Bull. Aust. Math. Soc. **105** (2022), 303–313
doi:10.1017/S0004972721000538

ON φ-AMENABILITY OF DUAL BANACH ALGEBRAS

ALIREZA JABERI and AMIN MAHMOODI

(Received 20 May 2021; accepted 1 June 2021; first published online 9 July 2021)

Abstract

Generalising the concept of injectivity, we study the notion of φ-injectivity for dual Banach algebras. It provides a framework for studying φ-amenability of enveloping dual Banach algebras.

2020 *Mathematics subject classification*: primary 46H25; secondary 43A07.

Keywords and phrases: φ-amenability, φ-injectivity, dual Banach algebras.

1. Introduction

In his memoir, Johnson [3] introduced the cohomological notion of an amenable Banach algebra. The concept of φ-amenability, which is a modification of Johnson's amenability, was introduced by Kaniuth *et al.* [4] and independently by Monfared [8]. By way of background, φ-amenability is a generalisation of the notion of (left) amenability for *Lau algebras* (or F-algebras); these are Banach algebras that are preduals of a von Neumann algebra where the identity element of the von Neumann algebra is a character [5]. The notion of injectivity for dual Banach algebras was introduced by Daws [1]. We recall the definitions in Definitions 2.1 and 2.4 below.

Motivated by these concepts, we define and study φ-injective dual Banach algebras. In Section 2, we recall some background definitions and notation. In Section 3, we introduce and investigate φ-injectivity of a dual Banach algebra $\mathfrak{A}$. Among other things, we prove that φ-injectivity is equivalent to φ-amenability whenever $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ is a w^*-continuous homomorphism. In Section 4, using the idea of φ-injectivity, we discuss φ-amenability of the *enveloping dual Banach algebra* $\mathrm{WAP}(\mathfrak{A}^*)^*$ of a Banach algebra $\mathfrak{A}$. Besides examples, we will characterise φ-amenability of $\mathrm{WAP}(\mathfrak{A}^*)^*$ in terms of continuous representations from $\mathfrak{A}$ on reflexive Banach spaces. Section 5 is devoted to non-$\tilde{\varphi}$-amenability of the algebra $\mathrm{WAP}(\ell^1(\mathbb{N}_\wedge)^*)^*$ where φ is the augmentation character on $\ell^1(\mathbb{N}_\wedge)$. Finally, in Appendix A, we shall see that every nonzero homomorphism $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ becomes automatically a w^*-continuous homomorphism $\varphi : \mathrm{WAP}(\mathfrak{A}^*)^* \longrightarrow \mathbb{C}$.

2. Preliminaries

For a Banach algebra $\mathfrak{A}$, the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a canonical way. The *diagonal operator* $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $\pi(a \otimes b) = ab$ is an $\mathfrak{A}$-bimodule homomorphism. Let E be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \longrightarrow E$ is called a *derivation* if it satisfies $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \longrightarrow E$ is defined by $ad_x(a) = a \cdot x - x \cdot a$. We write $\Delta(\mathfrak{A})$ for the set of all homomorphisms from $\mathfrak{A}$ onto $\mathbb{C}$.

DEFINITION 2.1 [4]. Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. The algebra $\mathfrak{A}$ is *φ-amenable* if there exists a bounded linear functional m on $\mathfrak{A}^*$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$.

PROPOSITION 2.2 [4, Theorem 1.1]. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. Then $\mathfrak{A}$ is φ-amenable if and only if every derivation $D : \mathfrak{A} \longrightarrow E^*$ is inner, where E is a Banach $\mathcal{A}$-bimodule such that $a \cdot x = \varphi(a)x$ for all $a \in \mathfrak{A}$ and $x \in E$.*

Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A Banach algebra $\mathfrak{A}$ is *dual* if it is dual as a Banach $\mathfrak{A}$-bimodule. We write $\mathfrak{A} = (\mathfrak{A}_*)^*$ if we wish to stress that $\mathfrak{A}$ is a dual Banach algebra with predual $\mathfrak{A}_*$.

Let $\mathfrak{A}$ be a dual Banach algebra and let E be a Banach $\mathfrak{A}$-bimodule. Then $\sigma wc(E)$ stands for the set of all elements $x \in E$ such that the maps

$$\mathfrak{A} \longrightarrow E, \quad a \longmapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases} \tag{$*$}$$

are w^*- wk-continuous. It is well known that $\sigma wc(E)$ is a closed submodule of E.

Suppose that $\mathfrak{A}$ is a Banach algebra and that E is a Banach $\mathfrak{A}$-bimodule. An element $x \in E$ is *weakly almost periodic* if the maps in $(*)$ are weakly compact. The set of all weakly almost periodic elements in E is denoted by $\mathrm{WAP}(E)$.

Let $\mathfrak{A}$ be a Banach algebra. For $\varphi \in \mathrm{WAP}(\mathfrak{A}^*)$ and $\Psi \in \mathrm{WAP}(\mathfrak{A}^*)^*$, define $\Psi \cdot \varphi \in \mathrm{WAP}(\mathfrak{A}^*)$ by $\langle a, \Psi \cdot \varphi\rangle = \langle \varphi \cdot a, \Psi\rangle$ for all $a \in \mathfrak{A}$. This turns $\mathrm{WAP}(\mathfrak{A}^*)^*$ into a Banach algebra by letting

$$\langle \varphi, \Phi\Psi\rangle = \langle \Psi \cdot \varphi, \Phi\rangle \quad (\Phi, \Psi \in \mathrm{WAP}(\mathfrak{A}^*)^*,\ \varphi \in \mathrm{WAP}(\mathfrak{A}^*)).$$

More precisely, $\mathrm{WAP}(\mathfrak{A}^*)^*$ is a dual Banach algebra and there is a (continuous) homomorphism $\iota : \mathfrak{A} \longrightarrow \mathrm{WAP}(\mathfrak{A}^*)^*$ whose range is w^*-dense. Indeed, the map ι is obtained by composing the canonical inclusion $\mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ with the adjoint of the inclusion map $\mathrm{WAP}(\mathfrak{A}^*) \hookrightarrow \mathfrak{A}^*$ [10].

PROPOSITION 2.3 [10, Theorem 4.10]. *Let $\mathfrak{A}$ be a Banach algebra, let $\mathfrak{B}$ be a dual Banach algebra and let $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a (continuous) homomorphism. Then there exists a unique w^*-continuous homomorphism $\tilde{\theta} : WAP(\mathfrak{A}^*)^* \longrightarrow \mathfrak{B}$ such that $\theta = \tilde{\theta} \circ \iota$. In particular, every w^*-continuous homomorphism from $WAP(\mathfrak{A}^*)^*$ into $\mathfrak{B}$ is uniquely determined by its restriction to $\mathfrak{A}$.*

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

Let $\mathcal{S}$ be a subset of an algebra $\mathcal{H}$. We use $\mathcal{S}^c$ to denote the *commutant* of $\mathcal{S}$ in $\mathcal{H}$, that is, $\mathcal{S}^c = \{h \in \mathcal{H} \ : \ hs = sh, \ s \in \mathcal{S}\}$. It is obvious that $\mathcal{S}^c$ is a closed subalgebra of $\mathcal{H}$. For Banach spaces E and F, we write $\mathcal{L}(E, F)$ for the set of all bounded linear maps from E into F and $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$. We also write I_E for the identity map on E.

Let E be a Banach space and let $\mathcal{S} \subseteq \mathcal{L}(E)$ be a subalgebra. A *quasi expectation* for $\mathcal{S}$ is a projection $Q : \mathcal{L}(E) \longrightarrow \mathcal{S}^c$ such that $Q(cTd) = cQ(T)d$ for $c, d \in \mathcal{S}^c$ and $T \in \mathcal{L}(E)$.

DEFINITION 2.4 [1, Definition 6.12]. A dual Banach algebra $\mathfrak{A}$ is *injective* if, whenever $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous unital representation, then there is a quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^c$.

Connes amenable dual Banach algebras were systematically introduced by Runde in [9]. The remarkable point is that injectivity and Connes amenability are the same notions [1, Theorem 6.13].

3. On φ-injectivity of dual Banach algebras

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras and let $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism. For $\varphi \in \Delta(\mathfrak{A})$, we define

$$\theta(\mathfrak{A})^\varphi = \{b \in \mathfrak{B} : \theta(a)b = \varphi(a)b \ \ (a \in \mathfrak{A})\}.$$

Obviously, $\theta(\mathfrak{A})^\varphi$ is a (closed) right ideal of $\mathfrak{B}$. One may see Lemma 5.1 below as a concrete example of such a set.

DEFINITION 3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, let $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism and let $\varphi \in \Delta(\mathfrak{A})$. A *$\varphi$-quasi expectation* $Q : \mathfrak{B} \longrightarrow \theta(\mathfrak{A})^\varphi$ is a projection from $\mathfrak{B}$ onto $\theta(\mathfrak{A})^\varphi$ satisfying $Q(cbd) = cQ(b)d$ for $c, d \in \theta(\mathfrak{A})^c$ and $b \in \mathfrak{B}$.

It is standard that $\mathcal{L}(E) = (E^* \hat{\otimes} E)^*$ is a dual Banach algebra whenever E is a reflexive Banach space [9]. For a dual Banach algebra $\mathfrak{A}$, we denote by $\Delta_{w^*}(\mathfrak{A})$ the set of all w^*-continuous homomorphisms from $\mathfrak{A}$ onto $\mathbb{C}$.

DEFINITION 3.2. Let $\mathfrak{A}$ be a dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathfrak{A})$. We say that $\mathfrak{A}$ is *φ-injective* if, whenever $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on a reflexive Banach space E, then there is a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^\varphi$.

It should be stressed that Definition 3.2 is in fact a generalisation of the classical definition of injectivity (see Corollary 3.7 below).

Let $\mathfrak{A}$ be a dual Banach algebra. It is known that its *unitisation* $\mathfrak{A}^\sharp = \mathfrak{A} \oplus \mathbb{C}$ is a dual Banach algebra as well. Let $\varphi \in \Delta_{w^*}(\mathfrak{A})$ and let $\varphi^\sharp$ be its unique extension to $\mathfrak{A}^\sharp$. It is obvious that $\varphi^\sharp \in \Delta_{w^*}(\mathfrak{A}^\sharp)$.

THEOREM 3.3. *Suppose that $\mathfrak{A}$ is a dual Banach algebra and that $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then $\mathfrak{A}$ is φ-injective if and only if $\mathfrak{A}^\sharp$ is $\varphi^\sharp$-injective.*

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

Proof. Let $\mathfrak{A}$ be φ-injective and let $\varrho : \mathfrak{A}^\sharp \longrightarrow \mathcal{L}(E)$ be a w^*-continuous representation where E is a reflexive Banach space. Clearly, $\hat{\varrho} = \varrho|_{\mathfrak{A}}$ is a w^*-continuous representation for $\mathfrak{A}$. Hence, there is a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \hat{\varrho}(\mathfrak{A})^\varphi$. Since $\varrho(\mathfrak{A}^\sharp)^{\varphi^\sharp} = \hat{\varrho}(\mathfrak{A})^\varphi$ and $\varrho(\mathfrak{A}^\sharp)^c = \hat{\varrho}(\mathfrak{A})^c$, we are done.

Conversely, suppose that $\mathfrak{A}^\sharp$ is $\varphi^\sharp$-injective and that $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on a reflexive Banach space E. We extend ϱ to $\hat{\varrho}$ from $\mathfrak{A}$ into $\mathfrak{A}^\sharp$ by setting $\hat{\varrho}(a, \lambda) = \varrho(a) + \lambda I_E$ for $a \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. It is readily seen that $\hat{\varrho}$ is a w^*-continuous representation. By the assumption, there is a $\varphi^\sharp$-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \hat{\varrho}(\mathfrak{A}^\sharp)^{\varphi^\sharp}$. Because $\hat{\varrho}(\mathfrak{A}^\sharp)^c = \varrho(\mathfrak{A})^c$ and $\hat{\varrho}(\mathfrak{A}^\sharp)^{\varphi^\sharp} = \varrho(\mathfrak{A})^\varphi$, we conclude that $\mathfrak{A}$ is φ-injective. □

Theorem 3.4. *Suppose that $\mathfrak{A} = (\mathfrak{A}_*)^*$ and $\mathfrak{B} = (\mathfrak{B}_*)^*$ are dual Banach algebras, $\varphi \in \Delta_{w^*}(\mathfrak{A})$ and that $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a w^*-continuous homomorphism. If $\mathfrak{A}$ is φ-amenable, then there exists a φ-quasi expectation $Q : \mathfrak{B} \longrightarrow \theta(\mathfrak{A})^\varphi$.*

Proof. Here we follow the standard argument in [11, Theorem 5.1.24]. Let $E = \mathfrak{B}\hat{\otimes}\mathfrak{B}_*$ be equipped with the $\mathfrak{A}$-bimodule operation given through

$$a \cdot (b \otimes f) = \varphi(a)(b \otimes f) \quad \text{and} \quad (b \otimes f) \cdot a = b \otimes f \cdot \theta(a)$$

for $a \in \mathfrak{A}$, $f \in \mathfrak{B}_*$ and $b \in \mathfrak{B}$. Identifying E^* with $\mathcal{L}(\mathfrak{B})$ as

$$T(b \otimes f) = \langle f, T(b)\rangle \quad (T \in \mathcal{L}(\mathfrak{B}),\ f \in \mathfrak{B}_*,\ b \in \mathfrak{B}),$$

we obtain as the corresponding dual $\mathfrak{A}$-bimodule operation on $\mathcal{L}(\mathfrak{B})$

$$(a \cdot T)(b) = \theta(a)T(b) \quad \text{and} \quad (T \cdot a)(b) = \varphi(a)T(b) \quad (a \in \mathfrak{A},\ b \in \mathfrak{B},\ T \in \mathcal{L}(\mathfrak{B})).$$

Let F be the subspace of E^* consisting of those $T \in E^*$ such that

$$\langle zb \otimes f - b \otimes f \cdot z, T\rangle = 0, \quad \langle bz \otimes f - b \otimes z \cdot f, T\rangle = 0 \quad \text{and} \quad \langle z' \otimes f, T\rangle = 0$$

for all $b \in \mathfrak{B}$, $f \in \mathfrak{B}_*$, $z \in \theta(\mathfrak{A})^c$ and $z' \in \theta(\mathfrak{A})^\varphi$. It is routine to verify that F is a w^*-closed $\mathfrak{A}$-submodule of E^* and thus a dual Banach $\mathfrak{A}$-bimodule in its own right. Considering the derivation $D = ad_{I_{\mathfrak{B}}} : \mathfrak{A} \longrightarrow \mathcal{L}(\mathfrak{B})$, we claim that D attains its values in F. To see this, let $b \in \mathfrak{B}$, $f \in \mathfrak{B}_*$, $z \in \theta(\mathfrak{A})^c$, $z' \in \theta(\mathfrak{A})^\varphi$ and $a \in \mathfrak{A}$. Then

$$\langle z' \otimes f, Da\rangle = \langle z' \otimes f \cdot \theta(a), I_{\mathfrak{B}}\rangle - \varphi(a)\langle z' \otimes f, I_{\mathfrak{B}}\rangle = \langle f, \theta(a)z'\rangle - \langle f, \varphi(a)z'\rangle = 0$$

and

$$\begin{aligned}\langle zb \otimes f - b \otimes f \cdot z, Da\rangle &= \langle zb \otimes f - b \otimes f \cdot z, a \cdot I_{\mathfrak{B}} - I_{\mathfrak{B}} \cdot a\rangle \\ &= \langle (zb \otimes f) \cdot a - (b \otimes f \cdot z) \cdot a - a \cdot (zb \otimes f) + a \cdot (b \otimes f \cdot z), I_{\mathfrak{B}}\rangle \\ &= \langle zb \otimes f \cdot \theta(a) - b \otimes f \cdot z\theta(a) - \varphi(a)zb \otimes f + \varphi(a)b \otimes f \cdot z, I_{\mathfrak{B}}\rangle \\ &= \langle zb, f \cdot \theta(a)\rangle - \langle b, f \cdot z\theta(a)\rangle - \varphi(a)\langle zb, f\rangle + \varphi(a)\langle b, f \cdot z\rangle \\ &= \langle \theta(a)zb, f\rangle - \langle z\theta(a)b, f\rangle - \varphi(a)\langle zb, f\rangle + \varphi(a)\langle zb, f\rangle = 0,\end{aligned}$$

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

because $z \in \theta(\mathfrak{A})^c$. Also,

$$\begin{aligned}\langle bz \otimes f - b \otimes z \cdot f, Da\rangle &= \langle bz \otimes f - b \otimes z \cdot f, a \cdot I_{\mathfrak{B}} - I_{\mathfrak{B}} \cdot a\rangle \\ &= \langle (bz \otimes f) \cdot a - (b \otimes z \cdot f) \cdot a - a \cdot (bz \otimes f) + a \cdot (b \otimes z \cdot f), I_{\mathfrak{B}}\rangle \\ &= \langle bz \otimes f \cdot \theta(a) - b \otimes z \cdot f \cdot \theta(a) - \varphi(a) bz \otimes f + \varphi(a) b \otimes z \cdot f, I_{\mathfrak{B}}\rangle \\ &= \langle bz, f \cdot \theta(a)\rangle - \langle b, z \cdot f \cdot \theta(a)\rangle - \varphi(a)\langle bz, f\rangle + \varphi(a)\langle b, z \cdot f\rangle \\ &= \langle \theta(a) bz, f\rangle - \langle \theta(a) bz, f\rangle - \varphi(a)\langle bz, f\rangle + \varphi(a)\langle bz, f\rangle = 0.\end{aligned}$$

Then, by Proposition 2.2, there exists $\rho \in F$ such that $D = ad_\rho$. Setting $Q = I_{\mathfrak{B}} - \rho$, we see that $a \cdot Q = Q \cdot a$ for all $a \in \mathfrak{A}$. Hence, $\theta(a)Q(b) = \varphi(a)Q(b)$ for $b \in \mathfrak{B}$ and so Q takes values in $\theta(\mathfrak{A})^\varphi$.

Because $\rho \in F$, we have $0 = \langle z' \otimes f, \rho\rangle = \langle \rho(z'), f\rangle$ for $f \in \mathfrak{B}_*$, $z' \in \theta(\mathfrak{A})^\varphi$. That is, $\rho(z') = 0$ and thus $Q(z') = z'$ for each $z' \in \theta(\mathfrak{A})^\varphi$. Therefore, Q is the identity on $\theta(\mathfrak{A})^\varphi$ and thus a projection onto $\theta(\mathfrak{A})^\varphi$.

Next, for each $b \in \mathfrak{B}$, $f \in \mathfrak{B}_*$ and $z \in \theta(\mathfrak{A})^c$,

$$0 = \langle zb \otimes f - b \otimes f \cdot z, \rho\rangle = \langle \rho(zb), f\rangle - \langle \rho(b), f \cdot z\rangle = \langle \rho(zb) - z\rho(b), f\rangle$$

and so $\rho(zb) = z\rho(b)$. Similarly,

$$0 = \langle bz \otimes f - b \otimes z \cdot f, \rho\rangle = \langle \rho(bz), f\rangle - \langle \rho(b), z \cdot f\rangle = \langle \rho(bz) - \rho(b)z, f\rangle,$$

so that $\rho(bz) = \rho(b)z$. Thus,

$$Q(zb) = zb - \rho(zb) = zb - z\rho(b) = zQ(b), \quad Q(bz) = bz - \rho(bz) = bz - \rho(b)z = Q(b)z.$$

We then have $Q(z_1 b z_2) = z_1 Q(b z_2) = z_1 Q(b) z_2$ for $z_1, z_2 \in \theta(\mathfrak{A})^c, b \in \mathfrak{B}$. Therefore, Q is a φ-quasi expectation. □

To establish Theorem 3.6 below, we need some preliminaries from [1, pages 253–255]. Let $\mathfrak{A}$ be a Banach algebra. First, recall that $(\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* = \mathcal{L}(\mathfrak{A}, \mathfrak{A}^*)$, where we choose the convention that $\langle a \otimes b, T\rangle = \langle a, T(b)\rangle$ for $a, b \in \mathfrak{A}$, $T \in \mathcal{L}(\mathfrak{A}, \mathfrak{A}^*)$. Next, let $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ be a (continuous) representation on a reflexive Banach space E. Then $\mathcal{L}(E)$ becomes a Banach $\mathfrak{A}$-bimodule with actions $a \cdot T = \varrho(a)T$ and $T \cdot a = T\varrho(a)$ for $a \in \mathfrak{A}$, $T \in \mathcal{L}(E)$. Also, $\mathcal{L}(E)$ is a Banach $\varrho(\mathfrak{A})^c$-bimodule in the obvious way. We write $\mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$ for the collection of all $\varrho(\mathfrak{A})^c$-bimodule homomorphisms, that is, maps $Q \in \mathcal{L}(\mathcal{L}(E))$ such that $Q(ST) = SQ(T)$ and $Q(TS) = Q(T)S$ for all $S \in \varrho(\mathfrak{A})^c$ and $T \in \mathcal{L}(E)$. We turn $\mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$ into a Banach $\mathfrak{A}$-bimodule by setting

$$(a \,.\, Q)(T) = \varrho(a)Q(T) \quad \text{and} \quad (Q \,.\, a)T = Q(T)\varrho(a)$$

for $a \in \mathfrak{A}$, $T \in \mathcal{L}(E)$ and $Q \in \mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$. We notice that $\mathcal{L}(\mathcal{L}(E))$ is a dual Banach algebra with predual $\mathcal{L}(E)\widehat{\otimes}(E\widehat{\otimes}E^*)$. Let $X \subseteq \mathcal{L}(E)\widehat{\otimes}(E\widehat{\otimes}E^*)$ be the closure of the linear span of the set consisting of all elements of the form $ST \otimes x \otimes \mu - T \otimes x \otimes S^*(\mu)$ and $TS \otimes x \otimes \mu - T \otimes S(x) \otimes \mu$ for all $S \in \varrho(\mathfrak{A})^c$, $T \in \mathcal{L}(E)$, $x \in E$, $\mu \in E^*$. Because $X^\perp = \mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$, we see that $\mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$ is a dual Banach algebra with the predual

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

$Y = \mathcal{L}(E)\widehat{\otimes}E\widehat{\otimes}E^*/X$. Now define $\psi : Y \longrightarrow \mathcal{L}(\mathfrak{A}, \mathfrak{A}^*)$ via

$$\langle a \otimes b, \psi(T \otimes x \otimes \mu + X)\rangle = \langle \varrho(a)T\varrho(b)(x), \mu\rangle \quad (a, b \in \mathfrak{A},\ x \in E,\ \mu \in E^*,\ T \in \mathcal{L}(E)).$$

We turn $\mathcal{L}(E)\widehat{\otimes}E\widehat{\otimes}E^*$ into a Banach $\mathfrak{A}$-bimodule through

$$a \cdot (T \otimes x \otimes \mu) = T \otimes \varrho(a)(x) \otimes \mu \quad \text{and} \quad (T \otimes x \otimes \mu) \cdot a = T \otimes x \otimes \varrho(a)^*(\mu)$$

for $a \in \mathfrak{A}, x \in E, \mu \in E^*, T \in \mathcal{L}(E)$. Observe that ψ is an $\mathfrak{A}$-bimodule homomorphism.

The next proposition shows that it is possible to choose E to make ψ a bijection onto $\sigma wc(\mathcal{L}(\mathfrak{A}, \mathfrak{A}^*))$.

PROPOSITION 3.5 [1, Theorem 6.11]. *Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a unital dual Banach algebra. There exist a reflexive normal Banach left $\mathfrak{A}$-module E and an isometric w^*-continuous representation $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ such that ψ (associated with ϱ as above) maps into $\sigma wc(\mathcal{L}(\mathfrak{A}, \mathfrak{A}^*))$ and is a bijection. In particular, $\psi^* : \sigma wc(\mathcal{L}(\mathfrak{A}, \mathfrak{A}^*))^* \longrightarrow \mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$ is an isomorphism.*

Let $\mathfrak{A}$ be a dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathfrak{A})$. From [6], $\mathfrak{A}$ is *φ-Connes amenable* if there exists a bounded linear functional m on $\sigma wc(\mathfrak{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f \,.\, a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \sigma wc(\mathfrak{A}^*)$.

The following result could be compared with [1, Theorem 6.13].

THEOREM 3.6. *Suppose that $\mathfrak{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then the following are equivalent:*

(i) *$\mathfrak{A}$ is φ-amenable;*
(ii) *$\mathfrak{A}$ is φ-contractible (in the sense of [2]);*
(iii) *$\mathfrak{A}$ is φ-Connes amenable;*
(iv) *$\mathfrak{A}$ is φ-injective.*

PROOF. The equivalence (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) is [7, Theorem 2.4].

(i) $\Longrightarrow$ (iv) Suppose that $\mathfrak{A}$ is φ-amenable and $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on some reflexive Banach space E. By Theorem 3.4, there is a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^\varphi$, that is, $\mathfrak{A}$ is φ-injective.

(iv) $\Longrightarrow$ (iii) Suppose that $\mathfrak{A}$ is φ-injective. By Theorem 3.3 and [4, Lemma 3.2], without loss of generality, we may suppose that $\mathfrak{A}$ is unital. Take the w^*-continuous representation $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ and the map ψ as in Proposition 3.5. By the assumption, there exists a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^\varphi$. Notice that $Q \in \mathcal{L}_{\mathfrak{A}}(\mathcal{L}(E))$. Define $M := (\psi^*)^{-1}(Q) \in \sigma wc((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)^*$. As Q maps into $\varrho(\mathfrak{A})^\varphi$, it follows that $a \,.\, Q = \varphi(a)Q$ for $a \in \mathfrak{A}$, so that $a \,.\, M = \varphi(a)M$. Next, for some $\alpha \in \mathbb{C}$, we have $\langle \varphi \otimes \varphi, M\rangle = \alpha$. Hence, putting $N = (1/\alpha)M$, it is readily seen that $\langle \varphi \otimes \varphi, N\rangle = 1$ and $a \,.\, N = \varphi(a)N$ for $a \in \mathfrak{A}$. On the other hand, from [10], $\pi^*(\sigma wc(\mathfrak{A}^*)) \subseteq \sigma wc((\mathfrak{A}\hat{\otimes}\mathfrak{A})^*)$. We then set $m := (\pi^*|_{\sigma wc(\mathfrak{A}^*)})^*(N) \in \sigma wc(\mathfrak{A}^*)^*$. One may check that $m(\varphi) = 1$ and $m(f \,.\, a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \sigma wc(\mathfrak{A}^*)$. Thus, $\mathfrak{A}$ is φ-Connes amenable. □

COROLLARY 3.7. *An injective dual Banach algebra $\mathfrak{A}$ is φ-injective for all $\varphi \in \Delta_{w^*}(\mathfrak{A})$.*

PROOF. Since $\mathfrak{A}$ is injective, it is Connes amenable [1, Theorem 6.13]. It then follows from [6, Theorem 2.2] that $\mathfrak{A}$ is φ-Connes amenable for each $\varphi \in \Delta_{w^*}(\mathfrak{A})$. The result is now immediate by Theorem 3.6. □

4. Application to WAP$(\mathfrak{A}^*)^*$ and examples

The following result is analogous to [4, Proposition 3.5].

THEOREM 4.1. *Suppose that $\mathfrak{A}$ is a Banach algebra, $\mathfrak{B} = (\mathfrak{B}_*)^*$ is a dual Banach algebra, $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a continuous homomorphism with w^*-dense range and $\varphi \in \Delta_{w^*}(\mathfrak{B})$. If $\mathfrak{A}$ is $\varphi \circ \theta$-amenable, then $\mathfrak{B}$ is φ-amenable.*

PROOF. Take $m \in \mathfrak{A}^{**}$ with $m(\varphi \circ \theta) = 1$ and $m(f \cdot a) = (\varphi \circ \theta)(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. Define $n \in \sigma wc(\mathfrak{B}^*)^*$ by $n(g) = m(g \circ \theta)$ for $g \in \sigma wc(\mathfrak{B}^*)$. Note that $\varphi \in \sigma wc(\mathfrak{B}^*)$ as $\varphi \in \mathfrak{B}_*$ (see also [7, Lemma 2.3]). Then $n(\varphi) = m(\varphi \circ \theta) = 1$. For $a, a' \in \mathfrak{A}$ and $g \in \sigma wc(\mathfrak{B}^*)$,

$$n(g \cdot \theta(a)) = m((g \cdot \theta(a)) \circ \theta) = m((g \circ \theta) \cdot a) = (\varphi \circ \theta)(a)m(g \circ \theta) = (\varphi \circ \theta)(a)n(g),$$

because

$$\langle (g \cdot \theta(a)) \circ \theta, a' \rangle = \langle g, \theta(a)\theta(a') \rangle = \langle (g \circ \theta) \cdot a, a' \rangle.$$

Next, for an arbitrary element $b \in \mathfrak{B}$, there is a net $(a_i)_i \subseteq \mathfrak{A}$ such that $\theta(a_i) \xrightarrow{w^*} b$. For each $g \in \sigma wc(\mathfrak{B}^*)$, we then have $g \cdot \theta(a_i) \xrightarrow{wk} g \cdot b$. Hence,

$$n(g \cdot b) = \lim_i n(g \cdot \theta(a_i)) = \lim_i \varphi(\theta(a_i))n(g) = \varphi(b)n(g).$$

Thus, $\mathfrak{B}$ is φ-amenable, by Theorem 3.6(iii). □

REMARK 4.2. Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. By Proposition 2.3, there exists a unique element $\tilde{\varphi} \in \Delta_{w^*}(\mathrm{WAP}(\mathfrak{A}^*)^*)$ extending φ. We shall henceforth keep the notation $\tilde{\varphi}$.

COROLLARY 4.3. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is φ-amenable, then WAP$(\mathfrak{A}^*)^*$ is $\tilde{\varphi}$-amenable.*

PROOF. As $\varphi = \tilde{\varphi} \circ \iota$, this is a consequence of Remark 4.2 and Theorem 4.1. □

EXAMPLE 4.4. Let G be a locally compact group and let $A(G)$ and $VN(G) = A(G)^*$ be the Fourier algebra and the von Neumann algebra of G, respectively. From [4, Example 2.6], $A(G)$ is φ_t-amenable for every $t \in G$, where φ_t is the point evaluation at $t \in G$, that is, $\varphi_t(f) = f(t)$, $f \in A(G)$. So, by Corollary 4.3, WAP$(VN(G))^*$ is $\tilde{\varphi}_t$-amenable for every $t \in G$.

The converse of Corollary 4.3 holds for dual Banach algebras as follows.

THEOREM 4.5. *Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then $\mathfrak{A}$ is φ-amenable if and only if WAP$(\mathfrak{A}^*)^*$ is $\tilde{\varphi}$-amenable.*

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

PROOF. Since $\mathfrak{A}_* \subseteq \sigma wc(\mathfrak{A}^*) \subseteq \mathrm{WAP}(\mathfrak{A}^*)$ from [10], there exists an inclusion map $\varepsilon : \mathfrak{A}_* \longrightarrow \mathrm{WAP}(\mathfrak{A}^*)$. Then ε^* is an $\mathfrak{A}$-bimodule homomorphism from $\mathrm{WAP}(\mathfrak{A}^*)^*$ onto $\mathfrak{A}$.

Suppose that $\mathrm{WAP}(\mathfrak{A}^*)^*$ is $\tilde{\varphi}$-amenable. Let E be a Banach $\mathfrak{A}$-bimodule for which $a\,.\,x = \varphi(a)x$ for all $a \in \mathfrak{A}$ and $x \in E$ and let $D : \mathfrak{A} \longrightarrow E^*$ be a derivation. We turn E into a Banach $\mathrm{WAP}(\mathfrak{A}^*)^*$-bimodule through

$$\Lambda\,.\,x := \tilde{\varphi}(\Lambda)x \quad \text{and} \quad x\,.\,\Lambda := \varepsilon^*(\Lambda)\,.\,x \quad (x \in E,\ \Lambda \in \mathrm{WAP}(\mathfrak{A}^*)^*)\,.$$

Now, by Proposition 2.2, the derivation $D\varepsilon^* : \mathrm{WAP}(\mathfrak{A}^*)^* \longrightarrow E^*$ is inner. Thus, there exists $x \in E$ such that $(D\varepsilon^*)(\Lambda) = \Lambda\,.\,x - x\,.\,\Lambda$ for all $\Lambda \in \mathrm{WAP}(\mathfrak{A}^*)^*$. Consequently, $Da = a\,.\,x - x\,.\,a$, $a \in \mathfrak{A}$. Again by Proposition 2.2, $\mathfrak{A}$ is φ-amenable. □

We write $\mathbb{D}$ for the open unit disk. For the discrete convolution algebra $\ell^1(\mathbb{Z}^+)$, it is known that $\Delta(\ell^1(\mathbb{Z}^+)) \equiv \bar{\mathbb{D}}$ under the bijective map $z \longmapsto \varphi_z$, where φ_z is the point evaluation at z, that is, $\varphi_z(\sum_{n=0}^{\infty} c_n\delta_n) = \sum_{n=0}^{\infty} c_n z^n$. It is not hard to see that $\Delta_{w^*}(\ell^1(\mathbb{Z}^+)) = \mathbb{D}$. It was shown in [4, Example 2.5] that $\ell^1(\mathbb{Z}^+)$ is φ_z-amenable when $|z| = 1$ and it is not φ_z-amenable if $z \in \mathbb{D}$. Hence, by Corollary 4.3, $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is $\tilde{\varphi}_z$-amenable when $|z| = 1$. As $\ell^1(\mathbb{Z}^+)$ is a dual Banach algebra, we conclude from Theorem 4.5 that $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is not $\tilde{\varphi}_z$-amenable for each $z \in \mathbb{D}$. Notice that $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is not amenable. To see this, we first observe that there exists a continuous homomorphism from $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ onto $\ell^1(\mathbb{Z}^+)$ by the universal property (with $\ell^1(\mathbb{Z}^+)$ and the identity map in place of $\mathfrak{B}$ and θ, respectively). Therefore, amenability of $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ forces $\ell^1(\mathbb{Z}^+)$ to be amenable, which is not the case.

Putting all these results together gives the following example.

EXAMPLE 4.6. (i) $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is not amenable;
(ii) $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is not $\tilde{\varphi}_z$-amenable for each $z \in \mathbb{D}$;
(iii) $\mathrm{WAP}(\ell^\infty(\mathbb{Z}^+))^*$ is $\tilde{\varphi}_z$-amenable when $|z| = 1$.

Let $\mathfrak{A}$ be a Banach algebra and let $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ be a continuous representation on a Banach space E. We use $\tilde{\varrho} : \mathrm{WAP}(\mathfrak{A}^*)^* \longrightarrow \mathcal{L}(E)$ for the unique w^*-continuous representation obtained by Proposition 2.3.

LEMMA 4.7. *Let $\mathfrak{A}$ be a Banach algebra, let $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ be a continuous representation and $\varphi \in \Delta(\mathfrak{A})$. Then every φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^\varphi$ is exactly a $\tilde{\varphi}$-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \tilde{\varrho}(WAP(\mathfrak{A}^*)^*)^{\tilde{\varphi}}$ and vice versa.*

PROOF. The same argument as that in the proof of [1, Proposition 6.15] shows that $\varrho(\mathfrak{A})^c = \tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^c$. To complete the proof, we show that $\tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^{\tilde{\varphi}} = \varrho(\mathfrak{A})^\varphi$. It is obvious that $\tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^{\tilde{\varphi}} \subseteq \varrho(\mathfrak{A})^\varphi$. For the converse, suppose that $T \in \varrho(\mathfrak{A})^\varphi$. Thus, $\langle \varrho(a)T, \eta\rangle = \varphi(a)\langle T, \eta\rangle$ for each $a \in \mathfrak{A}$ and $\eta \in E^*\hat{\otimes}E$. Take $\Psi \in \mathrm{WAP}(\mathfrak{A}^*)^*$ and take a bounded net $(a_i) \subseteq \mathfrak{A}$ which converges to Ψ in the w^*-topology on $\mathrm{WAP}(\mathfrak{A}^*)^*$. Then, for $x \in E$, $\mu \in E^*$ and $T \in \varrho(\mathfrak{A})^\varphi$,

$$\begin{aligned}\langle \mu, \tilde{\varrho}(\Psi)T(x)\rangle &= \langle \mu \otimes T(x), \tilde{\varrho}(\Psi)\rangle = \langle \Psi, \varrho_*(\mu \otimes T(x))\rangle = \lim_i \langle \varrho_*(\mu \otimes T(x)), a_i\rangle \\ &= \lim_i \langle \mu, \varrho(a_i)T(x)\rangle = \lim_i \varphi(a_i)\langle \mu, T(x)\rangle = \langle \mu, \tilde{\varphi}(\Psi)T(x)\rangle,\end{aligned}$$

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

so that $T \in \tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^{\tilde{\varphi}}$, as required. □

The next step is a useful characterisation.

THEOREM 4.8. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. Then the following are equivalent:*

(i) *$WAP(\mathfrak{A}^*)^*$ is $\tilde{\varphi}$-amenable;*
(ii) *whenever $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a continuous representation on a reflexive Banach space E, there exists a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^{\varphi}$.*

PROOF. (i) $\Longrightarrow$ (ii) Let $\varrho : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ be a continuous representation on a reflexive Banach space E and let $\tilde{\varrho} : \mathrm{WAP}(\mathfrak{A}^*)^* \longrightarrow \mathcal{L}(E)$ be its unique extension to a w^*-continuous representation. By Theorem 3.6, $\mathrm{WAP}(\mathfrak{A}^*)^*$ is $\tilde{\varphi}$-injective and there exists a $\tilde{\varphi}$-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^{\tilde{\varphi}}$ by Definition 3.2. Now, by Lemma 4.7, $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^{\varphi}$ is indeed a φ-quasi expectation.

(ii) $\Longrightarrow$ (i) Suppose that $\varrho : \mathrm{WAP}(\mathfrak{A}^*)^* \longrightarrow \mathcal{L}(E)$ is a w^*-continuous representation on a reflexive Banach space E. Thus, $\varrho|_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ is a continuous representation. By the assumption, there exists a φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\mathfrak{A})^{\varphi}$. Again by Lemma 4.7, $Q : \mathcal{L}(E) \longrightarrow \tilde{\varrho}(\mathrm{WAP}(\mathfrak{A}^*)^*)^{\tilde{\varphi}}$ is a $\tilde{\varphi}$-quasi expectation, as required. □

5. For $\mathrm{WAP}(\ell^1(\mathbb{N}_\wedge)^*)^*$

Let $\mathbb{N}_\wedge$ be the semigroup $\mathbb{N}$ with the product $m \wedge n = \min\{m, n\}$ for $m, n \in \mathbb{N}$. In this section, we write φ for the *augmentation character* on $\ell^1(\mathbb{N}_\wedge)$, which is given by $\varphi(\sum_{n=1}^{\infty} \alpha_i \delta_i) = \sum_{n=1}^{\infty} \alpha_i$. In the light of Theorem 4.8, we will show that $\mathrm{WAP}(\ell^1(\mathbb{N}_\wedge)^*)^*$ is not $\tilde{\varphi}$-amenable. To this end, some preliminaries are needed.

Let E be a Banach space with a normalised basis $(e_n)_n$. For each $n \in \mathbb{N}$, we consider the linear functional $f_n \in E^*$, $n \in \mathbb{N}$, given by $\langle f_n, \sum \alpha_i e_i \rangle = \alpha_n$. Throughout the section, we use the notation ϱ for the representation $\varrho : \ell^1(\mathbb{N}_\wedge) \longrightarrow \mathcal{L}(E)$ given by

$$\varrho(\delta_n)(e_m) = \begin{cases} e_m & \text{for } m \leq n \\ 0 & \text{for } m > n \end{cases} \qquad (m, n \in \mathbb{N})$$

and linearity. In fact, $\varrho(\delta_n)$ is the projection onto the linear span of $\{e_1, \ldots, e_n\}$. It is standard that each element of $\mathcal{L}(E)$ can be considered as a matrix with respect to the basis $(e_n)_n$. We denote by $\mathcal{E}_{i,j}$ the matrix with 1 in the (i, j)th place and 0 elsewhere.

The next result shows that the set $\varrho(\ell^1(\mathbb{N}_\wedge))^{\varphi}$ consists of all matrices in $\mathcal{L}(E)$ with zero entries from the second row on.

LEMMA 5.1. $\varrho(\ell^1(\mathbb{N}_\wedge))^{\varphi} = \{T = (a_{i,j})_{i,j} \in \mathcal{L}(E) : a_{i,j} = 0$ *for* $i \geq 2\}$.

PROOF. Set $\mathcal{R} = \{T = (a_{i,j})_{i,j} \in \mathcal{L}(E) : a_{i,j} = 0 \text{ for } i \geq 2\}$ and notice that $\varrho(\ell^1(\mathbb{N}_\wedge))^{\varphi} = \{T \in \mathcal{L}(E) : \varrho(\delta_n)T = \varphi(\delta_n)T = T\}$. It is easily checked that $\mathcal{R} \subseteq \varrho(\ell^1(\mathbb{N}_\wedge))^{\varphi}$.

Conversely, for $T \in \varrho(\ell^1(\mathbb{N}_\wedge))^{\varphi}$,

$$\mathcal{E}_{n,n}T = (\varrho(\delta_n) - \varrho(\delta_{n-1}))T = \varrho(\delta_n)T - \varrho(\delta_{n-1})T = T - T = 0 \quad (n \geq 2).$$

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

A simple verification then shows that

$$0 = \mathcal{E}_{n,n}T(e_m) = \langle f_n, T(e_m)\rangle e_n \quad (m \geq 1,\ n \geq 2)$$

and therefore $\langle f_n, T(e_m)\rangle = 0$ for $m \geq 1, n \geq 2$. So, $T(e_m) \in \mathbb{C}e_1$ for each $m \geq 1$, which proves that $T \in \mathcal{R}$. □

REMARK 5.2. Compared to Lemma 5.1, $\varrho(\ell^1(\mathbb{N}_\wedge))^c$ is exactly the set of all diagonal matrices in $\mathcal{L}(E)$ [1].

We write $P_\varphi : \mathcal{L}(E) \longrightarrow \varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$ for the *canonical projection onto* $\varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$ defined by $T = (a_{i,j})_{i,j} \longmapsto P_\varphi(T) = (b_{i,j})_{i,j}$, where $b_{1,j} = a_{1,j}$ and $b_{i,j} = 0$ for $i \geq 2$ and all j. Next, we show that every φ-quasi expectation $Q : \mathcal{L}(E) \longrightarrow \varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$ must be the canonical projection onto $\varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$.

LEMMA 5.3. *Let $Q : \mathcal{L}(E) \longrightarrow \varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$ be a φ-quasi expectation. Then $Q = P_\varphi$.*

PROOF. Let $m, n \in \mathbb{N}$ and $T \in \mathcal{L}(E)$. From Remark 5.2, $\mathcal{E}_{n,n}, \mathcal{E}_{m,m} \in \varrho(\ell^1(\mathbb{N}_\wedge))^c$. Then

$$\langle f_m, T(e_n)\rangle Q(\mathcal{E}_{m,n}) = Q(\mathcal{E}_{m,m}T\mathcal{E}_{n,n}) = \mathcal{E}_{m,m}Q(T)\mathcal{E}_{n,n} = \langle f_m, Q(T)(e_n)\rangle \mathcal{E}_{m,n}.$$

As $\mathcal{E}_{m,n}$ is not in $\varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$ for $m > 1$ by Lemma 5.1, it follows that $\langle f_m, Q(T)(e_n)\rangle = 0$ for $m > 1$. Thus, $Q(T)(e_n) \in \mathbb{C}e_1$ for each n. Next, since $\mathcal{E}_{1,n} \in \varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$,

$$\begin{aligned}\langle f_1, T(e_n)\rangle \mathcal{E}_{1,n} = \langle f_1, T(e_n)\rangle Q(\mathcal{E}_{1,n}) &= Q(\mathcal{E}_{1,1}T\mathcal{E}_{n,n}) \\ &= \mathcal{E}_{1,1}Q(T)\mathcal{E}_{n,n} = \langle f_1, Q(T)(e_n)\rangle \mathcal{E}_{1,n}\end{aligned}$$

and hence $Q(T)(e_n) = \langle f_1, T(e_n)\rangle e_1$ for each $n \in \mathbb{N}$, as required. □

THEOREM 5.4. *The algebra $WAP(\ell^1(\mathbb{N}_\wedge)^*)^*$ is not $\tilde{\varphi}$-amenable.*

PROOF. By Theorem 4.8 and Lemma 5.3, it suffices to find a reflexive Banach space E such that P_φ is not bounded. It is clear that there is an isometric isomorphism Θ from $\varrho(\ell^1(\mathbb{N}_\wedge))^c$ onto $\varrho(\ell^1(\mathbb{N}_\wedge))^\varphi$. From [1, Theorem 7.6], there is a reflexive Banach space E for which the canonical projection $P_c : \mathcal{L}(E) \longrightarrow \varrho(\ell^1(\mathbb{N}_\wedge))^c$ is not bounded. Thus, $P_\varphi = \Theta \circ P_c$ is not bounded, as required. □

A combination of Corollary 4.3 and Theorem 5.4 yields the following result.

COROLLARY 5.5. *The algebra $\ell^1(\mathbb{N}_\wedge)$ is not φ-amenable.*

Appendix A

In this section, we show that $\Delta_{w^*}(\mathrm{WAP}(\mathfrak{A}^*)^*)$ contains $\Delta(\mathfrak{A})$ as a subset, as pointed out by an anonymous referee in response to a previous version of this work.

PROPOSITION A.1. *Let $\mathfrak{A}$ be a Banach algebra. Then $\Delta(\mathfrak{A}) \subseteq \Delta_{w^*}(WAP(\mathfrak{A}^*)^*)$.*

PROOF. Take $\varphi \in \Delta(\mathfrak{A})$, so that $\varphi \in \mathfrak{A}^*$. Then $\langle b, a \cdot \varphi\rangle = \varphi(a)\varphi(b)$ for every $a, b \in \mathfrak{A}$, so that $a \cdot \varphi = \varphi(a)\varphi$. Similarly, $\varphi \cdot a = \varphi(a)\varphi$. So, obviously, $\varphi \in \mathrm{WAP}(\mathfrak{A}^*)$. Hence, we may treat φ as a bounded linear map on $\mathrm{WAP}(\mathfrak{A}^*)^*$. As a consequence, φ is

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press

w^*-continuous. Next, for $\Psi \in \mathrm{WAP}(\mathfrak{A}^*)^*$ and $a \in \mathfrak{A}$, it follows that $\langle a, \Psi \cdot \varphi \rangle = \langle \varphi \cdot a, \Psi \rangle = \varphi(a) \langle \varphi, \Psi \rangle$, so that $\Psi \cdot \varphi = \langle \varphi, \Psi \rangle \varphi$. Then, for each $\Phi, \Psi \in \mathrm{WAP}(\mathfrak{A}^*)^*$,

$$\langle \varphi, \Phi\Psi \rangle = \langle \Psi \cdot \varphi, \Phi \rangle = \langle \langle \varphi, \Psi \rangle \varphi, \Phi \rangle = \langle \varphi, \Psi \rangle \langle \varphi, \Phi \rangle.$$

Thus, $\varphi \in \Delta_{w^*}(\mathrm{WAP}(\mathfrak{A}^*)^*)$. □

The following consequence should be compared with Corollary 4.3.

COROLLARY A.2. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is $\varphi \circ \iota$-amenable, then WAP$(\mathfrak{A}^*)^*$ is φ-amenable.*

PROOF. This is immediate by Proposition A.1 and Theorem 4.1. □

References

[1] M. Daws, 'Dual Banach algebras: representations and injectivity', *Studia Math.* **178** (2007), 231–275.

[2] Z. Hu, M. S. Monfared and T. Traynor, 'On character amenable Banach algebras', *Studia Math.* **193** (2009), 53–78.

[3] B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs of the American Mathematical Society, 127 (American Mathematical Society, Providence, RI, 1972).

[4] E. Kaniuth, A. T. Lau and J. Pym, 'On φ-amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 85–96.

[5] A. T. Lau, 'Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups', *Fund. Math.* **118** (1983), 161–175.

[6] A. Mahmoodi, 'On φ-Connes amenability of dual Banach algebras', *J. Linear Topol. Algebra* **3** (2014), 211–217.

[7] A. Mahmoodi, 'φ-contractibility and φ-Connes amenability coincide with some older notions', *Bull. Aust. Math. Soc.* **97** (2018), 274–278.

[8] M. S. Monfared, 'Character amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 697–706.

[9] V. Runde, 'Amenability for dual Banach algebras', *Studia Math.* **148** (2001), 47–66.

[10] V. Runde, 'Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule', *Math. Scand.* **95** (2004), 124–144.

[11] V. Runde, *Amenable Banach Algebras: A Panorama*, Springer Monographs in Mathematics (Springer, New York, 2020).

ALIREZA JABERI, Department of Mathematics, Central Tehran Branch,
Islamic Azad University, Tehran, Iran
e-mail: alirezajaberii1350@gmail.com

AMIN MAHMOODI, Department of Mathematics, Central Tehran Branch,
Islamic Azad University, Tehran, Iran
e-mail: a_mahmoodi@iauctb.ac.ir

https://doi.org/10.1017/S0004972721000538 Published online by Cambridge University Press