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ON THE REGULAR GRAPH RELATED TO THE G-CONJUGACY CLASSES

ADAM B. R. and Z. AKHLAGHI

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Abstract

Given a finite group G with a normal subgroup N, the simple graph $\Gamma_G(N)$ is a graph whose vertices are of the form $|x^G|$, where $x \in N \setminus Z(G)$ and x^G is the G-conjugacy class of N containing the element x. Two vertices $|x^G|$ and $|y^G|$ are adjacent if they are not coprime. We prove that, if $\Gamma_G(N)$ is a connected incomplete regular graph, then $N = P \times A$ where P is a p-group, for some prime p, $A \leq Z(G)$ and $\mathbf{Z}(N) \neq N \cap \mathbf{Z}(G)$.

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1. Introduction

Given a finite group G, by $\mathrm{cs}(G)$ we mean the set of conjugacy class sizes of the group G. It is well known that strong results can be obtained from $\mathrm{cs}(G)$ about the structure of G (see, for example, [5]).

Certain graphs have been introduced in order to study properties of a given finite group G. We will discuss the graphs which are constructed on the set $\mathrm{cs}(G)$. The common divisor graph on conjugacy class sizes, which we denote by $\Gamma(G)$ (see [2]), is a graph whose vertex set is $\mathrm{cs}(G) \setminus \{1\}$ and the vertices v and w are adjacent if $\gcd(v, w) > 1$. The properties of this graph associated to the structure of the group G have been thoroughly investigated in the last few decades. We refer to [6] for a survey on this topic.

Given a normal subgroup N of the finite group G, the set $\mathrm{cs}_G(N)$ denotes the G-conjugacy class sizes of N. Investigating the structure of N based on $\mathrm{cs}_G(N)$ could potentially extend the results based on $\mathrm{cs}(G)$, and these properties have also been actively studied in recent years. It is natural to define analogous graphs based on $\mathrm{cs}_G(N)$. Denote by $\Gamma(\mathrm{cs}_G(N))$ or $\Gamma_G(N)$ the graph whose vertex set contains elements of the form $|x^G|$, where $x \in N \setminus Z(G)$ and the vertices v and w are adjacent in $\Gamma_G(N)$ if and only if they are adjacent in $\Gamma(G)$ (see [1]).

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In [4] it is proved that *if* $\Gamma(G)$ *is* k*-regular then it must be a complete graph of order* $k+1$ for $k = 2, 3$. The result is extended for any $k \in \mathbb{N}$ in [3].

It seems reasonable to ask whether the same results hold when $\Gamma_G(N)$ is a regular graph. Note that if $\Gamma_G(N)$ is a disconnected graph, then by [1, Theorems A, B, E], $\Gamma_G(N)$ has two complete components and the structure of N in that case is determined. It remains to discuss the connected case. In this paper, we aim to prove the following theorem.

THEOREM 1.1 (Main Theorem). *Let* G *be a finite group and* N *a normal subgroup of* G *such that* $\Gamma_G(N)$ *is a connected incomplete regular graph. Then* $N \cong P \times A$*, where* P *is a* p*-group, for some prime* p*,* $A \leq \mathbf{Z}(G)$ *and* $\mathbf{Z}(N) \neq N \cap \mathbf{Z}(G)$*.*

This result shows that if $\Gamma_G(N)$ is a connected regular graph, then either $\Gamma_G(N)$ is complete as we expected, or $N = P \times A$, where A is a central subgroup of G and P is a p-group. As an application of the main theorem we re-prove the main result in [3].

2. Preliminaries

DEFINITION 2.1. For a given vertex v of the graph Γ, define the neighbourhood of v as the set of vertices adjacent to v, including v itself, and denote it by $\mathcal{N}_\Gamma(v)$.

DEFINITION 2.2. Two distinct vertices v_1, v_2 of the graph Γ are said to be *partners* if

$$\mathcal{N}_\Gamma(v_1) = \mathcal{N}_\Gamma(v_2).$$

Partnership defines an equivalence relation on the set of vertices of the graph.

LEMMA 2.3. *Let* $|x^G|$ *be a vertex of* $\Gamma_G(N)$ *for some* $x \in N$ *and assume* $\Gamma_G(N)$ *is regular. If* $y = x^a$ *is noncentral for some integer* a*, then either* $|y^G| = |x^G|$ *or* $|x^G|$ *and* $|y^G|$ *are partners.*

PROOF. If $|x^G| \neq |y^G|$, then $C_G(x) < C_G(y)$. Therefore $|x^G|$ is divisible by $|y^G|$ and so $\mathcal{N}_\Gamma(|y^G|) \subseteq \mathcal{N}_\Gamma(|x^G|)$. As $|y^G|$ and $|x^G|$ have the same degrees in $\Gamma_G(N)$, we conclude that they are partners. □

We say that G is quasi-Frobenius if $G/\mathbf{Z}(G)$ is Frobenius. The inverse images in G of the kernel and a complement of $G/\mathbf{Z}(G)$ are then called the kernel and a complement of G.

LEMMA 2.4 [2, Theorem 2]. *For a finite group* G*, the divisor graph* $\Gamma(G)$ *has exactly two connected components if and only if* G *is quasi-Frobenius with abelian kernel and complement.*

3. Main results

LEMMA 3.1. *Let* G *be a finite group and let* N *be a normal subgroup of* G *such that* $|N/(N \cap Z(G))|$ *is divisible by two distinct primes* p_1 *and* p_2*. Let* $x_0, y_0 \in N \setminus (N \cap \mathbf{Z}(G))$ *be* p_1*- and* p_2*-elements, respectively, such that* $x_0 \in C_G(y_0)$*. Also,*

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assume that $\Gamma_G(N)$ *is a connected incomplete regular graph. Denote by* v_0, w_0 *and* z_0 *the sizes of the conjugacy classes of* G *associated with* x_0, y_0 *and* x_0y_0*, respectively. Then the following statements hold.*

(a) *There exist* p_1*- and* p_2*-elements* $x_1, y_1 \in N \setminus (N \cap \mathbf{Z}(G))$*, respectively, such that* $v_1 = |x_1^G|, w_1 = |y_1^G| \in \mathcal{N}_{\Gamma_G(N)}(z_0)$*, where* v_1 *and* w_1 *are not adjacent in* $\Gamma_G(N)$*,* $(v_1, p_1p_2) = p_2$ *and* $(w_1, p_1p_2) = p_1$*.*

(b) v_0 *is divisible by* p_2*,* w_0 *is divisible by* p_1 *and* z_0 *is divisible by* p_1p_2*.*

PROOF. By Lemma 2.3, $S = \{v_0, w_0, z_0\}$ is a subset of $cs_G(N)$, with at most three elements, and any two elements in S (in case of existence) are partners in $\Gamma_G(N)$.

Since $\Gamma_G(N)$ is a connected incomplete regular graph, for each vertex v in $\Gamma_G(N)$, there exist two distinct nonadjacent vertices in $\mathcal{N}_{\Gamma_G(N)}(v)$. Therefore noncentral elements x_1 and y_1 in N exist such that $v_1 = |x_1^G|$, $w_1 = |y_1^G|$ with $(v_1, w_1) = 1$, and both are connected to $z_0 = |(x_0y_0)^G|$. By Lemma 2.3, we may assume that $o(x_1)$ and $o(y_1)$ are both powers of primes.

Note that p_1 cannot divide both $|x_1^G|$ and $|y_1^G|$. Assume that $p_1 \nmid |x_1^G|$, so that $C_G(x_1)$ must contain some Sylow p_1-subgroup of G. Without loss of generality, we may assume $C_G(x_1)$ contains x_0 and so $x_1x_0 = x_0x_1$. If $o(x_1)$ is not a power of p_1, then there exist integers a_0 and a_1 such that $x_0 = (x_0x_1)^{a_0}$ and $x_1 = (x_0x_1)^{a_1}$ and, by Lemma 2.3, v_1 and v_0 must be equal or partners. On the other hand, z_0 and v_0 are equal or partners, which means that v_1 and z_0 are partners, implying that v_1 and w_1 must be adjacent, a contradiction. Therefore x_1 must be a p_1-element. Note that if p_2 does not divide $|x_1^G|$, by the same argument x_1 is a p_2-element, which is not possible. Hence p_2 does not divide $|y_1^G|$, and by the same argument as before, $o(y_1)$ must be a power of p_2.

We now prove (b). Assume that $p_2 \nmid v_0$ so that $C_G(x_0)$ contains a Sylow p_2-subgroup of G. Without loss of generality, we may assume $C_G(x_0)$ contains y_1 and so $y_1x_0 = x_0y_1$. Then x_0 and $y_1 \in \langle x_0y_1 \rangle$, and by Lemma 2.3, w_1 and v_0 are partners or equal. Since v_0 and z_0 are partners or equal, it follows that v_1 must be adjacent to w_1, a contradiction. Therefore, v_0 is divisible by p_2. Similarly, w_0 is divisible by p_1. Since v_0 and w_0 divide z_0, it follows that z_0 is divisible by p_1p_2, as required. □

THEOREM 3.2. *Let* G *be a finite group and* N *a normal subgroup of* G *such that* $|N/(N \cap Z(G))|$ *is divisible by two distinct primes* p_1 *and* p_2*. Let* x_0 *by a* p_1*-element and* y_0 *a* p_2*-element such that* $x_0, y_0 \in N \setminus (\mathbf{Z}(G) \cap N)$ *and* $x_0y_0 = y_0x_0$*. If* $\Gamma_G(N)$ *is a connected regular graph, then* $\Gamma_G(N)$ *is complete.*

PROOF. Assume that $\Gamma_G(N)$ is not complete so that Lemma 3.1 applies. Accordingly there exist vertices $z_0 := |(x_0y_0)^G|$, $v_0 := |x_0^G|$ and $w_0 := |y_0^G|$ as described in the statement of Lemma 3.1. Let v_1 and w_1 be vertices as described in the statement of Lemma 3.1(a) and define $A = \mathcal{N}_{\Gamma_G(N)}(v_1) \setminus \{z_0, v_1\}$.

First, we prove that every element in A is divisible by either p_1 or p_2. Suppose not and assume there exists an element $s \in N$ such that $|s^G| \in A$ and $(p_1p_2, |s^G|) = 1$. According to Lemma 2.3, we may assume s is an r-element for some prime r. Without loss of generality, we may assume $r \neq p_1$.

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Since $(p_1p_2, |s^G|) = 1$, we may assume that $x_0s = sx_0$ and hence, by Lemma 2.3, $|(sx_0)^G|$ and $|s^G|$ must be partners or equal. Also v_0 and $|(sx_0)^G|$ are partners or equal, hence v_0 and $|s^G|$ are partners or equal. Replacing x_0 by x_1 and using the same argument, we deduce that v_1 and $|s^G|$ are partners, which implies that v_1 and w_1 are adjacent, as v_0 and z_0 are partners or equal, a contradiction. Accordingly, every vertex in A is divisible by p_1 or p_2.

Observe that $d(v_1) = |A| + 1$. By the regularity of $\Gamma_G(N)$, we have $d(z_0) = |A| + 1$. By Lemma 3.1(b), z_0 is adjacent to all vertices in $\{v_1, w_1\} \cup A$, which implies that $w_1 \in A$, again a contradiction by the definition of A. □

PROOF OF THE MAIN THEOREM. Let $\pi(N/(N \cap \mathbf{Z}(G))) = \{p_1, \ldots, p_n\}$ where the p_i are distinct primes. By Theorem 3.2, there is no element of order divisible by p_ip_j in $N/(N \cap \mathbf{Z}(G))$, for $i \neq j$. So, if $n > 1$, then $\mathbf{Z}(N) \leq \mathbf{Z}(G)$ and, for every p_i-element a_i, $|a_i^G|$ is divisible by $(\prod_{j=1}^n p_j)/p_i$.

If $n > 2$, then $\Gamma_G(N)$ is complete. Therefore we discuss the case where $n \leq 2$. First, assume $n = 2$. From the above discussion, $\mathbf{Z}(N) = N \cap \mathbf{Z}(G)$ and $|N/\mathbf{Z}(N)| = p_1^{n_1}p_2^{n_2}$, for integers n_1 and n_2.

We may assume $|x^G|$ is not divisible by p_1p_2 for every noncentral element $x \in N$, since otherwise $\Gamma_G(N)$ is complete. Therefore, by the argument in the first paragraph of the proof, $|x^N| = p_j^{n_j}$ for every noncentral p_i-element x, where $j \neq i \in \{1, 2\}$. So, $\Gamma(N)$ is a disconnected graph with two vertices of sizes $p_i^{n_i}$, for $i = 1, 2$. From Lemma 2.4, N is a quasi-Frobenius group with abelian kernel and complement. Therefore, the Frobenius complements of $N/\mathbf{Z}(N) = N/(N \cap \mathbf{Z}(G))$ are cyclic p_i-groups, for $i = 1, 2$. Without loss of generality, we assume that the Frobenius complements of $N/(N \cap \mathbf{Z}(G))$ are cyclic p_2-groups. We remark that $S = \{|y^G| : 1 \neq y\mathbf{Z}(N) \in K/\mathbf{Z}(N)\}$ and $T = \{|y^G| : 1 \neq y\mathbf{Z}(N) \in H/\mathbf{Z}(N)\}$ induce a complete subgraph of $\Gamma_G(N)$, where $K/\mathbf{Z}(N)$ and $H/\mathbf{Z}(N)$ are the Frobenius kernel and a Frobenius complement of $G/\mathbf{Z}(G)$. Further, $S \cup T$ is the set of all vertices of $\Gamma_G(N)$.

Let x be a noncentral p_2-element of N, such that $\langle x(N \cap \mathbf{Z}(G))\rangle$ is a Sylow p_2-subgroup of $N/(N \cap \mathbf{Z}(G))$. The conjugacy class size of every noncentral p_2-element of N is the partner of $|x^G|$ or equal to $v = |x^G|$, by Lemma 2.3. As $\Gamma_G(N)$ is connected, $\mathcal{N}_{\Gamma_G(N)}(v) \cap S \neq \emptyset$ and $T \subseteq \mathcal{N}_{\Gamma_G(N)}(v)$. Now, $\mathcal{N}_{\Gamma_G(N)}(v)$ induces a complete subgraph of $\Gamma_G(N)$, so the degree of each element in $\mathcal{N}_{\Gamma_G(N)}(v) \cap S$ is $|S \cup T| - 1$. Therefore, the completeness of $\Gamma_G(N)$ follows from its regularity. This proves the case $n = 2$ of the theorem.

Finally, consider the case $n = 1$. If $\mathbf{Z}(N) \leq \mathbf{Z}(G)$, the graph would be complete, since each vertex is divisible by p. Therefore, it must be the case that $\mathbf{Z}(N) \nleq \mathbf{Z}(G)$ and the proof is complete. □

As a consequence of the Main Theorem, we re-prove the main result in [3].

COROLLARY 3.3. *If G is a finite group and $\Gamma(G)$ is a k-regular graph for some $k \geq 1$, then $\Gamma(G)$ is complete.*

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PROOF. If $\Gamma(G)$ is disconnected, then by Lemma 2.4, $G/\mathbf{Z}(G)$ is a Frobenius group with abelian kernel and complement and so $\Gamma(G)$ has exactly two isolated vertices, namely the order of the Frobenius kernel and a Frobenius complement of $G/\mathbf{Z}(G)$. Therefore, as $k \geq 1$, it follows that $\Gamma(G)$ is connected. Now using the Main Theorem and replacing N by G itself, we see that $G = P \times A$, where A is a central subgroup of G and P is a p-group. Therefore all nontrivial conjugacy class sizes are p-numbers, which leads to the completeness of $\Gamma(G)$. □

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ADAM B. R., Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic), 15914 Tehran, Iran
e-mail: adamr21@ru.is

Z. AKHLAGHI, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic), 15914 Tehran, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran
e-mail: z_akhlaghi@aut.ac.ir