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A Multipartite Version of the Hajnal–Szemerédi Theorem for Graphs and Hypergraphs

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A perfect K_t-matching in a graph G is a spanning subgraph consisting of vertex-disjoint copies of K_t. A classic theorem of Hajnal and Szemerédi states that if G is a graph of order n with minimum degree $\delta(G) \geqslant (t-1)n/t$ and $t|n$, then G contains a perfect K_t-matching. Let G be a t-partite graph with vertex classes $V_1, \ldots, V_t$ each of size n. We show that, for any $\gamma > 0$, if every vertex $x \in V_i$ is joined to at least $\big((t-1)/t+\gamma\big)n$ vertices of V_j for each $j \neq i$, then G contains a perfect K_t-matching, provided n is large enough. Thus, we verify a conjecture of Fischer [6] asymptotically. Furthermore, we consider a generalization to hypergraphs in terms of the codegree.

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1. Introduction

Given a graph G and an integer $t \geqslant 3$, a *K_t-matching* is a set of vertex-disjoint copies of K_t in G. A *perfect K_t-matching* (or *K_t-factor*) is a spanning K_t-matching. Clearly, if G contains a perfect K_t-matching then t divides $|G|$. A classic theorem of Hajnal and Szemerédi [8] states a relationship between the minimum degree and the existence of a perfect K_t-matching.

Theorem 1.1 (Hajnal–Szemerédi theorem [8]). *Let $t > 2$ be an integer. Let G be a graph of order n with minimum degree $\delta(G) \geqslant (t-1)n/t$ and $t|n$. Then G contains a perfect K_t-matching.*

Let G be a t-partite graph with vertex classes $V_1, \ldots, V_t$. We say that G is *balanced* if $|V_i| = |V_j|$ for $1 \leqslant i < j \leqslant t$. Write $G[V_i, V_j]$ for the induced bipartite subgraph on vertex

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classes V_i and V_j. Define $\widetilde{\delta}(G)$ to be $\min_{1\leqslant i<j\leqslant t} \delta(G[V_i, V_j])$. Fischer [6] conjectured the following multipartite version of the Hajnal–Szemerédi theorem.

Conjecture 1.2 (Fischer [6]). *Let G be a balanced t-partite graph with each class of size n. Then there exists an integer $a_{n,t}$ such that if $\widetilde{\delta}(G) \geqslant (t-1)n/t + a_{n,t}$, then G contains a perfect K_t-matching.*

Note that the $+a_{n,t}$ term was not presented in Fischer's original conjecture, but it was shown to be necessary for odd t in [19]. For $t = 2$, the conjecture can be easily verified by Hall's theorem. For $t = 3$, Johansson [11] proved that $\widetilde{\delta}(G) \geqslant 2n/3 + \sqrt{n}$ suffices for all n. Using the regularity lemma, Magyar and Martin [19] and Martin and Szemerédi [20] proved Conjecture 1.2 for $t = 3$ and $t = 4$ respectively for n sufficiently large, where $a_{n,t} = 1$ if both t and n are odd, and $a_{n,t} = 0$ otherwise. For $t \geqslant 5$, Csaba and Mydlarz [4] proved that $\widetilde{\delta}(G) \geqslant c_t n/(c_t + 1)$ is sufficient, where $c_t = t - 3/2 + (1 + 1/2 + \cdots + 1/t)/2$. In this paper, we show that Conjecture 1.2 is true asymptotically.

Theorem 1.3. *Let $t \geqslant 2$ be an integer and let $\gamma > 0$. Then there exists an integer $n_0 = n_0(t, \gamma)$ such that if G is a balanced t-partite graph with each class of size $n \geqslant n_0$ and $\widetilde{\delta}(G) \geqslant \big((t-1)/t + \gamma\big)n$, then G contains a perfect K_t-matching.*

Independently, Theorem 1.3 also has been proved by Keevash and Mycroft [13]. Their proof involves the hypergraph blow-up lemma [12], so n_0 is extremely large, whereas our proof gives a much smaller n_0. Since the submission of this paper, Keevash and Mycroft [14] have proved Conjecture 1.2, provided n is large enough. Also, Han and Zhao [10] gave a different proof of Conjecture 1.2 for $t = 3, 4$, again provided n is large enough.

We further generalize Theorem 1.3 to hypergraphs. For $a \in \mathbb{N}$, we refer to the set $\{1, \ldots, a\}$ as $[a]$. For a set U, we denote by $\binom{U}{k}$ the set of k-sets of U. A *k-uniform hypergraph*, or *k-graph* for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and $E(H) \subset \binom{V(H)}{k}$ is a family of k-sets of $V(H)$. We simply write V to mean $V(H)$ if it is clear from the context. For a k-graph H and an l-set $T \in \binom{V}{l}$, let $N^H(T)$ be the set of $(k-l)$-sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in H. Let $\deg^H(T) = |N^H(T)|$. Define the *minimum l-degree* $\delta_l(H)$ of H to be the minimal $\deg^H(T)$ over all $T \in \binom{V}{l}$. For $U \subset V$, we denote by $H[U]$ the induced subgraph of H on vertex set U.

A k-graph H is *t-partite* if there exists a partition of the vertex set V into t classes $V_1, \ldots, V_t$ such that every edge intersects every class in at most one vertex. Similarly, H is *balanced* if $|V_1| = \cdots = |V_t|$. An l-set $T \in \binom{V}{l}$ is said to be *legal* if $|T \cap V_i| \leqslant 1$ for $i \in [t]$. For $I \subset [t]$, $T \subset V$ is *I-legal* if $|T \cap V_i| = 1$ for $i \in I$ and $|T \cap V_i| = 0$ otherwise. We write V_I to be the set of I-legal sets. For disjoint sets I, J such that $I \cup J \in \binom{[t]}{k}$ and an I-legal set $T \in V_I$, denote by $N_J^H(T)$ the set of J-legal sets S such that $S \cup T$ is an edge in H and write $\deg_J^H(T) = |N_J^H(T)|$. For $l \in [k-1]$ and $I \in \binom{[t]}{l}$, define $\widetilde{\delta}_I(H) = \min\{\deg_J^H(T) : T \in V_I$ and $J \in \binom{[t]\setminus I}{k-|I|}\}$. Finally, we set $\widetilde{\delta}_l(H) = \min\{\widetilde{\delta}_I(H) : I \in \binom{[t]}{l}\}$. If H is clear from the

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context, we drop the superscript of H. Note that for graphs, when $k = 2$, $\widetilde{\delta}_1(G) = \widetilde{\delta}(G)$ as defined earlier.

Let K_t^k be the complete k-graph on t vertices. It is easy to see that a t-partite k-graph H contains a perfect K_t^k-matching only if H is balanced.

Definition. Let $1 \leqslant l < k \leqslant t$ and $n \geqslant 1$ be integers. Define $\phi_l^k(t,n)$ to be the smallest integer d such that every t-partite k-graph H with each class of size n and $\widetilde{\delta}_l(H) \geqslant d$ contains a perfect K_t^k-matching. Equivalently,

$$\phi_l^k(t,n) = \min\{d : \ \widetilde{\delta}_l(H) \geqslant d \Rightarrow H \text{ contains a perfect } K_t^k\text{-matching}\},$$

where H is a t-partite k-graph H with each class of size n. Write $\phi^k(t,n)$ for $\phi_{k-1}^k(t,n)$.

Note that Theorem 1.3 implies that $\phi^2(t,n) \sim (t-1)n/t$. Various cases of $\phi_l^k(k,n)$ have been studied. Daykin and Häggkvist [5] showed that $\phi_1^k(k,n) \leqslant (k-1)n^{k-1}/k$, which was later improved by Hán, Person and Schacht [9]. Kühn and Osthus [15] showed that $n/2 - 1 < \phi^k(k,n) = \phi_{k-1}^k(k,n) \leqslant n/2 + \sqrt{2n\log n}$. Aharoni, Georgakopoulos and Sprüssel [1] then reduced the upper bound to $\phi^k(k,n) \leqslant \lceil (n+1)/2 \rceil$. For $k/2 \leqslant l < k-1$, Pikhurko [21] showed that $\phi_l^k(k,n) \leqslant n^{k-l}/2$. The exact value of $\phi_1^3(3,n)$ has been determined by the authors in [17]. In this paper, we give an upper bound on $\phi^k(t,n)$ for $3 \leqslant k < t$.

Theorem 1.4. *For $3 \leqslant k < t$ and $\gamma \geqslant 0$, there exists an integer $n_0 = n_0(k,t,\gamma)$ such that, for all $n \geqslant n_0$,*

$$\phi^k(t,n) \leqslant \left(1 - \left(\binom{t-1}{k-1} + 2\binom{t-2}{k-2}\right)^{-1} + \gamma\right)n.$$

We do not believe the upper bound is best possible. For $k = 3$ and $t = 4$, it was shown, independently in [16] and [13], that for any $\gamma > 0$ if H is a 3-graph (not 3-partite) with $\delta_2(H) = (3/4 + \gamma)n$, then H contains a perfect K_4^3-matching, provided n is large enough. (Moreover, in [13], Keevash and Mycroft have determined the exact value of the $\delta_2(H)$-threshold for the existence of perfect K_4^3-matchings.) Thus, it is natural to believe that $\phi^3(4,n)$ should be $3n/4 + o(n)$.

Our proofs of Theorems 1.3 and 1.4 use the absorption technique introduced by Rödl, Ruciński and Szemerédi [22]. We now present an outline of the absorption technique. First, we remove a set U of disjoint copies of K_t^k from H satisfying the conditions of the absorption lemma, Lemma 3.2, and call the resulting graph H'. Next, we find a K_t^k-matching covering almost all vertices of H'. Let W be the set of 'leftover' vertices. By the absorption property of U, there is a perfect K_t^k-matching in $H[U \cup W]$. Hence, we obtain a perfect K_t^k-matching in H as required.

In order to find a K_t^k-matching covering almost all vertices of H', we follow the approach of Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [2], who consider fractional matchings. Let $\mathcal{K}_t^k(H)$ be the set of K_t^k in a k-graph H. A *fractional K_t^k-matching* in a k-graph H is a function $w : \mathcal{K}_t^k(H) \to [0,1]$ such that for each $v \in V$ we

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have

$$\sum\{w(T) : v \in T \in \mathcal{K}_t^k(H)\} \leqslant 1.$$

Then $\sum_{T\in\mathcal{K}_t^k(H)} w(T)$ is the *size* of w. If the size is $|H|/t$, then w is *perfect*. We are interested in perfect fractional K_t^k-matchings w in a t-partite k-graph H with each class of size n. Note that $|H| = tn$, so if w is a perfect fractional K_t^k-matching in H, then

$$\sum\{w(T) : v \in T \in \mathcal{K}_t^k(H)\} = 1 \text{ for } v \in V \text{ and } \sum_{T\in\mathcal{K}_t^k(H)} w(T) = n.$$

Define $\phi_l^{*,k}(t,n)$ to be the fractional analogue of $\phi_l^k(t,n)$.

Theorem 1.5. *For $2 \leqslant k \leqslant t$ and $n \geqslant 1$,*

$$\lceil (t-k+1)n/t \rceil \leqslant \phi^{*,k}(t,n) \leqslant \begin{cases} \lceil (t-1)n/t \rceil & \text{for } k = 2, \\ \lceil \left(1 - \binom{t-1}{k-1}^{-1}\right) n \rceil + 1 & \text{for } k \geqslant 3. \end{cases}$$

In particular, $\phi^{*,2}(t,n) = \lceil (t-1)n/t \rceil$.

Notice that Theorem 1.5 is only tight for $k = 2$. The upper bound on $\phi^{*,k}(t,n)$ given in Theorem 1.5 is sufficient for our purpose, that is, to prove Theorems 1.3 and 1.4. In addition, we also obtain the following result.

Theorem 1.6. *Let $2 \leqslant k \leqslant t$ be integers. Then, given any $\varepsilon, \gamma > 0$, there exists an integer n_0 such that every k-graph H of order $n > n_0$ with*

$$\delta_{k-1}(H) \geqslant t\phi^{*,k}(t, \lceil n/t \rceil) + \gamma n$$

contains a K_t^k-matching $\mathcal{T}$ covering all but at most εn vertices.

Together with Theorem 1.5, we obtain the following corollary for general k-graphs.

Corollary 1.7. *Let $3 \leqslant k \leqslant t$ be integers. Then, given any $\varepsilon, \gamma > 0$, there exists an integer n_0 such that every k-graph H of order $n > n_0$ with*

$$\delta_{k-1}(H) \geqslant \left(1 - \binom{t-1}{k-1}^{-1} + \gamma\right) n$$

contains a K_t^k-matching $\mathcal{T}$ covering all but at most εn vertices.

Observe that Corollary 1.7 is a stronger statement than Lemma 6.1 in [16]. Thus, by replacing Lemma 6.1 in [16] with Theorem 1.6, we improve the bounds of Theorem 1.4 in [16].

In the next section, we prove Theorem 1.5. Theorems 1.3 and 1.4 are proved simultaneously in Section 3. Finally, Theorem 1.6 is proved in Section 4.

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2. Perfect fractional K_t^k-matchings

In this section we are going to prove Theorem 1.5. We require the Farkas lemma.

Lemma 2.1 (Farkas lemma (see [18], p. 257)). *A system of equations $yA = b$, $y \geqslant 0$ is solvable if and only if the system $Ax \geqslant 0$, $bx < 0$ is unsolvable.*

First we prove the lower bounds on $\phi^{*,k}(t,n)$.

Proposition 2.2. *Let $2 \leqslant k \leqslant t$ and $n \geqslant 1$ be integers. There exists a t-partite k-graph H with each class of size n with $\widetilde{\delta}_{k-1}(H) = \lceil (t-k+1)n/t \rceil - 1$ without a perfect fractional K_t^k-matching.* □

Proof. We fix t, k and n. Let $V_1, \ldots, V_t$ be disjoint vertex sets each of size n. For $i \in [t]$, fix a $(\lceil (t-k+1)n/t \rceil - 1)$-set $W_i \subset V_i$. Define H to be the t-partite k-graph on vertex classes $V_1, \ldots, V_t$ such that every edge in H meets W_i for some i. Clearly, $\widetilde{\delta}_{k-1}(H) = \lceil (t-k+1)n/t \rceil - 1$. Thus, it suffices to show that H does not contain a perfect fractional K_t^k-matching. Let A be the matrix of H with rows representing the $K_t^k(H)$ and columns representing the vertices of H such that $A_{T,v} = 1$ if and only if $v \in T$ for $T \in \mathcal{K}_t^k(H)$ and $v \in V$. By the Farkas lemma, Lemma 2.1, taking $y = (w(T) : T \in \mathcal{K}_t^k(H))$ and $b = (1, \ldots, 1)$, there is no perfect fractional K_t^k-matching in H if and only if there is a weighting function $w : V \to \mathbb{R}$ such that

$$\sum_{v \in T} w(v) \geqslant 0, \text{ for all } T \in \mathcal{K}_t^k(H) \quad \text{and} \quad \sum_{v \in V} w(v) < 0. \tag{2.1}$$

Set $w(v) = (k-1)/(t-k+1)$ if $v \in \bigcup_{i \in [t]} W_i$ and $w(v) = -1$ otherwise. Clearly,

$$\sum w(v) = \frac{k-1}{t-k+1} t \left(\left\lceil \frac{(t-k+1)n}{t} \right\rceil - 1 \right) - t \left(n - \left\lceil \frac{(t-k+1)n}{t} \right\rceil + 1 \right) < 0.$$

For $T \in \mathcal{K}_t^k(H)$, T contains at least $t-k+1$ vertices in $\bigcup_{i \in [t]} W_i$ and so $\sum_{v \in T} w(v) \geqslant 0$. Thus, w satisfies (2.1), so H does not contain a perfect fractional K_t^k-matching. □

Proof of Theorem 1.5. By Proposition 2.2 it is sufficient to prove the upper bound on $\phi^{*,k}(t,n)$. Fix k, t and n. Suppose to the contrary that there exists a t-partite k-graph H with each class of size n and

$$\widetilde{\delta}_{k-1}(H) \geqslant \widetilde{\delta}$$

that does not contain a perfect fractional K_t^k-matching, where $\widetilde{\delta}$ is the upper bound on $\phi^{*,k}(t,n)$ stated in the theorem. By an argument similar to that in the proof of Proposition 2.2, there is a weighting function $w : V \to \mathbb{R}$ satisfying (2.1). Let $V_1, \ldots, V_t$ be the vertex classes of H with $V_i = \{v_{i,1}, \ldots, v_{i,n}\}$ for $i \in [t]$. We identify the t-tuple $(j_1, \ldots, j_t) \in [n]^t$ with the $[t]$-legal set $\{v_{1,j_1}, \ldots, v_{t,j_t}\}$ and write $w(j_1, \ldots, j_t)$ to mean $\sum_{i \in [t]} w(v_{i,j_i})$. Without loss of generality we may assume that for $i \in [t]$, $(w(v_{i,j}))_{j \in [n]}$ is a decreasing sequence, *i.e.*,

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$w(v_{i,j}) \geqslant w(v_{i,j'})$ for $1 \leqslant j < j' \leqslant n$. By considering the vertex weighting w' such that

$$w'(v) = \begin{cases} w(v)+\varepsilon & \text{if } v \in V_i, \\ w(v)-\varepsilon & \text{if } v \in V_{i'}, \\ w(v) & \text{otherwise,} \end{cases}$$

with $\varepsilon > 0$, we may assume that $w(v_{i,n}) = w(v_{i',n})$ for all $i, i' \in [t]$. By (2.1), $w(v_{i,n})$ is negative as $w(v_{i,j}) \geqslant w(v_{i,n}) = w(v_{i',n})$ for all $j \in [n]$ and $i, i' \in [t]$. Thus, by multiplying through by a suitable constant we may assume that $w(v_{i,n}) = -1$ for all $i \in [t]$. We further assume that $w(v) \leqslant t-1$ for all $v \in V$, because (2.1) still holds after we replace $w(v)$ with $\min\{w(v), t-1\}$. Finally, we apply the linear transformation $(w(v)+1)/t$ for $v \in V$, which scales w so that it now lies in the interval $[0,1]$, and w satisfies the following inequalities:

$$\sum_{v\in T} w(v) \geqslant 1 \text{ for all } T \in \mathcal{K}_t^k(H) \quad \text{and} \quad \sum_{v \in V} w(v) < n. \tag{2.2}$$

For $j \in [t]$, set $r(j) = n - \binom{j-1}{k-1}(n-\widetilde{\delta})$. Given a J-legal set $T \in \mathcal{K}_j^k(H)$ with $J \in \binom{[t]}{j}$ and $j < k$, for each $i \in [t]\backslash J$ there are at least $r(j+1)$ vertices $v \in V_i$ such that $T \cup v$ forms a K_{j+1}^k. Note that $r(j) = n$ for $j \in [k-1]$ and $r(k) = \widetilde{\delta}$. By the definition of $\widetilde{\delta}$, we know that $r(t) \geqslant 1$. Hence, we can find a K_t^k $(j_1, j_2, \ldots, j_t)$ with $j_i \geqslant r(i)$ for $i \in [t]$.

Recall that for $i \in [t]$ and $1 \leqslant j < j' \leqslant n$, $w(v_{i,j}) \geqslant w(v_{i,j'})$. Therefore,

$$\sum_{i\in[t]} w(v_{i,r(i)}) = w(r(1), r(2), \ldots, r(t)) \geqslant w(j_1, j_2, \ldots, j_t) \geqslant 1$$

by (2.2). By a similar argument, for any permutation σ of $[t]$ we have

$$\sum_{i\in[t]} w(v_{i,r(\sigma(i))}) \geqslant 1.$$

Setting $\sigma = (1, 2, \ldots, t)$, we have

$$\sum_{i\in[t]}\sum_{j\in[t]} w(v_{i,r(j)}) = \sum_{j\in[t]}\sum_{i\in[t]} w(v_{i,r(\sigma^j(i))}) \geqslant t. \tag{2.3}$$

Observe that $w(v_{i,r(j)}) \leqslant w(v_{i,r(j+1)})$ for $i \in [t]$ and $j \in [t-1]$. Since $r(j) = n$ for $j \in [k-1]$ and $w(v_{i,n}) = 0$ for $i \in [t]$,

$$\begin{aligned} \sum_{i\in[t]} w(v_{i,r(t)}) &= \frac{1}{t-k+1}\sum_{i\in[t]}\left(\sum_{j\in[k-1]} w(v_{i,r(j)}) + (t-k+1)w(v_{i,r(t)})\right) \\ &\geqslant \frac{1}{t-k+1}\sum_{i\in[t]}\sum_{j\in[t]} w(v_{i,r(j)}) \geqslant \frac{t}{t-k+1}, \end{aligned} \tag{2.4}$$

where the last inequality is due to (2.3).

Claim 2.3.

$$\sum_{i\in[t]}\left(\sum_{j\in[t-1]} (r(j)-r(j+1))w(v_{i,r(j)}) + \frac{r(k)-r(t)}{t-k} w(v_{i,r(t)})\right) \geqslant \frac{t(r(k)-r(t))}{t-k}.$$

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Proof of claim. Consider the multiset A containing $(t-k)(r(j)-r(j+1))$ copies of j for $k \leqslant j \leqslant t-1$ and $r(k)-r(t)$ copies of t. In order to prove the claim (by multiplying though by $(t-k)$), it is enough to show that

$$\sum_{i\in[t]}\sum_{j\in A} w(v_{i,r(j)}) \geqslant t(r(k)-r(t)).$$

First note that

$$\sum_{k\leqslant j\leqslant t-1} (r(j)-r(j+1)) = r(k)-r(t),$$

so the number of elements j (with multiplicity) in A with $k \leqslant j \leqslant t-1$ is exactly $(t-k)(r(k)-r(t))$. Note that $r(j)-r(j+1) = \binom{j-1}{k-2}(n-\widetilde{\delta})$. Hence, for $k \leqslant j < j' \leqslant t-1$, there are more copies of j' than copies of j in A. Recall that A contains precisely $r(k)-r(t)$ copies of t. It follows that we can replace some elements by smaller elements to obtain a multiset A' containing each of $k,\dots,t$ exactly $r(k)-r(t)$ times. Since $w(v_{i,r(j)})$ is increasing in j and $w(v_{i,r(j)}) = 0$ for $j \in [k-1]$, it follows that

$$\begin{aligned}\sum_{i\in[t]}\sum_{j\in A} w(v_{i,r(j)}) &\geqslant \sum_{i\in[t]}\sum_{j\in A'} w(v_{i,r(j)}) = (r(t)-r(k))\sum_{i\in[t]}\sum_{k\leqslant j\leqslant t} w_{v_{i,r(j)}}\\ &= (r(t)-r(k))\sum_{i\in[t]}\sum_{j\in[t]} w(v_{i,r(j)}) \geqslant t(r(t)-r(k))\end{aligned}$$

as required, where the last inequality is due to (2.3). □

Recall that $r(k) = \widetilde{\delta}$ and $r(1) = n$. Since $w(v_{i,j'})$ is decreasing in j', $w(v_{i,j'}) \geqslant w(v_{i,r(j)})$ for $r(j+1) < j' \leqslant r(j)$ and $j \in [t]$, where we take $r(t+1) = 0$. Hence,

$$\sum_{i\in[t]}\sum_{j\in[n]} w(v_{i,j}) \geqslant \sum_{i\in[t]}\Bigg(\sum_{j\in[t-1]} (r(j)-r(j+1))w(v_{i,r(j)}) + r(t)w(v_{i,r(t)})\Bigg).$$

By Claim 2.3 and (2.4), this is at least

$$\begin{aligned}&\frac{t(r(k)-r(t))}{t-k} + \sum_{i\in[t]}\left(r(t) - \frac{r(k)-r(t)}{t-k}\right)w(v_{i,r(t)})\\ &\geqslant \frac{t(r(k)-r(t))}{t-k} + \left(r(t) - \frac{r(k)-r(t)}{t-k}\right)\frac{t}{t-k+1}\\ &= \frac{tr(k)}{t-k+1} = \frac{t\widetilde{\delta}}{t-k+1} \geqslant n,\end{aligned}$$

contradicting (2.2). The proof of Theorem 1.5 is completed. □

Note that the inequality above suggests that for $k \geqslant 3$, we would have $\phi^{*,k}(t,n) = \widetilde{\delta} \leqslant \lceil (t-k+1)n/t \rceil$. However, our proof requires that $1 \leqslant r(t) = n - \binom{t-1}{k-1}(n-\widetilde{\delta})$, implying that $\widetilde{\delta} \geqslant \left(1 - \binom{t-1}{k-1}^{-1}\right)n + 1$.

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3. Proof of Theorems 1.3 and 1.4

First we need the following simple proposition.

Proposition 3.1. *Let $\gamma > 0$. Let H be a balanced t-partite k-graph with partition classes $V_1, \dots, V_t$, each of size n with*

$$\widetilde{\delta}_{k-1}(H) \geqslant \left(1 - \left(\binom{t-2}{k-1} + 2\binom{t-2}{k-2}\right)^{-1} + \gamma\right) n.$$

Then, for $i \in [t]$ and distinct vertices $u, v \in V_i$, there are at least $(\gamma n)^{t-1}$ legal $[t]\backslash i$-sets T such that $T \cup u$ and $T \cup v$ span copies of K_t^k in H. □

Proof. Let $u, v \in V_1$. For $2 \leqslant i \leqslant t$, we pick $w_i \in V_i$ such that $w_i \in N(T)$ for all legal $(k-1)$-sets $T \subset \{u, v, w_2, \dots, w_{i-1}\}$. By the definition of $\widetilde{\delta}_{k-1}(H)$, there are at least γn choices for each w_i. The proposition easily follows. □

Using Proposition 3.1, we obtain an absorption lemma. Its proof can be easily obtained by modifying the proof of Lemma 4.2 in [17]. For the sake of completeness, it is included in the Appendix.

Lemma 3.2 (Absorption lemma). *Let $2 \leqslant k < t$ be integers and let $\gamma > 0$. Then, there is an integer n_0 satisfying the following. For each balanced t-partite k-graph H with each class of size $n \geqslant n_0$ and*

$$\widetilde{\delta}_{k-1}(H) \geqslant \left(1 - \left(\binom{t-2}{k-1} + 2\binom{t-2}{k-2}\right)^{-1} + \gamma\right) n,$$

there exists a balanced vertex subset $U \subset V(H)$ of size $|U| \leqslant \gamma^{t(t-1)} n/(t^2 2^{t+2})$ such that there exists a perfect K_t^k-matching in $H[U \cup W]$ for every balanced vertex subset $W \subset V \backslash U$ of size $|W| \leqslant \gamma^{2t(t-1)} n/(t^2 2^{2t+5})$.

Our next task is to find a large K_t^k-matching in H covering all but at most εn vertices, which requires a theorem of Frankl and Rödl [7] and Chernoff's inequality. The proof of Lemma 3.5 is based on Claim 4.1 in [2]. For constants $a, b, c > 0$, write $a = b \pm c$ for $b - c \leqslant a \leqslant b + c$.

Theorem 3.3 (Frankl and Rödl [7]). *For all $t, \varepsilon \geqslant 0$ and $a > 3$, there exists $\tau = \tau(\varepsilon)$, $D = D(n)$, and $n_0 = n_0(\tau)$ such that if $n \geqslant n_0$ and H is a t-graph of order n satisfying*

(a) $\deg^H(v) = (1 \pm \tau)D$ *for all $v \in V$, and*
(b) $\Delta_2(H) = \max_{T \in \binom{V(H)}{2}} \deg^H(T) < D/(\log n)^a$,

then Hcontains a matching M covering all but at most εn vertices.

Lemma 3.4 (Chernoff's inequality (see, *e.g.*, [3])). *Let $X \sim \mathrm{Bin}(n, p)$. Then, for $0 < \lambda \leqslant np$,*

$$\mathbb{P}(|X - np| \geqslant \lambda) \leqslant 2 \exp\left(-\frac{\lambda^2}{4np}\right) \quad \text{and} \quad \mathbb{P}(X \leqslant np - \lambda) \leqslant \exp\left(-\frac{\lambda^2}{4np}\right).$$

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Lemma 3.5. *Let $2 \leqslant k \leqslant t$ be integers. Then, for any given $\varepsilon, \gamma > 0$, there exists an integer n_0 such that every t-partite k-graph H with partition classes $V_1, \ldots, V_t$, each of size $n > n_0$, with*

$$\widetilde{\delta}_{k-1}(H) \geqslant \phi^{*,k}(t,n) + \gamma n$$

contains a K_t^k-matching $\mathcal{T}$ covering all but at most εn vertices.

Proof. Fix k, t and ε. If $k = t = 2$, then the lemma easily holds and so we may assume that $t \geqslant 3$. Write $\phi^* = \phi^{*,k}(t,n)/n$. We assume that n is sufficiently large throughout the proof. Let H be a balanced t-partite k-graph H with partition classes $V_1, \ldots, V_t$, each of size n, with $\widetilde{\delta}_{k-1}(H) \geqslant (\phi^* + \gamma)n$. Our aim is to define a t-graph H^* on vertex set $V(H)$ satisfying the condition of Theorem 3.3, where every edge in H^* corresponds to a K_t^k in H. Hence, by Theorem 3.3, there exists a matching M covering all but at most εn vertices of H^* corresponding to a K_t^k-matching in H.

We are going to construct H^* via two rounds of randomization. For $i \in [t]$, let R_i be a random binomial subset of V_i with probability $p = n^{-0.9}$. Let $R = (R_1, \ldots, R_t)$. Then, by Chernoff's inequality (Lemma 3.4),

$$\mathbb{P}(|R_i - n^{0.1}| \geqslant n^{0.075}) \leqslant 2\exp(-n^{0.05}/2). \tag{3.1}$$

For each $I \in \binom{[t]}{k-1}$, each I-legal set $T \subset R$ and $i \in [t]\backslash I$,

$$\mathbb{E}(\deg_i^{H[R]}(T)) \geqslant (\phi^* + \gamma)n \times n^{-0.9} = (\phi^* + \gamma)n^{0.1}.$$

Again, by Chernoff's inequality (Lemma 3.4),

$$\mathbb{P}(\deg_i^{H[R]}(T) < (\phi^* + \gamma/2)n^{0.1}) \leqslant \exp(-\gamma^2 n^{0.1}/(16(\phi^* + \gamma))) = e^{-\Omega(n^{0.1})}. \tag{3.2}$$

Let $m = n^{0.1} - n^{0.075}$. Let R_i' be a randomly chosen m-set in R_i and let $R' = (R_1', \ldots, R_t')$. By (3.1) and (3.2), we have with probability $1 - e^{-\Omega(n^{0.05})}$

$$\widetilde{\delta}_{k-1}(H[R']) \geqslant (c + \gamma/2)n^{0.1} - 2n^{0.075} \geqslant (c + \gamma/4)m.$$

Since R_i' is chosen randomly from R_i, which is also chosen randomly, a given element is chosen in R_i' with probability $m/n = n^{-0.9} - n^{-0.925}$ minus an exponentially small correction term. Hence we may assume that, for $v \in V$,

$$n^{-0.9} \geqslant \mathbb{P}(v \in R') \geqslant (1 - 2n^{-0.025})n^{-0.9}.$$

Now, we take $n^{1.1}$ independent copies of R' and denote them by $R'(1), R'(2), \ldots, R'(n^{1.1})$. For a subset of vertices $S \subset V$, let

$$Y_S = |\{i : S \subset R'(i)\}|.$$

Since the probability that a particular R^i (not $R'(i)$) contains S is $n^{-0.9n}$, $\mathbb{E}(Y_S) \leqslant n^{1.1-0.9|S|}$. With probability at least $1 - 2\exp(-9n^{1.5}/2)$ by Lemma 3.4, $Y_v = n^{0.2} \pm 3n^{0.175}$ for every $v \in V$, where recall that $y = x \pm c$ means $x - c \leqslant y \leqslant x + c$. Let $Z_2 = |\{S \in \binom{V}{2} : Y_S \geqslant 3\}|$, and observe that

$$\mathbb{E}(Z_2) < n^2 \left(n^{1.1}\right)^3 \left(n^{-0.9}\right)^6 = n^{-0.1}.$$

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Let $Z_3 = |\{S \in \binom{V}{3} : Y_S \geqslant 2\}|$ and observe that

$$\mathbb{E}(Z_3) < n^3 \left(n^{1.1}\right)^2 \left(n^{-0.9}\right)^6 = n^{-0.2}.$$

The latter implies that every 3-set $S \in \binom{V}{3}$ lies in at most one $R'(i)$ with high probability. In summary, there exist $n^{1.1}$ vertex sets $R'(1), \dots, R'(n^{1.1})$ such that

(i) for every $v \in V$, $Y_v = n^{0.2} \pm 3n^{0.175}$,
(ii) every 2-set $S \in \binom{V}{2}$ is in at most two sets $R'(i)$,
(iii) every 3-set $S \in \binom{V}{3}$ is in at most one set $R'(i)$,
(iv) for $i \in [n^{1.1}]$, $R'(i) = (R'_1, \dots, R'_t)$ with $R'_j \subset V_j$ and $|R'_j| = m$ for $j \in [t]$,
(v) for $i \in [n^{1.1}]$, $\widetilde{\delta}_{k-1}(H[R'(i)]) \geqslant (\phi^* + \gamma/4)m$.

Fix one such sequence $R'(1), \dots, R'(n^{1.1})$.

By (v) and the definition of ϕ^*, there exists a fractional perfect K_t^k-matching w^i in $H[R'(i)]$ for $i \in [n^{1,1}]$. Now we conduct our second round of random process by defining a random t-graph H^* on vertex classes V such that each $[t]$-legal set T is randomly independently chosen with

$$\mathbb{P}(T \in H^*) = \begin{cases} w^{i_T}(T) & \text{if } T \in \mathcal{K}_t^k(H[R'(i_T)]) \text{ for some } i_T \in [t], \\ 0 & \text{otherwise.} \end{cases}$$

Note that i_T is unique by (iii) (as $t \geqslant 3$) and so H^* is well defined. For $v \in V$, let $I_v = \{i : v \in R'(i)\}$ and so $|I_v| = Y_v = n^{0.2} \pm 3n^{0.175}$ by (i). For every $v \in V$, let E_v^i be the set of K_t^k in $H[R'(i)]$ containing v. Thus, for $v \in V$, $\deg^{H^*}(v)$ is a generalized binomial random variable with expectation

$$\mathbb{E}(\deg^{H^*}(v)) = \sum_{i \in I_v} \sum_{T \in E_v^i} w^i(T) = |I_v| = n^{0.2} \pm 3n^{0.175}.$$

Similarly, for every 2-set $\{u, v\}$,

$$\mathbb{E}(\deg^{H^*}(u, v)) = \sum_{i \in I_v \cap I_u} \sum_{T \in E_v^i \cap E_u^i} w^i(T) \leqslant |I_v \cap I_u| \leqslant 2,$$

by (ii). Hence, again by Chernoff's inequality, Lemma 3.4, we may assume that for every $v \in V$ and every 2-set $\{u, v\}$,

$$\deg^{H^*}(v) = n^{0.2} \pm 4n^{0.2-\varepsilon}, \quad \deg^{H^*}(u, v) < n^{0.1}.$$

Thus, H^* satisfies the hypothesis of Theorem 3.3 and the proof is completed. □

Next we prove Theorems 1.3 and 1.4.

Proof of Theorems 1.3 and 1.4. Fix k and t and $\gamma > 0$. Let

$$d = \begin{cases} (t-1)n/t & \text{if } k = 2, \\ \left(1 - \left(\binom{t-1}{k-1} + 2\binom{t-2}{k-2}\right)^{-1}\right)n & \text{if } k \geqslant 3. \end{cases}$$

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Note that $d \geqslant \phi^{*,k}(t,n)$ by Theorem 1.5. Let H be a t-partite k-graph with vertex classes $V_1,\ldots,V_t$ each of size $n \geqslant n_0$ and $\widetilde{\delta}_{k-1}(H) \geqslant d + \gamma n$. We are going to show that H contains a perfect K_t^k-matching. Throughout this proof, n_0 is assumed to be sufficiently large. By Lemma 3.2, there exists a balanced vertex set U in V of size $|U| \leqslant \gamma^{t(t-1)}n/(t^2 2^{t+2})$, such that there exists a perfect K_t^k-matching in $H[U \cup W]$ for every balanced vertex subset $W \subset V \backslash U$ of size $|W| \leqslant \gamma^{2t(t-1)}n/(t^2 2^{2t+5})$. Set $H' = H[V \backslash U]$ and note that $\widetilde{\delta}_{k-1}(H') \geqslant d + \gamma n/2 \geqslant (\phi^{*,k}(t,n) + \gamma/2)n$. By Lemma 3.5, there exists a K_t^k-matching $\mathcal{T}$ in H' covering all but at most εn vertices of H', where $\varepsilon = \gamma^{2t(t-1)}/(t^2 2^{2t+5})$. Let $W = V(H') \backslash V(\mathcal{T})$, so W is balanced. Since $H[U \cup W]$ contains a perfect K_t^k-matching $\mathcal{T}'$ by the choice of U, $\mathcal{T} \cup \mathcal{T}'$ is a perfect K_t^k-matching in H. □

4. Proof of Theorem 1.6

Note that together Lemma 3.5 and the lemma below imply Theorem 1.6. Hence all that remains is to prove Lemma 4.1.

Lemma 4.1. *For integers $t \geqslant k \geqslant 2$, there exists n_0 such that the following holds. Suppose that H is a k-graph with $n \geqslant n_0$ vertices with $t|n$. Then there exists a partition $V_1,\ldots,V_t$ of $V(H)$ into sets of size n/t such that for every $l \in [k-1]$, every $I \in \binom{[t]}{l}$, every legal I-set T and $J \in \binom{[t]\backslash I}{k-l}$, we have*

$$\frac{t^{k-l}}{(k-l)!}\deg_J^{H'}(T) \geqslant \deg^H(T) - 2(t\ln n)^{1/2}n^{k-l-1/2},$$

where H' is the induced t-partite k-subgraph of H with vertex classes $V_1,\ldots,V_t$.

Proof. First set $m = k-l$ and let $U_1,\ldots,U_t$ be a random partition of V, where each vertex appears in vertex class U_j independently with probability $1/t$. For a fixed l-set $T = \{v_1,\ldots,v_l\}$, let $N^H(T)$ be the link hypergraph of T. Thus, $N^H(T)$ is an m-graph with $\deg^H(T)$ edges. We decompose $N^H(T)$ into $i_0 \leqslant mn^{m-1}$ non-empty pairwise edge-disjoint matchings, which we denote by $M_1,\ldots,M_{i_0}$. To see that this is possible consider the auxiliary graph G with $V(G) = E(N^H(T))$, in which, for $A, B \in N^H(T)$, A and B are joined in G if and only if $A \cap B \neq \emptyset$. Since G has maximum degree at most $m\binom{n-1}{m-1}$, G can be properly coloured using at most mn^{m-1} colours, where each colour class corresponds to a matching.

For every edge $E \in N^H(T)$, and every index set $J \in \binom{[t]}{m}$, we say that E is J-*good* if E is J-legal with respect to $U_1,\ldots,U_t$. Since the partition $U_1,\ldots,U_t$ was chosen randomly, we have, for fixed $J \in \binom{[t]}{m}$,

$$\mathbb{P}(E \text{ is } J\text{-good}) = m!t^{-m}.$$

Thus, for $X_{i,J} = X_{i,J}(T) = |\{E \in M_i : E \text{ is } J\text{-good}\}|$ we have

$$\mu_{i,J} = \mu_{i,J}(T) = \mathbb{E}(X_{i,J}) = \frac{m!}{t^m}|M_i|.$$

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Now call a matching M_i *bad* (with respect to $U_1, \dots, U_t$) if there exists a set $J \in \binom{[t]}{m}$ such that

$$X_{i,J} \leqslant \left(1 - \left(\frac{2(2k-1)\ln n}{\mu_{i,J}}\right)^{1/2}\right)\mu_{i,J},$$

and call T a *bad set* if there is at least one bad $M_i = M_i(T)$. Otherwise call T a *good set*. For a fixed M_i the events 'E is J-good' with $E \in M_i$ are jointly independent, and hence by Chernoff's inequality, Lemma 3.4,

$$\mathbb{P}(M_i \text{ is bad}) \leqslant \binom{t}{m}\exp(-(2k-1)\ln n) = \binom{t}{m} n^{-2k+1}.$$

Recall that $i_0 \leqslant mn^{m-1}$ and $m \leqslant k-1$, so we have

$$\mathbb{P}(T \text{ is bad}) \leqslant i_0 \binom{t}{m} n^{-2k+1} \leqslant n^{-k}.$$

By summing over all l-sets T, we obtain that

$$\mathbb{P}(\text{there exists a bad } l\text{-set}) \leqslant n^{-1}.$$

Moreover, Chernoff's inequality, Lemma 3.4, yields

$$\mathbb{P}(|U_j| \geqslant n/t + n^{1/2}(\ln n)^{1/4}/t) \leqslant \exp(-(\ln n)^{1/2}/4t).$$

Thus with positive probability there is a partition $U_1, \dots, U_t$ such that all l-sets T are good and

$$|U_j| \leqslant n/t + n^{1/2}(\ln n)^{1/4}/t, \quad \text{for all } j \in [t].$$

Consequently, by redistributing at most $n^{1/2}(\ln n)^{1/4}$ vertices of the partition $U_1, \dots, U_t$, we obtain an equipartition $V_1, \dots, V_t$ with

$$|V_j| = n/t \quad \text{and} \quad |U_j \backslash V_j| \leqslant n^{1/2}(\ln n)^{1/4}/t, \quad \text{for all } j \in [t].$$

Let H' be the induced t-partite k-subgraph with vertex classes $V_1, \dots, V_t$. Note that for an l-set $I \in \binom{[t]}{l}$, an I-legal set T and an m-set $J \in \binom{[t]\backslash I}{m}$,

$$\begin{aligned}\deg_J^{H'}(T) &\geqslant \sum_{i \in [i_0]} \left(1 - \left(\frac{2(2k-1)\ln n}{\mu_{i,J}}\right)\right)\mu_{i,J} - m\frac{n^{1/2}(\ln n)^{1/4}}{t} n^{m-1} \\ &\geqslant \frac{m!}{t^m}\deg_J^H(T) - \big(2(2k-1)\ln n\big)^{1/2} \sum_{i \in [i_0]} \mu_{i,J}^{1/2} - m\frac{n^{1/2}(\ln n)^{1/4}}{t} n^{m-1}.\end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain that

$$\sum_{i \in [i_0]} \mu_{i,J}^{1/2} \leqslant \left(i_0 \sum_{i \in [i_0]} \mu_{i,J}\right)^{1/2} \leqslant \left(mn^{m-1}\frac{m!}{t^m}\binom{n}{m}\right)^{1/2} \leqslant n^{m-1/2}.$$

Therefore,

$$\deg_J^H(T) \geqslant \frac{m!}{t^m}\deg_J^H(T) - 2(k\ln n)^{1/2} n^{m-1/2},$$

as required. □

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Appendix: Proof of Lemma 3.2

Proof. Throughout the proof we may assume that n_0 is chosen sufficiently large. Let H be a balanced t-partite k-graph with partition classes $V_1, \dots, V_t$ each of size n, and $\widetilde{\delta}_{k-1}(H) \geqslant \widetilde{\delta}$, where $\widetilde{\delta}$ is the lower bound on $\widetilde{\delta}_{k-1}(H)$ stated in the lemma. Let H' be the t-partite t-graph on $V_1, \dots, V_t$ in which $v_1 v_2 \cdots v_t \in E(H')$ if and only if $v_1 v_2 \cdots v_t$ is a K_t^k in H. Furthermore, set $m = t(t-1)$, and call a balanced m-set A an *absorbing* m-set for a balanced t-set T if A spans a matching of size $t-1$ in H' and $A \cup T$ spans a matching of size t in H'. In other words, $A \cap T = \emptyset$ and both $H'[A]$ and $H'[A \cup T]$ contain a perfect matching. Denote by $\mathcal{L}(T)$ the set of all absorbing m-sets for T. Next, we show that for every balanced t-set T there are many absorbing m-sets for T.

Claim A.1. *For every balanced t-set T, $|\mathcal{L}(T)| \geqslant \gamma^m \binom{n}{t-1}^t / 2^t$.*

Proof. Let $T = \{v_1, \dots, v_t\}$ be fixed with $v_i \in V_i$ for $i \in [t]$. By Proposition 3.1 it is easy to see that there exist at least $(\gamma n)^{t-1}$ edges in H' containing v_1. Since n_0 was chosen large enough, there are at most $(t-1)n^{t-2} \leqslant (\gamma n)^{t-1}/2$ edges in H' which contain v_1 and v_j for some $2 \leqslant j \leqslant t$. Fix an edge $v_1 u_2 \cdots u_t$ in H' with $u_j \in V_j \backslash \{v_j\}$ for $2 \leqslant j \leqslant t$. Set $U_1 = \{u_2, \dots, u_t\}$ and $W_0 = T$. For each $2 \leqslant j \leqslant t$, suppose we succeed in choosing a $(t-1)$-set U_j such that U_j is disjoint from $W_{j-1} = U_{j-1} \cup W_{j-2}$ and both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H'. Then for a fixed $2 \leqslant j \leqslant t$ we call such a choice U_j *good*, motivated by $A = \bigcup_{j \in [t]} U_j$ being an absorbing m-set for T.

Note that in each step $2 \leqslant j \leqslant t$ there are precisely $t + (j-1)(t-1)$ vertices in W_{j-1}. More specifically, for $i \in [t]$, there are at most $j \leqslant t$ vertices in $V_i \cap W_{j-1}$. Thus, the number of edges in H' intersecting u_j (or v_j respectively) and at least one other vertex in W_j is at most $(t-1)jn^{t-2} < t^2 n^{t-2} \leqslant (\gamma n)^{t-1}/2$. For each $2 \leqslant j \leqslant t$, by Proposition 3.1 there are at least $(\gamma n)^{t-1} - (\gamma n)^{t-1}/2 = (\gamma n)^{t-1}/2$ choices for U_j and in total we obtain $(\gamma n)^m / 2^t$ absorbing m-sets for T with multiplicity at most $((t-1)!)^t$. □

Now, choose a family F of balanced m-sets by selecting each of the $\binom{n}{t-1}^t$ possible balanced m-sets independently with probability

$$p = \gamma^m n \Big/ \left(t^3 2^{t+3} \binom{n}{t-1}^t \right).$$

Then, by Chernoff's inequality, Lemma 3.4, with probability $1 - o(1)$ as $n \to \infty$, the family F satisfies the properties

$$|F| \leqslant \gamma^m n / (t^3 2^{t+2}) \tag{A.1}$$

and

$$|\mathcal{L}(T) \cap F| \geqslant \frac{\gamma^{2m} n}{t^3 2^{2t+4}}, \tag{A.2}$$

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for all balanced t-sets T. Furthermore, we can bound the expected number of intersecting m-sets in F by

$$\binom{n}{t-1}^t \times t(t-1) \times \binom{n}{t-2}\binom{n}{t-1}^{t-1} \times p^2 \leqslant \frac{\gamma^{2m} n}{t^3 2^{2t+6}}.$$

Thus, using Markov's inequality, we derive that, with probability at least 1/2,

$$F \text{ contains at most } \frac{\gamma^{2m} n}{t^3 2^{2t+5}} \text{ intersecting pairs.} \tag{A.3}$$

Hence, with positive probability the family F has all properties stated in (A.1), (A.2) and (A.3). By deleting all the intersecting balanced m-sets and non-absorbing m-sets in such a family F, we get a subfamily F' consisting of pairwise disjoint balanced m-sets, which satisfies

$$|\mathcal{L}(T) \cap F'| \geqslant \frac{\gamma^{2m} n}{t^3 2^{2t+4}} - \frac{\gamma^{2m} n}{t^3 2^{2t+5}} = \frac{\gamma^{2m} n}{t^3 2^{2t+5}}$$

for all balanced t-sets T. Let $U = V(F')$ and so U is balanced. Moreover, U is of size at most $t|V(F')| \leqslant t|V(F)| \leqslant \gamma^m n/(t^2 2^{t+2})$ by (A.1). For a balanced set $W \subset V \backslash V(M)$ of size $|W| \leqslant \gamma^{2m} n/(t^2 2^{2t+5})$, W can be partitioned into at most a $\gamma^{2m} n/(t^3 2^{2t+5})$ balanced t-set. Each balanced t-set can be successively absorbed using a different absorbing m-set in F', so there exists a perfect matching in $H'[U \cup W]$. Hence, there is a perfect K_t^k-matching in $H[U \cup W]$. □

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