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BIT FLIPPING AND TIME TO RECOVER

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Abstract

We call 'bits' a sequence of devices indexed by positive integers, where every device can be in two states: 0 (idle) and 1 (active). Start from the 'ground state' of the system when all bits are in 0-state. In our first binary flipping (BF) model the evolution of the system behaves as follows. At each time step choose one bit from a given distribution $\mathcal{P}$ on the positive integers independently of anything else, then flip the state of this bit to the opposite state. In our second damaged bits (DB) model a 'damaged' state is added: each selected idling bit changes to active, but selecting an active bit changes its state to damaged in which it then stays forever. In both models we analyse the recurrence of the system's ground state when no bits are active. We present sufficient conditions for both the BF and DB models to show recurrent or transient behaviour, depending on the properties of the distribution $\mathcal{P}$. We provide a bound for fractional moments of the return time to the ground state for the BF model, and prove a central limit theorem for the number of active bits for both models.

Keywords: Binary system; bit flipping; random walk on a group; Markov chain recurrence; critical behaviour

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1. Introduction and model description

In many areas of engineering and science one faces an array of devices which possess a few states. In the simplest case these could be on-off or idle-active states, in other situations a damaged state is also possible. By the analogy with computer science, such a two-state device can be called a *bit* which in some cases can also be 'damaged'. Assuming the bits change their states in a random fashion, a natural question to ask is when, if at all, the system of bits recovers to the state when none of the bits are active. We call such a state with only idling or damaged bits a *ground state* of the system. The time to recover may be finite, but, in general, may also assume infinite values when the system actually does not recover. In the latter case we speak of a *transient* behaviour of the system. In the former case, depending on whether the average recovery time is finite or not, we speak of a *positive*- or of a *null-recurrence*. Similarly to random walk models, this classification is tightly related to the exact random mechanism governing the change of the bits' states.

In this paper we consider two basic models: binary flipping and damaged bits. In both models we deal with a countably infinite array of bits which we index by the positive integers $\mathbb{N} = \{1, 2, \dots\}$. Initially, at step 0, the system is in the ground state, i.e. all the bits are idling.

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At each next step the index of the bit to change its state is sampled independently of the current state of the bits from a given probability distribution on $\mathbb{N}$,

$$\mathcal{P} = (p_1, p_2, \dots)\colon \sum_{i=1}^{\infty} p_i = 1.$$

We assume that the rates are all positive, otherwise our models are described by a finite-state Markov chain with an evident behaviour. Renumbering the bits, if necessary, we may assume that the rates are nonincreasing: $p_1 \ge p_2 \ge p_3 \ge \cdots > 0$, so that the bits most likely to change their state are put first. The main quantities of interest are the time to recovery τ—the number of steps until the first return to the ground state, and η_n—the number of bits active at step n.

Binary flipping. In the binary flipping (BF) model each bit alternates between the two states: idle and active. At step $n = 0$ all of the bits are idling. Let $\chi_1, \chi_2, \dots$ be independent and identically distributed (i.i.d.) random variables with distribution $\mathcal{P}$. At each step $n = 1, 2, \dots,$ the bit with index χ_n is *flipped*, i.e. its state is changed to the opposite. If 0 and 1 represent, respectively, the idling and the active states, the evolution of the system is described by a discrete-time Markov chain $\{\zeta_n\}_{n\ge 0} = \{(\zeta_n^1, \zeta_n^2, \zeta_n^3, \dots)\}_{n\ge 0}$ with the state space $\mathcal{X} = \{x \in \{0, 1\}^{\mathbb{N}}\colon x \text{ has finitely many nonzeros}\}$ such that $\zeta_0 = \mathbf{0}$ is the vector of all 0s, and

$$\zeta_{n+1}^k = \begin{cases} \zeta_n^k, & k \ne \chi_{n+1}, \\ 1 - \zeta_n^k, & k = \chi_{n+1}, \end{cases} \qquad k = 1, 2, \dots, \ n = 0, 1, 2, \dots. \tag{1.1}$$

The main quantity of interest is the first time of return to the ground state with no active bits, i.e. the stopping time

$$\tau_{\mathrm{BF}} = \min\{n \ge 1\colon \zeta_n^k = 0 \text{ for all } k = 1, 2, \dots\}.$$

Damaged bits. This damaged bits (DB) model elaborates on the first one by adding a damaged state to the bits. As in the BF model above, we start with a sequence of idling bits and then consecutively sample from $\mathcal{P}$ for the index of the bit to change its state. When selected, an idle bit becomes active, however, an active bit becomes damaged, and a damaged bit remains damaged, so no reversal is possible. If 0, 1, 2 encode the idle, active, and damaged states, respectively, the corresponding Markov chain $\{\zeta_n\}_{n\ge 0}$ with the state space $\mathcal{Y} = \{y \in \{0, 1, 2\}^{\mathbb{N}}\colon y \text{ has finitely many nonzeros}\}$ is defined by

$$\zeta_{n+1}^k = \begin{cases} \zeta_n^k, & k \ne \chi_{n+1}, \\ \min\{2, 1 + \zeta_n^k\}, & k = \chi_{n+1}, \end{cases} \qquad k, n \in \mathbb{N}, \tag{1.2}$$

with the starting configuration ζ_0 being the vector of all 0s, $\zeta_0 = \mathbf{0}$. Here again we are looking for the number of steps to return to the ground state which is now understood as the subset of $\mathcal{Y}$ with no active bits, i.e.

$$\tau_{\mathrm{DB}} = \min\{n \ge 1\colon \zeta_n^k \in \{0, 2\} \text{ for all } k = 1, 2, \dots\}.$$

In contrast to the BF model, the ground state in the DB model, in general, cannot be identified with any one particular state of the Markov chain $\{\zeta_n\}$.

Continuous-time version. Consider continuous-time versions of both the BF and DB models. Let $\{\zeta_t\}_{t\ge 0} = \{(\zeta_t^1, \zeta_t^2, \dots)\}_{t\ge 0}$ be a continuous-time Markov jump process with the state space $\mathcal{X}$ in the BF case, and $\mathcal{Y}$ in the DB case. The process has jump rate 1, at each jump a random index is sampled from $\mathcal{P}$, then the state of the respective coordinate is changed according to the BF or the DB dynamics. Define the renewal process $\{t_n\}_{n\geq 0}$ of jump times and let $t_0 = 0$. The embedded Markov chain $\{\zeta_{t_n}\}_{n\geq 0}$ is then a distributional copy of the discrete-time version $\{\zeta_n\}_{n\geq 0}$ of the model. One of the advantages of this representation, sometimes referred to as *Poissonisation* and widely used since at least 1968 [2], is the independence of the marginal processes $\{\zeta_t^k\}_{t\geq 0}$ for different $k = 1, 2, \ldots$, which leads to explicitly computable probabilities as we demonstrate here. The notion of recurrence/transience stays the same for both the discrete- and continuous-time implementations.

A directly related class of models was studied in [4], where the main question is which properties of random sequences are preserved under independent dynamic resampling of individual terms at ticks of Poisson clocks of different rates. In a somewhat similar vein, one could interpret the BF model as a dynamical percolation process on $\mathbb{Z}$, where, starting with all edges 'open', every edge is switching between 'open' and 'closed' states. The question of recurrence is then equivalent to the question of existence of a sequence of percolation times when all the edges are open and, thus, 0 is connected to ∞. For a recent survey of the dynamical percolation, see [7].

The Markov chains (1.1) and (1.2) describing our models can be regarded as random walks on an infinite-dimensional group; see, e.g. [6]. Typically the analysis of random walks on discrete groups assumes a finite generator set, so that the underlying Cayley graph is locally finite, as, for example, in [8]. However, the state spaces in our models are not finitely generated groups, so analysis of a random walk in such a space is interesting in its own right. Practical applications are also envisaged: in addition to an evident relation to modelling reliability of a complex system with multiple components prone to fail at different rates, one can also mention computer science and information encryption techniques. The term 'bit flipping' is borrowed from the literature on randomised simplex algorithms [3], where a similar model was analysed: each flipped bit there makes all of the bits to the right change their states as well.

2. Main results

For the above models we prove the following main result: each model exhibits a transient or recurrent behaviour, depending on how fast the rates $\{p_k\}$ decay. We start with the BF model: there turns out to be a critical decay separating the two regimes.

Theorem 2.1. *If the distribution $\mathcal{P}$ is such that*

(i) $\limsup_{k\to\infty} 2^k p_k < \infty$*, then the BF model is recurrent, i.e.*

$$\mathbb{P}(\tau_{\mathrm{BF}} < \infty) = 1;$$

(ii) $\liminf_{k\to\infty} (2-\varepsilon)^k p_k > 0$ *for some* $\varepsilon > 0$*, then the BF model is transient, i.e.*

$$\mathbb{P}(\tau_{\mathrm{BF}} = \infty) > 0.$$

Furthermore, the BF model is never positive recurrent, as we show in the next theorem.

Theorem 2.2. *The expected time of recovery in the BF model is infinite, i.e.* $\mathbb{E}\tau_{\mathrm{BF}} = \infty$.

Although the first moment of τ_{BF} is infinite, it is reasonable to ask for which values of $r < 1$ the rth moment becomes finite. The next theorem presents bounds for such r in the case of asymptotically geometrically decaying $\{p_k\}$, these are presented graphically in Figure 1.

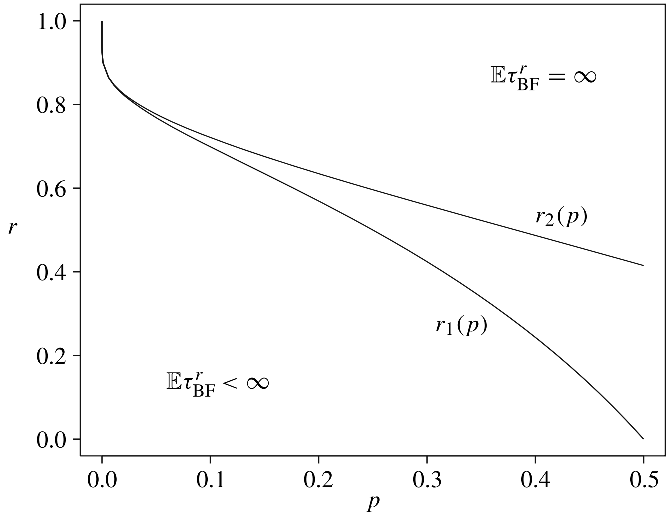

FIGURE 1: Integrability of τ_{BF}^r as given by Theorem 2.3.

Theorem 2.3. *Consider a recurrent BF model in discrete time with* $p_k \sim C_1 p^k$ *for some fixed constant* $C_1 > 0$ *and* $p \in (0, \frac{1}{2})$. *Then*

(i) $\mathbb{E}\tau_{\mathrm{BF}}^r < \infty$ *for any positive* $r < r_1(p) := 1 - \log 2/\log(1/p)$. *Moreover, for any such* r, *if the Markov chain (1.1) starts from an arbitrary initial state* $\zeta_0 \in \mathcal{X}$ *with the largest active bit* M_0, *then there exists a constant* $C_2 = C_2(C_1, p, r)$ *such that*

$$\mathbb{E}[\tau_{\mathrm{BF}}^r \mid M_0 = m] \le C_2 \left(\frac{1}{2p}\right)^m;$$

(ii) $\mathbb{E}\tau_{\mathrm{BF}}^r = \infty$ *for any* $r > r_2(p) := 1 - \log(2 - p)/\log(1/p)$.

Remark 2.1. There is an obvious coupling of the DB model with the BF model: just declare the bits which flipped more than once in the BF model damaged in the DB model. Then $\tau_{\mathrm{DB}} \le \tau_{\mathrm{BF}}$ almost surely and the upper bound of Theorem 2.3(i) also holds for τ_{DB}.

The DB model can also be recurrent or transient, depending on $\mathcal{P}$. The recurrence/transience now does not correspond to the recurrence/transience of the Markov chain (1.2), because the ground state of the DB model is an infinite collection of states of $\{\zeta_n\}$. We call the DB model recurrent if $\tau_{\mathrm{DB}} < \infty$ with probability 1, and transient otherwise. Denote by Q_k the tail of the distribution $\mathcal{P}$: $Q_k = \sum_{j=k+1}^{\infty} p_j$.

Theorem 2.4. *If the distribution* $\mathcal{P}$ *is such that*

(i) $\limsup_{k\to\infty}(Q_{k+1}/Q_k) = p < 1$, *then the DB model is recurrent;*

(ii) $p_k \sim C \exp(-\alpha k^{\gamma})$, $k \to \infty$, *for some* $\alpha > 0$, $\gamma \in (0, \frac{1}{2})$, *then the DB model is transient.*

Denote by η_t the total number of active bits in the continuous version of each model at time $t \ge 0$. In both the BF and DB models, whenever $\mathbb{E}\eta_t \to \infty$, conditions of the central limit theorem are fulfilled for η_t. We prove the following fact.

Theorem 2.5. *For both the BF and DB models, whenever* $\mathbb{E}\eta_t \to \infty$*, then*

$$\operatorname{var} \eta_t \to \infty \quad \text{and} \quad \frac{\eta_t - \mathbb{E}\eta_t}{\sqrt{\operatorname{var} \eta_t}} \xrightarrow{\mathrm{D}} \mathcal{N}(0, 1) \quad \text{as } t \to \infty,$$

where '$\xrightarrow{\mathrm{D}}$' *denotes convergence in distribution. In the BF model the condition* $\mathbb{E}\eta_t \to \infty$ *is always fulfilled, and in the DB model a sufficient condition for* $\mathbb{E}\eta_t \to \infty$ *is* $p_k \sim C \exp(-\alpha k^\gamma)$*,* $k \to \infty$*, for some constants* $C > 0, \alpha > 0, \gamma \in (0, 1)$.

3. Proofs

3.1. Transience and recurrence of the BF model

Proof of Theorem 2.1. First, we are going to prove the theorem for a particular case of a geometric decrease $p_k = Cp^k$ for some $p \in (0, 1)$ and then extend it using monotonicity arguments.

Consider the continuous-time BF model. Recall that $\zeta_t = (\zeta_t^k)_{k\geq 1}$ is a continuous-time Markov jump process on $\mathcal{X}$ representing the configuration of the bits at time $t \geq 0$, and $\zeta_0 = \mathbf{0} = (0, 0, \ldots)$. Denote by ν_{total} the total time $\{\zeta_t\}$ spends in the state $\mathbf{0}$ for $t > 0$. Since the process $\{\zeta_t\}$ is irreducible, recurrence of the BF model implies that the state $\mathbf{0}$ is recurrent. Since the holding times at state $\mathbf{0}$ are i.i.d. exponential with parameter 1, we obtain $\mathbb{E}\nu_{\text{total}} = \infty$. When the BF model is transient, i.e. when

$$q = \mathbb{P}\{\zeta_t = \mathbf{0} \text{ for some finite } t > t_1 \mid \zeta_0 = \mathbf{0}\} < 1,$$

where t_1 is the time of the first jump of the process ζ_t, then ν_{total} is distributed as the sum $\sum_{i=1}^{\nu} \varepsilon_i$, where ν has a geometric distribution with parameter q and the ε_i are i.i.d. exponentially distributed with parameter 1 random variables representing holding times at state $\mathbf{0}$. In that case, $\mathbb{E}\nu_{\text{total}} = \mathbb{E}\nu\mathbb{E}\varepsilon_i = 1/q < \infty$. Thus, $\mathbb{E}\nu_{\text{total}} = \infty$ is equivalent to recurrence of $\zeta(t)$ and of the BF model. Since

$$\mathbb{E}\nu_{\text{total}} = \mathbb{E}\int_0^\infty \prod_{k=1}^\infty \mathbf{1}_{\{k\text{th bit is idle at time } t\}} \,\mathrm{d}t = \int_0^\infty \prod_{k=1}^\infty \mathbb{P}\{\zeta_t^k = 0\} \,\mathrm{d}t$$

and

$$\mathbb{P}\{\zeta_t^k = 0\} = \sum_{j=0}^\infty \mathbb{P}\{k\text{th bit flipped } 2j \text{ times by time } t\} = \mathrm{e}^{-p_k t} \sum_{j=0}^\infty \frac{(p_k t)^{2j}}{(2j)!} = \tfrac{1}{2}(1 + \mathrm{e}^{-2p_k t}),$$

the transience is equivalent to the convergence of the integral

$$\mathbb{E}\nu_{\text{total}} = \int_0^\infty \prod_{k=1}^\infty \tfrac{1}{2}(1 + \mathrm{e}^{-2p_k t}) \,\mathrm{d}t = \int_0^\infty \prod_{k=1}^\infty (1 - f(p_k t)) \,\mathrm{d}t, \tag{3.1}$$

where $f(x) = (1 - \mathrm{e}^{-2x})/2$. Now we establish lower and upper bounds for the infinite product under the integral.

Fix an arbitrary small $\theta > 0$. Note that the function $1 - f(x)$ is continuous, decreasing in x, and $1 - f(x) = 1/(2 - \theta)$ has only one root, call this root x_θ. Decompose the right-hand side of (3.1) into two parts, i.e.

$$\prod_{k=1}^{\infty}(1-f(p_k t)) = \underbrace{\prod_{\{k\colon p_k t\le x_\theta\}}(1-f(p_k t))}_{\Phi_1(t)}\ \underbrace{\prod_{\{k\colon p_k t> x_\theta\}}(1-f(p_k t))}_{\Phi_2(t)}.$$

The first factor $\Phi_1(t)$ stays between the two positive constants C_1, C_2 for all $t \ge 0$. The upper bound $\Phi_1(t) \le 1 =: C_2$ is obvious. For the lower bound, make an exponential change of timescale $t = (x_\theta/p_1)(1/p)^s$, $s \in (-\infty, +\infty)$. The key observation is that for a geometric $\mathcal{P}$, the rescaled function $\Psi(s) = \Phi_1((x_\theta/p_1)(1/p)^s)$ is periodic with period 1 on the positive half-line $s \ge 0$. Since $\Psi(s)$ is left-continuous and nonincreasing on its intervals of continuity, its global minima are attained at the discontinuities $s = 0, 1, 2, \dots$, and, therefore, for any $t \ge 0$,

$$\Phi_1(t) \ge \Psi(0) = \Phi\biggl(\frac{x_\theta}{p_1}\biggr) = \prod_{\{k\colon p_k/p_1\le 1\}}\biggl(1 - f\biggl(\frac{p_k}{p_1}x_\theta\biggr)\biggr) = \prod_{k=0}^{\infty}(1-f(p^k x_\theta)) =: C_1.$$

Thus defined, C_1 is positive, since

$$\sum_{k=0}^{\infty} f(p^k x_\theta) = \sum_{k=0}^{\infty}\frac{1-\mathrm{e}^{-2p^k x_\theta}}{2} \le \sum_{k=0}^{\infty} p^k x_\theta = \frac{x_\theta}{1-p} < \infty.$$

To estimate the second factor $\Phi_2(t)$, introduce $A(t) = \{k\colon p_k t > x_\theta\}$. Then, for any $k \in A(t)$, we have $1 - f(p_k t) \le 1/(2-\theta)$, and, thus,

$$\biggl(\frac{1}{2}\biggr)^{|A(t)|} \le \Phi_2(t) \le \biggl(\frac{1}{2-\theta}\biggr)^{|A(t)|}.$$

Since

$$|A(t)| = \operatorname{card}\biggl\{k\colon p_k > \frac{x_\theta}{t}\biggr\} = \operatorname{card}\biggl\{k\colon k < \frac{\log x_\theta}{\log p} - \frac{\log Ct}{\log p}\biggr\},$$

we obtain

$$\frac{\log t}{\log(1/p)} + C_3 \le |A(t)| \le \frac{\log t}{\log(1/p)} + C_3 + 1,$$

and, hence,

$$C_4\biggl(\frac{1}{2}\biggr)^{\log t/\log(1/p)} < \Phi_2(t) < C_5\biggl(\frac{1}{2-\theta}\biggr)^{\log t/\log(1/p)},$$

$$C_6 t^{-\log 2/\log(1/p)} < \prod_{k=1}^{\infty}(1-f(p_k t)) < C_7 t^{-\log(2-\theta)/\log(1/p)}, \tag{3.2}$$

proving the theorem for a geometric $\{p_k\}$ by choosing a sufficiently small θ.

Consider now the case of a general distribution $\mathcal{P}$. In case (i), for all sufficiently large k, $p_k < C_8 2^{-k} < 2^{C_9-k}$, and since $1 - f(x)$ is nonincreasing in x and $1 - f(x) > \frac{1}{2}$ for $x > 0$,

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we can choose a sufficiently large M so that

$$\begin{aligned}\int_0^\infty \prod_{k=1}^\infty (1 - f(p_k t))\,\mathrm{d}t &\geq C_{10} \int_0^\infty \prod_{k=M}^\infty (1 - f(p_k t))\,\mathrm{d}t \\ &\geq C_{10} \int_0^\infty \prod_{k=M}^\infty (1 - f(2^{C_9 - k} t))\,\mathrm{d}t \\ &= C_{10} \int_0^\infty \prod_{k=1}^\infty (1 - f(2^{-k} 2^{C_9 + M - 1} t)) \frac{\mathrm{d}(2^{C_9+M-1} t)}{2^{C_9+M-1}} \\ &= C_{11} \int_0^\infty \prod_{k=1}^\infty (1 - f(2^{-k} t))\,\mathrm{d}t.\end{aligned}$$

Similarly, in case (ii), for all sufficiently large k, $p_k > C_{12}(2-\varepsilon)^{-k} > (2-\varepsilon)^{C_{13}-k}$, and $1 - f(x) \leq 1,\ x > 0$, yielding

$$\int_0^\infty \prod_{k=1}^\infty (1 - f(p_k t))\,\mathrm{d}t \leq C_{14} \int_0^\infty \prod_{k=1}^\infty (1 - f((2-\varepsilon)^{-k} t))\,\mathrm{d}t,$$

and both theorem statements follow from (3.2). □

Proof of Theorem 2.2. The Markov chain $\{\zeta_n\}_{n\geq 0}$ as defined by (1.1) is irreducible, has period 2, and a symmetric transition matrix; hence, the chain $\{\zeta_{2n}\}_{n\geq 0}$ is irreducible, aperiodic, and also has a symmetric (and, thus, doubly stochastic) transition matrix. Therefore, the chain $\{\zeta_{2n}\}_{n\geq 0}$ has a unique (up to a multiplicative constant) nonnegative nondegenerate invariant measure π, uniform on its state space $\{x \in \mathcal{X} \colon x \text{ has an even number of 1s}\}$. The latter, however, is countably infinite; thus, the total mass of π is necessarily infinite and $\{\zeta_{2n}\}_{n\geq 0}$ cannot be positive recurrent, in particular, for $\mathbb{E}\tau_{\mathrm{BF}} = \infty$. □

In order to prove Theorem 2.3, we make use of [1, Theorem 1 and Corollary 1].

Theorem 3.1. ([1, Theorem 1].) *Suppose that $\{Y_n\}_{n\geq 0}$ is an $\{\mathcal{F}_n\}$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$. Introduce $\tau_A = \inf\{n \geq 0 \colon Y_n \leq A\}$. Suppose that there exist positive constants A and ε such that, for every n, Y_n^{2r} is integrable and*

$$Y_n^{2-2r}\mathbb{E}[Y_{n+1}^{2r} - Y_n^{2r} \mid \mathcal{F}_n] \leq -\varepsilon \quad \text{on } \{\tau_A \geq n\}.$$

Then for any r^* satisfying $0 < r^* < r$ there exists a constant $c = c(\varepsilon, r^*, r)$ such that, for any $x \geq 0$, $\mathbb{E}\tau_A^{r^*} \leq cx^{2r}$ whenever $Y_0 = x$ almost surely.

Theorem 3.2. ([1, Corollary 1].) *Let $\{Y_n\}_{n\geq 0}$, τ_A be as in Theorem 3.1. Suppose that there exist positive constants A, ε, and J such that, for any n,*

$$\mathbb{E}[Y_{n+1}^2 - Y_n^2 \mid \mathcal{F}_n] \geq -\varepsilon \quad \text{on } \{\tau_A > n\},$$

and, for some $\rho > 1$,

$$Y_n^{2-2\rho}\mathbb{E}[Y_{n+1}^{2\rho} - Y_n^{2\rho} \mid \mathcal{F}_n] \leq J \quad \text{on } \{\tau_A > n\}.$$

Suppose also that $Y_0 = x > A$ and for some positive r_0 the process $\{Y_{n\wedge\tau_A}^{2r_0}\}_{n\geq 0}$ is a submartingale. Then, for any $r > r_0$, $\mathbb{E}\tau_A^r = \infty$.

We will also need the following technical lemma.

Lemma 3.1. *Let $\{\zeta_n\}_{n\geq 0}$ be a discrete-time BF model starting from the ground state $\zeta_0 = \mathbf{0}$ with the parameter distribution $\mathcal{P} = \{p_1, p_2, \dots\}$ possibly with a finite support $p_1 \geq p_2 \geq p_3 \geq \cdots \geq 0$. Then for $K = \min\{k\colon \sum_{i=k}^{\infty} p_i \leq \frac{1}{2}\}$ and any $n = 1, 2, \dots$, the vector $(\zeta_n^K, \zeta_n^{K+1}, \dots)$ is stochastically dominated by the vector $(\check{\zeta}^K, \check{\zeta}^{K+1}, \dots)$ of i.i.d.* $\mathrm{Bern}(\frac{1}{2})$ *random variables.*

Proof. Assume that $\sum_{k=K}^{\infty} p_k > 0$, otherwise the lemma statement is trivial. Let $\{\zeta_n\}_{n\geq 0}$ and $\{\check{\zeta}_n\}_{n\geq 0}$ be two discrete-time BF models with the same transition probabilities where the first one starts from the ground state $\zeta_0 = \mathbf{0}$ and the second one starts from stationarity: $\check{\zeta}_0 = (\check{\zeta}_0^1, \check{\zeta}_0^2, \dots)$ is an infinite vector of i.i.d. $\mathrm{Bern}(\frac{1}{2})$ random variables. Our goal is to couple the Markov chains $\{\zeta_n\}$ and $\{\check{\zeta}_n\}$ on $\{0,1\}^{\mathbb{N}}$ preserving the almost sure coordinatewise domination $\zeta_n^k \leq \check{\zeta}_n^k$ for $k = K, K+1, \dots$, and all $n = 0, 1, 2, \dots$.

The idea is to treat the first $K-1$ bits of both Markov chains as a 'buffer' for which the domination does not generally hold. This is an expense to pay for the domination for all the large coordinates. On every step, if one of the chains is flipped at some coordinate $k \geq K$, where the chains agree, the other one does the same. If, otherwise, they disagree at such k, then the other one is flipped at one of the coordinates of the buffer, thus removing the discrepancy at k. As a result, no new discrepancies are created for $k \geq K$ and the coordinatewise domination is preserved almost surely outside of the buffer.

Specifically, we define the joint transition dynamics for $\{\zeta_n\}$ and $\{\check{\zeta}_n\}$ inductively, for $n = 0, 1, 2, \dots$. Denote by D_n the (random) set of discrepancies at time n, i.e. the set of indices $k \geq K$ at which ζ_n, $\check{\zeta}_n$ disagree. The induction assumption is that the coordinatewise domination is preserved on step n: $\zeta_n^k \leq \check{\zeta}_n^k$ for all $k \geq K$ and, hence, only discrepancies of the form $\zeta_n^k = 0$, $\check{\zeta}_n^k = 1$ are possible, i.e.

$$D_n = \{k \geq K : \zeta_n^k = 0, \check{\zeta}_n^k = 1\}.$$

Let $F^{-1}(u)\colon (0,1) \to \mathbb{N}$ be the quantile function for the distribution $\mathcal{P}$, i.e.

$$F^{-1}(u) = \min\bigg\{k\colon \sum_{i=1}^{k} p_i > u\bigg\}, \qquad u \in (0,1).$$

The key element of the coupling is a map $s_n(u)\colon (0,1) \to (0,1)$ which swaps the parts of $(0,1)$ mapped by F^{-1} to D_n with the parts of $(0,1)$ of the same length, mapped to the buffer, i.e.

$$s_n(u) = \begin{cases} 1-u & \text{if } F^{-1}(u) \in D_n \quad \text{or} \quad F^{-1}(1-u) \in D_n, \\ u & \text{otherwise.} \end{cases}$$

The condition $\sum_{i=K}^{\infty} p_i \leq \frac{1}{2}$ ensures that there is always enough buffer space for such a swap.

We now introduce a common source of randomness for the chains: the sequence $U_1, U_2, \dots$ of i.i.d. random variables distributed uniformly on the interval $(0,1)$. The indices of the bits to flip on step $n+1$ in ζ_n and $\check{\zeta}_n$, $n = 0, 1, \dots$, are defined, respectively, as

$$\chi_{n+1} = F^{-1}(U_{n+1}), \qquad \check{\chi}_{n+1} = F^{-1}(s_n(U_{n+1})).$$

Since $s_n(u)$ preserves the Lebesgue measure, $s_n(U_{n+1})$ is also uniformly distributed implying that both chains have correct transition probabilities: $\mathbb{P}(\chi_{n+1} = k) = \mathbb{P}(\check{\chi}_{n+1} = k) = p_k$, $k = 1, 2, \dots$, $n = 0, 1, 2, \dots$. The coordinatewise domination $\zeta_n^k < \check{\zeta}_n^k$ obviously holds for all k for $n = 0$, and on each step $n = 1, 2, \dots$ it is preserved for $k \geq K$ by the construction of s_n, χ_n, $\check{\chi}_n$. $\square$

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Proof of Theorem 2.3. (i) Select an arbitrary $y \in ((1/p)^r, 1/2p)$, which is always possible to do, because $p < \frac{1}{2}$ and $r < 1 - \log 2/\log(1/p)$ given the assumptions. Denote by M_n the index of the rightmost active bit at time n, i.e. $M_n = \max\{k \colon \zeta_n^k = 1\}$ with the convention $M_n = 0$ for $\zeta_n = \mathbf{0}$. In the formulation of Theorem 3.1, put $Y_n^{2r} = y^{M_n}$. Define the filtration $\mathcal{F}_n = \sigma(\boldsymbol{\zeta}_0, M_1, \dots, M_n)$. The process $\{Y_n\}$ is obviously $\{\mathcal{F}_n\}$-adapted. Recall that χ_k is an index of a bit flipped on step k, $\chi_k \sim \mathcal{P}$. We have

$$\mathbb{E}(Y_n^{2r}) = \mathbb{E}(y^{M_n}) \le \mathbb{E}(y^{\sum_{k=1}^n \chi_k}) = (\mathbb{E}(y^{\chi_1}))^n. \tag{3.3}$$

The inequality above follows, since $M_n \le \max\{\chi_1, \chi_2, \dots, \chi_n\} \le \sum_{k=1}^n \chi_k$, so that the right-hand side of (3.3) is finite since $py < p(1/2p) = \frac{1}{2} < 1$. Next,

$$\begin{aligned}\mathbb{E}[Y_{n+1}^{2r} - Y_n^{2r} \mid M_n = m] \\ = \underbrace{\mathbb{E}[(Y_{n+1}^{2r} - Y_n^{2r})\,\mathbf{1}_{\{\chi_{n+1}=m\}} \mid M_n = m]}_{E_1} + \underbrace{\mathbb{E}[(Y_{n+1}^{2r} - Y_n^{2r})\,\mathbf{1}_{\{\chi_{n+1}>m\}} \mid M_n = m]}_{E_2}.\end{aligned}$$

Introduce $\psi(x_K, \dots, x_{m-1}) = y^{\max\{j \colon x_j = 1,\, j=K,\dots,m-1\}} - y^m$. Then

$$E_1 \le \mathbb{E}[\psi(\zeta_n^K, \dots, \zeta_n^{m-1})\,\mathbf{1}_{\{\chi_{n+1}=m\}} \mid M_n = m] = p_m \mathbb{E}[\psi(\zeta_n^K, \dots, \zeta_n^{m-1}) \mid M_n = m]. \tag{3.4}$$

Our claim is that the vector $(\zeta_n^K, \dots, \zeta_n^{m-1})$ conditionally on $\{M_n = m\}$ is stochastically dominated by a vector of i.i.d. Bernoulli random variables $(\check{\zeta}^K, \dots, \check{\zeta}^{m-1})$. Introduce an embedded Markov chain $\{\widetilde{\zeta}_l\}_{l\ge 0} = \{(\widetilde{\zeta}_l^1, \dots, \widetilde{\zeta}_l^{m-1})\}_{l\ge 0}$ tracking the state of the first $m-1$ coordinates of $\{\zeta_n\}$ considered at the times when one of those coordinates changes. We set $\widetilde{\zeta}_0 = (\zeta_0^1, \dots, \zeta_0^{m-1})$ and define $\widetilde{\zeta}_l = (\zeta_{t_l(m)}^1, \dots, \zeta_{t_l(m)}^{m-1})$, where $t_l(m)$ is the lth time when one of the first $m-1$ coordinates of ζ_n is flipped. Lemma 3.1 applied to the BF model $\{\widetilde{\zeta}_l\}_{l\ge 0}$ with the flipping probabilities

$$\widetilde{\mathcal{P}} = \left\{\frac{p_1}{S_{m-1}}, \dots, \frac{p_{m-1}}{S_{m-1}}, 0, 0, \dots\right\}, \qquad S_{m-1} = \sum_{k=1}^{m-1} p_k,$$

implies for every $l = 0, 1, 2, \dots$ the stochastic domination

$$(\widetilde{\zeta}_l^{\tilde{K}}, \dots, \widetilde{\zeta}_l^{m-1}) \le_{\mathrm{st}} (\check{\zeta}^{\tilde{K}}, \dots, \check{\zeta}^{m-1}),$$

where $\check{\zeta}^{\tilde{K}}, \dots, \check{\zeta}^{m-1}$ are i.i.d. Bern$(\frac{1}{2})$ random variables. Note that

$$\tilde{K} = \min\left\{k \colon \sum_{i=k}^{\infty} p_i \le \frac{S_{m-1}}{2}\right\} \le K = \min\left\{k \colon \sum_{i=k}^{\infty} p_i \le \frac{1}{2}\right\},$$

therefore, for every $l = 0, 1, 2, \dots,$

$$(\widetilde{\zeta}_l^K, \dots, \widetilde{\zeta}_l^{m-1}) \le_{\mathrm{st}} (\check{\zeta}^K, \dots, \check{\zeta}^{m-1}). \tag{3.5}$$

Introduce the series of events: $A(n, m, l) = \{\sum_{k=1}^n \mathbf{1}_{\{1\le \chi_k \le m-1\}} = l\}$ for $n = 0, 1 \dots$ and $l = 0, \dots, n$. Conditionally on $A(n, m, l)$, the first $m-1$ coordinates of vector ζ are flipped l times

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and, hence, the distribution of $(\zeta_n^1, \dots, \zeta_n^{m-1})$ is the same as that of $(\widetilde{\zeta}_l^1, \dots, \widetilde{\zeta}_l^{m-1})$, so we can continue (3.4) with

$$\begin{aligned}
E_1 &\le p_m \sum_{l=0}^{n} \mathbb{E}[\psi(\zeta_n^K, \dots, \zeta_n^{m-1})\, \mathbf{1}_{A(n,m,l)} \mid M_n = m] \\
&= p_m \sum_{l=0}^{n} \mathbb{E}[\psi(\zeta_n^K, \dots, \zeta_n^{m-1})\, \mathbf{1}_{\{M_n=m\}} \mid A(n,m,l)] \frac{\mathbb{P}(A(n,m,l))}{\mathbb{P}(M_n = m)}.
\end{aligned}$$

Conditionally on $A(n, m, l)$, the random variables

$$\psi(\zeta_n^K, \dots, \zeta_n^{m-1}) = \psi(\widetilde{\zeta}_l^K, \dots, \widetilde{\zeta}_l^{m-1}) \quad \text{and} \quad \mathbf{1}_{\{M_n=m\}}$$

are independent. Indeed, on $A(n, m, l)$, the first variable is a function of the chain $\widetilde{\zeta}$ after l steps which is governed by transition probabilities $\widetilde{\mathcal{P}}$; while the event $\{M_n = m\}$ relates to the configuration of the bits $m, m + 1, \dots$ after $n - l$ steps of the BF model with parameter distribution $\{p_k/(1 - S_{m-1}),\ k = m, m + 1, \dots\}$. Thus,

$$\begin{aligned}
E_1 &\le p_m \sum_{l=0}^{n} \mathbb{E}[\psi(\zeta_n^K, \dots, \zeta_n^{m-1}) \mid A(n,m,l)]\mathbb{P}(M_n = m \mid A(n,m,l)) \frac{\mathbb{P}(A(n,m,l))}{\mathbb{P}(M_n = m)} \\
&= p_m \sum_{l=0}^{n} \mathbb{E}[\psi(\widetilde{\zeta}_l^K, \dots, \widetilde{\zeta}_l^{m-1})]\mathbb{P}(A(n,m,l) \mid M_n = m).
\end{aligned}$$

The function ψ is nondecreasing with respect to the coordinatewise order on its argument, so the stochastic domination (3.5) implies that

$$\begin{aligned}
E_1 &\le p_m \mathbb{E}\psi(\check{\zeta}^K, \dots, \check{\zeta}^{m-1}) \underbrace{\sum_{l=0}^{n} \mathbb{P}(A(n,m,l) \mid M_n = m)}_{=1} \\
&= p_m \mathbb{E}\psi(\check{\zeta}^K, \dots, \check{\zeta}^{m-1}) \\
&= \sum_{k=K}^{m-1} (y^k - y^m) p_m \left(\frac{1}{2}\right)^{m-k}.
\end{aligned}$$

Because of the assumption (i) for an arbitrary small $\varepsilon > 0$ we can, if necessary, increase K so that $p_k \ge C_1(1 - \varepsilon)p^k$ for any $k \ge K$, and continue, thus,

$$\begin{aligned}
E_1 &\le C_1(1-\varepsilon)(py)^m \sum_{k=K}^{m-1} ((2y)^{-m+k} - 2^{-m+k}) \\
&= C_1(1-\varepsilon)(py)^m \left(\frac{2-2y}{2y-1} - \frac{(2y)^{-m+K}}{2y-1} + 2^{-m+K}\right) \\
&\le C_1(1-\varepsilon)(py)^m \left(\frac{2-2y}{2y-1} + 2^{-m+K}\right). \qquad (3.6)
\end{aligned}$$

Before choosing a particular value of ε, we make the following three observations.

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First, because of the condition $py < \frac{1}{2}$, and the asymptotic equivalence $p_k \sim C_1 p^k$, $k \to \infty$, for an arbitrary small $\varepsilon > 0$, we can choose a large $M = M(\varepsilon)$ so that, for $m \geq M$,

$$\begin{aligned} E_2 &= \sum_{k=1}^{\infty} p_{m+k}(y^{m+k} - y^m) \\ &\leq C_1(1+\varepsilon)(py)^m \bigg(\sum_{k=1}^{\infty} (py)^k - \sum_{k=1}^{\infty} p^k \bigg) \\ &= C_1(1+\varepsilon)(py)^m \bigg(\frac{py}{1-py} - \frac{p}{1-p} \bigg). \end{aligned} \tag{3.7}$$

Second, introduce

$$Q(p, y, \varepsilon) = (1-\varepsilon)\frac{2-2y}{2y-1} + (1+\varepsilon)\bigg(\frac{py}{1-py} - \frac{p}{1-p} \bigg).$$

Given the appropriate choice of ε, M, K, the definition $Y_n^{2r} = y^{M_n}$, together with (3.6) and (3.7), implies that

$$Y_n^{2-2r}\mathbb{E}[Y_{n+1}^{2r} - Y_n^{2r} \mid M_n = m] \leq C_1(Q(p, y, \varepsilon) + 2^{-m+K})(py^{1/r})^m \quad \text{for } m > M. \tag{3.8}$$

Because of our choice of y, we have $py^{1/r} > 1$ and, therefore, given $Q(p, y, \varepsilon) < 0$ we can further increase M so that the right-hand side of (3.8) is negative and uniformly separated from 0 for all $m > M$.

Third, to find ε satisfying $Q(p, y, \varepsilon) < 0$, note that for any fixed p_0, y_0, $Q(p_0, y_0, \varepsilon)$ is a continuous function of ε in a small neighbourhood of $\varepsilon = 0$, whenever $Q(p_0, y_0, 0)$ is well defined and nonzero. The inequality $Q(p, y, 0) < 0$ can be written as

$$\frac{(2py-1)(y-1)(2-p)}{(1-py)(1-p)(2y-1)} > 0,$$

which is satisfied due to our choice of y.

Now, we can choose a small $\varepsilon > 0$ so that $Q(p, y, \varepsilon) < 0$, then fix large K, M so that (3.6) and (3.7) are satisfied, and so the right-hand side of (3.8) is negative, uniformly separated from 0, for all $m > M$, as required by Theorem 3.1.

Denote $\tau_x = \inf\{n \geq 1 \colon M_n \leq x\}$. Theorem 3.1 implies that for $p < \frac{1}{2}$ and $r < 1 - \log 2/\log(1/p)$ there exists $C = C(p, r)$ such that for our particular choice of y and M, we have

$$\mathbb{E}[\tau_M^r \mid M_0 = x] \leq Cy^x \leq C\bigg(\frac{1}{2p}\bigg)^x. \tag{3.9}$$

We now prove that $\tau_{\mathrm{BF}}^r = \tau_0^r$ is integrable and satisfies the same asymptotic bound. In $\mathbb{E}[\tau_0^r \mid M_0 = x]$, τ_0 is the first time when the process M_n reaches 0 starting from the state x. For any $M \geq 0$, we have $\tau_0 = \tau_M + (\tau_M - \tau_0)$. By simple coupling arguments, the law of $(\tau_M - \tau_0)$, conditional on $\{M_0 = x\}$, is stochastically dominated by the law of τ_0, conditional on $\{M_0 = M\}$. That, together with the inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for $0 < r < 1$ and nonnegative a, b, implies the bound

$$\mathbb{E}[\tau_0^r \mid M_0 = x] \leq 2^r(\mathbb{E}[\tau_M^r \mid M_0 = x] + \mathbb{E}[\tau_0^r \mid M_0 = M]\,).$$

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An asymptotic upper bound for the first conditional expectation on the right-hand side is given by (3.9). It is left to the reader to derive an upper bound for the second expectation. Conditionally on $\{M_0 = M\}$, τ_0 is stochastically dominated from above by the sum of two terms. The first one is the time needed for $\boldsymbol{\zeta}_n$ to reach $\mathbf{0}$ not leaving the finite sub-cube $\{0, 1\}^M$, which is, in turn, dominated by $\tau_0^{\wedge M} = \inf\{n\colon \zeta_n^{\wedge M} = \mathbf{0}^{\wedge M}\}$. The second one is a geometrically distributed number of excursions $\gamma \sim \text{geom}(\pi)$ from $\{0, 1\}^M$. Thus,

$$\mathbb{E}[\tau_0^r \mid M_0 = M] \le \sum_{k=1}^{\infty} \mathbb{E}[\tau_0^r \mid M_0 = M,\ \gamma = k]\mathbb{P}\{\gamma = k\}.$$

Now, conditionally on $\{\gamma = k\}$,

$$\begin{aligned}\mathbb{E}[\tau_0^r \mid M_0 = M,\ \gamma = k] &\le \mathbb{E}\bigg[\bigg(\tau_0^{\wedge M} + \sum_{j=1}^{k} \psi_j\bigg)^r \ \bigg|\ M_0 = M\bigg] \\ &\le k^{1+r}(\mathbb{E}[(\tau_0^{\wedge M})^r \mid M_0 = M] + \mathbb{E}\psi^r),\end{aligned} \tag{3.10}$$

where ψ_j is the length of excursion $j = 1, \dots, \gamma$ and ψ denotes the length of a typical excursion. The first expectation on the right-hand side of (3.10) is a finite constant. As for the second, for some finite constant $C_4 > 0$, we have

$$\mathbb{E}\psi^r = 1 + \sum_{k=1}^{\infty} p_{k+M}\mathbb{E}[\tau_M^r \mid M_0 = k + M] \le 1 + \sum_{k=1}^{\infty} C_4 p^{k+M}\bigg(\frac{1}{2p}\bigg)^{k+M} < \infty.$$

Thus, for some $C_5 > 0$, $\mathbb{E}[\tau_0^r \mid M_0 = M] \le \sum_{k=1}^{\infty} C_5 k^{1+r}\pi(1-\pi)^{k-1} < \infty$,completing the proof of (i).

(ii) Put $Y_n^2 = y^{M_n}$ for some $y > 1$ and verify the conditions of Theorem 3.2. As before, Y_n is adapted and for an arbitrary small $\varepsilon > 0$, we can choose $M = M(\varepsilon)$ large enough so that

$$\begin{aligned}\mathbb{E}[Y_{n+1}^2 - Y_n^2 \mid M_n = m] &\ge -p_m y^m + \sum_{k=1}^{\infty} p_{m+k}(y^{m+k} - y^m) \\ &\ge -C_1(1-\varepsilon)p^m y^m + \sum_{k=1}^{\infty} C_1(1+\varepsilon)p^{m+k}(y^{m+k} - y^m) \\ &= C_1(py)^m\bigg(-1 + \varepsilon + (1+\varepsilon)\sum_{k=1}^{\infty} p^k(y^k - 1)\bigg) \\ &= C_1(py)^m\bigg(-1 + \varepsilon + \frac{(1+\varepsilon)p(-1+y)}{(1-p)(1-py)}\bigg) \\ &= C_1(py)^m R(p, y, \varepsilon),\end{aligned}$$

where $R(p, y, \varepsilon) = (-1 + \varepsilon + (1+\varepsilon)p(-1+y)/(1-p)(1-py))$. It is then possible to choose a small enough $\varepsilon > 0$ and a large M so that the latter expression is bounded from below for all $m > M$, when $py < 1$. Furthermore, for such p, y, we have, as before,

$$Y_n^{2-2\rho}\mathbb{E}[Y_{n+1}^{2\rho} - Y_n^{2\rho} \mid M_n = m] \le C_1 y^{m(1-\rho)}(py^\rho)^m(Q(p, y^\rho, \varepsilon) + (1-\varepsilon)2^{-m+K}),$$

which is bounded from above when ρ is such that $py^\rho < 1$ (such a $\rho > 1$ exists whenever $py < 1$).

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Finally, find a suitable value r_0 such that the process $Y^{2r_0}_{n\wedge\tau_M}$ is a submartingale. Since

$$\mathbb{E}[Y^{2r_0}_{n+1} - Y^{2r_0}_n \mid M_n = m] \geq C_1(py^{r_0})^m R(p, y^{r_0}, \varepsilon),$$

we can choose $\varepsilon > 0$ so that the latter is greater than 0 for any $m > M$, if

$$r_0 \in \left(\frac{\log(1/(2p-p^2))}{\log y}, 1\right).$$

Recalling that we can take y arbitrary close to $1/p$, we conclude that the conditions of Theorem 3.2 are satisfied for any r_0 such that

$$r_0 \in \left(1 - \frac{\log(2-p)}{\log(1/p)}, 1\right).$$

This together with the results of Theorem 2.2 implies that none of the fractional moments of τ_M (and, hence, of τ_0) of order higher than $1 - \log(2-p)/\log(1/p)$ exist, completing the proof of (ii). □

3.2. Transience and recurrence of the DB model

Proof of Theorem 2.4. (i) We first consider the discrete-time version of the DB model. Introduce R_n, the index of the rightmost bit (i.e. with the largest index) to ever have been flipped by time n. The sequence $\{R_n\}$ is almost surely nondecreasing. We aim to prove that almost surely for infinitely many terms of the sequence $\{R_n\}$, each of the bits $1, 2, \dots, R_n$ is flipped at least twice before the next flip of some bit with an index larger than R_n. That would guarantee that the ground state of the DB model, corresponding to the set of states $\{y \in \{0,1,2\}^{\mathbb{N}} \colon y$ has no 1s and only a finite number of 2s$\}$ of Markov chain $\{\zeta_n\}$, is visited infinitely often.

It is convenient to use the continuous-time version now. Let $\Pi_1(t), \Pi_2(t), \dots$ the independent Poisson processes (clocks) describing the times at which, respectively, the first, the second, etc. bits are chosen for flipping. Introduce

$$\tau_{>k} = \inf\left\{t > 0 \colon \sum_{j=k+1}^{\infty} \Pi_j(t) > 0\right\},$$

the time of the first flip of a bit with an index greater than k. Note that $\tau_{>k}$ is a stopping time for each $k = 0, 1, 2, \dots$, and, moreover, $\tau_{>0} \leq \tau_{>1} \leq \tau_{>2} \leq \cdots$. Introduce the events

$$\begin{aligned} A_k &= \{k \text{ appears in the sequence } \{R_n\}\}, \\ B_k &= A_k \cap \{\text{starting from the first tick of } \Pi_k, \text{ each of the clocks } \Pi_1, \Pi_2, \dots, \Pi_k \\ &\qquad\qquad \text{ticks at least twice before the first tick of one of the clocks } \Pi_{k+1}, \Pi_{k+2}, \dots\}. \end{aligned}$$

Our aim is to prove that the events B_k happen infinitely often. In terms of a continuous-time notation, we can write these as

$$\begin{aligned} A_k &= \{\tau_{>k-1} < \tau_{>k}\}, \\ B_k &= \bigcap_{j\leq k} \{\Pi_j([\tau_{>k-1}, \tau_{>k})) \geq 2\}. \end{aligned} \tag{3.11}$$

Since $\{\tau_{>k}\}$ is a sequence of stopping times, it is not difficult now to see that the events B_k are independent of each other. By the Borel–Cantelli lemma it suffices to prove that the series $\sum_{k\geq 1}\mathbb{P}\{B_k\}$ diverges.

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The probability of A_k (probability of an index k to ever appear in the sequence $\{R_n\}$) is $p_k/(p_k+Q_k) = 1 - Q_k/Q_{k-1}$, which is uniformly bounded away from 0 given assumption (i).

As follows from (3.2), the probability $\mathbb{P}(B_k \mid A_k)$ is equal to the probability for each of the first k Poisson clocks $\Pi_1(t), \dots, \Pi_k(t)$ to tick at least twice before the time of the first tick of one of the clocks $\Pi_{k+1}(t), \Pi_{k+2}(t), \dots$. We have

$$\mathbb{P}(B_k \mid A_k) = \mathbb{P}\biggl(\bigcap_{j=1}^{k}\{\Pi_j(\tau_{>k}) \ge 2\}\biggr) = \int_0^\infty \prod_{j=1}^{k} \mathbb{P}\{\Pi_j(t) \ge 2\}\, \mathrm{d}\mathbb{P}(\tau_{>k} \le t). \tag{3.12}$$

Introduce $g(x) = \mathrm{e}^{-x}(1+x)$. Due to (i), there exists a large K such that

$$\frac{p_j}{Q_k} = \frac{p_j}{Q_{j-1}} \frac{Q_{j-1}}{Q_j} \cdots \frac{Q_{k-1}}{Q_k} \ge \biggl(\frac{1}{p} - 1\biggr) \underbrace{\frac{1}{p} \cdots \frac{1}{p}}_{k-j+1} \ge C_2 p^{j-k} \quad \text{for any } k \ge j \ge K.$$

The function $g(x)$ is strictly decreasing in x, so we can continue and write that (3.12) is equal to

$$\begin{aligned}\int_0^\infty \prod_{j=1}^{k} (1 - g(p_j t)) Q_k \mathrm{e}^{-Q_k t}\, \mathrm{d}t &= \int_0^\infty \prod_{j=1}^{k} \biggl(1 - g\biggl(\frac{p_j}{Q_k} t\biggr)\biggr) \mathrm{e}^{-t}\, \mathrm{d}t \\ &\ge C_1 \int_0^\infty \prod_{j=1}^{k-K} (1 - g(C_2 p^{-j} t)) \mathrm{e}^{-t}\, \mathrm{d}t \quad \text{for } k \ge K,\end{aligned}$$

where C_1 and C_2 are positive constants. Next,

$$\prod_{j=1}^{k-K} (1 - g(C_2 p^{-j} t)) \ge \prod_{j=1}^{\infty} (1 - g(C_2 p^{-j} t)).$$

The latter is strictly positive, i.e.

$$\sum_{j=1}^{\infty} g(C_2 t p^{-j}) = \sum_{j=1}^{\infty} \mathrm{e}^{-C_2 t p^{-j}} (1 + C_2 t p^{-j}) \le C_3 \sum_{j=1}^{\infty} \mathrm{e}^{-C_4 t p^{-j}} < \infty \quad \text{for all } t;$$

thus, $\prod_{j=1}^{k-K}(1 - g(C_2 p^{-k} t))$ is bounded away from 0 uniformly in k, $k \ge K$, by

$$h(t) = \prod_{j=1}^{\infty} (1 - g(C_2 p^{-j} t)) > 0, \quad \text{and} \quad \mathbb{P}(B_k \mid A_k) \ge C_1 \int_0^\infty h(t) \mathrm{e}^{-t}\, \mathrm{d}t > 0,$$

so the series $\sum_{k=1}^{\infty} \mathbb{P}(B_k)$ diverges and the DB model is recurrent given assumption (i).

(ii) Now, assume that $p_k \sim C\mathrm{e}^{-\alpha k^\gamma}$. Consider the total time ν spent in the ground state, when none of the bits are active. We are going to prove for this particular choice of p_k that the expected time spent in the ground state $\mathbb{E}\nu = \int_0^\infty \prod_{k=1}^{\infty} (1 - p_k t \mathrm{e}^{-p_k t})\, \mathrm{d}t$ is finite. The product under the integral is bounded by

$$\prod_{k=1}^{\infty} (1 - p_k t \mathrm{e}^{-p_k t}) \le \exp\biggl\{\mathrm{card}\{k \colon l_{1,\varepsilon} \le p_k t \le l_{2,\varepsilon}\} \log\biggl(1 - \frac{1}{\mathrm{e}} + \varepsilon\biggr)\biggr\}.$$

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Here, $l_{1,\varepsilon}$ and $l_{2,\varepsilon}$ are the left and the right boundaries of the interval, where the function $x\mathrm{e}^{-x}$ is greater than or equal to $1/\mathrm{e} - \varepsilon$. Taking into account the particular choice of p_k, we write

$$\begin{aligned}\operatorname{card}\{k\colon l_{1,\varepsilon} \le p_k t \le l_{2,\varepsilon}\} &\sim \bigg(\frac{1}{\alpha}\log\frac{tC}{l_{1,\varepsilon}}\bigg)^{1/\gamma} - \bigg(\frac{1}{\alpha}\log\frac{tC}{l_{2,\varepsilon}}\bigg)^{1/\gamma} \\ &\sim \frac{\log l_{2,\varepsilon} - \log l_{1,\varepsilon}}{\gamma\alpha^{1/\gamma-1}}(\log(tC))^{1/\gamma-1}; \end{aligned} \tag{3.13}$$

hence, the infinite product in question is integrable for $\gamma < \frac{1}{2}$. □

Remark 3.1. The condition of Theorem 2.4(i),

$$\limsup_{k\to\infty}\frac{Q_{k+1}}{Q_k} = p < 1, \tag{3.14}$$

is stronger than a condition in the style of Theorem 2.1(i), i.e.

$$\limsup_{k\to\infty}\beta^k p_k < \infty \quad \text{for some constant } \beta > 1. \tag{3.15}$$

It is not difficult to see that (3.14) implies (3.15) with

$$\beta = \frac{1}{p+\varepsilon} \quad \text{for any } \varepsilon \in (0, 1-p).$$

The converse implication does not hold in general: for a counterexample, define $\kappa(k) = \min\{j^2\colon j \in \mathbb{N} \text{ and } j^2 > k\}$ and put $p_k = C2^{-\kappa(k)}$, $k = 1, 2, \ldots$, where C is a normalizing constant. Then (3.15) holds with $\beta = 2$. However, (3.14) does not hold: setting $k_i = i^2$, $i = 1, 2, \ldots$, we obtain, for the subsequence $\{k_i\}$,

$$\begin{aligned}\frac{Q_{k_i}}{Q_{k_i-1}} &= 1 - \frac{p_{k_i}}{Q_{k_i-1}} \\ &= 1 - \frac{p_{i^2}}{\sum_{j=k_i}^{\infty} p_j} \\ &\ge 1 - \frac{p_{i^2}}{\sum_{j=i^2}^{(i+1)^2-1} p_j} \\ &= 1 - \frac{C2^{-(i+1)}}{((i+1)^2 - i^2)C2^{-(i+1)}} \\ &\ge 1 - \frac{1}{2i+1} \\ &\to 1, \qquad i\to\infty. \end{aligned}$$

For (3.15) to imply (3.14) we need additional constraints on $\mathcal{P}$; for example, it is enough to require the sequence $\{Q_k/Q_{k+1}\}$ to be monotone.

3.3. The central limit theorem

For the proof of the central limit theorem for the number of active bits in the BF and DB models we use the following general central limit theorem for a triangular array.

Theorem 3.3. ([5, Chapter 8, Theorem 5].) *Let* $\{\xi_{k,n}\}$, $1 \le k \le r_n$, $1 \le n \le \infty$, *be a triangular array of random variables such that* $\mathbb{E}\xi_{k,n} = 0$ *and that the random variables* $(\xi_{k,n})_{1\le k\le r_n}$ *are mutually independent inside of every row* $n = 1, 2, \dots$. *Assume that*

(i) $\sum_{k=1}^{r_n} \mathbb{E}\xi_{k,n}^2 = 1$;

(ii) $\sum_{k=1}^{r_n} \mathbb{E}[\xi_{k,n}^2; |\xi_{k,n}| > M] \to 0$, $n \to \infty$, *for every* $M > 0$.

Then $\sum_{k=1}^{r_n} \xi_{k,n} \xrightarrow{\mathrm{D}} \mathcal{N}(0,1)$ *as* $n \to \infty$.

Proof of Theorem 2.5. The expected number of active bits $\mathbb{E}\eta_t$ in both models tends to ∞, given the assumptions. For the BF model, we have

$$\mathbb{E}\eta_t = \sum_{k=1}^{\infty} \mathbb{P}\{\zeta_t^k = 1\} = \sum_{k=1}^{\infty} \frac{1}{2}(1 - \mathrm{e}^{-2p_k t}).$$

Every term in the latter sum monotonically approaches $\frac{1}{2}$ as $t \to \infty$; thus, the whole sum tends to ∞.

Next, for the DB model, given the assumption $p_k \sim C\exp(-\alpha k^{\gamma})$, $k \to \infty$, if we fix a small $\varepsilon > 0$ and take $l_{1,\varepsilon}$ and $l_{2,\varepsilon}$, which are the left and the right boundaries of the interval where the function $x\mathrm{e}^{-x}$ is greater than $1/\mathrm{e} - \varepsilon$ to be as in (3.13), then, by the same reasoning as in (3.13), we obtain

$$\begin{aligned}\mathbb{E}\eta_t &= \sum_{k=1}^{\infty} \mathbb{P}\{\zeta_t^k = 1\} \\ &= \sum_{k=1}^{\infty} p_k t \mathrm{e}^{-p_k t} \\ &\ge (\mathrm{e}^{-1} - \varepsilon)\,\mathrm{card}\{k : \lambda_{1,\varepsilon} \le p_k t \le \lambda_{2,\varepsilon}\} \\ &\ge C_1(\log(tC))^{1/\gamma - 1} \\ &\to \infty\end{aligned}$$

for a constant C_1 depending on ε, γ, and α.

The rest of the proof works for both the BF and DB models. It is sufficient to prove the central limit theorem for the embedded discrete-time process $\{\eta_{T_n}\}_{n\ge1}$ for an arbitrary nonrandom time sequence $\{T_n\}_{n\ge1}$ going to ∞. Let us fix such a sequence and denote $\zeta_n := \zeta_{T_n}$ and $\eta_n := \eta_{T_n}$, for short. Introduce the random variables

$$Z_{n,k} = \mathbf{1}_{\{\zeta_n^k = 1\}}, \qquad \xi_{n,k} = \begin{cases} \dfrac{Z_{n,k} - \mathbb{E}Z_{n,k}}{\sqrt{\mathrm{var}\,\eta_n}}, & k < r_n, \\[2ex] \dfrac{\sum_{k\ge r_n}(Z_{n,k} - \mathbb{E}Z_{n,k})}{\sqrt{\mathrm{var}\,\eta_n}}, & k = r_n. \end{cases}$$

We choose a suitable sequence $\{r_n\}$ later. The random variables $\{\xi_{n,k}\}_{k=1}^{r_n}$ are mutually independent for every n. Theorem 3.3(i) holds trivially. For (ii), we have

$$\sum_{1\le k\le r_n} \mathbb{E}[\xi_{n,k}^2; |\xi_{n,k}| > M] = \underbrace{\sum_{1\le k\le r_n - 1} \mathbb{E}[\xi_{n,k}^2; |\xi_{n,k}| > M]}_{S_1} + \underbrace{\mathbb{E}[\xi_{n,r_n}^2; |\xi_{r_n,n}| > M]}_{S_2}. \tag{3.16}$$

By the assumptions, $\mathbb{E}\eta_t \to \infty$ as $t \to \infty$. Moreover,

$$C_2\mathbb{E}\eta_t \le \operatorname{var}\eta_t = \sum_{k\ge 1} f(p_k t)(1 - f(p_k t)) \le \mathbb{E}\eta_t,$$

where $f(x) = \frac{1}{2}(1 - \mathrm{e}^{-x})$ in the BF model, $f(x) = x\mathrm{e}^{-x}$ in the DB model, and $C_2 = (1 - \sup_{x\in\mathbb{R}^+} f(x))$, with the respective f, so that $0 < C_2 < 1$ in both cases. By the construction of $\xi_{n,k}$, the sum S_1 in (3.16) tends to 0 as n goes to ∞, because almost surely $\xi_{n,k} \le 1/\operatorname{var}\eta_n \to 0$ and every term in S_1 is eventually 0. Finally,

$$\mathbb{E}\xi^2_{r_n,n} = \frac{1}{\operatorname{var}\eta_n} \sum_{k\ge r_n} f(p_k T_n)(1 - f(p_k T_n))$$

and so we can choose r_n such that the latter sum is no larger than, for instance, $\sqrt{\operatorname{var}\eta_n}$, thus satisfying Theorem 3.3(ii) and completing the proof. □

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