*Combinatorics, Probability and Computing* (2019), **28**, pp. 871–880
doi:10.1017/S0963548319000221

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ARTICLE

# On the Size-Ramsey Number of Cycles

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(Received 28 January 2017; revised 19 April 2019; first published online 17 July 2019)

**Abstract**

For given graphs $G_1, \ldots, G_k$, the size-Ramsey number $\hat{R}(G_1, \ldots, G_k)$ is the smallest integer $m$ for which there exists a graph $H$ on $m$ edges such that in every $k$-edge colouring of $H$ with colours $1, \ldots, k$, $H$ contains a monochromatic copy of $G_i$ of colour $i$ for some $1 \leqslant i \leqslant k$. We denote $\hat{R}(G_1, \ldots, G_k)$ by $\hat{R}_k(G)$ when $G_1 = \cdots = G_k = G$.

Haxell, Kohayakawa and Łuczak showed that the size-Ramsey number of a cycle $C_n$ is linear in $n$, that is, $\hat{R}_k(C_n) \leqslant c_k n$ for some constant $c_k$. Their proof, however, is based on Szemerédi's regularity lemma so no specific constant $c_k$ is known.

In this paper, we give various upper bounds for the size-Ramsey numbers of cycles. We provide an alternative proof of $\hat{R}_k(C_n) \leqslant c_k n$, avoiding use of the regularity lemma, where $c_k$ is exponential and doubly exponential in $k$, when $n$ is even and odd, respectively. In particular, we show that for sufficiently large $n$ we have $\hat{R}_2(C_n) \leqslant 10^5 \times cn$, where $c = 6.5$ if $n$ is even and $c = 1989$ otherwise.

**2010 MSC Codes:** Primary 05C55; Secondary 05D10

## 1. Introduction

For given graphs $G_1, \ldots, G_k$ and a graph $H$, we say that $H$ is *Ramsey* for $(G_1, \ldots, G_k)$ and we write $H \longrightarrow (G_1, \ldots, G_k)$ if, no matter how one colours the edges of $H$ with $k$ colours $1, \ldots, k$, there exists a monochromatic copy of $G_i$ of colour $i$ in $H$, for some $1 \leqslant i \leqslant k$. Ramsey's theorem [16] states that for given graphs $G_1, \ldots, G_k$, there exists a graph $H$ that is Ramsey for $(G_1, \ldots, G_k)$. Note that, if a graph $H$ is Ramsey for $(G_1, \ldots, G_k)$ and $H$ is a subgraph of $H'$, then $H'$ is also Ramsey for $(G_1, \ldots, G_k)$. In this view, in order to study the collection of graphs which are Ramsey for $(G_1, \ldots, G_k)$, it suffices to study the collection $\mathcal{F}(G_1, \ldots, G_k)$ of graphs which are minimal subject to being Ramsey for $(G_1, \ldots, G_k)$. These graphs are called *Ramsey minimal* for $(G_1, \ldots, G_k)$.

Many interesting problems in graph theory concern the study of various parameters related to Ramsey minimal graphs for $(G_1, \ldots, G_k)$. The most well-known and well-studied one is the smallest number of vertices of a graph in $\mathcal{F}(G_1, \ldots, G_k)$, which is referred to as the *Ramsey number* of

† Research partially supported by INSF grant no. 95844679.
‡ Research partially carried out in the IPM-Isfahan Branch and in part supported by a grant from IPM (no. 95050217).
§ Research partially supported by SNSF grant 200021-149111.

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https://doi.org/10.1017/S0963548319000221 Published online by Cambridge University Press

$(G_1, \ldots, G_k)$ and is denoted by $R(G_1, \ldots, G_k)$. In the diagonal case, where $G = G_1 = \cdots = G_k$, we may write $R_k(G)$ for $R(G_1, \ldots, G_k)$. Estimating $R(K_n) = R_2(K_n)$ is one of the main open problems in Ramsey theory. Erdős [8] and Erdős and Szekeres [10] showed that $2^{n/2} \leqslant R(K_n) \leqslant 2^{2n}$, and despite a lot of effort, there have not been many improvements to the exponents of the bounds. For further results about the Ramsey numbers of graphs, see [5, 15] and the references therein.

In this paper we consider another well-studied parameter called the *size-Ramsey number* $\hat{R}(G_1, \ldots, G_k)$ of the given graphs $G_1, \ldots, G_k$, which is defined as the minimum number of edges of a graph in $\mathcal{F}(G_1, \ldots, G_k)$. When $G = G_1 = \cdots = G_k$, it is denoted by $\hat{R}_k(G)$. The investigation of the size-Ramsey numbers of graphs was initiated by Erdős, Faudree, Rousseau and Schelp [9] in 1978. Since then, the size-Ramsey numbers of graphs have been studied with particular focus on the case of trees, bounded-degree graphs and sparse graphs. The survey paper due to Faudree and Schelp [11] collects some results about size-Ramsey numbers.

One of the most studied directions in this area is the size-Ramsey number of paths. In 1983, Beck [3] showed that $\hat{R}_2(P_n) < 900\, n$ for sufficiently large $n$, where $P_n$ is a path on $n$ vertices. This verifies the linearity of the size-Ramsey number of paths in terms of the number of vertices, and since then, different approaches have been attempted by several authors to reduce the constant coefficient in the upper bound: see [4, 6, 14]. Most of these approaches are based on the classic models of random graphs. Currently, the best known upper bound is due to Dudek and Prałat [7] which proved that $\hat{R}_2(P_n) \leqslant 74\, n$, for sufficiently large $n$.

In this paper, we investigate the size-Ramsey number of cycles. The linearity of $\hat{R}_k(C_n)$ (in terms of $n$) follows from the earlier result by Haxell, Kohayakawa and Łuczak [13]. Nevertheless, their proof is based on the regularity lemma and therefore is unable to determine a specific constant coefficient. The standard techniques for proving linear bounds for paths, avoiding the use of the regularity lemma, seem to be insufficient to prove a linear bound for cycles. Here, we give such a proof for the following theorem.

**Theorem 1.1.** *Let $n_1, n_2, \ldots, n_t$ be a sequence of positive integers with $t_e$ even numbers and $t_o$ odd numbers. Let $c = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e}$, $n = \max(n_1, \ldots, n_t)$ and suppose that for all i, we have $n_i \geqslant 2\lceil \log(nc)\rceil + 2$. Then*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leqslant (\ln c + 1)\, c^2\, n.$$

The above theorem is proved by showing that an Erdős–Renyi random graph with suitable edge probability is almost surely a Ramsey graph for a collection of cycles. By considering the binomial random bipartite graph model we will give further improvement on the bound in Theorem 1.1 (see Theorems 3.2 and 3.4).

Throughout the paper, the notations $\log x$ and $\ln x$ refer to the logarithms to the bases 2 and Euler's number $e$, respectively. Also, for a graph $G$ and a subset $S \subseteq V(G)$, $N_G(S)$ stands for the set of all vertices of $G$ which have at least one neighbour in $S$.

## 2. Cycles versus a complete bipartite graph

In this section we prove some auxiliary results which will later be used to bound the size-Ramsey numbers of cycles. Specifically, we prove some linear upper bounds (in terms of the number of vertices) for the Ramsey and bipartite Ramsey numbers of cycles versus a complete bipartite graph. First, we give some definitions and lemmas.

A rooted tree with at most two children for each vertex is called a *binary tree*. The *depth* of a vertex in a binary tree $T$ is the distance from the vertex to the root of $T$ and the maximum depth of any vertex is called the *height* of $T$. If a tree has only one vertex (the root), the height is zero.

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A *perfect binary tree* is a binary tree with all leaves at the same depth where every internal vertex (non-leaf vertex) has exactly two children. Now we begin with the following lemma.

**Lemma 2.1.** *For every positive integer $n \geqslant 2$, there is a binary tree of height $\lceil \log n \rceil$ and at most $2n + \lceil \log n \rceil - 2$ vertices which has exactly $n$ leaves, all of the same depth.*

**Proof.** If $n = 2^t$ for some $t$, then clearly the perfect binary tree of height $t$ has exactly $n$ leaves and $2n - 1$ vertices and we are done. Now, assume that $n = 2^{t_1} + \cdots + 2^{t_r}$, where $r \geqslant 2$ and $t_1 > \cdots > t_r \geqslant 0$. For each $1 \leqslant i \leqslant r$, let $T_i$ be the perfect binary tree of height $t_i$ with $2^{t_i+1} - 1$ vertices and $2^{t_i}$ leaves. Now, we construct a binary tree $T$ as follows. Consider the vertex-disjoint binary trees $T_1, \ldots, T_r$ with roots $x_1, \ldots, x_r$ and a new path $P = v_1 \ldots v_{t_1 - t_r + 1}$ and add an edge from $v_{t_1 - t_i + 1}$ to the root $x_i$ of $T_i$ for each $1 \leqslant i \leqslant r$. One can easily check that $T$ is a binary tree rooted at $v_1$ of height $t_1 + 1$ with $n = 2^{t_1} + \cdots + 2^{t_r}$ leaves and

$$|V(T)| = \sum_{i=1}^{r} 2^{t_i+1} - r + t_1 - t_r + 1 \leqslant 2n + t_1 - 1$$

vertices. Clearly $\lceil \log n \rceil = t_1 + 1$, so $T$ is a binary tree with $n$ leaves of depth $\lceil \log n \rceil$ and at most $2n + \lceil \log n \rceil - 2$ vertices. □

We also need the following tree-universality result due to Haxell and Kohayakawa.

**Theorem 2.2** (**[12]**)**.** *Let $1 \leqslant d \leqslant t$ be fixed integers. Suppose that $G$ is a bipartite graph with associated bipartition $(V_1, V_2)$, such that for every subset $S \subseteq V_i$ $(i \in \{1, 2\})$ with $|S| \leqslant 2t/d$, we have $|N_G(S)| \geqslant 2d|S|$. Then $G$ contains as a subgraph every tree with maximum degree at most $d$ whose each bipartition class has at most $t$ vertices.*

The above theorem is used to prove the following lemma about finding red paths in a 2-coloured balanced complete bipartite graph. The proof technique of this lemma is similar to the techniques from [2].

**Lemma 2.3.** *Let $n, m_1, m_2$ be positive integers such that $n$ is even and*

$$\min\{m_1, m_2\} \geqslant n \geqslant (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 1).$$

*Suppose that we have a 2-edge-coloured $K_{7m_1+8m_2, 8m_1+7m_2}$ with colours red and blue and bipartition classes $V_1$ and $V_2$ which has no blue $K_{m_1,m_2}$. Then for each $i \in \{1, 2\}$, there is $V_i' \subseteq V_i$ with $|V_i'| = m_i$ such that for every $x \in V_1'$ and $y \in V_2'$ there is a red path of length $n - 1$ from $x$ to $y$.*

**Proof.** Assume that the edges of $H = K_{N_1,N_2}$ are coloured by red and blue and $(V_1, V_2)$ is the bipartition of $H$ with $|V_1| = N_1 = 7m_1 + 8m_2$ and $|V_2| = N_2 = 8m_1 + 7m_2$. Let $H_r$ and $H_b$ be the subgraphs of $H$ induced on the red and blue edges, respectively. By our assumption, $H_b$ is $K_{m_1,m_2}$-free. Thus, we have

$$\text{for all } i, j \in \{1, 2\},\ |N_{H_r}(S)| > N_{i+1} - m_{j+1},\ \text{for every } S \subseteq V_i \text{ with } |S| \geqslant m_j, \tag{2.1}$$

(reading $i + 1$ and $j + 1$ modulo 2).

**Claim 1.** *There is an induced subgraph $G \subseteq H_r$ with parts $\hat{V}_i \subseteq V_i$, $i = 1, 2$, which satisfies*

$$\text{for all } i \in \{1, 2\},\ |N_G(S)| \geqslant 6|S|,\ \text{for every } S \subseteq \hat{V}_i \text{ with } |S| \leqslant m_1 + m_2. \tag{2.2}$$

**Proof of the claim.** Define

$$\mathcal{E} = \{(X_1, X_2) \mid X_i \subseteq V_i, |X_i| \leqslant m_1 + m_2, |N_{H_r \setminus X_{i+1}}(X_i)| \leqslant 6|X_i|,\ \text{for each } i \in \{1, 2\}\},$$

https://doi.org/10.1017/S0963548319000221 Published online by Cambridge University Press

(reading $i+1$ modulo 2) and let $(A_1, A_2)$ be a pair in $\mathcal{E}$ with largest $|A_1|+|A_2|$. We claim that $|A_i| < m_{i+1}$ ($i \in \{1,2\}$), since otherwise, by (2.1), we have

$$N_{i+1} - m_i - |A_{i+1}| < |N_{H_r \setminus A_{i+1}}(A_i)| \leqslant 6|A_i|,$$

which implies that $N_{i+1} < 6|A_i| + |A_{i+1}| + m_i \leqslant 7m_1 + 7m_2 + m_i = N_{i+1}$, a contradiction. Therefore, $|A_i| < m_{i+1}$ ($i \in \{1,2\}$).

Now, for each $i \in \{1,2\}$, let $\hat{V}_i = V_i \setminus A_i$ and $G$ be the induced subgraph of $H_r$ on $\hat{V}_1 \cup \hat{V}_2$. To see (2.2), for some $i \in \{1,2\}$, let $S \subseteq \hat{V}_i$ be a subset with $|S| \leqslant m_1 + m_2$. To the contrary, suppose that $|N_G(S)| < 6|S|$. Then

$$|N_{H_r \setminus A_{i+1}}(S \cup A_i)| \leqslant |N_G(S)| + |N_{H_r \setminus A_{i+1}}(A_i)| < 6|S| + 6|A_i| = 6|S \cup A_i|.$$

Also, $|N_{H_r \setminus (S \cup A_i)}(A_{i+1})| \leqslant |N_{H_r \setminus A_i}(A_{i+1})| \leqslant 6|A_{i+1}|$. Thus, by maximality of $(A_1, A_2)$, we have $|S \cup A_i| > m_1 + m_2$. On the other hand, we have $|A_i| < m_{i+1}$. Therefore, $|S| > m_i$. Hence, using (2.1), we have

$$N_{i+1} - m_{i+1} < |N_{H_r}(S)| \leqslant |N_G(S)| + |A_{i+1}| < 6|S| + m_i \leqslant 6m_1 + 6m_2 + m_i,$$

which implies that $N_{i+1} < 7m_1 + 7m_2$, again a contradiction. This proves the claim. □

Now, let $G$ be the subgraph of $H_r$ which satisfies (2.2). In light of Theorem 2.2, $G$ (and so $H_r$) contains a copy of any tree $T$ with maximum degree 3 whose bipartition classes are of size at most $\lfloor 3(m_1+m_2)/2 \rfloor$. Now, for $i \in \{1,2\}$, let $T_i$ be a binary tree with $m_i$ leaves of depth $\lceil \log m_i \rceil$ and at most $2m_i + \lceil \log m_i \rceil - 2$ vertices (which exists due to Lemma 2.1). Also, let $T$ be a tree on at most $n + 2m_1 + 2m_2 - 6$ vertices formed by attaching the roots of $T_1$ and $T_2$ by a path of length $n - 1 - \lceil \log m_1 \rceil - \lceil \log m_2 \rceil$. Note that $T$ has maximum degree 3 with leaves $x_1, \ldots, x_{m_1}$, $y_1, \ldots, y_{m_2}$, where there is a path of length $n-1$ from $x_i$ to $y_j$ for every $1 \leqslant i \leqslant m_1$ and $1 \leqslant j \leqslant m_2$. Also note that since $n$ is even, $\{x_1, \ldots, x_{m_1}\}$ and $\{y_1, \ldots, y_{m_2}\}$ are contained in different parts of the bipartition of $T$. Without loss of generality, we can assume that $V_1{}' = \{x_1, \ldots, x_{m_1}\} \subseteq V_1$ and $V_2{}' = \{y_1, \ldots, y_{m_2}\} \subseteq V_2$. It can be seen that the size of the bipartition class of $T$ contained in $V_i$ ($i \in \{1,2\}$) is at most $n/2 + 3m_i/2 + m_{i+1} \leqslant 3(m_1+m_2)/2$. To see this, note that half of the vertices in the path from $x_1$ to $y_1$ are in $V_{i+1}$, also all vertices in $V'_{i+1}$ and all parents of vertices in $V'_i$ are in $V_{i+1}$. Thus, $|V_{i+1}| \geqslant n/2 + m_{i+1} + m_i/2 - 2$ and so $|V_i| \leqslant n/2 + m_{i+1} + 3m_i/2 - 4$. Hence, by Theorem 2.2, $H_r$ contains a copy of $T$. This completes the proof. □

Given bipartite graphs $G_1, \ldots, G_k$, the *bipartite Ramsey number* $BR(G_1, \ldots, G_k)$ is defined as the smallest integer $b$ such that, for any edge colouring of the complete bipartite graph $K_{b,b}$ with $k$ colours $1, \ldots, k$, there exists a monochromatic copy of $G_i$ of colour $i$ in $K_{b,b}$, for some $1 \leqslant i \leqslant k$. In other words, it is the smallest integer $b$ such that $K_{b,b} \to (G_1, \ldots, G_k)$. The above lemma can be used to give an upper bound for the bipartite Ramsey number of an even cycle versus a complete bipartite graph.

**Lemma 2.4.** *Let $n, m$ be positive integers such that $n$ is even with $m \geqslant n \geqslant 2\lceil \log m \rceil + 1$. Then, $BR(C_n, K_{m,m}) \leqslant 15m$.*

**Proof.** Let $H$ be a 2-edge-coloured complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = |V_2| = 15m$. Suppose that $H$ contains no blue $K_{m,m}$. To prove the lemma, it is enough to show that $H$ contains a red $C_n$.

By Lemma 2.3, there are $V_i{}' \subseteq V_i$ with $|V_i{}'| = m$ ($i \in \{1,2\}$) such that for every $x \in V_1{}'$ and $y \in V_2{}'$ there is a red path of length $n-1$ from $x$ to $y$. Since $H$ has no blue $K_{m,m}$, there is a red edge $xy$ for some $x \in V_1{}'$ and $y \in V_2{}'$. Adding this edge to the red path of length $n-1$ from $x$ to $y$ gives a red cycle of length $n$ as required. □

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The following corollary is an immediate consequence of Lemma 2.4.

**Corollary 2.5.** *Let $m$ and $n_1, \ldots, n_t$ be positive integers such that, for every $1 \leqslant i \leqslant t$, $n_i$ is even and $m \geqslant n_i \geqslant 2\lceil \log (15^{t-1}m)\rceil + 1$. Then, $BR(C_{n_1}, \ldots, C_{n_t}, K_{m,m}) \leqslant 15^t m$.*

**Proof.** We give a proof by induction on $t$. The case $t = 1$ follows from Lemma 2.4. Now, assuming the assertion holds for $t < t_0$, we are going to prove it for $t = t_0$. To see this, consider the graph $H = K_{15^{t_0}m, 15^{t_0}m}$ whose edges are coloured by the colours $1, 2, \ldots, t_0 + 1$ and suppose that there is no copy of $C_{n_i}$ of colour $i$ in $H$ for all $1 \leqslant i \leqslant t_0$. We show that there is a copy of $K_{m,m}$ of colour $t_0 + 1$ in $H$. By the induction hypothesis, we have $BR(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{15m,15m}) \leqslant 15^{t_0}m$ and so there is a copy of $K_{15m,15m}$ in $H$ whose edges are coloured by colours $t_0$ and $t_0 + 1$. Now using Lemma 2.4, this copy contains either a copy of $C_{n_{t_0}}$ of colour $t_0$, or a copy of $K_{m,m}$ of colour $t_0 + 1$. By the assumption, the earlier case does not hold. Hence, there is a copy of $K_{m,m}$ of colour $t_0 + 1$ in $H$ and we are done. □

For the case of odd cycles, we need a variant of Lemma 2.4 for 3-partite graphs which is stated as follows.

**Lemma 2.6.** *Let $n, m_1, m_2$ be positive integers, where*

$$\min\{m_1, m_2\} \geqslant n \geqslant (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 2).$$

*Then*

$$K_{X,Y,Z} \longrightarrow (C_n, K_{m_1,m_2}),$$

*where $K_{X,Y,Z}$ is a complete 3-partite graph with colour classes $X, Y, Z$ of sizes $|X| = 7m_1 + 8m_2$, $|Y| = 8m_1 + 7m_2$ and $|Z| = m_1 + m_2 - 1$.*

**Proof.** The case when $n$ is even follows from Lemma 2.3 (it just suffices to consider the subgraph $K_{X,Y}$ of $K_{X,Y,Z}$ and apply Lemma 2.3). Now, let $n$ be odd. Consider a 2-edge colouring of $K_{X,Y,Z}$ and suppose that there is no blue $K_{m_1,m_2}$.

By Lemma 2.3, there are sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = m_1$ and $|Y'| = m_2$ such that, for every $x \in X'$ and $y \in Y'$, there is a red path of length $n - 2$ from $x$ to $y$ contained in $X \cup Y$.

Now, since there is no blue $K_{m_1,m_2}$ in the 2-edge-coloured $K_{X,Z}$, we have $|N^r_Z(X')| \geqslant m_1$, where $N^r_Z(S)$ is the set of all vertices in $Z$ which have a neighbour in $S$ in the red subgraph of $K_{X,Y,Z}$. Similarly, since there is no blue $K_{m_1,m_2}$ in the 2-edge-coloured $K_{Y,Z}$, we have $|N^r_Z(Y')| \geqslant m_2$. Therefore, since $|Z| = m_1 + m_2 - 1$, we have $N^r_Z(X') \cap N^r_Z(Y') \neq \emptyset$. Hence, there are some vertices $x \in X'$, $y \in Y'$, and $z \in N^r_Z(x) \cap N^r_Z(y)$. Now, the concatenation of the edges $yz$ and $zx$ and the path of length $n - 2$ from $x$ to $y$ in $X \cup Y$ comprises a red $C_n$, as required. □

Lemma 2.6 and the fact that the graph $K_{X,Y,Z}$ in Lemma 2.6 is a subgraph of $K_{16m_1+16m_2-1}$ immediately imply the following result.

**Corollary 2.7.** *Let $n, m_1, m_2$ be positive integers, where*

$$\min\{m_1, m_2\} \geqslant n \geqslant (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 2).$$

*Then, $R(C_n, K_{m_1,m_2}) \leqslant 16m_1 + 16m_2 - 1$.*

Let $f_1(m_1, m_2) = 16m_1 + 16m_2 - 1$, and for every $t \geqslant 2$, define

$$f_t(m_1, m_2) = f_{t-1}(f_{t-1}(m_1 + m_2 - 1, 8m_1 + 7m_2), 7m_1 + 8m_2). \tag{2.3}$$

In the following, we show that $f_t(m_1, m_2)$ is an upper bound for the Ramsey number of $t$ cycles (with some restrictions on their sizes) versus the graph $K_{m_1,m_2}$.

**Theorem 2.8.** *Let $m_1, m_2$ and $n_1, \ldots, n_t$ be positive integers such that*

$$\min\{m_1, m_2\} \geqslant n_i \geqslant 2\lceil \log (f_t(m_1, m_2))\rceil + 2 \quad \textit{for each } 1 \leqslant i \leqslant t.$$

*Then,*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m_1,m_2}) \leqslant f_t(m_1, m_2).$$

**Proof.** We give a proof by induction on $t$. The case $t = 1$ follows from Corollary 2.7. Now, assuming correctness of the assertion for $t < t_0$, we are going to prove it for $t = t_0$. Consider the $(t_0 + 1)$-edge-coloured graph $H = K_N$ with colours $1, 2, \ldots, t_0 + 1$, where $N = f_{t_0}(m_1, m_2)$. We assume that $H$ contains no copy of $C_{n_i}$ of colour $i$ for each $1 \leqslant i \leqslant t_0$ and we show that there is a copy of $K_{m_1,m_2}$ of colour $t_0 + 1$. By the induction hypothesis, we have $R(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{N_1,N_2}) \leqslant N$ for $N_1 = f_{t_0-1}(m_1 + m_2 - 1, 8m_1 + 7m_2)$ and $N_2 = 7m_1 + 8m_2$. Thus, there is a copy of 2-edge-coloured $K_{N_1,N_2}$ with parts $X$ and $Y$ by colours $t_0$ and $t_0 + 1$ in $K_N$. Now, again by the induction hypothesis, we have

$$|X| = N_1 = f_{t_0-1}(m_1 + m_2 - 1, 8m_1 + 7m_2) \geqslant R(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{m_1+m_2-1,8m_1+7m_2}).$$

Therefore, there is a copy of a 2-edge-coloured $K_{m_1+m_2-1,8m_1+7m_2}$ by colours $t_0$ and $t_0 + 1$ with parts $X'$ and $X''$ in the induced subgraph of $K_N$ on $X$. Thus, the edges of the complete 3-partite graph with the colour classes $Y$, $X'$ and $X''$ are coloured by colours $t_0$ and $t_0 + 1$ and so by Lemma 2.6, there is a copy of $K_{m_1,m_2}$ of colour $t_0 + 1$ in $H$ and we are done. □

The following corollary follows from Theorem 2.8 and the fact that $f_2(m_1, m_2) = 2416m_1 + 2176m_2 - 273$.

**Corollary 2.9.** *Let $m_1, m_2, n_1, n_2$ be positive integers such that*

$$\min\{m_1, m_2\} \geqslant n_1, n_2 \geqslant 2\lceil \log (2416m_1 + 2176m_2 - 273)\rceil + 2.$$

*Then, we have*

$$R(C_{n_1}, C_{n_2}, K_{m_1,m_2}) \leqslant 2416m_1 + 2176m_2 - 273.$$

By calculating the function $f_t(m_1, m_2)$ and using Theorem 2.8, we can prove the following theorem.

**Theorem 2.10.** *Let $t \geqslant 3$ and $m_1, m_2$ and $n_1, \ldots, n_t$ be positive integers such that*

$$\min\{m_1, m_2\} \geqslant n_i \geqslant 2\lceil \log (10^{2^t-1}(m_1 + m_2))\rceil + 2 \quad \textit{for each } 1 \leqslant i \leqslant t.$$

*Then,*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m_1,m_2}) \leqslant 10^{2^t-1}(m_1 + m_2).$$

**Proof.** Using Theorem 2.8, it just suffices to show that $f_t(m_1, m_2) \leqslant 10^{2^t-1}(m_1 + m_2)$. To see this, let $f_t(m_1, m_2) = a_t m_1 + b_t m_2 + c_t$, where $a_t$, $b_t$ and $c_t$ are three functions in terms of $t$. From (2.3), one can easily see that for each $t \geqslant 2$,

$$\begin{aligned} a_t &= a_{t-1}^2 + 8a_{t-1}b_{t-1} + 7b_{t-1}, \\ b_t &= a_{t-1}^2 + 7a_{t-1}b_{t-1} + 8b_{t-1}, \quad \text{and} \\ c_t &= -a_{t-1}^2 + a_{t-1}c_{t-1} + c_{t-1}. \end{aligned}$$

Clearly, for every $i \geqslant 2$ we have $b_i < a_i$, and so for every $t \geqslant 3$,

$$a_t < a_{t-1}^2 + 8a_{t-1}^2 + 7a_{t-1} \leqslant 10a_{t-1}^2.$$

Therefore, by induction on $t$ we can see that for every $t \geqslant 3$,

$$b_t \leqslant a_t \leqslant 10^{2^t-1}.$$

On the other hand, again by induction on $t$, we have $c_t \leqslant 0$. Therefore

$$f_t(m_1, m_2) \leqslant a_t m_1 + b_t m_2 \leqslant 10^{2^t-1}(m_1 + m_2). \qquad \square$$

With all these results in hand, we can prove the main result of this section, as follows.

**Theorem 2.11.** *Let $t_e$ and $t_o$, respectively, be the number of even and odd integers in the sequence $(n_1, \ldots, n_t)$ and suppose that $m \geqslant n_i \geqslant 2(\lceil \log N \rceil + 1)$ for each $1 \leqslant i \leqslant t$, where $N = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e} m$. Then*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m,m}) \leqslant N.$$

**Proof.** The case $t_o = 0$ follows from Corollary 2.5 (note that in this case the complete graph $K_N$ has a complete bipartite graph $K_{15^{t_e}m,15^{t_e}m}$ as a subgraph). So, let $t_o \geqslant 1$. Also, without loss of generality, assume that $n_i$ is odd for all $1 \leqslant i \leqslant t_o$. Consider a $(t+1)$-edge-coloured $K_N$ with colours $1, 2, \ldots, t+1$. Assume that there is no copy of $C_{n_i}$ of colour $i$ for each $1 \leqslant i \leqslant t$. Our goal is to show that there is a copy of $K_{m,m}$ of colour $t+1$. Using Corollary 2.7 and Corollary 2.9 when $t_o = 1, 2$ and Theorem 2.10 when $t_o \geqslant 3$, we have $R(C_{n_1}, \ldots, C_{n_{t_o}}, K_{15^{t_e}m,15^{t_e}m}) \leqslant N$ and so there is a copy of $K_{15^{t_e}m,15^{t_e}m}$ in $K_N$ whose edges are coloured by $t_e + 1$ colours $t_o + 1, \ldots, t+1$. Now, Corollary 2.5 implies that $BR(C_{n_{t_o+1}}, \ldots, C_{n_t}, K_{m,m}) \leqslant 15^{t_e}m$ and so there is a copy of $K_{m,m}$ of colour $t+1$ in the $(t_e+1)$-edge-coloured $K_{15^{t_e}m,15^{t_e}m}$, as desired. $\square$

## 3. Random graphs and upper bounds

In this section, we will apply the obtained results in Section 2 on random graphs to give some linear upper bounds in terms of the number of vertices for the size-Ramsey number of large cycles. For this purpose, we deploy two random structure models, namely binomial random graphs and binomial random bipartite graphs.

The first model, called *binomial random graph* $\mathcal{G}(n, p)$, is the random graph $G$ with the vertex set $[n] := \{1, 2, \ldots, n\}$ in which every pair $\{i, j\} \subseteq [n]$ appears independently as an edge in $G$ with probability $p$. Note that an event in a probability space is said to hold *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as $n$ goes to infinity. To see more about random graphs we refer the reader to [1, 4]. We will state some results that hold a.a.s. and we always assume that $n$ is large enough.

The first lemma asserts that there is a graph on $N$ vertices whose number of edges is linear in terms of $N$, while it has no large hole (a pair of disjoint subsets of vertices with no edge between them). It should be noted that a very similar fact has been proved in [6] to prove a linear upper bound for the size-Ramsey number of paths. Here, we give a proof for completeness. For two subsets of vertices $S, T$, the number of edges with one end in $S$ and one end in $T$ is denoted by $e(S, T)$.

**Lemma 3.1.** *Let $c \in \mathbb{R}_+$ and let $d = d(c)$ be such that*

$$(1 - 2c) \ln (1 - 2c) + 2c \ln (c) + c^2 d \geqslant 0. \qquad (3.1)$$

*Then, in the graph $G \in \mathcal{G}(N, d/N)$, a.a.s. for every two disjoint sets of vertices $S$ and $T$ with $|S| = |T| = cN$, we have $e(S, T) \geqslant 1$.*

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**Proof.** Let $S$ and $T$ with $|S| = |T| = cN$ be fixed and let $X = X_{S,T} = e(S, T)$. Clearly,

$$\mathbb{P}(X = 0) = \left(1 - \frac{d}{N}\right)^{c^2N^2} \leqslant \exp\left( - c^2\, dN\right).$$

Thus, by the union bound over all choices of $S$ and $T$ we have

$$\mathbb{P}\left(\bigcup_{S,T} (X_{S,T} = 0)\right) \leqslant \binom{N}{cN}\binom{(1-c)N}{cN} \exp\left( - c^2\, dN\right)$$
$$= \frac{N!}{(cN)!(cN)!((1-2c)N)!} \exp\left( - c^2\, dN\right).$$

Using Stirling's formula ($x! \sim \sqrt{2\pi x}(x/e)^x$) we get

$$\mathbb{P}\left(\bigcup_{S,T} (X_{S,T} = 0)\right) \leqslant \frac{1}{2\pi c\sqrt{1-2c}N}\cdot\left(\frac{(1-2c)^{2c-1}\exp\left( - c^2 d\right)}{c^{2c}}\right)^N$$
$$\leqslant \frac{1}{2\pi c\sqrt{1-2c}N} = o(1),$$

where the last inequality is due to (3.1). This completes the proof. □

Combining Theorem 2.11 and Lemma 3.1 gives some information on the size-Ramsey numbers of cycles. Roughly speaking, these two facts imply that for sufficiently large $N$,

$$\mathcal{G}(N, d/N) \longrightarrow (C_{n_1}, \ldots, C_{n_t})$$

when we have some restrictions on the parameters. In the following result, which is the main result of this paper, we use this fact to give a linear upper bound for the size-Ramsey number of large cycles.

**Theorem 3.2.** *Let $f = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e}$, where $t_e$ and $t_o$, respectively, are the number of even and odd integers in the sequence $(n_1, \ldots, n_t)$. Also let $c = \min\{19773, f\}$ if $t = 2$ and $c = f$ otherwise. Suppose that $n = \max\{n_1, \ldots, n_t\}$, and for each $1 \leqslant i \leqslant t$, we have $n_i \geqslant 2\lceil\log(nc)\rceil + 2$. Then, for sufficiently large $n$, we have*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leqslant (\ln c + 1)\, c^2\, n.$$

**Proof.** Let $N = nc$, $d = ((2c^{-1} - 1)\ln(1 - 2c^{-1}) - 2c^{-1}\ln(c^{-1}))/c^{-2}$ and $G = \mathcal{G}(N, d/N)$. By Lemma 3.1, a.a.s. for every two disjoint sets of vertices $S$ and $T$ in $V(G)$ with $|S| = |T| = n$, we have $e(S, T) \geqslant 1$. Therefore, a.a.s. the complement of $G$ does not contain $K_{n,n}$ as a subgraph. On the other hand, the expected number of edges of $G$ is $(d/N)\binom{N}{2} \leqslant Nd/2$, and the concentration around the expectation follows immediately from the Chernoff bound. Hence, for sufficiently large $N$, there exists a graph $H$ on $N$ vertices with at most $Nd/2$ edges whose complement does not contain $K_{n,n}$ as a subgraph. Hence, by Corollary 2.9 and Theorem 2.11, we have $H \longrightarrow (C_{n_1}, \ldots, C_{n_t})$. This means that for sufficiently large $N$ we have

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leqslant \frac{Nd}{2} \leqslant \frac{c\ln c - (c-2)\ln(c-2)}{2}\, c^2\, n \leqslant (\ln c + 1)\, c^2\, n,$$

where the last inequality follows by applying the mean value theorem to the function $x\ln x$. □

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Based on Theorem 3.2, for sufficiently large $n$, we have

$$\hat{R}(C_n, C_n) \leqslant \begin{cases} 1989 \times 10^5\, n & \text{if } n \text{ is odd,} \\ 86 \times 10^5\, n & \text{if } n \text{ is even.} \end{cases}$$

It should be noted that other random graph models can be used in the above method to improve the bounds obtained. One of these models is that of *random regular graphs*, which gives slightly better results, but we omit the computations because it does not give much improvement. See [7] for an application of this method to size-Ramsey numbers. Another model is the binomial random bipartite graphs, described below, which give better upper bounds when all the cycles are even.

The *binomial random bipartite graph* $\mathcal{G}(n, n, p)$ (where $p$ may be a function of $n$) is the random bipartite graph $G = (V_1 \cup V_2, E)$ whose partite sets $V_1, V_2$ are of order $n$ and each pair $(i, j) \in V_1 \times V_2$ appears independently as an edge in $G$ with probability $p$. The following is the counterpart of Lemma 3.1 for the random bipartite graphs. Once again, it is well known and we include its proof for completeness.

**Lemma 3.3.** *Let* $0 < c < 1$ *and let* $d = d(c)$ *be such that*

$$2(1-c)\ln(1-c) + 2c\ln c + c^2 d \geqslant 0.$$

*Then, a.a.s. for every two sets of vertices* $S$ *and* $T$ *in different colour classes of* $G \in \mathcal{G}(N, N, d/N)$ *with* $|S| = |T| = cN$, *we have* $e(S, T) \geqslant 1$.

**Proof.** Let $S$ and $T$ with $|S| = |T| = cN$ be fixed and let $X = X_{S,T} = e(S, T)$. Clearly,

$$\mathbb{P}(X = 0) = \left(1 - \frac{d}{N}\right)^{c^2N^2} \leqslant \exp(-c^2 dN).$$

Thus, by the union bound over all choices of $S$ and $T$ we have

$$\mathbb{P}\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leqslant \binom{N}{cN}^2 \exp(-c^2 dN) = \left(\frac{N!}{(cN)!((1-c)N)!}\right)^2 \exp(-c^2 dN).$$

Using Stirling's formula we get

$$\mathbb{P}\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leqslant \frac{1}{2\pi c(1-c)N} \cdot \left(\frac{\exp(-c^2 d/2)}{c^c(1-c)^{1-c}}\right)^{2N} \leqslant \frac{1}{2\pi c(1-c)N} = o(1),$$

as desired. □

The following theorem gives an improvement of Theorem 3.2 when the lengths of all cycles are even.

**Theorem 3.4.** *Assume that* $n_1, \ldots, n_t$ *are even positive integers and* $n = \max\{n_1, \ldots, n_t\}$. *Also, suppose that for each* $1 \leqslant i \leqslant t$ *we have* $n_i \geqslant 2\lceil \log(15^t n) \rceil + 2$. *Then, for sufficiently large* $n$, *we have*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leqslant 2 \times 15^{2t}(t \ln 15 + 1)\, n.$$

**Proof.** Let $c = 15^{-t}$, $N = n/c$, $d = (-2(1-c)\ln(1-c) - 2c\ln c)/c^2$ and $G = \mathcal{G}(N, N, d/N)$. By Lemma 3.3, a.a.s. for every two sets of vertices $S$ and $T$ in different colour classes of $G$ with $|S| = |T| = n$, we have $e(S, T) \geqslant 1$. Therefore a.a.s. the complement of $G$ with respect to $K_{N,N}$ does not contain $K_{n,n}$ as a subgraph and so by Corollary 2.5, we have $G \longrightarrow (C_{n_1}, \ldots, C_{n_t})$. On the other

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hand, the expected number of edges of $G$ is $Nd$ and concentration around the expectation follows immediately from the Chernoff bound. This means that for sufficiently large $n$ we have

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leqslant Nd = 2c^{-2}(c^{-1}\ln c^{-1} - (c^{-1}-1)\ln(c^{-1}-1))\, n \leqslant 2c^{-2}(\ln c^{-1} + 1)\, n,$$

where the last inequality is due to the mean value theorem. □

As a consequence of Theorem 3.4, for sufficiently large even $n$, we have

$$\hat{R}(C_n, C_n) \leqslant 65 \times 10^4 n.$$

**Acknowledgements.** The authors would like to thank two anonymous referees for many suggestions improving this paper.

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**Cite this article:** Javadi R, Khoeini F, Omidi GR and Pokrovskiy A (2019). On the Size-Ramsey Number of Cycles. *Combinatorics, Probability and Computing* **28**, 871–880. https://doi.org/10.1017/S0963548319000221

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