
\end{aligned}$$

It follows from (3.26) and arguments similar to those in [7] that, for $t \in (T, \infty)\backslash\{t_{i,j}^k\}_{k=1}^\infty$,

$$|z_j^i(t)| = O(ie^{\nu_2^{(i)}(t-T)}). \tag{3.27}$$

This implies that our claim (3.24) holds for $z_j^i(t) \neq 0$. If $z_j^i(t) = 0$, we can easily see that (3.24) holds. Therefore, our claim (3.24) holds.

For $i = 1$ and $1 \leqslant j \leqslant m_1$, we know that

$$\nu_4^{(1)} < \nu_2^{(1)} < \nu_3^{(1)} = -1 < 0 < \nu_1^{(1)}.$$

Therefore,

$$\begin{aligned}
z_j^1(t) =& A_{j,2}^1 e^{\nu_2^{(1)}t} + A_{j,3}^1 e^{\nu_3^{(1)}t} + A_{j,4}^1 e^{\nu_4^{(1)}t} \\
&- B_1^1 \int_t^\infty \mathrm{e}^{\nu_1^{(1)}(t-\tau)} f_j^1(\tau)\mathrm{d}\tau + B_3^1 \int_T^t \mathrm{e}^{\nu_3^{(1)}(t-\tau)} f_j^1(\tau)\mathrm{d}\tau \\
&+ B_2^1 \int_T^t \mathrm{e}^{\nu_2^{(i)}(t-\tau)} f_j^1(\tau)\mathrm{d}\tau + B_4^1 \int_T^t \mathrm{e}^{\nu_4^{(i)}(t-\tau)} f_j^1(\tau)\mathrm{d}\tau.
\end{aligned}$$

Arguments similar to those in the proof of (3.24) imply that, for $1 \leqslant j \leqslant m_1$ and $t > T$,

$$|z_j^1(t)| = O(e^{-(t-T)}). \tag{3.28}$$

Since $Z(t) = [\Sigma_{i=1}^\infty \Sigma_{j=1}^{m_i} (z_j^i(t))^2]^{1/2}$, we have

$$\begin{aligned}
Z(t) \leqslant& \Sigma_{i=1}^\infty \Sigma_{j=1}^{m_i} |z_j^i(t)| \\
\leqslant& O(\Sigma_{i=2}^\infty i m_i e^{\nu_2^{(i)}(t-T)}) + O(e^{-(t-T)}).
\end{aligned}$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

Let $T^* = 10T$. We obtain that, for $t > T^*$,

$$\Sigma_{i=2}^{\infty} i m_i e^{\nu_2^{(i)}(t-T)} = O(e^{\nu_2^{(2)}(t-T)}). \tag{3.29}$$

To see (3.29), we notice that, for any $t > T^*$ (we may enlarge $T^*$),

$$\lim_{i\to\infty} \frac{(i+1)m_{i+1}e^{\nu_2^{(i+1)}(t-T)}}{im_i e^{\nu_2^{(i)}(t-T)}} = e^{-(t-T)} \lim_{i\to\infty} \frac{(i+1)m_{i+1}}{im_i} = e^{-(t-T)} < \frac{1}{2}.$$

Since $\nu_2^{(2)} < -1$, we easily have that, for $t > T_*$,

$$Z(t) = O(e^{-t}). \tag{3.30}$$

Let $s_0 = e^{-T^*}$. We see from (3.30) that there exists $C > 0$ such that for $0 < s < s_0$,

$$W(s) \leqslant Cs. \tag{3.31}$$

This completes the proof of this proposition. □

LEMMA 3.3. *Let $v$ be a solution of (3.2). Then there exist constants $0 < s_0 < 1/10$ and $M = M(v) > 0$ such that for $N \geqslant 5$ and $s \in (0, s_0)$,*

$$|\overline{v}(s)| \leqslant Ms^2, \quad |\overline{v}'(s)| \leqslant Ms, \quad |\overline{v}''(s)| \leqslant M \tag{3.32}$$

*and*

$$\int_{S^{N-1}} v^2(s,\theta)\mathrm{d}\theta \leqslant Ms^2. \tag{3.33}$$

*Proof.* Let $\overline{z}(t) = \overline{v}(s)$, $t = -\ln s$. Then $\overline{z}(t)$ satisfies the equation

$$\overline{z}^{(4)} + 2N(\overline{z})_{ttt} + (N^2 + 2N - 4)(\overline{z})_{tt} + 2N(N-2)(\overline{z})_t = \overline{f}(t), \tag{3.34}$$

where $|\overline{f}(t)| = o(e^{-D/2\, e^{2t}})$. The corresponding polynomial of (3.34) is

$$\nu^4 + 2N\nu^3 + (N^2 + 2N - 4)\nu^2 + 2N(N-2)\nu = 0. \tag{3.35}$$

The four roots of (3.35) are:

$$\nu_1^{(0)} = 0, \ \ \nu_2^{(0)} = 2 - N, \ \ \nu_3^{(0)} = -2, \ \ \nu_4^{(0)} = -N. \tag{3.36}$$

The ODE theory implies

$$\begin{aligned} \overline{z}(t) &= M_1 + A_2 e^{-2t} + A_3 e^{-(N-2)t} + A_4 e^{-Nt} \\ &\quad - B_1 \int_t^{\infty} \overline{f}(\tau)\mathrm{d}\tau + B_2 \int_T^t \mathrm{e}^{-2(t-\tau)}\overline{f}(\tau)\mathrm{d}\tau \\ &\quad + B_3 \int_T^t \mathrm{e}^{-(N-2)(t-\tau)}\overline{f}(\tau)\mathrm{d}\tau + B_4 \int_T^t \mathrm{e}^{-N(t-\tau)}\overline{f}(\tau)\mathrm{d}\tau. \end{aligned} \tag{3.37}$$

The fact that $\overline{z}(t) \to 0$ as $t \to \infty$ implies that $M_1 = 0$. Arguments similar to those in the proof of proposition 3.2 imply that for $t > T^*$,

$$|\overline{z}(t)| = O(e^{-2t}).$$

This implies that $(3.32)_1$ holds. Differentiating (3.37) with respect to $t$ once and twice respectively and noticing $\overline{v}'(s) = -\overline{z}'(t)e^t$ and $\overline{v}''(s) = [\overline{z}''(t) + \overline{z}'(t)]e^{2t}$, we

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

easily see that $(3.32)_2$ and $(3.32)_3$ hold. Note that $v(s,\theta) = w(s,\theta) + \overline{v}(s)$, we obtain (3.33). This completes the proof of this lemma. □

LEMMA 3.4. *Let $\tau \geqslant 0$ be an integer and let $v$ be a solution of (3.2). Then there exist $0 < s_0 < 1/10$ and $M = M(v,\tau,s_0) > 0$ such that for $s \in (0, s_0)$,*

$$\max_{|y|=s} |D^\tau v(y)| \leqslant M s^{1-\tau}. \tag{3.38}$$

*Proof.* We first obtain (3.38) for the case of $\tau = 0$. If we define $z(t,\theta) = w(s,\theta)$, we see that

$$\max_{\theta\in S^{N-1}} |z(t,\theta)| \leqslant \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} |z_j^i(t)| \max_{\theta\in S^{N-1}} |Q_j^i(\theta)| \leqslant \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} D_i |z_j^i(t)|,$$

where $D_i$ is given in (3.7). Arguments similar to those in the proof of proposition 3.2 imply that there exist $C > 0$ independent of $t$ and $T^* \gg 1$ such that, for $t \geqslant T^*$,

$$\Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} D_i |z_j^i(t)| = O\left(\Sigma_{i=2}^{\infty} i m_i D_i e^{\nu_2^{(i)}(t-T)}\right) + O(e^{-(t-T)}) \leqslant Ce^{-t}$$

(note that $\lim_{i\to\infty}(((i+1)m_{i+1}D_{i+1})/(im_iD_i)) = 1$) and hence

$$\max_{\theta\in S^{N-1}} |z(t,\theta)| \leqslant Ce^{-t},$$

$$\max_{\theta\in S^{N-1}} |w(s,\theta)| \leqslant Cs \tag{3.39}$$

for $0 < s < s_0 := e^{-T^*}$. Therefore, (3.38) with $\tau = 0$ can be obtained from (3.39) and the fact that $v(s,\theta) = w(s,\theta) + \overline{v}(s)$.

We only show (3.38) for $\tau = 1$, the rest is essentially the same by differentiating $w(s,\theta) = \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} w_j^i(s) Q_j^i(\theta)$. We only need to show $|\nabla w(y)| \leqslant C$. Since $|\nabla w|^2 = w_s^2 + ((1)/(s^2))|w_\theta|^2$, we need to present the estimates of $w_s^2$ and $|w_\theta|^2$. We see that $w_s(s,\theta) = \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} (w_j^i)'(s) Q_j^i(\theta)$, then

$$\max_{\theta\in S^{N-1}} |w_s(s,\theta)| \leqslant \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} D_i |(w_j^i)'(s)|. \tag{3.40}$$

For each $\sigma_i = i(N+i-2)$ and $1 \leqslant j \leqslant m_i$, we see from the expression of $z_j^i(t)$ in (3.23) and $(w_j^i)'(s) = -(z_j^i)'(t)e^t$ $(t = -\ln s)$ that for $0 < s < s_0$,

$$D_i |(w_j^i)'(s)| \leqslant \tilde{M}_i s^{-(\nu_2^{(i)}+1)} \quad \text{for } i \geqslant 2$$

and

$$D_1 |(w_j^1)'(s)| \leqslant \tilde{M}_1.$$

These and (3.40) imply that there is $M_1 = M_1(v,s_0) > 0$ independent of $s$ such that, for $s \in (0, s_0)$,

$$\max_{\theta\in S^{N-1}} |w_s(s,\theta)| \leqslant M_1. \tag{3.41}$$

Note that $\nu_2^{(i)} + 1 < 0$ for $i \geqslant 2$. Since $|w_\theta(s,\theta)| = \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i} |w_j^i(s)||(Q_j^i)_\theta|$, we also obtain that there exists $M_2 = M_2(v,s_0) > 0$ independent of $s$ such that for

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

$s \in (0, s_0)$,

$$\max_{\theta \in S^{N-1}} |w_\theta(s,\theta)| \leqslant M_2 s. \tag{3.42}$$

(Note that $\lim_{i\to\infty}(((i+1)m_{i+1}E_{i+1})/(im_iE_i)) = 1$.) Therefore, for $s \in (0, s_0)$,

$$\max_{|y|=s} |\nabla w(y)|^2 = \max_{|y|=s}[w_s^2 + \frac{1}{s^2}|w_\theta|^2] \leqslant \hat{M}, \tag{3.43}$$

where $\hat{M} = M_1^2 + M_2^2$. This, (3.32)$_2$ and the fact that $v(s,\theta) = w(s,\theta) + \overline{v}(s)$ imply that (3.38) holds for $\tau = 1$. This completes the proof of this lemma. □

Let

$$\tilde{w}(s,\theta) = \frac{w(s,\theta)}{s}.$$

Then $\tilde{w}(s,\theta)$ satisfies the equation

$$\begin{aligned} &\tilde{w}^{(4)} - 2(N-5)s^{-1}\tilde{w}_{sss} + (N-3)(N-7)s^{-2}\tilde{w}_{ss} + (N-1)(N-3)s^{-3}\tilde{w}_s \\ &\quad - (N-1)(N-3)s^{-4}\tilde{w} + 2s^{-4}\Delta_\theta\tilde{w} - 2(N-3)s^{-3}\Delta_\theta\tilde{w}_s \\ &\quad + 2s^{-2}\Delta_\theta\tilde{w}_{ss} + s^{-4}\Delta_\theta^2\tilde{w} - s^{-4}\hat{g}(s,\xi,\tilde{w}) = 0, \end{aligned} \tag{3.44}$$

where

$$\hat{g}(s,\xi,\tilde{w}) = \lambda_0 s^{-4}[e^{s^{-2}(\xi(s,\theta)-D)}\tilde{w}(s,\theta) - \overline{e^{s^{-2}(\xi(s,\theta)-D)}\tilde{w}(s,\theta)}].$$

We also have

$$\tilde{w}(s,\theta) = \Sigma_{i=1}^{\infty}\Sigma_{j=1}^{m_i}\tilde{w}_j^i(s)Q_j^i(\theta), \quad \tilde{w}_j^i(s) = \frac{w_j^i(s)}{s}.$$

Then, for any integer $i \geqslant 1$ and $1 \leqslant j \leqslant m_i$, $\tilde{w}_j^i(s)$ satisfies the equation:

$$\begin{aligned} &(\tilde{w}_j^i)^{(4)} - 2(N-5)s^{-1}(\tilde{w}_j^i)_{sss} + [(N-3)(N-7) - 2\sigma_i]s^{-2}(\tilde{w}_j^i)_{ss} \\ &\quad + [(N-1)(N-3) + (2N-6)\sigma_i]s^{-3}(\tilde{w}_j^i)_s \\ &\quad + [-(N-1)(N-3) - 2\sigma_i + \sigma_i^2]s^{-4}\tilde{w}_j^i = s^{-4}\hat{g}_j^i(s), \end{aligned} \tag{3.45}$$

where $\hat{g}_j^i(s) = \lambda_0 s^{-3}\int_{S^{N-1}} \mathrm{e}^{s^{-2}(v-D)}Q_j^i(\theta)\mathrm{d}\theta$. Note that $|\hat{g}_j^i(s)| = o\Big(e^{-((D)/(2))s^{-2}}\Big)$ for $s$ near 0.

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

Let $\tilde{z}_j^i(t) = \tilde{w}_j^i(s)$, $t = -\ln s$. We see that $\tilde{z}_j^i(t)$ satisfies the equation (for $t$ near $\infty$):

$$\begin{aligned}(\tilde{z}_j^i)^{(4)} &+ 2(N-2)(\tilde{z}_i)_{ttt} + (N^2 - 4N + 2 - 2\sigma_i)(\tilde{z}_j^i)_{tt} \\ &- 2[N - 2 + (N-2)\sigma_i](\tilde{z}_j^i)_t + [-(N-1)(N-3) - 2\sigma_i + \sigma_i^2]\tilde{z}_j^i \\ &= \hat{f}_j^i(t),\end{aligned} \tag{3.46}$$

where $|\hat{f}_j^i(t)| = o(e^{-((D)/(2))\, e^{2t}})$. The corresponding polynomial of (3.46) is

$$\begin{aligned}&\nu^4 + 2(N-2)\nu^3 + (N^2 - 4N + 2 - 2\sigma_i)\nu^2 - 2[N - 2 + (N-2)\sigma_i]\nu \\ &+ [-(N-1)(N-3) - 2\sigma_i + \sigma_i^2] = 0\end{aligned} \tag{3.47}$$

which has four roots:

$$\tilde{\nu}_k^{(i)} = \nu_k^{(i)} + 1, \quad k = 1, 2, 3, 4$$

that is,

$$\tilde{\nu}_1^{(i)} = i + 1, \quad \tilde{\nu}_2^{(i)} = 3 - N - i, \quad \tilde{\nu}_3^{(i)} = i - 1, \quad \tilde{\nu}_4^{(i)} = 1 - N - i. \tag{3.48}$$

Thus we see that for $i = 2, 3, 4, \ldots$,

$$\tilde{\nu}_4^{(i)} < \tilde{\nu}_2^{(i)} < -1 < 0 < \tilde{\nu}_3^{(i)} < \tilde{\nu}_1^{(i)}.$$

For $i = 1$, the four roots are given by

$$\tilde{\nu}_1^{(1)} = 2, \quad \tilde{\nu}_2^{(1)} = 2 - N, \quad \tilde{\nu}_3^{(1)} = 0, \quad \tilde{\nu}_4^{(1)} = -N.$$

Then

$$\tilde{\nu}_4^{(1)} < \tilde{\nu}_2^{(1)} < -1 < 0 = \tilde{\nu}_3^{(1)} < \tilde{\nu}_1^{(1)}.$$

Since $|w_j^i(s)| = O(is^{-\nu_2^{(i)}})$ for $i \geqslant 2$, $1 \leqslant j \leqslant m_i$ and $|w_j^1(s)| = O(s)$ for $1 \leqslant j \leqslant N$, we have $\lim_{s\to 0} \tilde{w}_j^i(s) = 0$ for $i \geqslant 2$, $1 \leqslant j \leqslant m_i$ and $|\tilde{w}_j^1(s)|$ is bounded for $s$ near 0 and $1 \leqslant j \leqslant N$. Then $|\tilde{z}_j^1(t)|$ is bounded for $t$ near $\infty$ and $1 \leqslant j \leqslant N$ and for any $T \gg 1$,

$$\begin{aligned}\tilde{z}_j^1(t) = C_j &+ A_{j,2}^1 e^{\tilde{\nu}_2^{(1)} t} + A_{j,4}^1 e^{\tilde{\nu}_4^{(1)} t} \\ &- B_1^1 \int_t^\infty \mathrm{e}^{\tilde{\nu}_1^{(1)}(t-\tau)} \hat{f}_j^1(\tau)\mathrm{d}\tau - B_3^1 \int_t^\infty \hat{f}_j^1(\tau)\mathrm{d}\tau \\ &+ B_2^1 \int_T^t \mathrm{e}^{\tilde{\nu}_2^{(1)}(t-\tau)} \hat{f}_j^1(\tau)\mathrm{d}\tau + B_4^1 \int_T^t \mathrm{e}^{\tilde{\nu}_4^{(1)}(t-\tau)} \hat{f}_j^1(\tau)\mathrm{d}\tau,\end{aligned}$$

where $C_j$ is a constant. This implies that $\tilde{z}_j^1(t) \to A_j$ (a constant) as $t \to \infty$. Since $Q_j^1$ $(1 \leqslant j \leqslant N)$ are the eigenfunctions corresponding to $\sigma_1 = N - 1$, we see that

$$\lim_{s\to 0} \tilde{w}(s, \theta) = V(\theta), \tag{3.49}$$

where $V(\theta)$ is 0 or one of the eigenfunctions of $-\Delta$ on $S^{N-1}$ corresponding to $N-1$ and also one of the eigenfunction of $\Delta^2$ on $S^{N-1}$ corresponding to $(N-1)^2$, that is,

$$\Delta_\theta V + (N-1)V = 0, \quad \overline{V} = 0,$$

and

$$\Delta_\theta^2 V = (N-1)^2 V, \quad \overline{V} = 0.$$

In conclusion, we have the following theorem.

THEOREM 3.5. *Let $v$ be a solution of (3.2) and $\tilde{w}$ be given in (3.44). Then we have*

(i) $v(y) = \overline{v}(s) + s\tilde{w}(s,\theta)$ *satisfies*

$$|\overline{v}(s)| = O(s^2), \quad |\overline{v}'(s)| = O(s), \quad |\overline{v}''(s)| = O(1).$$

(ii) *For any non-negative integers $\tau$ and $\tau_1$, there exists $M = M(v,\tau,\tau_1) > 0$ such that*

$$|s^\tau D_\theta^{\tau_1} D_s^\tau \tilde{w}(y)| \leqslant M, \quad y \in B_{s_0}, \quad y \neq 0, \tag{3.50}$$

*where $B_{s_0} = \{y \in \mathbb{R}^N : |y| < s_0\}$. Moreover, $\tilde{w}$ satisfies*

$$\lim_{s\to 0} \tilde{w}(s,\theta) = V(\theta), \tag{3.51}$$

*uniformly in $C^\tau(S^{N-1})$, where $V(\theta)$ is 0 or one of the eigenfunctions of $-\Delta$ on $S^{N-1}$, that is,*

$$\Delta_\theta V + (N-1)V = 0, \quad \overline{V} = 0$$

*(or one of the eigenfunction of $\Delta^2$ on $S^{N-1}$, that is, $\Delta_\theta^2 V = (N-1)^2 V$, $\overline{V} = 0$.)*

It is known from lemma 8.1 of [**13**] that

$$V(\theta) = \theta \cdot x_0 \tag{3.52}$$

for some $x_0 \in \mathbb{R}^N$ fixed and $\theta = ((x)/(|x|)) \in S^{N-1}$.

We obtain from theorem 3.5 the asymptotic expansion of $u(x)$ near $|x| = \infty$.

THEOREM 3.6. *Let $N \geqslant 5$ and $u$ be a solution of (1.1) satisfying (1.5). Then $u$ admits the expansion:*

$$u(x) = r^2 \left[ -D + \xi(r) + \frac{\eta(r,\theta)}{r} \right], \tag{3.53}$$

$$w(x) := -\Delta u(x) = 2ND + \xi_1(r) + \frac{\eta_1(r,\theta)}{r} \tag{3.54}$$

*where*

$$\xi_1(r) = -[r^2\xi'' + (N+3)r\xi' + 2N\xi],$$

$$\eta_1(r,\theta) = -[r^2\eta_{rr} + (N+1)r\eta_r + (N-1)\eta + \Delta_\theta \eta].$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

*Moreover, the following properties are satisfied:*

(i) $\xi(r) = r^{-2}\overline{u}(r) + D$ *and there exist* $R_0\ (:= s_0^{-1}) > 0$ *and a constant* $M = M(u) > 0$ *such that, for* $r > R_0$,

$$|\xi(r)| \leqslant Mr^{-2}, \quad |\xi'(r)| \leqslant Mr^{-3}, \quad |\xi''(r)| \leqslant Mr^{-4}, \tag{3.55}$$

$$|\xi_1(r)| \leqslant Mr^{-2}. \tag{3.56}$$

(ii) *Let* $\tau$ *and* $\tau_1$ *be two non-negative integers. Then there exists a positive constant* $M := M(u, \tau, \tau_1)$ *such that, for* $r > R_0$,

$$|r^{\tau} D_{\theta}^{\tau_1} D_r^{\tau} \eta(r, \theta)| \leqslant M, \tag{3.57}$$

$$|\eta_1(r, \theta)| \leqslant M. \tag{3.58}$$

(iii) *Let* $\tau$ *be a non-negative integer. Then* $\eta(r, \theta)$ *tends to* $V(\theta)$ *uniformly in* $C^{\tau}(S^{N-1})$ *as* $r \to \infty$, *where* $V(\theta)$ *is given in* (3.51)

*and*

$$V(\theta) = \theta \cdot x_0 \tag{3.59}$$

*for some* $x_0 \in \mathbb{R}^N$ *fixed and* $\theta = ((x)/(|x|)) \in S^{N-1}$.

## 4. Proof of Theorem 1.1

We present the proof of our main theorem in this section.

Without loss of generality, we assume $x^0 = 0$ in theorem 1.1. The necessity follows from proposition 2.1.

We will use moving-plane arguments of the system of equations (see [**10**]) to prove the sufficiency.

We first write (1.1) to a system of equations:

$$\begin{cases} -\Delta u = w, & \text{in } \mathbb{R}^N, \\ -\Delta w = \lambda_0 e^u, & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

We now start the procedure of moving-plane. For any $\gamma \in \mathbb{R}$, let $\Sigma_\gamma$ be the following hyperplane:

$$\Sigma_\gamma = \{x \in \mathbb{R}^N :\ x_1 = \gamma\}.$$

For $x \in \mathbb{R}^N$, denote $x^\gamma$ to be the reflection point of $x$ about $\Sigma_\gamma$, that is,

$$x^\gamma := (2\gamma - x_1, x_2, \ldots, x_N).$$

As a consequence of the expansions of $u(x)$ in theorem 3.6, we have the following lemma.

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

LEMMA 4.1. *Let $N \geqslant 5$ and $u$ be a solution of* (1.1) *satisfying* (1.5). *Then,*

(i) *If $\gamma^j \in \mathbb{R} \to \gamma$ and $\{x^j\} \to \infty$ with $x_1^j < \gamma^j$, then*

$$\lim_{j\to\infty} \frac{1}{\gamma^j - x_1^j}[u(x^j) - u((x^j)^\gamma)] = 4D\gamma - 2(x_0)_1, \tag{4.2}$$

*where $(x_0)_1$ is the first component of $x_0$ given in* (3.59).

(ii) *Define*

$$\gamma_0 = \frac{(x_0)_1}{2D}. \tag{4.3}$$

*Then there exists a constant $M = M(u) > 0$ such that*

$$\frac{\partial u}{\partial x_1} \leqslant 0 \quad \textit{if } x_1 \geqslant \gamma_0 + 1 \quad \textit{and} \quad |x| \geqslant M. \tag{4.4}$$

*Proof.* To prove (4.2), without loss of generality, we assume that

$$\lim_{j\to\infty} \frac{x_j}{|x_j|} = \overline{\theta} \in S^{N-1}.$$

For simplicity, we also assume that $\gamma^j = \gamma$, $j = 1, 2, \ldots$ since the following arguments work equally well for the sequence $\{\gamma^j\}$. Using the the expansion of $u$ in (3.53), we have

$$\begin{aligned}\frac{1}{\gamma - x_1^j}[u(x^j) - u((x^j)^\gamma)] &= \frac{1}{\gamma - x_1^j}[-D(|x^j|^2 - |(x^j)^\gamma|^2)] \\ &\quad + \frac{1}{\gamma - x_1^j}[|x^j|^2\xi(|x^j|) - |(x^j)^\gamma|^2\xi(|(x^j)^\gamma|)] \\ &\quad + \frac{1}{\gamma - x_1^j}[|x^j|\eta(|x^j|, \theta^j) - |(x^j)^\gamma|\eta(|(x^j)^\gamma|, (\theta^j)^\gamma)] \\ &= I + II + III.\end{aligned}$$

We have

$$-D(|x^j|^2 - |(x^j)^\gamma|^2) = 4D\gamma(\gamma - x_1^j)$$

and hence

$$I = 4D\gamma.$$

We also have that there is $\beta_j$ between $|x^j|$ and $|(x^j)^\gamma|$ such that

$$|x^j|^2\xi(|x^j|) - |(x^j)^\gamma|^2\xi(|(x^j)^\gamma|) = \left[2\beta_j\xi(\beta_j) + \beta_j^2\xi'(\beta_j)\right]\frac{-4\gamma(\gamma - x_1^j)}{|x^j| + |(x^j)^\gamma|},$$

and, in turn,

$$II = \frac{1}{\gamma - x_1^j}[2\beta_j\xi(\beta_j) + \beta_j^2\xi'(\beta_j)]\frac{-4\gamma(\gamma - x_1^j)}{|x^j| + |(x^j)^\gamma|} = O(|x_j|^{-2}) \to 0$$

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

as $j \to \infty$, since $((|(x^j)^\gamma|)/(|x^j|)) \to 1$ as $j \to \infty$. Here we have used the estimates of $\xi(r)$ and $\xi'(r)$ in (3.55). We now write

$$\begin{aligned} III &= \frac{\eta(|(x^j)^\gamma|, (\theta^j)^\gamma)}{\gamma - x_1^j}[|x^j| - |(x^j)^\gamma|] \\ &\quad + \frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|, (\theta^j)^\gamma) - \eta(|(x^j)^\gamma|, (\theta^j)^\gamma)] \\ &\quad + \frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|, \theta^j) - \eta(|x^j|, (\theta^j)^\gamma)] \\ &= III_1 + III_2 + III_3. \end{aligned}$$

As before, by (3.57) and arguments similar to those in the proof of (8.11) in lemma 5.2 of [**13**], we obtain that $III_1 = O(|x^j|^{-1}) \to 0$ as $j \to \infty$, $III_2 = O(|x^j|^{-1}) \to 0$ as $j \to \infty$ and $III_3 \to -2(x_0)_1$ as $j \to \infty$. These imply that (4.2) holds.

To prove (4.4), we use (4.2). Suppose that (4.4) is false. Then there exists a sequence $\{x^j\} \to \infty$ such that

$$\frac{\partial u}{\partial x_1}(x^j) > 0, \quad x_1^j \geqslant \gamma_0 + 1, \quad \forall j \in \mathbb{N}.$$

It follows that there exists a sequence of bounded positive numbers $\{d_j\}$ such that

$$u(x^j) < u(x_{d_j}), \quad x_{d_j} = x^j + (2d_j, 0, \ldots, 0), \quad \forall j \in \mathbb{N}.$$

Let

$$\gamma^j = x_1^j + d_j > x_1^j.$$

We have

$$\frac{1}{\gamma^j - x_1^j}\left[u(x^j) - u((x^j)^\gamma)\right] < 0, \quad \forall j \in \mathbb{N}. \tag{4.5}$$

There are two possibilities:

$$\liminf_{j\to\infty} \gamma^j < \infty, \qquad \lim_{j\to\infty} \gamma^j = \infty.$$

If the first case occurs, we choose a convergent subsequence of $\{\gamma^j\}$ (still denoted by $\{\gamma^j\}$) with the limit $\gamma \geqslant \gamma_0 + 1$ and apply (4.2) and (4.3) to obtain

$$\lim_{j\to\infty} \frac{1}{\gamma^j - x_1^j}\left[u(x^j) - u((x^j)^\gamma)\right] = 4D\gamma - 2(x_0)_1 \geqslant 4D > 0.$$

This contradicts (4.5). We can derive a contradiction for the second case similarly. The proof is a little variant of the proof of Lemma 8.2 of [**13**]. Thus, neither the first nor the second case can occur and (4.4) holds. This completes the proof of this lemma. □

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

**Completion of the proof of Theorem 1.1**

To complete the proof of the sufficiency, we use moving-plane arguments for the system of equations (4.1). The main idea of the proof is similar to that in the proof of theorem 1.1 of [7]. We notice that lemma 6.1 of [7] is true for our problem here, lemma 6.2 of [7] is also true for the entire solution $u$ of (1.1) satisfying (1.5) provided $N \geqslant 5$. (To obtain the conclusions of lemma 6.2 of [7] for our problem here, we need the expansions of $u(x)$ and $w(x)$ given in (3.53) and (3.54).)

We first claim that there exists $\gamma' > 0$ such that

$$u(x) > u(x^{\gamma}), \quad w(x) > w(x^{\gamma}) \quad \text{if } x_1 < \gamma \text{ and } \quad \gamma \geqslant \gamma'. \tag{4.6}$$

Suppose for contradiction that (4.6) is not true. A little variant of lemma 6.2 of [7] implies that there exist two sequences $\{\gamma^i\} \to \infty$ and $\{x^i\}$, with $x_1^i < \gamma^i$ such that

$$u(x^i) \leqslant u(y^i), \quad y^i = (x^i)^{\gamma^i}, \;\; i = 1, 2, \dots. \tag{4.7}$$

Obviously, $y^i \to \infty$, so $u(y^i) \to -\infty$. In turn $|x^i| \to \infty$. By lemma 4.1, we must have

$$x_1^i \leqslant \gamma_0 + 1 \quad \text{for } i \quad \text{large.}$$

It follows that, for any $\gamma_1 > \gamma_0 + 1$,

$$u(x^i) \leqslant u(y^i) \leqslant u((x^i)^{\gamma_1}) \quad \text{for } i \quad \text{large,}$$

since $(x^i)_1^{\gamma^i} \gg (x^i)_1^{\gamma_1}$ for $i$ large and $u(x) \to -\infty$ as $|x| \to \infty$. On the other hand, by lemma 4.1 again, we conclude that

$$0 \geqslant \frac{1}{(\gamma_1 - x_1^i)}[u(x^i) - u((x^i)^{\gamma_1})] \to 4D\gamma_1 - 2(x_0)_1 > 0$$

since $x_1^i \leqslant \gamma_0 + 1 < \gamma_1$. This is a contradiction and (4.6) follows.

Now let $\Gamma \subset \mathbb{R}$ be defined by

$$\Gamma = \{\gamma \in (\gamma_0, \infty) : \; (4.6) \text{ holds}\}.$$

We will prove that

$$\Gamma = (\gamma_0, \infty). \tag{4.8}$$

We first prove that $\Gamma$ is open. Suppose for contradiction that, for some $\gamma \in \Gamma$, there exist sequences $\{\gamma^i\}$ and $\{x^i\}$ with $\gamma^i \to \gamma$ as $i \to \infty$ and $x_1^i < \gamma^i$ such that (4.7) holds. Obviously, there is a subsequence of $\{x^i\}$ tending to either $\infty$ or $\hat{x} \in \mathbb{R}^N$ as $i \to \infty$. If the first case occurs, we simply use lemma 4.1 to derive a contradiction, since $\gamma > \gamma_0$. If the second case occurs, we can infer, from the definition of $\gamma$, lemma 6.1 of [7] and a variant of lemma 6.2 of [7], that $\hat{x}_1 = \gamma$. It follows that $((\partial u)/(\partial x_1))(\hat{x}) \geqslant 0$, $\hat{x}_1 = \gamma$. This simply cannot happen because of (6.3) of [7],

https://doi.org/10.1017/prm.2018.49 Published online by Cambridge University Press

that is, $\Gamma$ is open. Set

$$\tilde{\gamma} = \inf\{\gamma \in (\gamma_0, \infty) : \ (\gamma, \infty) \subset \Gamma\}.$$

We show that

$$\tilde{\gamma} = \gamma_0. \tag{4.9}$$

Suppose for contradiction that this is not true, that is, $\tilde{\gamma} > \gamma_0$. By continuity, we have

$$u(x) \geqslant u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}.$$

By lemma 6.1 of [7] and a lemma similar to lemma 6.2 of [7], we see that either

$$u(x) \equiv u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}$$

or

$$u(x) > u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}, \quad i.e., \ \tilde{\gamma} \in \Gamma.$$

The latter cannot occur because $(\tilde{\gamma}, \infty)$ is maximal and $\Gamma$ is open. The former cannot occur either because it contradicts lemma 4.1 since $\tilde{\gamma} > \gamma_0$ and (4.8) is obtained.

By continuity again, we have

$$u(x) \geqslant u(x^{\gamma_0}) \quad \text{for } x_1 < \gamma_0.$$

Reversing the $x_1$-axis, we conclude that

$$u(x) \geqslant u(x^{\gamma_0}) \quad \text{for } x_1 > \gamma_0.$$

That is, $u$ is symmetric about the plane $x_1 = \gamma_0$. Since this argument applies for any direction, we finally obtain the radial symmetry of $u$ about the point $x^0 := ((x_0)/(2D)) \in \mathbb{R}^N$ and $x_0$ is given in (3.59). Set $y = x - x^0$ and $\tilde{u}(y) := u(x)$. Then $\tilde{u}(y)$ is a radial entire solution of (1.1), that is, $\tilde{u}(y) = \tilde{u}(|y|)$ and satisfies

$$\lim_{|y|\to\infty} |y|^{-2}\tilde{u}(|y|) = \lim_{|x|\to\infty} |x|^{-2}u(x) = -D.$$

It is known from [1] that $\tilde{u}(|y|)$ is a non-maximal radial entire solution of (1.1) with the initial value $\tilde{u}(0) = u(x^0)$. The sufficiency of theorem 1.1 is proved and hence the proof of theorem 1.1 is complete. □

Remark 4.2. We conjecture that the following conclusion holds: *If $u \in C^4(\mathbb{R}^N)$ is an entire solution of (1.1) with $N \geqslant 5$, then $u$ is the maximal radial entire solution of (1.1) about some $x^0 \in \mathbb{R}^N$, if and only if*

$$|x|^{-2}u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{4.10}$$

This conjecture implies that if $u$ is an entire solution of (1.1) and (4.10) holds for $u$, then $u$ must have the exact asymptotic behaviour at $\infty$:

$$u(x) + 4\ln|x| \to 0 \quad \text{as } |x| \to \infty.$$

**Acknowledgements**

We thank the referee for his/her valuable suggestions, which improve the presentation of this paper. The research of the second author is supported by NSFC (11171092, 11571093)

## References

1 G. Arioli, F. Gazzola and H. C. Grunau. Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity. *J. Differ. Equ.* **230** (2006), 743–770.

2 E. Berchio, A. Farina, A. Ferrero and F. Gazzola. Existence and stability of entire solutions to a semilinear fourth order elliptic problem. *J. Differ. Equ.* **252** (2012), 2596–2616.

3 S. Y. A. Chang and W. X. Chen. A note on a class of higher order conformally covariant equations. *Discrete Contin. Dyn. Syst.* **7** (2001), 275–281.

4 I. Chavel. *Eigenvalues in Riemannian geometry* (New York: Academic Press, Inc., 1984).

5 L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault. The Gel'fand problem for the biharmonic operator. *Arch. Rational Mech. Anal.* **208** (2013), 725–752.

6 Z. M. Guo. Further study of entire solutions of a biharmonic equation with exponential nonlinearity. *Ann. Mat. Pura Appl.* **193** (2014), 187–201.

7 Z. M. Guo, X. Huang and F. Zhou. Radial symmetry of entire solutions of a bi-harmonic equation with exponential nonlinearity. *J. Funct. Anal.* **268** (2015), 1972–2004.

8 C. S. Lin. A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$. *Comment. Math. Helv.* **73** (1998), 206–231.

9 P. J. Mckenna and W. Reichel, Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry. *Electron J. Differ. Equ.* **37** (2003), 13.

10 W. C. Troy. Symmetry properties in systems of semilinear elliptic equations. *J. Differ. Equ.* **42** (1981), 400–413.

11 G. Warnault. Liouville theorems for stable radial solutions for the biharmonic operator. *Asymptotic Anal.* **69** (2010), 87–98.

12 J. C. Wei and X. W. Xu. Classification of solutions of higher order conformally invariant equations. *Math. Ann.* **313** (1999), 207–228.

13 H. H. Zou. Symmetry of positive solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$. *J. Differ. Equ.* **120** (1995), 46–88.