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On the Lagrange and Markov dynamical spectra

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Abstract. We consider the Lagrange and the Markov dynamical spectra associated to horseshoes on a surface with Hausdorff dimension greater than one. We show that for a 'large' set of real functions on the surface and for 'typical' horseshoes with Hausdorff dimension greater than one, both the Lagrange and the Markov dynamical spectra have persistently non-empty interior.

1. *Introduction*

Regular Cantor sets on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined by expansive maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see precise definition in Appendix A). Some background on the regular Cantor sets, relevant to our work, can be found in [**CF89**, **PT93**, **MY01**, **MY10**].

A mathematical object intimately related to our work (cf. [**CF89**]), is the classical Lagrange spectrum, which is defined as follows: given an irrational number α, according to Dirichlet's theorem, the inequality $|\alpha - p/q| < 1/q^2$ has infinite rational solutions p/q. Markov and Hurwitz improved this result (cf. [**CF89**]), by verifying that, for all irrational α, the inequality $|\alpha - p/q| < 1/(\sqrt{5}q^2)$ has an infinite number of rational solutions p/q.

Meanwhile, for a fixed irrational α, better results can be expected. This leads us to associate to each α its best constant of approximation (Lagrange value of α), given by

$$k(\alpha) = \sup\left\{k > 0 : \left|\alpha - \frac{p}{q}\right| < \frac{1}{kq^2} \text{ has infinitely many rational solutions } \frac{p}{q}\right\}$$
$$= \limsup_{\substack{|p|,q\to\infty \\ p\in\mathbb{Z},q\in\mathbb{N}}} |q(q\alpha - p)|^{-1} \in \mathbb{R}\cup\{+\infty\}.$$

Then we always have $k(\alpha) \geq \sqrt{5}$. Consider the set

$$L = \{k(\alpha) : \alpha \in \mathbb{R}\backslash\mathbb{Q} \text{ and } k(\alpha) < \infty\},$$

known as the *Lagrange spectrum* (for properties of L, cf. [**CF89**]).

In 1947, Hall (cf. [**Hal47**]) proved that for the regular Cantor set $C(4)$ of the real numbers in [0, 1], which only has the coefficients 1, 2, 3, 4 in its continued fraction,

$$C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2} - 1)].$$

Let α be an irrational number expressed in continued fractions by $\alpha = [a_0, a_1, \ldots]$. Define, for each $n \in \mathbb{N}$, $\alpha_n = [a_n, a_{n+1}, \ldots]$ and $\beta_n = [0, a_{n-1}, a_{n-2}, \ldots]$. Using elementary continued fraction techniques it can be proved that

$$k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n).$$

With this latter characterization of the Lagrange spectrum and from Hall's result, it follows that $L \supset [6, +\infty)$; the Lagrange spectrum contains a whole half-line, called *Hall's ray*.

In 1975, Freiman (cf. [**Fre75**, **CF89**]) proved some difficult results showing that the arithmetic sum of certain (regular) Cantor sets, related to continued fractions, contain intervals, and he used them to determined the precise beginning of Hall's ray (the biggest half-line contained in L), which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4, 52782956616\ldots.$$

Another interesting set is the classical *Markov spectrum* defined by (cf. [**CF89**])

$$M = \left\{\inf_{(x,y)\in\mathbb{Z}^2\backslash(0,0)} |f(x, y)|^{-1} : f(x, y) = ax^2 + bxy + cy^2 \text{ with } b^2 - 4ac = 1\right\}.$$

Both the Lagrange and Markov spectra have a dynamical interpretation. This fact is an important motivation for our work.

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and $\sigma : \Sigma \to \Sigma$, which is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. If $f : \Sigma \to \mathbb{R}$ is defined by $f((a_n)_{n\in\mathbb{Z}}) = \alpha_0 + \beta_0 = [a_0, a_1, \ldots] + [0, a_{-1}, a_{-2}, \ldots]$, then

$$L = \left\{\limsup_{n\to\infty} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma\right\}$$

and

$$M = \left\{\sup_{n\in\mathbb{Z}} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma\right\}.$$

This last interpretation, in terms of shift, admits a natural generalization of the Lagrange and Markov spectra in the context of hyperbolic dynamics (at least in dimension two, which is the focus of this work).

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We will define the Lagrange and Markov dynamical spectra as follows. Let $\varphi : M^2 \to M^2$ be a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ. Let $f : M^2 \to \mathbb{R}$ be a continuous real function: then the *Lagrange dynamical spectrum* associated to (f, Λ) is defined by

$$L(f, \Lambda) = \left\{ \limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda \right\},$$

and the *Markov dynamical spectrum* associated to (f, Λ) is defined by

$$M(f, \Lambda) = \left\{ \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) : x \in \Lambda \right\}.$$

The problem of finding intervals in the classical Lagrange and Markov spectra is closely related to the study of the fractal geometry of regular Cantor sets related to the Gauss map. Fractal geometry of Cantor sets is also the key to solving some problems about dynamical Lagrange and Markov spectra in dimension two. In fact, using results on stable intersections of two regular Cantor sets for which the sum of Hausdorff dimensions is greater than one (cf. [**MY01**, **MY10**]), we prove the following theorem.

MAIN THEOREM. *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then, arbitrarily close to φ, there is a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the continuation of Λ associated to $\psi \in W$, there is an open and dense set $H_\psi \subset C^1(M, \mathbb{R})$ such that for all $f \in H_\psi$,*

$$\operatorname{int} L(f, \Lambda_\psi) \neq \emptyset \quad \textit{and} \quad \operatorname{int} M(f, \Lambda_\psi) \neq \emptyset,$$

where int A *denotes the interior of* A.

Remark. In the previous statement, by horseshoe we mean a compact, locally maximal, transitive hyperbolic invariant set of saddle type (and so it contains a dense subset of periodic orbits).

2. *Preliminaries*

2.1. *Preliminaries from dynamical systems.*

If Λ is a hyperbolic set associated to a C^2-diffeomorphism, then the stable and unstable foliations $\mathcal{F}^s(\Lambda)$ and $\mathcal{F}^u(\Lambda)$ are C^1. Moreover, these foliations can be extended to C^1 foliations defined on a full neighborhood of Λ (cf. [**KH95**, p. 604]).

Let Λ be a horseshoe of φ and consider a finite collection $(R_a)_{a\in\mathbb{A}}$ of disjoint rectangles of M, which form a Markov partition of Λ (cf. [**Shu86**, p. 129]). The set $\mathbb{B} \subset \mathbb{A}^2$ of admissible transitions consist of pairs (a_0, a_1) such that $\varphi(R_{a_0}) \cap R_{a_1} \neq \emptyset$. We can define the following transition matrix B, which induces the same transitions as $\mathbb{B} \subset \mathbb{A}^2$, as

$$b_{a_i a_j} = 1 \quad \text{if } \varphi(R_{a_i}) \cap R_{a_j} \neq \emptyset, \qquad b_{a_i a_j} = 0 \quad \text{otherwise, for } (a_i, a_j) \in \mathbb{A}^2.$$

Let $\Sigma_{\mathbb{A}} = \{\underline{a} = (a_n)_{n\in\mathbb{Z}} : a_n \in \mathbb{A} \forall n \in \mathbb{Z}\}$. We can define the homeomorphism of $\Sigma_{\mathbb{A}}$ as the shift $\sigma : \Sigma_{\mathbb{A}} \to \Sigma_{\mathbb{A}}$ where $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

Let $\Sigma_B = \{\underline{a} \in \Sigma_{\mathbb{A}} : b_{a_n a_{n+1}} = 1\}$; this set is a closed and σ-invariant subspace of $\Sigma_{\mathbb{A}}$. Still denote by σ the restriction of σ to Σ_B. The pair (Σ_B, σ) is called a subshift of finite type of $(\Sigma_{\mathbb{A}}, \sigma)$. Given $x, y \in \mathbb{A}$, we denote by $N_n(x, y, B)$ the number of admissible strings for B of length $n+1$ that begin at x and end at y. Then

$$N_n(x, y, B) = b^n_{xy}.$$

In particular, since $\varphi|_\Lambda$ is transitive, given $x, y \in \mathbb{A}$, there always exists a minimum number $n(x, y) \in \mathbb{N}^*$ such that $N_{n(x,y)}(x, y, B) > 0$ and, putting $N_0 := \max\{n(x, y) : x, y \in \mathbb{A}\}$ for all $x, y \in \mathbb{A}$, there is a word beginning at x and ending at y of length less or equal to $N_0 + 1$.

Subshifts of finite type also have a sort of local product structure. First, we define the local stable and unstable sets (cf. [**Shu86**, Ch. 10])

$$\begin{aligned} W^s_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\, d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\, a_n = b_n\}, \\ W^u_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\, d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\, a_n = b_n\}, \end{aligned}$$

where $d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} 2^{-(2|n|+1)} \delta_n(\underline{a}, \underline{b})$, and $\delta_n(\underline{a}, \underline{b})$ is zero when $a_n = b_n$ and one otherwise. So, if $\underline{a}, \underline{b} \in \Sigma_B$ and $d(\underline{a}, \underline{b}) < 1/2$, then $a_0 = b_0$ and $W^u_{1/3}(\underline{a}) \cap W^u_{1/3}(\underline{b})$ is a unique point, denoted by the bracket $[\underline{a}, \underline{b}] = (\ldots, b_{-n}, \ldots, b_{-1}, b_0, a_1, \ldots, a_n, \ldots)$.

If φ is a diffeomorphism of a surface (2-manifold), then the dynamics of φ on Λ is topologically conjugate to a subshift Σ_B defined by B: namely, there is a homeomorphism $\Pi : \Sigma_B \to \Lambda$ such that the following diagram commutes.

$$\begin{array}{ccc} \Sigma_B & \xrightarrow{\sigma} & \Sigma_B \\ \Pi \downarrow & & \downarrow \Pi \\ \Lambda & \xrightarrow{\varphi} & \Lambda \end{array} \quad \text{i.e. } \varphi\circ\Pi = \Pi\circ\sigma.$$

Moreover, Π is a morphism of the local product structure: that is, $\Pi[\underline{a}, \underline{b}] = [\Pi(\underline{a}), \Pi(\underline{b})]$ (cf. [**Shu86**, Ch. 10]).

3. *The Lagrange and Markov dynamical spectra*

Let $\varphi : M \to M$ be a diffeomorphism of a compact 2-manifold M and let Λ be a horseshoe for φ.

Remark. We have $L(f, \Lambda) \subset M(f, \Lambda)$ for any $f \in C^0(M, \mathbb{R})$.

In fact, if we let $a \in L(f, \Lambda)$, then there is $x_0 \in \Lambda$ such that $a = \limsup_{n\to+\infty} f(\varphi^n(x_0))$. Since Λ is a compact set, there is a subsequence $(\varphi^{n_k}(x_0))$ of $(\varphi^n(x_0))$ such that $\lim_{k\to+\infty} \varphi^{n_k}(x_0) = y_0$ and

$$a = \limsup_{n\to+\infty} f(\varphi^n(x_0)) = \lim_{k\to+\infty} f(\varphi^{n_k}(x_0)) = f(y_0).$$

Claim. $f(y_0) \geq f(\varphi^n(y_0))$ for all $n \in \mathbb{Z}$. Otherwise, suppose there is $n_0 \in \mathbb{Z}$ such that $f(y_0) < f(\varphi^{n_0}(y_0))$. Put $\epsilon = f(\varphi^{n_0}(y_0)) - f(y_0)$. Then, since f is a continuous function, there is a neighborhood U of y_0 such that

$$f(y_0) + \frac{\epsilon}{2} < f(\varphi^{n_0}(z)) \quad \text{for all } z \in U.$$

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Thus, since $\varphi^{n_k}(x_0) \to y_0$, there is $k_0 \in \mathbb{N}$ such that $\varphi^{n_k}(x_0) \in U$ for $k \geq k_0$: therefore,

$$f(y_0) + \frac{\epsilon}{2} < f(\varphi^{n_0+n_k}(x_0)) \quad \text{for all } k \geq k_0.$$

This contradicts the definition of $a = f(y_0)$.

In the next section we give some tools to prove the Main Theorem.

3.1. *The 'large' subset of $C^1(M, \mathbb{R})$.* In this section we construct a 'large' set of functions in $C^1(M, \mathbb{R})$, which will be useful in the proof of Main Theorem.

THEOREM 1. *The set*

$$H_\varphi = \{f \in C^1(M, \mathbb{R}) : \#M_f(\Lambda) = 1 \textit{ and, for } z \in M_f(\Lambda),\ Df_z(e_z^{s,u}) \neq 0\} \tag{1}$$

is open and dense, where $M_f(\Lambda) = \{z \in \Lambda : f(z) \geq f(y) \forall y \in \Lambda\}$ and $e_z^{s,u}$ are unit vectors in $E_z^{s,u}$ of the definition of hyperbolicity, respectively.

Before proving this theorem we will present some auxiliary results.

We say that x is a boundary point of Λ in the unstable direction if x is a boundary point of $W_\epsilon^u(x) \cap \Lambda$: that is, if x is an accumulation point only from one side by points in $W_\epsilon^u(x) \cap \Lambda$. If x is a boundary point of Λ in the unstable direction, then, due to the local product structure, the same holds for all points in $W^s(x) \cap \Lambda$. So the boundary points in the *unstable* direction are local intersections of local *stable* manifolds with Λ. For this reason, we denote the set of boundary points in the unstable direction by $\partial_s\Lambda$. The boundary points in the stable direction are defined similarly. The set of these boundary points is denoted by $\partial_u\Lambda$.

The following theorem is due to Newhouse and Palis (cf. [**PT93**, p. 170]).

THEOREM [PN]. *For a horseshoe Λ, as above, there is a finite number of (periodic) saddle points $p_1^s, \ldots, p_{n_s}^s$ such that*

$$\Lambda \cap \left(\bigcup_i W^s(p_i^s)\right) = \partial_s\Lambda.$$

Similarly, there is a finite number of (periodic) saddle points $p_1^u, \ldots, p_{n_u}^u$ such that

$$\Lambda \cap \left(\bigcup_i W^u(p_i^u)\right) = \partial_u\Lambda.$$

Moreover, both $\partial_s\Lambda$ and $\partial_u\Lambda$ are dense in Λ.

LEMMA 1. *The set*

$$\mathscr{A}' = \{f \in C^2(M, \mathbb{R}) : \textit{ there is } z \in M_f(\Lambda) \textit{ with } Df_z(e_z^{s,u}) \neq 0\}$$

is dense in $C^2(M, \mathbb{R})$, where $e_z^{s,u}$ are unit vectors in $E_z^{s,u}$, respectively.

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Before proving Lemma 1, we recall the definition of Morse functions. Let $f : M \to \mathbb{R}$, C^r, $r \geq 2$. We say that f is a Morse function, if for all $x \in M$ such that $Df_x = 0$,

$$D^2 f(0) : T_x M \times T_x M \to \mathbb{R}$$

is non-degenerate: that is, if $D^2 f(0)(v, w) = 0$ for all $w \in T_x M$ implies $v = 0$. Denote this set by $\mathscr{M}$. A known result says that *the set of Morse functions is open and dense in* $C^2(M, \mathbb{R})$*,* $r \geq 2$. Note that, in this case, the set $\mathrm{Crit}(f) = \{x \in M : Df_x = 0\}$ is a discrete set. In particular, since Λ is a compact set, $\#(\mathrm{Crit}(f) \cap \Lambda) < \infty$.

Proof of Lemma 1. It is enough to show, simply, that $\mathscr{A}'$ is dense in $\mathscr{M}$ (*the Morse functions*). Let $f_1 \in \mathscr{M}$. Then $\#\,\mathrm{Crit}(f_1) < \infty$, so, since $\mathrm{int}\,\Lambda = \emptyset$, we can find $f \in \mathscr{M}$ C^2-close to f_1 such that $M_f(\Lambda) \cap \mathrm{Crit}(f) = \emptyset$. Therefore, if $z \in M_f(\Lambda)$, we have $Df_z(e_z^s) \neq 0$ or $Df_z(e_z^u) \neq 0$.

If, for some $z \in M_f(\Lambda)$, both $Df_z(e_z^s)$ and $Df_z(e_z^u)$ are non-zero, then $f \in \mathscr{A}'$.

If otherwise, suppose that $Df_z(e_z^s) = 0$ and $Df_z(e_z^u) \neq 0$. Then there is a C^2-neighborhood $\mathcal{V}$ of f and a neighborhood U of z, such that, if $x \in U \cap \Lambda$ and $g \in \mathcal{V}$, then $Dg_x(e_x^u) \neq 0$. Let $\mathcal{R}$ be a Markov partition of Λ, such that the element R_z of $\mathcal{R}$ containing z is contained in U. Without loss of generality, we can assume that U is contained in a C^2-local chart $\phi : \tilde{U} \subset M \to V \subset \mathbb{R}^2$ with $U \subset \tilde{U}$ and $\tilde{U} \cap R' = \emptyset$ for all $R' \in \mathcal{R} \backslash \{R_z\}$. Observe that, since $Df_z(e_z^s) = 0$, $z \in \partial_s \Lambda$. Therefore the possible maximum points of f in $\Lambda \cap R_z$ are on $W_{loc}^s(z) \cap \Lambda := K^s$ (stable regular Cantor set), which has zero Lebesgue measure. Consider the function $\psi^s : K^s \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$\psi^s(x, \alpha) = \nabla(f \circ \phi^{-1})(\phi(x)) - \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D\phi_x(e_x^s),$$

where the above matrix is the orthogonal rotation. Since ψ^s extends to a C^1-function, then the Lebesgue measure of $\psi^s(K^s \times \mathbb{R})$ is zero. Therefore, there is a $v \in \mathbb{R}^2$ with norm very small such that $v \notin \psi^s(K^s \times \mathbb{R})$. Put $h(y) = f \circ \phi^{-1}(y) - \langle v, y\rangle$ for $y \in V$. Thus $D(h \circ \phi)_x e_x^s = Dh_{\phi(x)} D\phi_x e_x^s \neq 0$ for all $x \in K^s$. Since v can be chosen with norm arbitrarily small, $h \circ \phi$ is C^2-close to f and, since the function increases in the direction of its gradient, the maximum points of h in $\Lambda \cap R_z$ still can only appear in K^s. Thus h satisfies the condition of the lemma.

The case in which $Df_z(e_z^u) = 0$ and $Df_z(e_z^s) \neq 0$ is obtained analogously using a function C^2-close to f and in $\mathscr{A}'$.

This concludes the proof of the lemma. □

LEMMA 2. *Let* $f \in C^1(M, \mathbb{R})$ *and* $z \in M_f(\Lambda)$ *such that* $Df_z(e_z^{s,u}) \neq 0$*. Then* $z \in \partial_s \Lambda \cap \partial_u \Lambda$.

Proof. Using local coordinates in z, we can assume that we are in $U \subset \mathbb{R}^2$ containing zero. The hypothesis of the lemma implies that $Df_z \neq 0$: that is, if $f(z)$ is a regular value of f, then $\alpha := f^{-1}(f(z))$ is a C^1-curve transverse to $W_\epsilon^s(z)$ and $W_\epsilon^u(z)$ in z; moreover, the gradient vector $\nabla f(z)$ is orthogonal to α at the point z.

Let U be a small neighborhood of z. Then α is subdivided into two regions of U, say U_1 and U_2 (see Figure 1). Now suppose that $\nabla f(z)$ is pointing in the direction of U_1. Then in the region I, II, III, IV and V (see Figure 1), there are no points of Λ.

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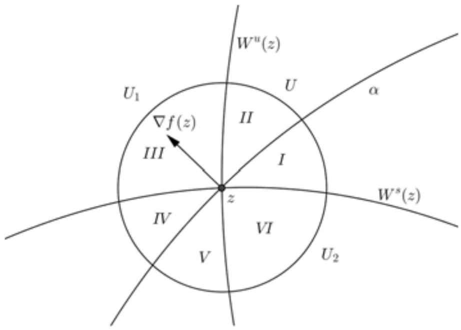

FIGURE 1. Localization of $z \in M_f(\Lambda)$.

In fact, as the function increases in the direction of its gradient, there are no points of Λ in the regions II, III and IV because $z \in M_f(\Lambda)$. If there were points in I of Λ, then, by the local product structure, there would be points in II of Λ, which we know cannot happen. Analogously, if there were points of Λ in V, then there would be points of Λ in IV, which we also know cannot happen. In conclusion, the only region where there are points of Λ is VI, so $z \in \partial_s\Lambda \cap \partial_u\Lambda$. $\square$

Remark 1. Since $C^s(M, \mathbb{R})$, $1 \le s \le \infty$ is dense in $C^r(M, \mathbb{R})$, $0 \le r < s$, Lemma 1 implies that $\mathscr{A}'$ is dense in $C^1(M, \mathbb{R})$.

LEMMA 3. *The set*

$$H_1 = \{f \in C^2(M, \mathbb{R}) : \#M_f(\Lambda) = 1 \textit{ and for } z \in M_f(\Lambda),\ Df_z(e_z^{s,u}) \neq 0\}$$

is dense in $C^2(M, \mathbb{R})$ *and therefore dense in* $C^1(M, \mathbb{R})$.

Proof. By Lemma 1, it is enough to show that H_1 is dense in $\mathcal{A}'$.

Let $f \in \mathcal{A}'$, then there is $z \in M_f(\Lambda)$ such that $Df_z(e_z^{s,u}) \neq 0$. Take U, a small neighborhood of z. Thus, given small $\epsilon > 0$, consider the function $\varphi_\epsilon \in C^2(M, \mathbb{R})$ such that φ_ϵ is C^2-close to the constant function zero. Also $\varphi_\epsilon = 0$ in $M \backslash U$, $\varphi_\epsilon(z) = \epsilon$ and z is a single maximum of φ_ϵ. In addition, $\varphi_\epsilon \xrightarrow{C^2} 0$ as $\epsilon \to 0$.

Define $g_\epsilon = f + \varphi_\epsilon$. Clearly, $g_\epsilon \xrightarrow{C^2} f$ as $\epsilon \to 0$. Since $z \in M_f(\Lambda)$, we have $g_\epsilon(z) = f(z) + \varphi_\epsilon(z) > f(x) + \varphi_\epsilon(x) = g_\epsilon(x)$ for all $x \in \Lambda$: that is, $z \in M_{g_\epsilon}(\Lambda)$ and $\#M_{g_\epsilon}(\Lambda) = 1$.

Also, $D(g_\epsilon)_z(e_z^{s,u}) = Df_z(e_z^{s,u}) \neq 0$: that is, $g_\epsilon \in H_1$. $\square$

LEMMA 4. *The set H_φ defined in (1) is open.*

Proof. Let $f \in H_\varphi$ and $z \in M_f(\Lambda)$ with $Df_z(e_z^{s,u}) \neq 0$, where $e_z^{s,u} \in E_z^{s,u}$ is a unit vector. Suppose that $\partial f/\partial e_z^{s,u} = \langle \nabla f(z), e_z^{s,u}\rangle = Df_z(e_z^{s,u}) > 0$ and $\nabla f(z)$ is the gradient vector of f at z.

Let $\mathcal{U} \subset C^1(M, \mathbb{R})$ be an open neighborhood of f such that, for all $g \in \mathcal{U}$, we have $\partial g/\partial e_z^{s,u} > 0$. The set $\{e_z^s, e_z^u\}$ is basis of T_zM. Let

$$V = \{v \in T_zM : v = a_v e_z^s + b_v e_z^u,\ a_v, b_v \ge 0\}.$$

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Also, let $v \in V\backslash\{0\}$. Then $\partial g/\partial v(z) = Dg_z(v) > 0$, for any $g \in \mathcal{U}$. Since, by Lemma 2, $z \in \partial_s \Lambda \cap \partial_u \Lambda$, this implies that there is an open set U of z such that $g(z) > g(x)$ for all $g \in \mathcal{U}$ and all $x \in U \cap \Lambda\backslash\{z\}$.

Let $\epsilon > 0$ such that $|f(z) - f(x)| > \epsilon/2$ for $x \in \Lambda\backslash U$. Let

$$V_{\epsilon/8}(f) = \left\{ g \in C^1(M, \mathbb{R}) : \|f - g\|_\infty < \frac{\epsilon}{8} \text{ and } \|Df - Dg\|_\infty < \frac{\epsilon}{8} \right\}$$

be a fundamental neighborhood of f. Then we claim that, for all $g \in V_{\epsilon/8}(f)$, the set $M_g(\Lambda) \subset U$. In fact, if $x \in \Lambda\backslash U$, then

$$\begin{aligned} g(z) - g(x) &= g(z) - f(z) + f(z) - g(x) - f(x) + f(x) \\ &\geq f(z) - f(x) - |g(z) - f(z)| - |g(x) - f(x)| \\ &\geq \frac{\epsilon}{2} - 2\frac{\epsilon}{8} = \frac{\epsilon}{4}. \end{aligned}$$

In particular, $g(z) > g(x)$ for all $x \in \Lambda\backslash U$, and so $M_g(\Lambda) = \{z\}$.

This implies that the open set $\mathcal{U}_1 = \mathcal{U} \cap V_{\epsilon/8}(f)$ is contained in H_φ. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Since $H_1 \subset H_\varphi$ and, by Lemma 3, the set H_1 is dense in $C^1(M, \mathbb{R})$, the set $H_\varphi \subset C^1(M, \mathbb{R})$ is dense and open in $C^1(M, \mathbb{R})$. □

4. *The Markov and Lagrange dynamical spectra and image of sub-horseshoes*

In this section, we prove that the Lagrange and Markov dynamical spectra contain the image of a sub-horseshoe by a real function.

Recall that the set H_φ (see (1)) is open and dense.

Let $f \in H_\varphi$ and $x_M \in M_f(\Lambda)$. Then, by Lemma 2, $x_M \in \partial_s \Lambda \cap \partial_u \Lambda$. By Theorem [PN], there are $p, q \in \Lambda$ periodic points such that

$$x_M \in W^s(p) \cap W^u(q).$$

Assume that p and q have the symbolic representation

$$(\ldots, a_1, \ldots, a_r, a_1, \ldots, a_r, \ldots) \quad \text{and} \quad (\ldots, b_1, \ldots, b_s, b_1, \ldots, b_s, \ldots),$$

respectively.

So there are l symbols $c_1, \ldots, c_l$ such that x_M is symbolically of the form

$$\begin{aligned} \Pi^{-1}(x_M) = (\ldots, b_1, \ldots, b_s, b_1, \ldots, b_s, c_1, \ldots, c_t, \ldots, c_l, \\ a_1, \ldots, a_r, a_1, \ldots, a_r, \ldots), \end{aligned}$$

where c_t is the zero position of $\Pi^{-1}(x_M)$.

Let $\underline{q}_{\tilde{s}} = (q_{-\tilde{s}}, \ldots, q_0, \ldots, q_{\tilde{s}})$ be an admissible word such that $x_M \in R_{\underline{q}_{\tilde{s}}} = \bigcap_{i=-\tilde{s}}^{\tilde{s}} \varphi^{-i}(R_{q_i})$, as in Figure 2, and put a sub-horseshoe $\tilde{\Lambda} := \bigcap_{n\in\mathbb{Z}} \varphi^n(\Lambda\backslash R_{\underline{q}_{\tilde{s}}})$. Thus there exists an open set U such that $U \cap \Lambda = \Lambda\backslash R_{\underline{q}_{\tilde{s}}}$ and

$$\tilde{\Lambda} := \bigcap_{n\in\mathbb{Z}} \varphi^n(U).$$

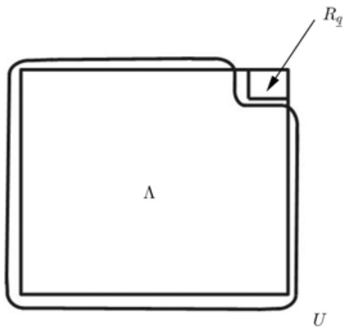

FIGURE 2. Removing the point of maximum.

Take $\tilde{s} \in \mathbb{N}$ sufficiently large so that the Hausdorff dimension of $\tilde{\Lambda}$ is close to the Hausdorff dimension of Λ (cf. Lemma 6).

Let $d \in \tilde{\Lambda}$, and call $\underline{d} = (\ldots, d_{-n}, \ldots, d_0, \ldots, d_n, \ldots)$ its symbolic representation. Given small $\epsilon > 0$, take $n_0 \in \mathbb{N}$ such that $\sum_{|n|\geq n_0} 2^{-(2|n|+1)} < \epsilon$ and put $\underline{d}_{n_0} = (d_{-n_0}, \ldots, d_{n_0})$, which is an admissible finite word. Define the cylinder $C_{\underline{d}_{n_0}} = \{\underline{w} \in \mathbb{A}^{\mathbb{Z}} : w_i = d_i \text{ for } i = -n_0, \ldots, n_0\}$. Then the set

$$C_{\underline{d}_{n_0},B} := \Sigma_B \cap C_{\underline{d}_{n_0}} = \{\underline{w} \in \Sigma_B : w_i = d_i \text{ for } i = -n_0, \ldots, n_0\}$$

is non-empty and contains a periodic point.

Using $N_{n(x,y)}(x, y, B) > 0$ for any $x, y \in \mathbb{A}$, there are admissible strings $\underline{e} = (e_1, \ldots, e_{k_0-1})$ and $\underline{f} = (f_1, \ldots, f_{j_0-1})$ joining d_0 with b_1 and a_r with d_1, respectively, with $k_0, j_0 < N_0$ (cf. §2.1).

Since x_M is a unique maximum point of f in Λ, if $\epsilon > 0$ is small enough, we can take $\tilde{\tilde{s}} > \tilde{s}$ and $\underline{q}_{\tilde{\tilde{s}}} = (q_{-\tilde{\tilde{s}}}, \ldots, q_0, \ldots, q_{\tilde{\tilde{s}}})$ to be an admissible word such that $x_M \in R_{\underline{q}_{\tilde{\tilde{s}}}} = \bigcap_{i=-\tilde{\tilde{s}}}^{\tilde{\tilde{s}}} \varphi^{-i}(R_{q_i}) \subset R_{\underline{q}_{\tilde{s}}}$ and

$$\sup \tilde{f}|_{\Pi^{-1}(\tilde{\Lambda})_\epsilon} < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)}, \tag{2}$$

where $\tilde{f} = f \circ \Pi$ and $\Pi^{-1}(\tilde{\Lambda})_\epsilon = \{\underline{x} \in \Sigma_B : d(\underline{x}, \Pi^{-1}(\tilde{\Lambda})) < \epsilon\}$.

Let $k \in \mathbb{N}$, $k > N_0$ and $k(s + r) + l > \tilde{\tilde{s}}$. Then, given the words $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$, we define the words

$$(a_1, \ldots, a_r)^k = \underbrace{(a_1, \ldots, a_r, \ldots, a_1, \ldots, a_r)}_{k \text{ times}}$$

and

$$(b_1, \ldots, b_s)^k = \underbrace{(b_1, \ldots, b_s, \ldots, b_1, \ldots, b_s)}_{k \text{ times}}.$$

Define the word

$$\alpha = ((b_1, \ldots, b_s)^k, c_1, \ldots, c_t, \ldots, c_l, (a_1, \ldots, a_r)^k),$$

where c_t is the zero position of the word α.

So, having fixed the words $\underline{e}$ and $\underline{f}$, we can define the following mapping, for all $\underline{x} \in C_{\underline{d}_{n_0},B}$, by

$$A(\underline{x}) = (\ldots, x_{-1}, x_0, e_1, \ldots, e_{k_0-1}, (b_1, \ldots, b_s)^k, c_1, \ldots, \\ c_t, \ldots, c_l, (a_1, \ldots, a_r)^k, f_1, \ldots, f_{j_0-1}, x_1, x_2, \ldots),$$

where c_t is the zero position of the word $A(\underline{x})$. Given a finite word $\underline{a} = (a_1, \ldots, a_n)$, we denote the length of the word $\underline{a}$ by $|\underline{a}| = n$. Then, since $k > N_0 \geq \max\{k_0, j_0\}$,

$$|\underline{e}|, |\underline{f}|, \tilde{\tilde{s}} < |\alpha| = k(s+r) + l,$$

where $\underline{e} = (e_1, \ldots, e_{k_0-1})$ and $\underline{f} = (f_1, \ldots, f_{j_0-1})$.

Now we may characterize $\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x})))$ for $\underline{x} \in C_{\underline{d}_{n_0},B} \cap \Pi^{-1}(\tilde{\Lambda})$.

Observe that $(\sigma^{l-t+kr+j_0-1}(A(\underline{x})))^+ = \underline{x}^+$ and call $\tau = l - t + kr + j_0 - 1$. Then, by the choice of n_0, $d(\sigma^{\tau+n_0+n}(A(\underline{x})), \sigma^{n_0+n}(\underline{x})) < \epsilon$ for all $n \geq 0$. Analogously, call $\eta = -(t + sk + k_0 - 1)$. Then $d(\sigma^{\eta-n_0-n}(A(\underline{x})), \sigma^{-n_0-n}(\underline{x})) < \epsilon$ for all $n \geq 0$. Moreover, since $\Pi^{-1}(\tilde{\Lambda})$ is a σ-invariant set, if $\underline{x} \in \Pi^{-1}(\tilde{\Lambda})$, (2) implies that

$$\tilde{f}(\sigma^{\tau+n_0+n}(A(\underline{x}))), \quad \tilde{f}(\sigma^{\eta-n_0-n}(A(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{a}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for all } n \geq 0.$$

The above inequality, implies that, for all $\underline{x} \in C_{\underline{d}_{n_0},B} \cap \Pi^{-1}(\tilde{\Lambda})$, there is $j \in \{\eta - n_0, \ldots, \tau + n_0\}$ such that $\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^j(A(\underline{x})))$.

Put $\Pi^{-1}(x) = \underline{x}$ and define the set

$$\tilde{\Lambda}_j := \{x \in \tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B}) : \sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = f(\sigma^j(A(\underline{x})))\}.$$

Thus,

$$\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B}) = \bigcup_{j=\eta-n_0}^{\eta+n_0} \tilde{\Lambda}_j. \tag{3}$$

This implies that there is $i_0 \in \{\eta - n_0, \ldots, \tau + n_0\}$ such that $\tilde{\Lambda}_{i_0}$ has non-empty interior in $\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B})$: so

$$HD(\tilde{\Lambda}) = HD(\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B})) = HD(\tilde{\Lambda}_{i_0}). \tag{4}$$

Therefore, for $\underline{x} \in \Pi^{-1}(\tilde{\Lambda}_{i_0})$,

$$\sup_{n\in\mathbb{Z}} \tilde{f}(\sigma^n(A(\underline{x}))) = \tilde{f}(\sigma^{i_0}(A(\underline{x}))). \tag{5}$$

The next goal is to show that $\tilde{A} = \Pi \circ A \circ \Pi^{-1}$ extends to a local diffeomorphism.

First, we show that $\tilde{A}$ extends to a local diffeomorphism in stable and unstable manifolds of d, $W^s_{loc}(d)$ and $W^u_{loc}(d)$.

As Λ is symbolically the product $\Sigma^-_{\mathbb{B}} \times \Sigma^+_{\mathbb{B}}$ (cf. Appendix A.2), we introduce the finite word β ($\beta = \underline{e}\alpha\underline{f}$). Using the notation of Appendix A, if

$$x^u \in W^u_{loc}(d) \cap \Lambda, \quad \text{then } f^u_\beta(x^u) \in W^u(d) \cap \Lambda \\ \text{and} \quad (\Pi^{-1}(f^u_\beta(x^u)))^+ = \beta(\Pi^{-1}(x^u))^+.$$

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Also, if

$$x^s \in W^s_{loc}(d) \cap \Lambda, \quad \text{then } f^s_\beta(x^s) \in W^s(d) \cap \Lambda$$
$$\text{and} \quad (\Pi^{-1}(f^s_\beta(x^s)))^- = (\Pi^{-1}(x^s))^- \beta.$$

The zero position of $\Pi^{-1}(\varphi^{-|\beta|+1}(f^s_\beta(x^s)))$ is equal to $(\beta)_0 = e_1$: that is

$$(\Pi^{-1}(\varphi^{-|\beta|+1}(f^s_\beta(x^s))))_0 = (\beta)_0 = (\Pi^{-1}(f^u_\beta(x^u)))_0.$$

So we can define the bracket

$$\begin{aligned}[\Pi^{-1}(f^u_\beta(x^u)), \Pi^{-1}(\varphi^{-|\beta|+1}(f^s_\beta(x^s)))] &= (\Pi^{-1}(x^s))^- \beta(\Pi^{-1}(x^u))^+ \\ &= A[\Pi^{-1}(x^u), \Pi^{-1}(x^s)].\end{aligned}$$

Note that, for x^u, x^s sufficiently close to d, the bracket $[\Pi^{-1}(x^u), \Pi^{-1}(x^s)]$ is well defined. As Π is a morphism of the local product structure,

$$\begin{aligned}[f^u_\beta(x^u), \varphi^{-|\beta|+1}(f^s_\beta(x^s))] &= \Pi([\Pi^{-1}(f^u_\beta(x^u)), \Pi^{-1}(\varphi^{-|\beta|+1}(f^s_\beta(x^s)))]) \\ &= \Pi(A[\Pi^{-1}(x^u), \Pi^{-1}(x^s)]) = \tilde{A}[x^u, x^s].\end{aligned} \tag{6}$$

Put $\tilde{A}_1(x^u) = f^u_\beta(x^u)$ and $\tilde{A}_1(x^s) = \varphi^{-|\beta|+1}(f^s_\beta(x^s))$: therefore, $\tilde{A}[x^u, x^s] = [\tilde{A}_1(x^u), \tilde{A}_2(x^s)]$. Thus we have the following lemma.

LEMMA 5. *If φ is a C^2-diffeomorphism, then $\tilde{A}$ extends to a local C^1-diffeomorphism defined in a neighborhood U_d of d. We may assume, without loss of generality (increasing n_0, if necessary), that $U_d \supset \tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0}, B})$.*

Proof. As φ is a C^2-diffeomorphism of a closed surface, then the stable and unstable foliations of the horseshoe Λ, $\mathscr{F}^s(\Lambda)$ and $\mathscr{F}^u(\Lambda)$ can be extended to C^1 invariant foliations defined on a full neighborhood of Λ. Also, if φ is a C^2-diffeomorphism, then f^s_β and f^u_β are at least C^1. Then, by (6), we have the result. □

An immediate consequence of Lemma 5 and (5) is the following corollary.

COROLLARY 1. *If $x \in \tilde{\Lambda}_{i_0}$, then* $\sup_{n\in\mathbb{Z}} f(\varphi^n(\tilde{A}(x))) = f(\varphi^{i_0}(\tilde{A}(x)))$.

This Corollary implies that $\{f(\varphi^{i_0}(\tilde{A}(x))) : x \in \tilde{\Lambda}_{i_0}\} \subset M(f, \Lambda)$.

Remark 2. We have $Df_{x_M}(e^{s,u}_{x_M}) \neq 0$, so this property is true in a neighborhood of x_M. Since, for every $x \in \tilde{\Lambda}_{i_0}$, $\varphi^{i_0}(\tilde{A}(x))$ belongs to a small neighborhood of x_M, for every $x \in \tilde{\Lambda}_{i_0}$, $Df_{\varphi^{i_0}(\tilde{A}(x))}(e^{s,u}_{\varphi^{i_0}(\tilde{A}(x))}) \neq 0$. Moreover, $D\varphi^{i_0}_{\tilde{A}(x)}(e^{s,u}_{\tilde{A}(x)}) \in E^{s,u}_{\varphi^{i_0}(\tilde{A}(x))}$, and since, by construction of $\tilde{A}$, $\partial\tilde{A}/\partial e^{s,u}_x$ is parallel to $e^{s,u}_{\tilde{A}(x)}$, for every $x \in \tilde{\Lambda}_{i_0}$ we have that $D(f \circ \varphi^{i_0} \circ \tilde{A})_x(e^{s,u}_x) \neq 0$.

Now we will prove the same for the Lagrange spectrum.

Let ϵ, n_0 and $\tilde{\tilde{s}}$ be as above. Then, using the above notation, let $x \in \tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0}, B})$ and $\Pi^{-1}(x) = (\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots)$. Thus there is an admissible string $E_i = (e^i_1, \ldots, e^i_{s_i})$ joining x_i with x_{-i} and the length $|E_i| = m_i - 1 < N_0$ for each i (cf. §2.1).

So we can define the following map for all $\underline{x} \in C_{\underline{d}_{n_0},B}$ by

$$A_1(\underline{x}) = (\ldots, x_3, E_3, x_{-3}, x_{-2}, x_{-1}, x_0, \beta, x_1, x_2, E_2, x_{-2}, x_{-1}, x_0, \beta, x_1, E_1, x_{-1}, x_0, \beta, x_1, E_1, x_{-1}, x_0, \beta, x_1, x_2, E_2, x_{-2}, x_{-1}, x_0, \beta, x_1, x_2, x_3, E_3, x_{-3}, \ldots),$$

where $\beta = \underline{e}\alpha \underline{f}$, as above.

Since $|E_i| < N_0$, then the set of words $\{E_i : i \in \mathbb{N}^*\}$ is finite. Therefore

$$\{E_i : i \in \mathbb{N}^*\} = \{D_1, \ldots, D_m\}$$

for some admissible words D_i with $|D_i| < N_0$. Now we can take $k > N_0 + 2n_0$ and, if necessary, by increasing $\tilde{\tilde{s}}$ we have that, since $|D_i| < N_0$ for each i, there exists a neighborhood $\mathcal{U}_i$ of D_i for which

$$\sup \tilde{f}|_{\sigma^r(\mathcal{U}_i)} < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for } |r| \le n_0 + |D_i| < n_0 + N_0. \tag{7}$$

Now we may characterize $\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x})))$ for $\underline{x} \in C_{\underline{d}_{n_0},B} \cap \Pi^{-1}(\tilde{\Lambda})$.

Let $m(n) \in \mathbb{N}$ such that

$$(\sigma^{m(n)}(A_1(\underline{x})))^+ = x_1, x_2, \ldots, x_n E_n x_{-n}, \ldots, x_0, \ldots \quad \text{and} \quad n \ge 2n_0.$$

Let k^* be such that $n - k^* = n_0$. Then, by definition of n_0,

$$d(\sigma^{m(n)+n_0+j}(A_1(\underline{x})), \quad \sigma^{n_0+j}(\underline{x})) < \epsilon \quad \text{for all } j = 0, \ldots, k^* - n_0,$$

and

$$d(\sigma^{m(n)+n+|E_n|+n_0+j}(A_1(\underline{x})), \quad \sigma^{-k^*+j}(\underline{x})) < \epsilon \quad \text{for all } j = 0, \ldots, k^* - n_0.$$

Moreover, since $\Pi^{-1}(\tilde{\Lambda})$ is a σ-invariant set, if $\underline{x} \in \Pi^{-1}(\tilde{\Lambda})$, then (2) implies that

$$\tilde{f}(\sigma^{m(n)+n_0+j}(A_1(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for all } j = 0, \ldots, k^* - n_0$$

and

$$\tilde{f}(\sigma^{m(n)+n+|E_n|+n_0+j}(A_1(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for all } j = 0, \ldots, k^* - n_0.$$

Also,

$$\sigma^{m(n)+k^*+s}(A_1(\underline{x})) \in \sigma^{|E_n^-|+n_0-s}(\mathcal{U}_{i(n)}) \quad \text{for all } s = 0, \ldots, n_0 + |E_n^-|$$

and

$$\sigma^{m(n)+n+|E_n^-|+s}(A_1(\underline{x})) \in \sigma^{-s}(\mathcal{U}_{i(n)}) \quad \text{for all } s = 0, \ldots, n_0 + |E_n^+|,$$

where $E_n = E_n^- E_n^+$ and $i(n) \in \{1, \ldots, m\}$. Therefore, (7) implies that

$$\tilde{f}(\sigma^{m(n)+k^*+s}(A_1(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for all } s = 0, \ldots, n_0 + |E_n^-|$$

and

$$\tilde{f}(\sigma^{m(n)+n+|E_n^-|+s}(A_1(\underline{x}))) < \inf \tilde{f}|_{\Pi^{-1}(R_{\underline{q}_{\tilde{\tilde{s}}}} \cap \Lambda)} \quad \text{for all } s = 0, \ldots, n_0 + |E_n^+|.$$

Note that, if $n_0 \le n < 2n_0$, then $k^* < n_0$ and the last two cases apply. Therefore, the last four inequalities above imply that, for all $\underline{x} \in C_{\underline{d}_{n_0},B} \cap \Pi^{-1}(\tilde{\Lambda})$, there is $j \in \{\eta - n_0, \ldots, \tau + n_0\}$ and a sequence $n_k(j)$ with

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \sup_k \tilde{f}(\sigma^{n_k(j)}(A_1(\underline{x}))) \quad \text{and} \quad (\sigma^{n_k(j)}(A_1(\underline{x})))_0 = (A_1(\underline{x}))_j$$

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for all k, where $\eta = -(t + ks + k_0 - 1)$ and $\tau = (l - t + kr + j_0 - 1)$, as above, are the length of the negative and positive parts of the finite word $\beta = \underline{e}\alpha \underline{f}$, respectively.

Put $\Pi^{-1}(x) = \underline{x}$ and define the set

$$\Lambda'_j := \{x \in \tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B}) : \limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \sup_k \tilde{f}(\sigma^{n_k(j)}(A_1(\underline{x})))\}.$$

Then

$$\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B}) = \bigcup_{j=\eta-n_0}^{\tau+n_0} \Lambda'_j.$$

Therefore, there is $j_0 \in \{\eta - n_0, \ldots, \tau + n_0\}$ such that Λ'_{j_0} has non-empty interior in $\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B})$: so

$$HD(\tilde{\Lambda}) = HD(\tilde{\Lambda} \cap \Pi(C_{\underline{d}_{n_0},B})) = HD(\Lambda'_{j_0}). \tag{8}$$

Thus, for $\underline{x} \in \Pi^{-1}(\Lambda'_{j_0})$,

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A(\underline{x}))) = \sup_k \tilde{f}(\sigma^{n_k(j_0)}(A_1(\underline{x}))).$$

So there is a subsequence $n_{k_m}(j_0)$ with $n_{k_m}(j_0) \to \infty$ as $m \to \infty$ such that

$$\sup_k \tilde{f}(\sigma^{n_k(j_0)}(A_1(\underline{x}))) = \lim_{m\to\infty} \tilde{f}(\sigma^{n_{k_m}(j_0)}(A_1(\underline{x}))).$$

By construction of A_1, it is true that

$$\lim_{m\to\infty} \sigma^{n_{k_m}(j_0)}(A_1(\underline{x})) = \sigma^{j_0}(A(\underline{x})),$$

where $A(\underline{x})$ is defined as before.

Therefore

$$\limsup_{n\to\infty} \tilde{f}(\sigma^n(A_1(\underline{x}))) = \tilde{f}(\sigma^{j_0}(A(\underline{x}))).$$

As an immediate consequence we have the following corollary.

COROLLARY 2. *If $x \in \Lambda'_{j_0}$, then*

$$\limsup_{n\to\infty} f(\varphi^n(\tilde{A}_1(x))) = f(\varphi^{j_0}(\tilde{A}(x))) \quad \textit{where } \tilde{A}_1 = \Pi \circ A_1 \circ \Pi^{-1}.$$

This Corollary implies that $\{f(\varphi^{j_0}(\tilde{A}(x)) : x \in \Lambda'_{j_0}\} \subset L(f, \Lambda)$.

5. *The image of the product of two regular Cantor sets by a real function and the behavior of the spectra*

In this section we give a condition for the image of a horseshoe by a 'typical' real function to have non-empty interior.

5.1. *Intersections of regular Cantor sets.* Assume we are given two sets of data $(\mathbb{A}, \mathbb{B}, \Sigma, g)$, $(\mathbb{A}', \mathbb{B}', \Sigma', g')$ defining regular Cantor sets K, K' (see Appendix A.1 for definitions and notations).

Let $r \in (1, +\infty]$. For $a \in \mathbb{A}$, denote by $\mathcal{P}^r(a)$ the space of C^r-embeddings of interval $I(a)$ into $\mathbb{R}$, endowed with the C^r topology. The affine group $\mathrm{Aff}(\mathbb{R})$ acts by composition on the left on $\mathcal{P}^r(a)$, with the quotient space being denoted by $\overline{\mathcal{P}}^r(a)$. We also consider $\mathcal{P}(a) = \bigcup_{r>1} \mathcal{P}^r(a)$ and $\overline{\mathcal{P}}(a) = \bigcup_{r>1} \overline{\mathcal{P}}^r(a)$, endowed with the inductive limit topologies.

Remark 3. In **[MY01]** $\mathcal{P}^r(a)$ is considered for $r \in (1, +\infty]$, but all the definitions and results involving $\mathcal{P}^r(a)$ can be obtained by considering $r \in [1, +\infty]$.

Let $\mathcal{A} = (\underline{\theta}, A)$, where $\underline{\theta} \in \Sigma^-$ and A is now an *affine* embedding of $I(\theta_0)$ into $\mathbb{R}$. We have a canonical map

$$\mathcal{A} \to \mathcal{P}^r = \bigcup_{\mathbb{A}} \mathcal{P}^r(a),$$
$$(\underline{\theta}, A) \mapsto A \circ k^{\underline{\theta}} (\in \mathcal{P}^r(\theta_0)).$$

We define, as before, the spaces $\mathcal{P} = \bigcup_{\mathbb{A}} \mathcal{P}(a)$ and $\mathcal{P}' = \bigcup_{\mathbb{A}'} \mathcal{P}(a')$.

A pair (h, h'), $(h \in \mathcal{P}(a), h' \in \mathcal{P}'(a'))$ is called a *smooth configuration* for $K(a) = K \cap I(a)$, $K'(a') = K' \cap I(a')$. Actually, rather than working in the product $\mathcal{P} \times \mathcal{P}'$, it is better to go to the quotient Q by the diagonal action of the affine group $\mathrm{Aff}(\mathbb{R})$. Elements of Q are called *smooth relative configurations* for $K(a)$, $K'(a')$.

We say that a smooth configuration $(h, h') \in \mathcal{P}(a) \times \mathcal{P}(a')$ is:

- *linked* if $h(I(a)) \cap h'(I(a')) \neq \emptyset$;
- *intersecting* if $h(K(\underline{a})) \cap h'(K(\underline{a}')) \neq \emptyset$, where $K(\underline{a}) = K \cap I(\underline{a})$ and $K(\underline{a}') = K \cap I(\underline{a}')$; and
- *stably intersecting* if it is still intersecting when we perturb it in $\mathcal{P} \times \mathcal{P}'$ and we perturb (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

All these definitions are invariant under the action of the affine group, and therefore make sense for smooth relative configurations.

As before, we can introduce the spaces $\mathcal{A}$, $\mathcal{A}'$ associated to the limit geometries of g, g', respectively. We denote by $\mathcal{C}$ the quotient of $\mathcal{A} \times \mathcal{A}'$ by the diagonal action on the left of the affine group. An element of $\mathcal{C}$, represented by $(\underline{\theta}, A) \in \mathcal{A}$, $(\underline{\theta}', A') \in \mathcal{A}'$, is called a relative configuration of the limit geometries determined by $\underline{\theta}$, $\underline{\theta}'$. We have canonical maps

$$\mathcal{A} \times \mathcal{A}' \to \mathcal{P} \times \mathcal{P}',$$
$$\mathcal{C} \to Q,$$

which allow us to define linked, intersecting and stably intersecting configurations at the level of $\mathcal{A} \times \mathcal{A}'$ or $\mathcal{C}$.

We consider the following subset V of $\Omega_\Sigma \times \Omega_{\Sigma'}$. A pair (g, g') belongs to V if, for any $[(\underline{\theta}, A), (\underline{\theta}', A')] \in \mathcal{A} \times \mathcal{A}'$, there is a translation R_t (in $\mathbb{R}$) such that $(R_t \circ A \circ k^{\underline{\theta}}, A' \circ k'^{\underline{\theta}'})$ is a stably intersecting configuration.

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THEOREM. (Cf. [**MY01**]) (1) *V is open in $\Omega_\Sigma \times \Omega_{\Sigma'}$, and $V \cap (\Omega_\Sigma^\infty \times \Omega_{\Sigma'}^\infty)$ is dense (for the C^∞-topology) in the set $\{(g, g'), HD(K) + HD(K') > 1\}$.*

(2) *Let $(g, g') \in V$. There exists $d^* < 1$ such that, for any $(h, h') \in \mathcal{P} \times \mathcal{P}'$, the set*

$$\mathcal{I}_s = \{t \in \mathbb{R}, (R_t \circ h, h') \text{ is a stably intersecting smooth configuration for } (g, g')\}$$

and is (open and) dense in

$$\mathcal{I} = \{t \in \mathbb{R}, (R_t \circ h, h') \text{ is an intersecting smooth configuration for } (g, g')\}.$$

Moreover, $HD(\mathcal{I} - \mathcal{I}_s) \le d^*$. The same d^* is also valid for $(\tilde{g}, \tilde{g}')$ in a neighborhood of (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

Keeping the previous notation, we have the following theorem.

THEOREM 2. *Let K and K' be two regular Cantor sets defined by expanding map g, g'. Suppose that $HD(K) + HD(K') > 1$ and $(g, g') \in V$. Let f be a C^1-function $f : U \to \mathbb{R}$ with $K \times K' \subset U \subset \mathbb{R}^2$ such that, in some point of $K \times K'$, its gradient is not parallel to any of the two coordinate axis. Then*

$$\operatorname{int} f(K \times K') \neq \emptyset.$$

Proof. By hypothesis, and by continuity of df, we find a pair of periodic points p_1, p_2 of K and K', respectively, with addresses $\overline{a}_1 = \underline{a_1 a_1 a_1} \ldots$ and $\overline{a}_2 = \underline{a_2 a_2 a_2} \ldots$, where $\underline{a}_1$ and $\underline{a}_2$ are finite sequences such that $df(p_1, p_2)$ is not a real multiple of dx nor of dy. There are increasing sequences of natural numbers (m_k) and (n_k) such that the intervals $I_{\underline{a}_1^{m_k}}$ and $I'_{\underline{a}_2^{n_k}}$, defined by the finite words $\underline{a}_1^{m_k}$ and $\underline{a}_2^{n_k}$, satisfy

$$\frac{|I_{\underline{a}_1^{m_k}}|}{|I'_{\underline{a}_2^{n_k}}|} \in (C^{-1}, C) \quad \text{for some } C > 1.$$

Thus we can assume that $|I_{\underline{a}_1^{m_k}}|/|I'_{\underline{a}_2^{n_k}}| \to \lambda \in [C^{-1}, C]$ as $k \to \infty$, and define $\tilde{\lambda} := -((\partial f/\partial x(p_1, p_2))/(\partial f/\partial y(p_1, p_2)))\lambda$.

As $(K, K') \in V$, there is $t \in \mathbb{R}$ such that $(\tilde{\lambda} k^{\overline{a}_1} + t, k'^{\overline{a}_2})$ is a stably intersecting configuration. So there are $\tilde{x} \in I((\underline{a}_1)_0)$ and $\tilde{y} \in I((\underline{a}_2)_0)$ such that $x_0 = k^{\overline{a}_1}(\tilde{x})$ and $y_0 = k^{\overline{a}_2}(\tilde{y})$ with $\tilde{\lambda} x_0 + t = y_0$, where $(\underline{a}_i)_0$ is the zero position of the finite word $\underline{a}_i$ for $i = 1, 2$. Moreover, $\tilde{x} = g^{m_k|\underline{a}_1|-1}(\bar{x})$ and $\tilde{y} = (g')^{n_k|\underline{a}_2|-1}(\bar{y})$ for some $\bar{x} \in I_{\underline{a}_1^{m_k}}$ and $\bar{y} \in I'_{\underline{a}_2^{n_k}}$.

Taking k large enough, we can assume that $df(\bar{x}, \bar{y})$ is not a real multiple of dx nor of dy. In particular, if $\partial f/\partial y(\bar{x}, \bar{y}) \neq 0$, then, by the local submersion theorem, there exists a C^1-diffeomorphism $H(x, y) = (x, g(x, y))$, defined in neighborhood of $(\bar{x}, \bar{y})$, such that $f(H(x, y)) = y$. Without loss of generality, we can suppose that H is defined in $I_{\underline{a}_1^{m_k}} \times I'_{\underline{a}_2^{n_k}}$. Put $g_s(x) := g(x, s)$; if s_0 is such that $f(\bar{x}, \bar{y}) = s_0$, then $g_{s_0}(\bar{x}) = \bar{y}$. Also, observe that $s \in f((K \cap I_{\underline{a}_1^{m_k}}) \times (K' \cap I'_{\underline{a}_2^{n_k}}))$ is equivalent to $g_s(K \cap I_{\underline{a}_1^{m_k}}) \cap (K' \cap I'_{\underline{a}_2^{n_k}}) \neq \emptyset$.

Thus, our problem reduces to proving that $g_s(K \cap I_{\underline{a}_1^{m_k}})$ and $K' \cap I'_{\underline{a}_2^{n_k}}$ have non-empty intersection for s close to $s_0 = f(\bar{x}, \bar{y})$.

Denote by $B_k : I'^{n_k}_{\underline{a}_2} \to [0, 1]$ and $T_k : I^{m_k}_{\underline{a}_1} \to [0, 1]$ the orientation-preserving affine maps given by $B_k(x) = 1/(b'_k - a'_k)(x - a'_k) = 1/|I'^{n_k}_{\underline{a}_2}|(x - a'_k)$ and $T_k(x) = 1/(b_k - a_k)(x - a_k) = 1/|I^{m_k}_{\underline{a}_1}|(x - a_k)$, where $I^{m_k}_{\underline{a}_1} = [a_k, b_k]$ and $I'^{n_k}_{\underline{a}_2} = [a'_k, b'_k]$.

Then, by definition of limit geometries (cf. Appendix A.1), $B_k(K' \cap I'^{n_k}_{\underline{a}_2})$ converges to $k^{\overline{a}_2}(K')$ and $T_k(K \cap I^{m_k}_{\underline{a}_1})$ converges to $k^{\overline{a}_1}(K)$ as regular Cantor sets.

Also, $B_k(g_{s_0}(K \cap I^{m_k}_{\underline{a}_1})) = B_k \circ g_{s_0} \circ T_k^{-1}(T_k(K \cap I^{m_k}_{\underline{a}_1}))$.

Claim. The map $B_k \circ g_{s_0} \circ T_k^{-1}$ converges to $\tilde{\lambda}x + t$ in the C^1 topology.

In fact, if we call $\epsilon_k = b_k - a_k = |I^{m_k}_{\underline{a}_1}|$ and $\epsilon'_k = b'_k - a'_k = |I'^{n_k}_{\underline{a}_2}|$,

$$\begin{aligned} B_k \circ g_{s_0} \circ T_k^{-1}(x) &= \frac{1}{\epsilon'_k}(g_{s_0}(\epsilon_k x + a_k) - a'_k) \\ &= \frac{1}{\epsilon'_k}(g_{s_0}(a_k) + g'_{s_0}(a_k)\epsilon_k x + r(\epsilon_k x) - a'_k) \\ &= B_k(g_{s_0}(a_k)) + g'_{s_0}(a_k)\frac{\epsilon_k}{\epsilon'_k}x + \frac{\epsilon_k}{\epsilon'_k}\frac{r(\epsilon_k x)}{\epsilon_k}. \end{aligned} \tag{9}$$

Since $g_{s_0}(\bar{x}) = \bar{y}$, $B_k(g_{s_0}(\bar{x})) = B_k(\bar{y}) = B_k \circ (g')^{-(n_k|\underline{a}_2|-1)}(\tilde{y})$ and the definition of limit geometries implies that $B_k(g_{s_0}(\bar{x}))$ converges to $k^{\overline{a}_2}(\tilde{y}) = y_0 = \tilde{\lambda}x_0 + t$ and $T_k(\bar{x}) = T_k \circ g^{-(m_k|\underline{a}_1|-1)}(\tilde{x})$ converges to $k^{\overline{a}_1}(\tilde{x}) = x_0$. Therefore, by (9),

$$B_k \circ g_{s_0}(\bar{x}) = B_k \circ g_{s_0} \circ T_k^{-1}(T_k(\bar{x})) = B_k(g_{s_0}(a_k)) + g'_{s_0}(a_k)\frac{\epsilon_k}{\epsilon'_k}T_k(\bar{x}) + \frac{\epsilon_k}{\epsilon'_k}\frac{r(\epsilon_k T_k(\bar{x}))}{\epsilon_k}.$$

So, if $k \to +\infty$, the left-hand side of the equation above converges to $\tilde{\lambda}x_0 + t$ and, since $g'_{s_0}(a_k) \to -((\partial f/\partial x(p_1, p_2))/(\partial f/\partial y(p_1, p_2)))$, $\epsilon_k/\epsilon'_k \to \lambda$, $T_k(\bar{x}) \to x_0$ and $(r(\epsilon_k T_k(\bar{x})))/\epsilon_k \to 0$, by definition of $\tilde{\lambda}$ and the above equation,

$$B_k \circ g_{s_0}(a_k) \to \tilde{\lambda}x_0 + t - \tilde{\lambda}x_0 = t. \tag{10}$$

Thus, by (9) and (10),

$$\lim_{k \to +\infty} B_k \circ g_{s_0} \circ T_k^{-1}(x) = \tilde{\lambda}x + t.$$

Moreover, since g_{s_0} is a C^1-function,

$$(B_k \circ g_{s_0} \circ T_k^{-1})'(x) = \frac{1}{\epsilon'_k}g'_{s_0}(T_k^{-1}(x)) \cdot \epsilon_k \to -\frac{\partial f/\partial x(p_1, p_2)}{\partial f/\partial y(p_1, p_2)}\lambda = \tilde{\lambda}.$$

This concludes the proof the claim.

Therefore,

$$B_k(g_{s_0}(K \cap I^{m_k}_{\underline{a}_1})) = B_k \circ g_{s_0} \circ T_k^{-1}(T_k(K \cap I^{m_k}_{\underline{a}_1})) \to \tilde{\lambda}k^{\overline{a}_1}(K) + t,$$

and

$$B_k(K' \cap I'^{n_k}_{\underline{a}_2}) \to k^{\overline{a}_2}(K').$$

Since $(\tilde{\lambda} k^{\underline{a}_1} + t, k^{\underline{a}_2})$ is a stably intersecting configuration, and this property is open (cf. Remark 3) and $g_s(\cdot)$ is C^1-close to $g_{s_0}(\cdot)$ for s close to s_0, for large enough k, the Cantor sets $B_k(g_s(K \cap I_{\underline{a}_1^{m_k}}))$ and $B_k(K' \cap I'_{\underline{a}_2^{n_k}})$ have non-empty intersection. Therefore $g_s(K \cap I_{\underline{a}_1^{m_k}})$ and $K' \cap I'_{\underline{a}_2^{n_k}}$ have non-empty intersection. □

The following example shows that the property V in the Theorem 2 is fundamental.

Example. Consider the regular Cantor set $K_\alpha := \bigcap_{n \geq 0} \psi^{-n}(I_1 \cup I_2)$, where

$$\psi(x) = \begin{cases} \dfrac{2}{1-\alpha} x & \text{if } x \in I_1 := \left[0, \dfrac{1-\alpha}{2}\right], \\ -\dfrac{2}{1-\alpha} x + \dfrac{2}{1-\alpha} & \text{if } x \in I_2 := \left[\dfrac{1+\alpha}{2}, 1\right]. \end{cases}$$

Hence $HD(K_\alpha) = -((\log 2)/(\log((1-\alpha)/2)))$ (cf. [**PT93**]). If $\alpha < 1/2$, then $HD(K_\alpha) > 1/2$, and for $1/3 < \alpha < 1/2$ it holds that $K_\alpha - K_\alpha$ has measure zero (cf. [**Mor99**]).

Moreover, $HD(K_\alpha \times K_\alpha) > 1$ and $f(x, y) = x - y$, which satisfies the hypothesis of Theorem 2. But, for $1/3 < \alpha < 1/2$,

$$\operatorname{int} f(K_\alpha \times K_\alpha) = \emptyset.$$

COROLLARY 3. *Let φ be a C^2-diffeomorphism, and Λ a horseshoe associated to φ. Suppose that K^s, K^u satisfy the hypotheses of Theorem 2 above. Let*

$$\mathcal{A}_\Lambda = \{f \in C^1(M, \mathbb{R}) : \exists z = (z^s, z^u) \in \Lambda \text{ such that } Df(z) \cdot e_z^{s,u} \neq 0\}.$$

Then, for all $f \in \mathcal{A}_\Lambda$, $\operatorname{int} f(\Lambda) \neq \emptyset$.

It is easy to prove that $\mathcal{A}_\Lambda$, given the above Corollary, is an open and dense set in $C^1(M, \mathbb{R})$.

6. *The Main Theorem*

The following theorem is a fundamental result, due to Moreira and Yoccoz in [**MY10**], on the existence of elements in V associated to the pairs of regular Cantor sets (K^s, K^u) defined by g^s, g^u, where g^s describes the geometric transverse of the unstable foliation $W^u(\Lambda, R)$ and g^u describes the geometric transverse of the stable foliation $W^s(\Lambda, R)$, as given in Appendix A.2.

THEOREM. (Cf. [**MY10**]) *Suppose that the sum of the Hausdorff dimensions of the regular Cantor sets K^s, K^u, defined by g^s, g^u, is greater than one. If the neighborhood $\mathcal{U}$ of φ_0 in* $\mathrm{Diff}^\infty(M)$ *is sufficiently small, there is an open and dense $\mathcal{U}^* \subset \mathcal{U}$ such that, for $\varphi \in \mathcal{U}^*$, the corresponding pair of expanding applications (g, g') belongs to V.*

We use the above result to show that the Markov and Lagrange spectra have typically non-empty interior in this context.

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Recall that, given a horseshoe Λ associated to a diffeomorphism φ, we defined, for $f \in H_\varphi$, the sub-horseshoe $\tilde{\Lambda}$ in §4 as $\tilde{\Lambda} := \bigcap_{n\in\mathbb{Z}} \varphi^n(\Lambda \backslash R_{\underline{a}_{\tilde{s}}})$.

The following lemma shows that the $HD(\tilde{\Lambda})$ does not change much compared to $HD(\Lambda)$. This is given more precisely in the following lemma.

LEMMA 6. *If Λ is a horseshoe associated to a C^2-diffeomorphism φ and $HD(\Lambda) > 1$, then $HD(\tilde{\Lambda}) > 1$ provided $\tilde{s}$ is large enough.*

Assuming Lemma 6, (4) and (8) imply the following corollary.

COROLLARY 4. *The sets $\tilde{\Lambda}_{i_0}$ and Λ'_{j_0} satisfy $HD(\tilde{\Lambda}_{i_0})$, $HD(\Lambda'_{j_0}) > 1$.*

Recalling that, as φ is a C^2-diffeomorphism, Λ is locally the product of stable and unstable regular Cantor sets $K^s \times K^u$. Then the previous lemma will be a consequence of the following lemma.

Let K be a regular Cantor set with expanding map ψ and Markov partition $\mathcal{R} = \{I_1, \ldots, I_k\}$ so that $K = \bigcap_{n\geq 0} \psi^{-n}(\bigcup_{i=1}^k I_i)$. Consider the transition matrix $A = (a_{ij})_{k\times k}$ associated to the partition $\mathcal{R}$, defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \psi(I_i) \supset I_j, \\ 0 & \text{if } \psi(I_i) \cap I_j = \emptyset. \end{cases}$$

To each admissible finite word of length m, $\underline{b} = (b_1, \ldots, b_m)$ such that $a_{b_i b_{i+1}} = 1$ for all $i < m$, we associate the interval $I_{\underline{b}} = I_{b_1} \cap \psi^{-1}(I_{b_2}) \cap \psi^{-2}(I_{b_3}) \cdots \cap \psi^{-(m-1)}(I_{b_m})$.

LEMMA 7. *Let K be a regular Cantor set with expanding map ψ and Markov partition $\mathcal{R} = \{I_1, \ldots, I_k\}$ so that $K = \bigcap_{n\geq 0} \psi^{-n}(\bigcup_{i=1}^k I_i)$. Given $\epsilon > 0$, there is a positive integer m_0 such that, for every $m \geq m_0$ and for every admissible finite word of length m, $\underline{b} = (b_1, \ldots, b_m)$,*

$$HD(K_{\underline{b}}) \geq HD(K) - \epsilon \quad \text{where } K_{\underline{b}} = \bigcap_{n\geq 0} \psi^{-n}\bigg(\bigcup_{i=1}^k I_i \Big\backslash I_{\underline{b}}\bigg).$$

Proof. Let $\mathcal{R}^n$ denote the set of connected components of $\psi^{-(n-1)}(I_i)$, $I_i \in \mathcal{R}$. Let B^n be the set of admissible words of length n, so that $\mathcal{R}^n = \{I_{\underline{b}}, \underline{b} \in B^n\}$. Fix $\tilde{i}, \tilde{j} \leq k$ such that $a_{\tilde{j}\tilde{i}} = 1$. Let $X^n = \{\underline{b} = (b_1, \ldots, b_n) \in B^n : b_1 = \tilde{i}, b_n = \tilde{j}\}$. For any positive integer r and $\underline{b_1}, \underline{b_2}, \ldots, \underline{b_r} \in X^n$, we have $\underline{b_1}\underline{b_2} \cdots \underline{b_r} \in X^{nr} \subset B^{nr}$. Let $\tilde{\mathcal{R}}^n = \{I_{\underline{b}}, \underline{b} \in X^n\}$.

For $R \in \mathcal{R}^n$ take $\Lambda_{n,R} = \sup |(\psi^n)'_{|R}|$. By the mixing condition, there is $c_1 > 0$ such that

$$\sum_{R\in\tilde{\mathcal{R}}^n} (\Lambda_{n,R})^{-d} \geq c_1 \sum_{R\in\mathcal{R}^n} (\Lambda_{n,R})^{-d} \quad \text{for all } d \geq 0, n \geq 1.$$

On the other hand, from [**PT93**, pp. 69–70], it follows that, if we define d_n implicitly by

$$\sum_{R\in\mathcal{R}^n} (\Lambda_{n,R})^{-d_n} = 1,$$

then $\lim d_n = HD(K)$, so in, particular, for large n, $d_n > HD(K)/2$. Notice, also, that there is $\lambda_1 > 1$ such that $\Lambda_{n,R} \geq \lambda_1^n$ for all $n \geq 1$.

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Let n be large so that $d_n > HD(K) - \varepsilon/2$ and $\lambda_1^{n\varepsilon/2} > 2/c_1$, and let $m_0 = 2n - 1$. Given $m \geq m_0$ and an admissible finite word of length m, $\underline{b} = (b_1, \ldots, b_m)$, we define words $\underline{c_j} = (b_j, b_{j+1}, \ldots, b_{j+n-1}) \in B^n$, $1 \leq j \leq n$. Let $L^n = \{\underline{c_j} : 1 \leq j \leq n\}$ and $\hat{\mathcal{R}}^n = \{I_{\underline{c}} : \underline{c} \in X^n \backslash L^n\}$. Then

$$\sum_{R \in \hat{\mathcal{R}}^n} (\Lambda_{n,R})^{-d_n} \geq \sum_{R \in \tilde{\mathcal{R}}^n} (\Lambda_{n,R})^{-d_n} - n\lambda_1^{-nd_n} \geq \sum_{R \in \tilde{\mathcal{R}}^n} (\Lambda_{n,R})^{-d_n} - n\lambda_1^{-nHD(K)/2}$$
$$\geq c_1 \sum_{R \in \mathcal{R}^n} (\Lambda_{n,R})^{-d_n} - n\lambda_1^{-nHD(K)/2} = c_1 - n\lambda_1^{-nHD(K)/2} > c_1/2,$$

and so

$$\sum_{R \in \hat{\mathcal{R}}^n} (\Lambda_{n,R})^{-(HD(K)-\varepsilon)} > \sum_{R \in \hat{\mathcal{R}}^n} (\Lambda_{n,R})^{-(d_n - \varepsilon/2)} > 1.$$

We may define the regular Cantor set (with expanding map ψ^n) as

$$\tilde{K} := \bigcap_{r \geq 0} \psi^{-nr}\Bigg(\bigcup_{\hat{I} \in \hat{\mathcal{R}}^n} \hat{I}\Bigg).$$

The previous estimate implies that

$$\sum_{R \in \overline{\mathcal{R}}^{nr}} (\Lambda_{nr,R})^{-(HD(K)-\varepsilon)} \geq \Bigg(\sum_{R \in \hat{\mathcal{R}}^n} (\Lambda_{n,R})^{-(HD(K)-\varepsilon)}\Bigg)^r > 1,$$

where $\overline{\mathcal{R}}^{nr} = \{I_{\underline{c_1}\underline{c_2}\cdots\underline{c_r}}, \underline{c_j} \in X^n \backslash L^n, \forall j \leq r\}$. Thus we conclude (as before) that $HD(\tilde{K}) \geq HD(K) - \epsilon$.

For any positive integer r and $\underline{b_1}, \underline{b_2}, \ldots, \underline{b_r} \in X^n \backslash L^n n$, the sequence $(a_1, a_2, \ldots, a_{nr}) = \underline{b_1}\underline{b_2} \cdots \underline{b_r}$ satisfies that, for all j, $1 \leq j \leq nr - m + 1$, $(a_j, a_{j+1} \ldots, a_{j+m-1}) \neq \underline{b}$, and so $\tilde{K} \subset K_{\underline{b}}$. In particular, $HD(K_{\underline{b}}) \geq HD(\tilde{K}) \geq HD(K) - \epsilon$. □

Proof of Lemma 6. Apply the previous Lemma to K^s and K^u and then use the fact that the Hausdorff dimension of the Cartesian product of regular Cantor sets is the sum of the Hausdorff dimensions of the Cantor sets. □

Note that, by Lemma 6 and the local structure of $\tilde{\Lambda}$, $HD(\tilde{\Lambda} \cap U_d) = HD(\tilde{\Lambda}) > 1$, where U_d is the small neighborhood of d given by Lemma 5.

6.1. *Proof of the Main Theorem.* Given a pair (φ, Λ) of a diffeomorphism φ and a horseshoe associated to φ with $HD(\Lambda) > 1$, we defined in §3.1 (cf. Lemma 4, Theorem 1 and (1)) the open dense set H_φ in $C^1(M, \mathbb{R})$. Recall that $\tilde{\Lambda}$ is a sub-horseshoe of Λ, as in Lemma 6, with $HD(\tilde{\Lambda} \cap U_d) = HD(\tilde{\Lambda}) > 1$. Then, by the theorem from [**MY10**] which we discussed above, there is a diffeomorphism φ_0 close to φ, a horseshoe Λ_0 associated to φ_0 and a sub-horseshoe $\tilde{\Lambda}_0 \subset \Lambda_0$ with $HD(\tilde{\Lambda}_0) > 1$ such that $\tilde{\Lambda}_0$ satisfies the hypotheses of Theorem 2 (we use the theorem to perturb the sub-horseshoe).

Given $f \in H_{\varphi_0}$, we can define a local diffeomorphism $\tilde{A}_{\varphi_0}(f)$; in coordinates given by the stable and unstable foliation, we can write, $\tilde{A}_{\varphi_0}(f)(x, y) = (\tilde{A}^1_{\varphi_0}(f)(x), \tilde{A}^2_{\varphi_0}(f)(y))$, as in §4 (see (6)).

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Let i_0 be such that Corollary 1 and (5) and (4) hold for $(\varphi_0, \tilde{\Lambda}_0)$. For $f \in H_{\varphi_0}$, Remark 2 implies that, for every $x \in \tilde{\Lambda}_{0,i_0}$, $D(f \circ \varphi_0^{i_0} \circ \tilde{A}_{\varphi_0}(f))_x(\tilde{e}_x^{s,u}) \neq 0$, where $\tilde{e}_x^{s,u}$ are the unit vectors in stable and unstable bundles of hyperbolic set $\tilde{\Lambda}_0$, respectively (here $\tilde{\Lambda}_{0,i_0}$ is defined as in (4) but for $\tilde{\Lambda}_0$ instead of $\tilde{\Lambda}$). So the function $f \circ \varphi_0^{i_0} \circ \tilde{A}_{\varphi_0}(f) \in \mathcal{A}_{\tilde{\Lambda}_0}$. Therefore, by Corollary 3,

$$\operatorname{int}(f \circ \varphi_0^{i_0} \circ \tilde{A}_{\varphi_0}(f))(\tilde{\Lambda}_0) \neq \emptyset. \tag{11}$$

Then, as in Corollary 1,

$$\sup_{n\in\mathbb{Z}} f(\varphi_0^n(\tilde{A}_{\varphi_0}(f)(x))) = (f \circ \varphi_0^{i_0} \circ \tilde{A}_{\varphi_0}(f))(x)$$

for all $x \in \tilde{\Lambda}_{0,i_0}$. This implies that $(f \circ \varphi_0^{i_0} \circ \tilde{A}_{\varphi_0}(f))(\tilde{\Lambda}_{0,i_0}) \subset M(f, \Lambda_0)$. Thus, by (11), $\operatorname{int} M(f, \Lambda_0) \neq \emptyset$.

Using Corollary 2 instead of Corollary 1, we get the analogous result for the Lagrange spectrum. This concludes the proof of the Main Theorem. □

A. *Appendix*

A.1. *Regular Cantor sets and limit geometries.* Let $\mathbb{A}$ be a finite alphabet, $\mathbb{B}$ a subset of $\mathbb{A}^2$ and $\Sigma_{\mathbb{B}}$ the subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions $\mathbb{B}$. We will always assume that $\Sigma_{\mathbb{B}}$ is topologically mixing, and that every letter in $\mathbb{A}$ occurs in $\Sigma_{\mathbb{B}}$.

An *expansive map of type* $\Sigma_{\mathbb{B}}$ is a map g with the following properties:

(i) the domain of g is a disjoint union $\bigcup_{\mathbb{B}} I(a, b)$, where, for each (a, b), $I(a, b)$ is a compact subinterval of $I(a) := [0, 1] \times \{a\}$; and

(ii) for each $(a, b) \in \mathbb{B}$, the restriction of g to $I(a, b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t.

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n\geq 0} g^{-n}\bigg(\bigcup_{\mathbb{B}} I(a, b)\bigg).$$

Let $\Sigma_{\mathbb{B}}^+$ be the unilateral subshift associated to $\Sigma_{\mathbb{B}}$. There exists a unique homeomorphism $h : \Sigma_{\mathbb{B}}^+ \to K$ such that

$$h(\underline{a}) \in I(a_0) \quad \text{for } \underline{a} = (a_0, a_1, \ldots) \in \Sigma_{\mathbb{B}}^+ \quad \text{and} \quad h \circ \sigma = g \circ h,$$

where, $\sigma^+ : \Sigma_{\mathbb{B}}^+ \to \Sigma_{\mathbb{B}}^+$ is defined as $\sigma^+((a_n)_{n\geq 0}) = (a_{n+1})_{n\geq 0}$. For $(a, b) \in \mathbb{B}$, let

$$f_{a,b} = [g|_{I(a,b)}]^{-1}.$$

This is a contracting diffeomorphism from $I(b)$ onto $I(a, b)$. If $\underline{a} = (a_0, \ldots, a_n)$ is a word of $\Sigma_{\mathbb{B}}$, we put

$$f_{\underline{a}} = f_{a_0,a_1} \circ \cdots \circ f_{a_{n-1},a_n}.$$

This is a diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ that we denote by $I(\underline{a})$. It has the property that, if z in the domain of $f_{\underline{a}}$,

$$f_{\underline{a}}(z) = h(\underline{a}h^{-1}(z)).$$

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Let $r > 1$ be a real number or $r = +\infty$. The space of C^r expansive maps of type Σ, endowed with the C^r topology, will be denoted by Ω^r_Σ. The union $\Omega_\Sigma = \bigcup_{r>1} \Omega^r_\Sigma$ is endowed with the inductive limit topology.

Let $\Sigma^- = \{(\theta_n)_{n\le 0}, (\theta_i, \theta_{i+1}) \in \mathbb{B} \text{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \ne \underline{\widetilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \underline{\widetilde{\theta}}) = \begin{cases} 1 & \text{if } \theta_0 \ne \widetilde{\theta}_0, \\ |I(\underline{\theta} \wedge \underline{\widetilde{\theta}})| & \text{otherwise,} \end{cases}$$

where $\underline{\theta} \wedge \underline{\widetilde{\theta}} = (\theta_{-n}, \dots, \theta_0)$ if $\widetilde{\theta}_{-j} = \theta_{-j}$ for $0 \le j \le n$ and $\widetilde{\theta}_{-n-1} \ne \theta_{-n-1}$.

Now, let $\underline{\theta} \in \Sigma^-$; for $n > 0$, let $\underline{\theta}^n = (\theta_{-n}, \dots, \theta_0)$ and let $B(\underline{\theta}^n)$ be the affine map from $I(\underline{\theta}^n)$ onto $I(\theta_0)$ such that the diffeomorphism $k_n^{\underline{\theta}} = B(\underline{\theta}^n) \circ f_{\underline{\theta}^n}$ is orientation preserving.

This well-known result (cf. [**Sul87**]) follows.

PROPOSITION. *Let $r \in (1, +\infty)$, $g \in \Omega^r_\Sigma$.*

(1) *For any $\underline{\theta} \in \Sigma^-$, there is a diffeomorphism $k^{\underline{\theta}} \in \mathrm{Diff}^r_+(I(\theta_0))$ such that $k_n^{\underline{\theta}}$ converges to $k^{\underline{\theta}}$ in $\mathrm{Diff}^{r'}_+(I(\theta_0))$ uniformly in $\underline{\theta}$, for any $r' < r$. The convergence is also uniform in a neighborhood of g in Ω^r_Σ.*

(2) *If r is an integer or $r = +\infty$, $k_n^{\underline{\theta}}$ converge to $k^{\underline{\theta}}$ in $\mathrm{Diff}^r_+(I(\theta_0))$. More precisely, for every $0 \le j \le r-1$, there is a constant C_j (independent of $\underline{\theta}$) such that*

$$|D^j \log D[k_n^{\underline{\theta}} \circ (k^{\underline{\theta}})^{-1}](x)| \le C_j |I(\underline{\theta}^n)|.$$

It follows that $\underline{\theta} \to k^{\underline{\theta}}$ is Lipschitz in the following sense: for $\theta_0 = \widetilde{\theta}_0$,

$$|D^j \log D[k^{\underline{\widetilde{\theta}}} \circ (k^{\underline{\theta}})^{-1}](x)| \le C_j d(\underline{\theta}, \underline{\widetilde{\theta}}).$$

A.2. *Expanding maps associated to a horseshoe.* Let Λ be a horseshoe associated to a C^2-diffeomorphism φ on a surface M and consider a finite collection $(R_a)_{a\in\mathbb{A}}$ of disjoint rectangles of M, which are a Markov partition of Λ. Consider the sets

$$W^s(\Lambda, R) = \bigcap_{n\ge 0} \varphi^{-n}\bigg(\bigcup_{a\in\mathbb{A}} R_a\bigg),$$

$$W^u(\Lambda, R) = \bigcap_{n\le 0} \varphi^{-n}\bigg(\bigcup_{a\in\mathbb{A}} R_a\bigg).$$

There is $r > 1$ and a collection of C^r-submersions $(\pi_a : R_a \to I(a))_{a\in\mathbb{A}}$ satisfying the property that if $z, z' \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ and $\pi_{a_0}(z) = \pi_{a_0}(z')$, then

$$\pi_{a_1}(\varphi(z)) = \pi_{a_1}(\varphi(z')).$$

In particular, the connected components of $W^s(\Lambda, R) \cap R_a$ are the level lines of π_a. Then we define a mapping g^u of class C^r (expansive of type $\Sigma_{\mathbb{B}}$) by the formula

$$g^u(\pi_{a_0}(z)) = \pi_{a_1}(\varphi(z))$$

for $(a_0, a_1) \in \mathbb{B}$, $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$. The regular Cantor set K^u, defined by g^u, describes the geometric transverse of the stable foliation $W^s(\Lambda, R)$. Moreover, there exists a unique homeomorphism $h^u : \Sigma^+_{\mathbb{B}} \to K^u$ such that

$$h^u(\underline{a}) \in I(a_0) \quad \text{for } \underline{a} = (a_0, a_1, \dots) \in \Sigma^+_{\mathbb{B}} \quad \text{and} \quad h^u \circ \sigma^+ = g^u \circ h^u,$$

where $\sigma^+ : \Sigma^+_{\mathbb{B}} \to \Sigma^+_{\mathbb{B}}$ is defined as $\sigma^+((a_n)_{n\ge 0}) = (a_{n+1})_{n\ge 0}$.

Given a finite word $\underline{a} = (a_0, \ldots, a_n)$, define $f^u_{\underline{a}}$, as in the previous section, such that

$$f^u_{\underline{a}}(z) = h^u(\underline{a}(h^u)^{-1}(z)).$$

Analogously, we can describe the geometric transverse of the unstable foliation $W^u(\Lambda, R)$, using a regular Cantor set K^s defined by a mapping g^s of class C^r (expansive of type $\Sigma_{\mathbb{B}}$).

Moreover, there exists a unique homeomorphism $h^s : \Sigma^-_{\mathbb{B}} \to K^s$ such that

$$h^s(\underline{a}) \in I(a_0) \quad \text{for } \underline{a} = (\ldots, a_1, a_0) \in \Sigma^-_{\mathbb{B}} \quad \text{and} \quad h^s \circ \sigma^- = g^s \circ h^s,$$

where $\sigma^- : \Sigma^-_{\mathbb{B}} \to \Sigma^-_{\mathbb{B}}$ is defined as $\sigma^-((a_n)_{n\le 0}) = (a_{n-1})_{n\le 0}$.

Given a finite word $\underline{a} = (a_{-n}, \ldots, a_0)$, define $f^s_{\underline{a}}$, as in the previous section, such that

$$f^s_{\underline{a}}(z) = h^s((h^s)^{-1}(z)\underline{a}).$$

Also, the horseshoe Λ is locally the product of two regular Cantor sets K^s and K^u. So the Hausdorff dimension of Λ, $HD(\Lambda)$, is equal to $HD(K^s \times K^u)$ but, for regular Cantor sets, $HD(K^s \times K^u) = HD(K^s) + HD(K^u)$. Thus $HD(\Lambda) = HD(K^s) + HD(K^u)$ (cf. [**PT93**, Ch. 4]).

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