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Ordered combinatory algebras and realizability†

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We propose the new concept of *Krivine ordered combinatory algebra* ($^{\mathcal{K}}\mathcal{OCA}$) as foundation for the categorical study of Krivine's classical realizability, as initiated by Streicher (2013).

We show that $^{\mathcal{K}}\mathcal{OCA}$'s are equivalent to Streicher's *abstract Krivine structures* for the purpose of modeling higher-order logic, in the precise sense that they give rise to the same class of *triposes*. The difference between the two representations is that the elements of a $^{\mathcal{K}}\mathcal{OCA}$ play both the role of truth values and realizers, whereas truth values are *sets* of realizers in $\mathcal{AKS}$s.

To conclude, we give a direct presentation of the realizability interpretation of a higher order language in a $^{\mathcal{K}}\mathcal{OCA}$, which showcases the dual role that is played by the elements of the $^{\mathcal{K}}\mathcal{OCA}$.

1. Introduction

Classical realizability was introduced in the mid 90's by Krivine as a complete reformulation of the principles of Kleene's (intuitionistic) realizability (see Kleene (1945)), to take into account the connection between control operators and classical reasoning discovered by Griffin (in Griffin (1990)). Initially developed in the framework of classical second-order Peano arithmetic (see Krivine (1994)), classical realizability was quickly extended to Zermelo-Fraenkel set theory in Krivine (2001b) using model-theoretic constructions reminiscent both to the construction of generic extensions in forcing and to the construction of intuitionistic realizability models of intuitionistic set theories, see Myhill (1973), Friedman (1973), McCarty (1983). In particular, Krivine showed in Krivine (2003) how to interpret the (classical) axiom of dependent choice in this framework. More recently, he also showed in Krivine (2001a) how to combine classical realizability with the method of forcing (in the sense of Cohen), in the very spirit of iterated forcing.

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Actually, Krivine's realizability (particularly in its most recent developments) was mostly developed independently of the long-standing tradition of intuitionistic realizability. And, as mentioned in Streicher (2013), it was dificult to see how Krivine's work could fit into the structural approach to realizability as initiated in Hyland (1982) and fully described in van Oosten (2008). One problem comes from the fact that the only realizability topos that validates classical logic is the one based on the trivial partial combinatory algebra ($\mathcal{PCA}$), and thus, is equivalent to **Set**, a fact that suggested for a long time that realizability and classical logic were incompatible.

To resolve this paradox, Streicher proposed in Streicher (2013) a categorical model for Krivine's realizability, still using the standard method that consists to combine the construction of a realizability tripos with the well known tripos-to-topos construction (see van Oosten (2008)). However, Streicher's construction of the realizability tripos departs from the standard construction from a $\mathcal{PCA}$ in several aspects.

First, Streicher does not use a $\mathcal{PCA}$, but a particular form of ordered combinatory algebra ($\mathcal{OCA}$), that is built from an abstract Krivine structure ($\mathcal{AKS}$) that provides the computational ingredients of Krivine realizability.

Second, the elements of the considered $\mathcal{OCA}$ (induced by the underlying $\mathcal{AKS}$) are not used as realizers, but directly as truth values, using the fact that the considered $\mathcal{OCA}$ has a meet-semilattice structure. In this way, Streicher can skip the step that consists in taking the powerset to define truth values, and more generally relations (as one would do if working with a $\mathcal{PCA}$).

A third ingredient of Streicher's construction is the introduction of a specific notion of filter, to distinguish the truth values that actually capture the notion of truth/provability. In practice, this filter is naturally defined from the pole of the $\mathcal{AKS}$ and the corresponding notion of proof-like terms.

Two warnings here concerning notations: following the usual trends we use sometimes the expression quasi–proof following the original french *quasi–preuve* instead of proof-like term; also the reader should be aware that the notion of filter used in this context is due to Hofstra (see Hofstra (2006)) and is different from the usual one.

In this paper, we revisit Streicher's work by showing that his construction can be performed working directly from a particular form of $\mathcal{OCA}$, which we call ${}^{\mathcal{K}}\mathcal{OCA}$, whose elements can be indifferently used as realizers (or conditions) and as truth values, similarily to the elements of a complete Boolean algebra in forcing.

In particular, we emphasize that complete Boolean algebras are particular cases of ${}^{\mathcal{K}}\mathcal{OCA}$'s (see Example 3.9), and that the realizability models represented by the associated triposes are essentially equivalent to 'Boolean valued models' known from set theory. In this sense the concept of ${}^{\mathcal{K}}\mathcal{OCA}$ can be seen as the common generalization of classical realizability and Cohen forcing. Since realizability over ${}^{\mathcal{K}}\mathcal{OCA}$'s is such a comprehensive notion, we can not expect it to always validate the *existence* and *disjunction* properties, which are often associated to realizability interpretations (see Remark 5.9).

Elaborating the analogy to the standard approach to categorical realizability we present the tripos construction starting from a slightly more general structure of ${}^{\mathcal{I}}\mathcal{OCA}$ that does not assume anything about the logic being classical.

Outline of the paper

In Section 2 we recall the definition of abstract Krivine structures ($\mathcal{AKS}$s), as defined in Streicher (2013), emphasizing the properties of the lattice of falsity values and their relationships with the ortogonality map.

In Section 3, we introduce our notion of *implicative ordered combinatory algebra* (${}^{\mathcal{I}}\mathcal{OCA}$) as a particular case of an $\mathcal{OCA}$, as well as the notion of *Krivine ordered combinatory algebra* (${}^{\mathcal{K}}\mathcal{OCA}$), the classical particular case of ${}^{\mathcal{I}}\mathcal{OCA}$. Both structures of ${}^{\mathcal{I}}\mathcal{OCA}$ and ${}^{\mathcal{K}}\mathcal{OCA}$ come with a complete meet-semilattice structure whose ordering intuitively represents subtyping, and with a (specific notion of) filter that captures both the notion of *pole* and the notion of *proof-like terms* in Krivine's realizability.

In Section 4, we show that the filter associated to each ${}^{\mathcal{I}}\mathcal{OCA}$ (${}^{\mathcal{K}}\mathcal{OCA}$) induces an ordering of entailment that is weaker than the primitive ordering of subtyping. Although application and implication in the ${}^{\mathcal{I}}\mathcal{OCA}$s and ${}^{\mathcal{K}}\mathcal{OCA}$s are related by a weak adjunction, the entailment preorder induced by the filter is a *Heyting preorder* (i.e.: the meet and the implication are related by a *full* adjunction).

In Section 5 we define the indexed entailment preorders, that lead to the construction of a tripos from an arbitrary ${}^{\mathcal{I}}\mathcal{OCA}$, and compare them with the corresponding construction based on Streicher's $\mathcal{AKS}$. In particular we show that considering all sets of stacks (as we do here) or restricting to the sets of stacks that are equal to their biortogonal (as Streicher does) amounts to the same, up to equivalence of indexed preorders. In this way, we get a tripos from the starting ${}^{\mathcal{I}}\mathcal{OCA}$, that happens to be classical in the case where the considered ${}^{\mathcal{I}}\mathcal{OCA}$ is a ${}^{\mathcal{K}}\mathcal{OCA}$.

The rest of section 5 is devoted to the comparison with Streicher's work. For that, we show that every $\mathcal{AKS}$ induces a ${}^{\mathcal{K}}\mathcal{OCA}$ (which is in fact the $\mathcal{OCA}$ built by Streicher) and prove that the tripos constructed using our method is isomorphic to the one constructed using Streicher's method. Conversely, we prove that any tripos constructed from a ${}^{\mathcal{K}}\mathcal{OCA}$ using our method is actually equivalent to a tripos constructed using Streicher's method from a particular $\mathcal{AKS}$ that can be deduced from the initial ${}^{\mathcal{K}}\mathcal{OCA}$. Thus, our (more algebraic method) and Streicher's method (that is closer to the calculus) are essentially equivalent to give a categorical characterization of classical realizability.

In Section 6, we present the internal classical realizability associated to a ${}^{\mathcal{K}}\mathcal{OCA}$. We prove that this realizability is adequate w.r.t. higher-order classic arithmetic. The salient feature of this definition of realizability is that it treats realizers and truth values at the same level, namely: as elements of the same ${}^{\mathcal{K}}\mathcal{OCA}$. For this reason we think that ${}^{\mathcal{K}}\mathcal{OCA}$s can be used as the basic building bricks in order to define classical realizability from a categorical perspective.

2. Streicher's Abstract Krivine Structures

As motivation for the introduction of the concept of *Krivine ordered combinatory algebra* (${}^{\mathcal{K}}\mathcal{OCA}$s), we recapitulate the definitions and basic ideas in Streicher (2013), regarding the notion of *Abstract Krivine Structure* ($\mathcal{AKS}$). These ideas were introduced by J.L. Krivine and reformulated categorically by T. Streicher –see Krivine (2008) and Streicher (2013) respectively–.

2.1. *Polarities*

Recall the definition of a polarity associated to a triple $(\Lambda, \Pi, \perp\!\!\!\perp)$ where Λ and Π are sets and $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ is a subset or *a relation* (see (Birkhoff 1955, Chapter V, Section 7)).

In our context of realizability, the elements of Λ and of Π are called respectively *terms* and *stacks* and the elements of $\Lambda \times \Pi$ are called *processes*. The processes are written as $t \star \pi$. Moreover if $t \star \pi \in \perp\!\!\!\perp$, we write $t \perp \pi$, and say that "t is orthogonal to π" or that "t realizes π". If $P \subseteq \Pi$ and $t \perp \pi$ for all $\pi \in P$ we say that "t realizes P" and write $t \perp P$.

Definition 2.1. Given a triple as above we define the following maps and sets:

(1)

$$\begin{aligned} (\quad)^{\perp} : \mathcal{P}(\Lambda) &\longrightarrow \mathcal{P}(\Pi) \\ \Lambda \supseteq L &\longmapsto L^{\perp} = \{\pi \in \Pi |\ \forall t \in L, t \star \pi \in \perp\!\!\!\perp\} = \{\pi \in \Pi |\ L \times \{\pi\} \subseteq \perp\!\!\!\perp\} \subseteq \Pi; \end{aligned}$$

$$\begin{aligned} {}^{\perp}(\quad) : \mathcal{P}(\Pi) &\longrightarrow \mathcal{P}(\Lambda) \\ \Pi \supseteq P &\longmapsto {}^{\perp}P = \{t \in \Lambda |\ \forall \pi \in P, t \star \pi \in \perp\!\!\!\perp\} = \{t \in \Lambda |\ \{t\} \times P \subseteq \perp\!\!\!\perp\} \subseteq \Lambda. \end{aligned}$$

The maps are called the polar maps and the subsets $L^{\perp}$ and ${}^{\perp}P$ are called the polars –or perpendiculars– of L and P respectively.

(2) The maps $L \mapsto {}^{\perp}(L^{\perp})$ and $\Pi \mapsto ({}^{\perp}\Pi)^{\perp}$ are called the closure operations associated to the polarity, and the family of closed subsets associated to these closure operations (see (Birkhoff 1955, Chapter V, Section 7, Theorem 19)) are denoted as:

$$\mathcal{P}_{\perp}(\Lambda) = \{L \subseteq \Lambda : {}^{\perp}(L^{\perp}) = L\} \subseteq \mathcal{P}(\Lambda), \quad \mathcal{P}_{\perp}(\Pi) = \{P \subseteq \Pi : ({}^{\perp}P)^{\perp} = P\} \subseteq \mathcal{P}(\Pi).$$

Remark 2.2. See (Birkhoff 1955, Chapter V, Section 7) for the considerations that follow:

(1) The maps $L \to L^{\perp}$ and $P \to {}^{\perp}P$ are antitonic with respect to the order given by the inclusion of sets and for $P_i \subseteq \Pi$, $L_i \subseteq \Lambda, i \in I$ we have:

$${}^{\perp}(\bigcap_{i\in I} P_i) \supseteq \bigcup_{i\in I} {}^{\perp}P_i,\ {}^{\perp}(\bigcup_{i\in I} P_i) = \bigcap_{i\in I} {}^{\perp}P_i; (\bigcap_{i\in I} L_i)^{\perp} \supseteq \bigcup_{i\in I} L_i^{\perp},\ (\bigcup_{i\in I} L_i)^{\perp} = \bigcap_{i\in I} L_i^{\perp}.$$

(2) For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that ${}^{\perp}(L^{\perp}) \supseteq L$ and $({}^{\perp}P)^{\perp} \supseteq P$. For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that $({}^{\perp}(L^{\perp}))^{\perp} = L^{\perp}$ and ${}^{\perp}(({}^{\perp}P)^{\perp}) = {}^{\perp}P$.

(3) The maps $(\quad)^{\perp} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ and ${}^{\perp}(\quad) : \mathcal{P}(\Pi) \to \mathcal{P}(\Lambda)$ when restricted respectively to $\mathcal{P}_{\perp}(\Lambda)$ and $\mathcal{P}_{\perp}(\Pi)$ are order reversing isomorphisms inverse to each other. This pair of maps is the *polarity* associated to the triple $(\Lambda, \Pi, \perp\!\!\!\perp)$.

(4) The following completeness properties hold. If $\mathcal{X}$ is a subset of $\mathcal{X} \subseteq \mathcal{P}_{\perp}(\Pi)$, we let:

$$\sup(\mathcal{X}) = \left({}^{\perp}\left(\bigcup\{P : P \in \mathcal{X}\}\right)\right)^{\perp},\ \inf(\mathcal{X}) = \bigcap\{P : P \in \mathcal{X}\}.$$

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In particular, $\sup(\mathcal{X})$ and $\inf(\mathcal{X})$ are the supremum and infimum of the set $\mathcal{X}$ in $\mathcal{P}_\perp(\Pi)$ with respect to the order given by the inclusion of sets. Moreover with respect to the order given by the inclusion, $\Lambda^\perp$ and Π; ${}^\perp\Pi$ and Λ are the minimal and maximal elements of $\mathcal{P}_\perp(\Pi)$ and $\mathcal{P}_\perp(\Lambda)$ respectively.

The only relevant structure at this point is the *lattice* structure in the sets $P_\perp(\Lambda)$ and $P_\perp(\Pi)$, where we take the (set theoretical) inclusion as the order, the intersection as "meet" and the union followed by taking double orthogonals as "join".

2.2. *The push map and realizability lattices*

In this section we add a *push map* to our basic structure, thus introducing the first binary operation that enriches the polarity. When the original structure is equipped with the new map it is called a *realizability lattice*.

Definition 2.3.

(1) A map push $:= (\cdot) : (t, \pi) \mapsto t \cdot \pi : \Lambda \times \Pi \to \Pi$ will be called a *push map*.

(2) We extend the push map to $(L, P) \mapsto L \cdot P : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ where $L \cdot P = \{t \cdot \pi : t \in L,\ \pi \in P\}$.

(3) The pair of lattices $\mathcal{P}(\Lambda), \mathcal{P}(\Pi)$ together with the push map, is called the *realizability lattice* associated to $(\Lambda, \Pi, \perp\!\!\!\perp, (\cdot))$. The family of realizability lattices is abbreviated as $\mathcal{RL}$.

(4) For $L \subseteq \Lambda$, $P \subseteq \Pi$ we define,

$$(\rightsquigarrow) : (L, P) \mapsto L \rightsquigarrow P : \mathcal{P}(\Lambda) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$$

as $L \rightsquigarrow P = \{\pi \in \Pi : L \cdot \pi \subseteq P\} \subseteq \Pi$. The subset $L \rightsquigarrow P \subseteq \Pi$ is called *the right conductor of L into P*.

Remark 2.4. For L and P as above: $L \cdot P \subseteq Q$ if and only if $P \subseteq L \rightsquigarrow Q$.

The above constructions of $L \rightsquigarrow P$ and $L.P$ combined with the operators $(\ )^\perp$ and ${}^\perp(\ )$ yield natural binary operations in $\mathcal{P}_\perp(\Pi)$ that are basic ingredients of the $\mathcal{OCA}$ associated to the $\mathcal{AKS}$ *à la Streicher*.

We define the following binary operations between subsets of Π.

Definition 2.5. Let $P, Q \subseteq \Pi$ then:

$$P \circ Q := {}^\perp Q \rightsquigarrow P \ \subseteq\ P \circ_\perp Q := ({}^\perp({}^\perp Q \rightsquigarrow P))^\perp, \tag{1}$$

$$P \Rightarrow Q := {}^\perp P \cdot Q \ \subseteq\ P \Rightarrow_\perp Q := ({}^\perp({}^\perp P \cdot Q))^\perp \in \mathcal{P}_\perp(\Pi). \tag{2}$$

Remark 2.6. The remark 2.4 can be written as: $P \circ Q \supseteq R$ if and only if $P \supseteq Q \Rightarrow R$. Hence, it follows that there is a *full* adjunction between $\circ$ and $\Rightarrow$.

Once the above definitions are established, we can deduce a crucial "half adjunction property" relating the operations $\circ_\perp$ and $\Rightarrow_\perp$ in $\mathcal{P}_\perp(\Pi)$.

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Theorem 2.7. *[Half adjunction property]* Assume that $P, Q, R \in \mathcal{P}_\perp(\Pi)$. If $Q \Rightarrow_\perp R \subseteq P$, then $R \subseteq P \circ_\perp Q$. In particular: $P \subseteq (Q \Rightarrow_\perp P) \circ_\perp Q$.

Proof. The following inclusions are equivalent: $Q \Rightarrow_\perp R \subseteq P$, $({}^\perp({}^\perp Q \cdot R))^\perp \subseteq P$, ${}^\perp Q \cdot R \subseteq P$ and $R \subseteq {}^\perp Q \rightsquigarrow P$. The last inclusion implies that $R \subseteq ({}^\perp({}^\perp Q \rightsquigarrow P))^\perp = P \circ_\perp Q$. □

2.3. *Abstract Krivine structures*

Next, –compare with Streicher (2013)– we complete the process of introducing the operations into a realizability lattice to obtain the concept of *Abstract Krivine Structure* abbreviated as $\mathcal{AKS}$. For that, we introduce the usual application map for terms, a store map from stacks to terms, the combinators K,S, and a distinguished term cc that is a realizer of Peirce's law. We introduce also a set of terms that we call quasi-proofs and assume that the three combinators above are quasi proofs.

Definition 2.8. An *Abstract Krivine Structure* (frequently written as $\mathcal{K}$) consists of the following data:

(1) A realizability lattice $(\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push})$,
(2) Functions
   (a) $\mathrm{app} : \Lambda \times \Lambda \to \Lambda$ is a function: $(t, u) \mapsto \mathrm{app}(t, u) = tu$,
   (b) $\mathrm{store} : \Pi \to \Lambda$ is a function: $\pi \mapsto \mathrm{store}(\pi) = k_\pi$,
(3) A set $\mathrm{QP} \subseteq \Lambda$ of "quasi–proofs", which is closed under application,
(4) Elementary combinators K,S,$\mathsf{cc} \in \mathrm{QP}$.
(5) The above elements are subject to the following axioms.
   (S1) If $t \perp s \cdot \pi$, then $ts \perp \pi$.
   (S2) If $t \perp \pi$, then for all $s \in \Lambda$ we have that $\mathsf{K} \perp t \cdot s \cdot \pi$.
   (S3) If $tu(su) \perp \pi$, then $\mathsf{S} \perp t \cdot s \cdot u \cdot \pi$.
   (S4) If $t \perp k_\pi \cdot \pi$, then $\mathsf{cc} \perp t \cdot \pi$.
   (S5) If $t \perp \pi$, then for all $\pi' \in \Pi$ we have that $k_\pi \perp t \cdot \pi'$.

Here –and in the rest of this paper– the product-like operations will not be associative and we assume that when parentheses are omitted, we associate to the left.

The elements of the structure above, named as:

$$\mathrm{store} : \pi \mapsto k_\pi : \Pi \to \Lambda \quad \text{and} \quad \mathsf{cc} \in \Lambda,$$

have a very special role in the sense they can be used to make the realizability theory *classical* as cc realizes Peirce's law. In this sense in the presence of the mentioned elements and the corresponding axioms (S4) and (S5), the $\mathcal{AKS}$ is *classical.*

Definition 2.9. For a general $\mathcal{AKS}$ we introduce the following definitions:

(1) For $L, M \subseteq \Lambda$ we define $LM = \{tu |\ t{\in}L \text{ and } u{\in}M\}$, which is $\mathrm{app}(L \times M)$.
(2) For $L, M \subseteq \Lambda$ we define $L \Rightarrow M = \{t \in \Lambda : tL \subseteq M\}$,

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(3) For $P, Q \subseteq \Pi$, $P \diamond Q := \big(({}^{\perp}P)({}^{\perp}Q)\big)^{\perp} \in \mathcal{P}_{\perp}(\Pi)$.
(4) $\mathsf{I} := \mathsf{SKK}, \mathsf{B} := \mathsf{S(KS)K}, \mathsf{E} := \mathsf{S(KI)} \in \mathrm{QP}$.

Lemma 2.10. In an $\mathcal{AKS}$, for $P, Q \in \mathcal{P}_{\perp}(\Pi)$, we have that condition *(S1)* in Definition 2.8, implies any of the two equivalent conditions below.

(1) $P \circ_{\perp} Q \subseteq ({}^{\perp}P{}^{\perp}Q)^{\perp} = P \diamond Q$
(2) If $t \perp P$ and $s \perp Q$, then $ts \perp P \circ_{\perp} Q$.

Proof. It is evident that the two conditions above are equivalent. Assuming *(S1)*, we want to prove that for all P, Q then: $\{\pi \in \Pi : {}^{\perp}Q.\pi \subseteq P\} \subseteq ({}^{\perp}P{}^{\perp}Q)^{\perp}$.

In other words we want to show that if $\pi \in \Pi$ is such that ${}^{\perp}Q.\pi \subseteq P$ then, for all $s \perp P$, $t \perp Q$ we have that $st \perp \pi$. It is clear that from the hypothesis ${}^{\perp}Q.\pi \subseteq P$ and $s \perp P$, $t \perp Q$, that $s \perp t.\pi$ and in this case the original condition *(S1)* implies that $st \perp \pi$. □

Lemma 2.11. For $P, Q, R \in \mathcal{P}(\Pi)$, $t, u, v \in \Lambda$, and $\pi \in \Pi$, we have:

(1) $t \perp P \Rightarrow_{\perp} Q$, $u \perp P$ implies $tu \perp Q$;
equivalently: $t \perp P \Rightarrow Q$, $u \perp P$ implies $tu \perp Q$;
(2) $t \perp (P \Rightarrow_{\perp} Q) \Rightarrow_{\perp} R$ if and only if $t \perp (P \Rightarrow Q) \Rightarrow R$.
Also if $t \perp P \Rightarrow_{\perp} (Q \Rightarrow_{\perp} R)$ then $t \perp P \Rightarrow (Q \Rightarrow R)$.
(3) $t \perp \pi$ implies that $\mathsf{I} \perp t{\cdot}\pi$;
(4) $t \perp uv{\cdot}\pi$ implies that $\mathsf{B} \perp t{\cdot}u{\cdot}v{\cdot}\pi$;
(5) $tu \perp \pi$ implies that $\mathsf{E} \perp t{\cdot}u{\cdot}\pi$.

Proof. (1) – and (2) – follow immediately from the fact that taking orthogonals three times is the same than taking them once. In particular, observe that $t \perp P \Rightarrow_{\perp} Q$ is equivalent to $t \perp P \Rightarrow Q$.

(3) – and (4) – are direct verifications.

(5) – The following chain of implications proves that $tu \perp \pi$ implies $\mathsf{E} := \mathsf{S(KI)} \perp t{\cdot}u{\cdot}\pi$:

$$tu \perp \pi \text{ by (2.11) implies } \mathsf{I} \perp tu \cdot \pi \text{ by } (S2) \text{ implies } \mathsf{K} \perp \mathsf{I} \cdot u \cdot tu \cdot \pi$$
$$\text{by } (S1) \text{ implies } \mathsf{KI}u(tu) \perp \pi \text{ by } (S3) \text{ implies } \mathsf{S} \perp \mathsf{KI} \cdot t \cdot u \cdot \pi$$
$$\text{which finally, by } (S1)\text{, implies } E := \mathsf{S(KI)} \perp t \cdot u \cdot \pi.$$ □

Next, we deduce some consequences or equivalent formulations of the basic axioms for an $\mathcal{AKS}$ in terms of elements of $\mathcal{P}(\Pi)$ and the operations $\Rightarrow$, $\Rightarrow_{\perp}$ and $\diamond$.

Lemma 2.12. For $P, Q, R \in \mathcal{P}(\Pi)$, $t, u, v \in \Lambda$, and $\pi \in \Pi$, we have:

(1) $\mathsf{K} \perp P \Rightarrow Q \Rightarrow P$;
(2) $\mathsf{S} \perp (P \Rightarrow Q \Rightarrow R) \Rightarrow (P \Rightarrow Q) \Rightarrow P \Rightarrow R$;
(3) $\mathsf{CC} \perp ((P \Rightarrow_{\perp} Q) \Rightarrow_{\perp} P) \Rightarrow_{\perp} P$ or equivalently
$\mathsf{CC} \perp ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$;
(4) $\mathsf{I} \perp P \Rightarrow_{\perp} P$;

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(5) $\mathsf{B} \perp (Q \Rightarrow R) \Rightarrow (P \Rightarrow Q) \Rightarrow P \Rightarrow R$;
(6) $\mathsf{E} \perp P \Rightarrow (Q \Rightarrow (P \diamond Q))$.

Proof. (1) – Let $t \in {}^{\perp}P$, $u \in {}^{\perp}Q$, $\pi \in P$. We have to show that $\mathsf{K} \perp t \cdot u \cdot \pi$. By *(S2)*, it is sufficient to show $t \perp \pi$, which follows from the definition of ${}^{\perp}P$.

(2) – Let $t \in {}^{\perp}(P \Rightarrow Q \Rightarrow R)$, $u \in {}^{\perp}(P \Rightarrow Q)$, $v \in {}^{\perp}P$, $\pi \in R$. By lemma 2.11 (2.11) we deduce $tv(uv) \in {}^{\perp}R$ and thus $tv(uv) \perp \pi$. Axiom *(S3)* implies $\mathsf{S} \perp t \cdot u \cdot v \cdot \pi$, as required.

(3) – Let $t \in {}^{\perp}((P \Rightarrow_{\perp} Q) \Rightarrow_{\perp} P)$ and $\pi \in P$. We have to show that $\mathsf{CC} \perp t \cdot \pi$, and by (S4) it is sufficient to show that $t \perp k_{\pi} \cdot \pi$. This would follow from $k_{\pi} \in {}^{\perp}(P \Rightarrow_{\perp} Q)$, so it remains to prove the latter.

Let $u \in {}^{\perp}P$, $\pi' \in Q$. We have to show that $k_{\pi} \perp u \cdot \pi'$, and by (S5) it is sufficient to show that $u \perp \pi$, which is true since $u \in {}^{\perp}P$ by assumption.

(4) – and (5) – are directly derived respectively from lemma 2.11, (3) and (4).

(6) – For all $t \perp P$, $u \perp Q$ and $\pi \in P \diamond Q = ({}^{\perp}P {}^{\perp}Q)^{\perp}$, we have that $tu \perp \pi$ and using 2.11(5) we deduce that $E \perp t \cdot u \cdot \pi$, that is what we wanted to prove. □

Theorem 2.13. Let $P, Q, R \in \mathcal{P}_{\perp}(\Pi)$. If $P \circ_{\perp} Q \supseteq R$ then $\{\mathsf{E}\}^{\perp} \circ_{\perp} P \supseteq (Q \Rightarrow_{\perp} R)$

Proof. Using the hypothesis and 2.10 we deduce that $R \subseteq P \diamond Q$. By the monotony of $\Rightarrow$ in the right argument it follows that $P \Rightarrow (Q \Rightarrow R) \subseteq P \Rightarrow (Q \Rightarrow (P \diamond Q))$ and by applying Lemma 2.12 (6) we get that $\{\mathsf{E}\}^{\perp} \supseteq P \Rightarrow (Q \Rightarrow R)$. Applying the adjunction property 2.6, we obtain that $\{\mathsf{E}\}^{\perp} \circ P \supseteq Q \Rightarrow R$. Finally, taking orthogonals twice, we conclude that $\{\mathsf{E}\}^{\perp} \circ_{\perp} P \supseteq Q \Rightarrow_{\perp} R$. □

Definition 2.14. In an $\mathcal{AKS}$ as above, the combinator $\mathsf{E} \in \mathrm{QP}$ is called an adjunctor.

Remark 2.15. Items (1)-(3) of Lemma 2.12 resemble the Hilbert style axiomatization of the implicational fragment of classical propositional logic. Using this analogy, it is easy to show the following:

Assume that $\varphi[X_1, \dots, X_n]$ is a propositional formula built up from propositional variables $X_1, \dots, X_n$ and implication. For arbitrary subsets $P_1, \dots, P_n \subseteq \Pi$ denote by $\varphi[P_1, \dots, P_n] \subseteq \Pi$ the evaluation of $\varphi[X_1, \dots, X_n]$ where $P_1, \dots, P_n$ are substituted for the variables, and implication is interpreted by the operation $\Rightarrow$ from Definition 2.5. If $\varphi[X_1, \dots, X_n]$ is provable in the Hilbert calculus then $\varphi[P_1, \dots, P_n]$ contains a quasi-proof, namely the element of QP obtained by evaluating the proof-term of $\varphi[X_1, \dots, X_n]$ in Λ.

3. Implicative and Krivine ordered combinatory algebras

In Streicher (2013), the author presented a construction of an *ordered combinatory algebra* *($\mathcal{OCA}$)* (see (van Oosten 2008, Section 1.8)) out of an $\mathcal{AKS}$, from which he constructed a tripos whose predicates are functions with values in the $\mathcal{OCA}$ (rather than in its powerset, which would be the usual approach in categorical realizability). However, Streicher's

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construction (where the elements of the $\mathcal{OCA}$ are directly used as truth values) does not give rise to a tripos in general, but only for some $\mathcal{OCA}$s – in particular for those induced by $\mathcal{AKS}$s. The notion of *implicative ordered combinatory algebra* abbreviated as ${}^{\mathcal{I}}\mathcal{OCA}$, is an axiomatization of the additional structure that we use in an $\mathcal{OCA}$ in order to guarantee that the induced indexed preorder is a tripos. The tripos will be classical in case the ${}^{\mathcal{I}}\mathcal{OCA}$ has an additional combinator, called c, that realizes Peirce's law.

In this section we focus our attention on *implicative ordered combinatory algebras*: ${}^{\mathcal{I}}\mathcal{OCA}$s and the modification consisting in adding the combinator c that produces a *Krivine ordered combinatory algebra*: ${}^{\mathcal{K}}\mathcal{OCA}$. The main features added to the usual structure of an ordered combinatory algebra –compare with Hofstra (2006)– are the following: a) we assume the existence of a distinguished element, that we call *adjunctor*; b) we assume that the ${}^{\mathcal{I}}\mathcal{OCA}$ is inf-complete; c) we have an implication mapping denoted as $\to$. These additions are present in the $\mathcal{OCA}$s that come from $\mathcal{AKS}$s, and will be crucial ingredients in the construction of the associated tripos, that we build up directly from the ${}^{\mathcal{I}}\mathcal{OCA}$ –compare with Streicher (2013)–. See for example Hofstra *et al.* (2004) and van Oosten (2008) for the standard approach to the subject.

3.1. *Ordered combinatory algebras*

Definition 3.1.

(1) An *ordered combinatory algebra* ($\mathcal{OCA}$) is a quintuple $\mathcal{A} = (A, \leqslant, \mathrm{app}, \mathsf{k}, \mathsf{s})$ –written frequently as A– where $(A, \leqslant)$ is a partial order,

$$\mathrm{app} : A \times A \to A, \qquad (a, b) \mapsto ab$$

is a monotone function, and k, s are elements of A satisfying

(a) $\mathsf{k}ab \leqslant a$

(b) $\mathsf{s}abc \leqslant ac(bc)$

for all $a, b, c \in A$.

(2) A *filter* in an $\mathcal{OCA}$ –called A– is a subset $\Phi \subseteq A$ which contains s and k and is closed under application. A pair $(\mathcal{A}, \Phi)$ is called a filtered $\mathcal{OCA}$.

Remark 3.2. Here –and in the rest of this paper– the product–like operations will not be associative and we assume that when parenthesis are omitted, we associate to the left.

In what follows, we will recall how to program directly in this $\mathcal{OCA}$, using the standard codifications in the combinatory algebras.

Definition 3.3. Let A be an $\mathcal{OCA}$ and take a denumerable set of *variables*: $\mathcal{V} = \{x_1, x_2, \ldots\}$ and consider $A(\mathcal{V})$ –called *the set of terms over* A– that is the set of formal expressions given by the following grammar: $p_1, p_2 ::= a \;\mid\; x \;\mid\; p_1p_2$ where $a \in A$ and $x \in \mathcal{V}$. As usual $A(x_1, \ldots, x_k)$ is the set of terms over A containing only the variables $x_1, \ldots, x_k$. One can naturally extend the order in A to an order in $A(\mathcal{V})$ in such a way that: if

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$p_1\ R\ p_2, q_1\ R\ q_2 \in A(\mathcal{V})$ then $p_1q_1\ R\ p_2q_2$ and if $p_1, p_2 \in A(\mathcal{V})$ then $\mathsf{k}\,p_1p_2\ R\ p_1$ and if $p_1, p_2, p_3 \in A(\mathcal{V})$ then $\mathsf{s}\,p_1p_2p_3\ R\ p_1p_3(p_2p_3)$.

We can prove that $(A(\mathcal{V}), R, \mathrm{j})$ where j is the concatenation is an $\mathcal{OCA}$ and $(A, \leqslant, \mathrm{app})$ is a sub-$\mathcal{OCA}$ of $(A(\mathcal{V}), R, \mathrm{j})$.

It is customary to denote the relation R as $\leqslant$ and the operation j as $\circ$ or as the concatenation of the factors. In this situation we say that $(A(\mathcal{V}), \leqslant, \circ)$ is an extension of $(A, \leqslant, \circ)$.

The following result is well known:

Theorem 3.4 (Combinatory completeness). For any finite set of variables $\{x_1, \dots, x_k, y\}$, there is a function $\lambda^* y : A(x_1, \dots, x_k, y) \to A(x_1, \dots, x_k)$ satisfying the following property:

$$\text{If} \quad t \in A(x_1, \dots, x_k, y)\ , \text{ and } \quad u \in A(x_1, \dots, x_k) \quad \text{then} \quad (\lambda^* y(t)) \circ u \leqslant t\{y := u\}.$$

Moreover if $X \subseteq A$ is an arbitrary subset and t is a term with all its coefficients in X, then $\lambda^* y(t)$ is a term with all its coefficients in $\langle X \rangle$, the closure of X by application. In particular if all the coefficients of t are in the filter Φ, then $\lambda^* y(t)$ is a polynomial with all the coefficients in Φ. Occasionally we write $\lambda^* y(t) = \lambda^* y.t$.

Proof. The function $\lambda^* y$ is defined recursively: i) If $y \neq x$, then $\lambda^* y(x) := \mathsf{k}\,x$; ii) $\lambda^* y(y) := \mathsf{s}\,\mathsf{k}\,\mathsf{k}$; iii) if p, q are polynomials, then: $\lambda^* y(pq) := \mathsf{s}\,(\lambda^* y(p))(\lambda^* y(q))$. From the fact that $\langle X \rangle$ contains k, s and it is closed under applications, we deduce the condition on the coefficients of $\lambda^* y(t)$. □

We use combinatory completeness to define some combinators that we will use later.

Definition 3.5. Let A be an $\mathcal{OCA}$, we can define the following combinators or combinatorial functions that are elements of Φ, or functions with codomain and domain Φ.

$$\mathsf{b} = \lambda^* x \lambda^* y \lambda^* z(x(yz)), \quad \mathsf{i} = \lambda^* x(x), \quad \mathsf{c} = \lambda^* x \lambda^* y \lambda^* z(xzy), \quad \mathsf{w} = \lambda^* x \lambda^* y(xyy);$$

$$\mathsf{t} = \lambda^* x \lambda^* y(x), \quad \mathsf{f} = \lambda^* x \lambda^* y(y), \quad \mathsf{p} = \lambda^* x \lambda^* y \lambda^* z(zxy), \quad \mathsf{p}_0 = \lambda^* x(x\,\mathsf{t}), \ \mathsf{p}_1 = \lambda^* x(x\,\mathsf{f}).$$

$$\mathsf{a} : \Phi \times \Phi \to \Phi \quad \mathsf{a}(r,s) = \lambda^* x(\mathsf{p}(rx)(sx)) \quad \mathsf{d} : \Phi \to \Phi \quad \mathsf{d}(f) = \lambda^* x(f(\mathsf{p}_0 x)(\mathsf{p}_1 x)).$$

Lemma 3.6. If A is an $\mathcal{OCA}$, the above definitions ensure that:

$$\mathsf{b}abc \leqslant a(bc)\,;\ \mathsf{i}a \leqslant a\,;\ \mathsf{c}abc \leqslant acb\,;\ \mathsf{w}ab \leqslant abb\,;\ \mathsf{p}_0(\mathsf{p}ab) \leqslant a\,;\ \mathsf{p}_1(\mathsf{p}ab) \leqslant b;$$

$$rc \leqslant a, sc \leqslant b \Rightarrow \mathsf{a}(r,s)c \leqslant \mathsf{p}ab\ ;\ \mathsf{d}(f)\ell \leqslant f(\mathsf{p}_0\ell)(\mathsf{p}_1\ell).$$

for all $a, b, c, \ell \in A$.

3.2. *Implicative ordered combinatory algebras*

Definition 3.7. An implicative ordered combinatory algebra –a $^{\mathcal{I}}\mathcal{OCA}$–, consists of an inf-complete partially ordered set $(A, \leqslant)$ equipped with:

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(1) binary operations

$$\mathrm{app} : A \times A \to A, \qquad (a, b) \mapsto ab$$

called *application*, monotone in both arguments, and

$$\mathrm{imp} : A^{\mathrm{op}} \times A \to A, \qquad (a, b) \mapsto a \to b$$

called *implication*, antitonic in the first argument and monotone in the second;
(2) a subset $\Phi \subseteq A$ (called *filter*) which is closed under application;
(3) distinguished elements $\mathsf{s}, \mathsf{k}, \mathsf{e} \in \Phi$ such that the following holds for all $a, b, c \in A$.

(PK) $\mathsf{k}ab \leqslant a$;

(PS) $\mathsf{s}abc \leqslant ac(bc)$;

(PA) $a \leqslant b \to c \quad \Rightarrow \quad ab \leqslant c$;

(PE) $ab \leqslant c \quad \Rightarrow \quad \mathsf{e}a \leqslant b \to c$.

3.3. *Krivine ordered combinatory algebras*

Definition 3.8. A Krivine ordered combinatory algebra –a ${}^{\mathcal{K}}\mathcal{OCA}$–, consists of an ${}^{\mathcal{I}}\mathcal{OCA}$ equipped with a distinguished element $\mathsf{c} \in \Phi$ such that for all $a, b \in A$,

$$(PC) \quad \mathsf{c} \leqslant ((a \to b) \to a) \to a.$$

Example 3.9. Any complete Boolean algebra $(B, \leqslant)$ (see e.g. (Gierz *et al.* 2003, O-2.6)) gives rise to a ${}^{\mathcal{K}}\mathcal{OCA}$ by defining

$$ab = a \wedge b \qquad a \to b = \neg a \vee b \qquad \Phi = \{\top\} \qquad \mathsf{s} = \mathsf{k} = \mathsf{e} = \mathsf{c} = \top.$$

We will comment on the realizability models associated to ${}^{\mathcal{K}}\mathcal{OCA}$'s of this form in Remark 5.9.

Next we show that in the definition of a ${}^{\mathcal{K}}\mathcal{OCA}$ (and also of a ${}^{\mathcal{I}}\mathcal{OCA}$) some of its elements are superfluous and can be obtained from the others. Here we present a minimal setup for the concept.

Definition 3.10. A quadruple $\mathcal{Q} = (A, \leqslant, \to, \Phi)$ where

(1) The relation $\leqslant$ is a partial order in A with the property that each $X \subseteq A$ has an infimum.
(2) The map $\to: A \times A \to A$ –*implication*– is antitonic in the first variable and monotone in the second.
(3) $\Phi \subset A$ –a *filter*– is a subset of A.

is said to be proper if Φ satisfies the following conditions:

(1) The set Φ is closed under the application: $\mathrm{app}(\Phi, \Phi) \subset \Phi$.
(2) The elements $\mathsf{k}, \mathsf{s}, \mathsf{e}, \mathsf{c} \in \Phi$.

Where the above maps and elements are defined as follows:

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(1) an *application map*: app : $A \times A \to A$ defined for all $a, b \in A$ as $\mathrm{app}(a, b) = ab := \inf\{c : a \leqslant (b \to c)\}$.
(2) $\mathsf{k} := \inf\{a \to (b \to a) : a, b \in A\}$.
(3) $\mathsf{s} := \inf\{a \to (b \to (c \to (ac)(bc))) : a, b, c \in A\}$.
(4) $\mathsf{e} := \inf\{a \to (b \to ab) : a, b \in A\}$.
(5) $\mathsf{c} := \inf\{(((a \to b) \to a) \to a) : a, b \in A\}$.

then

Theorem 3.11. If $\mathcal{Q} = (A, \leqslant, \to, \Phi)$ is proper (Definition 3.10), then $\mathcal{A}_{\mathcal{Q}} = (A, \leqslant, \to, \Phi, \mathrm{app}, \mathsf{k}, \mathsf{s}, \mathsf{e}, \mathsf{c})$ is a $^{\mathcal{K}}\mathcal{OCA}$.

Proof. To prove the *half adjunction property*, assume that for $a, b, c \in A$, $a \leqslant (b \to c)$ as ab is the infimum of the elements c with the above property, it is clear that in this situation $ab \leqslant c$ (condition (PA) of Definition 3.8: the "half adjunction property"). The fact that the application $(a, b) \mapsto ab$ is monotone in both variables follows directly from the definition by using the monotonicity properties of the implication. If $a, b, c \in A$ are such that $ab \leqslant c$, by definition we know that $\mathsf{e} \leqslant a \to (b \to ab)$ and then that $\mathsf{e} \leqslant a \to (b \to c)$. Applying the *half adjunction property* ((PA) in the previous definition) we obtain that $\mathsf{e}\,a \leqslant (b \to c)$. The satisfaction by k, s of the required properties follows by a direct application of (PA) and the condition for c is evidently satisfied. □

4. Indexed preorders and triposes

4.1. *Preorders, meet semi-lattices and Heyting preorders*

Definition 4.1. We denote by **Ord** the category of preorders and monotone maps. A preorder $(D, \leqslant)$ is a set D with a reflexive and transitive relation $\leqslant$. If $d \leqslant d'$ and $d' \leqslant d$, we say that d and d' are *isomorphic*, and write $d \cong d'$.

Definition 4.2. Let $(C, \leqslant)$ and $(D, \leqslant)$ be two preorders.

(1) For monotone maps $f, g : (C, \leqslant) \to (D, \leqslant)$, we define $f \leqslant g :\Leftrightarrow \forall d \in D\,.\, f(d) \leqslant g(d)$ and say that f and g are *isomorphic* (written $f \cong g$) if $f \leqslant g$ and $g \leqslant f$.
(2) A monotone map $f : (C, \leqslant) \to (D, \leqslant)$ is called an *equivalence*, if there exists a monotone map $g : (D, \leqslant) \to (C, \leqslant)$ such that $g \circ f \cong \mathrm{id}_D$, and $f \circ g \cong \mathrm{id}_E$ and g is called a *weak inverse* of f. In this situation we say that $(C, \leqslant)$ and $(D, \leqslant)$ are *equivalent* (written $(D, \leqslant) \simeq (E, \leqslant)$).
(3) Given monotone maps $f : (C, \leqslant) \to (D, \leqslant)$, $g : (D, \leqslant) \to (C, \leqslant)$, we say that '$f$ is left adjoint to g', or 'g is right adjoint to f', and write $f \dashv g$, if $\mathrm{id}_C \leqslant g \circ f$ and $f \circ g \leqslant \mathrm{id}_D$.

Remark 4.3. The following assertions are easy to prove.

(1) A monotone map $f : (C, \leqslant) \to (D, \leqslant)$ is an equivalence if and only if it is *order reflecting* and *essentially surjective*, i.e.
 (a) $\forall c, c' \in D\,.\, f(c) \leqslant f(c') \Rightarrow c \leqslant c'$, and
 (b) $\forall d \in D\; \exists c \in C\,.\, f(c) \cong d$.
(2) Let $f : (C, \leqslant) \to (D, \leqslant)$, $g : (D, \leqslant) \to (C, \leqslant)$ be monotone maps between preorders.

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(a) f is left adjoint to g, if and only if $\forall c \in C,\ d \in D\ .\ f(c) \leqslant d \Leftrightarrow c \leqslant g(d)$.

(b) Adjoints are unique up to isomorphism, i.e. when $f \dashv g$ and $f \dashv g'$, then $g \cong g'$ (and similarly for left adjoints).

Definition 4.4. A meet semi-lattice is a preorder $(D, \leqslant)$ equipped with a binary operation $\wedge$ and a distinguished element $\top$ such that for all $a, b, c \in D$:

(1) $a \wedge b \leqslant a$;
(2) $a \wedge b \leqslant b$;
(3) $c \leqslant a$ and $c \leqslant b \Rightarrow c \leqslant a \wedge b$;
(4) $a \leqslant \top$.

Remark 4.5. If $(D, \leqslant)$ is a meet semi-lattice, then the function $(d, d') \mapsto d \wedge d'$ is a monotone map of type $D \times D \to D$, which is right adjoint to the diagonal map $\delta : D \to D \times D$, $d \mapsto (d, d)$.

Definition 4.6. **SLat** is the category of meet semi-lattices, and *meet preserving monotone maps*, i.e. monotone maps $f : (D, \leqslant) \to (E, \leqslant)$ such that

(1) $f(d) \wedge f(d') \cong f(d \wedge d')$ for all $d, d' \in D$
(2) $f(\top) \cong \top$.

Definition 4.7. We define **HPO**, the category of Heyting preorders and morphisms.

(1) A *Heyting preorder* is a meet semi-lattice $(A, \leqslant)$ with a binary operation $\to : A \times A \to A$ (called *Heyting implication*) satisfying

$$a \wedge b \leqslant c \ \text{ if and only if } \ a \leqslant b \to c \tag{3}$$

for all $a, b, c \in A$.

(2) A *morphism of Heyting preorders* is a monotone map $f : (A, \leqslant) \to (B, \leqslant)$ such that

(a) $f(\top) \cong \top$

(b) $f(a \wedge b) \cong f(a) \wedge f(b)$

(c) $f(a \to b) \cong f(a) \to f(b)$

for all $a, b \in A$.

Remarks 4.8.

(1) The term ‘Heyting preorder’ is not standard, but it is the same as a ‘posetal Cartesian closed category’, or equivalently a preorder whose poset reflection is a ‘Heyting semi-lattice’ (Johnstone 2002, Part A1.5).
(2) A Heyting preorder with finite joins is what is called a *Heyting prealgebra*, e.g. in van Oosten (2008). The anti-symmetric version is the well known concept of *Heyting algebra*.
(3) Also, we don’t have to demand Heyting implication to be monotone – it follows from its definition that it is antitonic in the first, and monotonic in the second variable.

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4.2. *Preorders associated to $\mathcal{AKS}$s, $\mathcal{OCA}$s and $^{\mathcal{I}}\mathcal{OCA}$s*

$\mathcal{AKS}$

Definition 4.9. Let $\mathcal{K} = (\Lambda, \Pi, \dots)$ be an abstract Krivine structure. We define the relation $\sqsubseteq$ in $\mathcal{P}(\Pi)$ as follows:

$$P, Q \in \mathcal{P}(\Pi), \quad P \sqsubseteq Q \quad :\Leftrightarrow \quad \exists t \in \mathrm{QP}\ t \perp P \Rightarrow Q \tag{4}$$

for $P, Q \in \mathcal{P}(\Pi)$. An element $t \in \Phi$ as above is said to be "a realizer of the relation" $P \sqsubseteq Q$".

Remark 4.10. Notice that the relation above, could have been defined using the arrow $\Rightarrow_{\perp}$. Indeed, $t \perp P \Rightarrow Q$ if and only if $t \perp P \Rightarrow_{\perp} Q$.

Lemma 4.11. Let $\mathcal{K}$ be an abstract Krivine structure, then the relation $\sqsubseteq$ is a preorder on $\mathcal{P}(\Pi)$.

Proof. The combinator I is a realizer of $P \sqsubseteq P$ for any $P \in \mathcal{P}(\Pi)$, thus $\sqsubseteq$ is reflexive. For transitivity, assume that $P, Q, R \in \mathcal{P}(\Pi)$, and that $t, u \in \mathrm{QP}$ are realizers of $P \sqsubseteq Q$ and $Q \sqsubseteq R$, respectively. Then $\mathsf{B}tu$ is a realizer of $P \sqsubseteq R$. □

Lemma 4.12. The canonical inclusion $\mathcal{P}_{\perp}(\Pi) \hookrightarrow \mathcal{P}(\Pi)$ is an equivalence of preorders with respect to $\sqsubseteq$.

Proof. By Remark 4.3 it is suffices to show that the inclusion is order reflecting and essentially surjective. Since the order on $\mathcal{P}_{\perp}(\Pi)$ is defined as restriction of the order on $\mathcal{P}(\Pi)$ the first assertion is clear.

To prove that the inclusion is essentially surjective, we show that $P \sqsubseteq ({}^{\perp}P)^{\perp}$ and $({}^{\perp}P)^{\perp} \sqsubseteq P$ for all $P \in \mathcal{P}(\Pi)$. This holds since $\mathsf{I} \perp {}^{\perp}(({}^{\perp}P)^{\perp}) \cdot P = {}^{\perp}P \cdot P$ and $\mathsf{I} \perp {}^{\perp}P \cdot ({}^{\perp}P)^{\perp}$, for all $P \subseteq \Pi$. Both relations are realized by I as follows directly from Lemma 2.12, (2.12) applied in the cases of P and $({}^{\perp}P)^{\perp}$, respectively. □

$\mathcal{OCA}$

Definition 4.13. Let $(\mathcal{A}, \Phi)$ be a filtered $\mathcal{OCA}$ with maximum element $\top \in \Phi$. We define:

(1) The relation $\sqsubseteq_{\Phi}$ in A as follows: $a \sqsubseteq_{\Phi} b$, if and only if $\exists f \in \Phi$ such that $fa \leqslant b$.
(2) A map $\wedge : A \times A \to A$ as $a \wedge b := \mathsf{p}ab$ –see Definition 3.5.

Usually we omit the subscript Φ in the notation of the relation $\sqsubseteq_{\Phi}$ and write $a \sqsubseteq b$, also an element f as above is said to be "a realizer of the relation $a \sqsubseteq b$" and write this assertion as $f \Vdash a \sqsubseteq b$.

We establish some properties that will be of later use.

Lemma 4.14. If $(\mathcal{A}, \Phi)$ is a filtered $\mathcal{OCA}$ then in the notations of Definition 3.5 we have that:

(1) $\mathsf{p}_0 \Vdash a \wedge b \sqsubseteq a$
(2) $\mathsf{p}_1 \Vdash a \wedge b \sqsubseteq b$

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(3) If $r \Vdash c \sqsubseteq a$ and $s \Vdash c \sqsubseteq b$ then $\mathsf{a}(r,s) \Vdash c \sqsubseteq a \wedge b$
(4) $\mathsf{kk} \Vdash a \sqsubseteq \top$.

Hence, $(A, \wedge, \sqsubseteq)$ is a meet-semi-lattice.

Proof. All the assertions follow directly from Lemma 3.6. □

$^{\mathcal{I}}\mathcal{OCA}$

Next we show that in the case of the existence of an adjunctor, more precise assertions can be proved concerning the meet and the order $\sqsubseteq$.

Theorem 4.15. If $(\mathcal{A}, \Phi)$ is a $^{\mathcal{I}}\mathcal{OCA}$ then:

(1) If $a, b \in A$ then $a \sqsubseteq b$ if and only if there is an element $f \in \Phi$ such that $f \leqslant a \to b$.
(2) If $a, b, c \in A$:

$$a \wedge b \sqsubseteq c \Leftrightarrow a \sqsubseteq (b \to c).$$

In other words $(A, \sqsubseteq, \wedge, \to)$ is a Heyting preorder.

Proof.

(1) Assuming that $f \leqslant a \to b$ and using the half adjunction property we deduce that $fa \leqslant b$ i.e. that $a \sqsubseteq b$. In case that $a \sqsubseteq b$, first we deduce that $ga \leqslant b$ for some $g \in \Phi$. Using the adjunctor we deduce that $\mathsf{e}\, g \leqslant a \to b$.
(2) To see that the map $\to$ gives a Heyting implication on $(A, \sqsubseteq)$, we have to check that

$$a \wedge b \sqsubseteq c \quad \Leftrightarrow \quad a \sqsubseteq (b \to c)$$

where $a \wedge b = \mathsf{p}ab$.
If the right inequality holds, there exists an element $f \in \Phi$ such that $fa \leqslant b \to c$, and Definition 3.8, (PA) gives $fab \leqslant c$. In accordance with Lemma 3.6 there exists a function $\mathsf{d} : \Phi \to \Phi$ such that $\mathsf{d}(f)\ell \leqslant f(\mathsf{p}_0\ell)(\mathsf{p}_1\ell)$ for all $\ell \in A$, and this gives (substituting ℓ by $\mathsf{p}ab$)

$$\mathsf{d}(f)(a \wedge b) = \mathsf{d}(f)(\mathsf{p}ab) \leqslant fab \leqslant c.$$

Conversely, assume that the left hand side holds, i.e. there exists an $f \in \Phi$ such that $f(\mathsf{p}\, ab) \leqslant c$. Then we can produce the following chain of deductions:

$$f(\mathsf{p}\, ab) \leqslant c \Rightarrow \mathsf{b}f(\mathsf{p}\, a)\, b \leqslant c \Rightarrow \mathsf{b}(\mathsf{b}f)\mathsf{p}\, ab \leqslant c \Rightarrow \mathsf{e}(\mathsf{b}(\mathsf{b}f)\mathsf{p}\, a) \leqslant b \to c$$

$$\Rightarrow \mathsf{be}(\mathsf{b}(\mathsf{b}f)\mathsf{p})\, a \leqslant b \to c,$$

hence $\mathsf{be}(\mathsf{b}(\mathsf{b}f)\mathsf{p})$ is a realizer of $a \sqsubseteq b \to c$. □

For future use we prove the following property of the combinator c in the case that the $^{\mathcal{I}}\mathcal{OCA}$ is equipped with one.

Lemma 4.16. Let $\mathcal{A}$ be a $^{\mathcal{I}}\mathcal{OCA}$ and c an element of $\mathcal{A}$ such that $\mathsf{c} \leqslant ((a \to b) \to a) \to a$ for all $a, b \in \mathcal{A}$, then $\mathsf{c} \Vdash ((a \to \bot) \to \bot) \sqsubseteq a$ for all $a \in \mathcal{A}$.

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Proof. For $a \in \mathcal{A}$ we have the following chain of implications: $\mathsf{c} \leqslant ((a \to \bot) \to a) \to a$ $\Rightarrow \mathsf{c}((a \to \bot) \to a) \leqslant a \Rightarrow \mathsf{c}((a \to \bot) \to \bot) \leqslant \mathsf{c}((a \to \bot) \to a) \leqslant a \Rightarrow \mathsf{c} \Vdash ((a \to \bot) \to \bot)$ $\sqsubseteq a$ □

4.3. *Indexed preorders and indexed meet-semi-lattices*

Definition 4.17.

(1) An *indexed preorder* is a functor $\mathcal{D} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$.
(2) An *indexed meet-semi-lattice* is a functor $\mathcal{A} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{SLat}$.
(3) An *indexed Heyting preorder* is a functor $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HPO}$.

We only present the following definitions in the case of preorders, in the case of indexed meet-semilattices the concepts are similar.

Remarks 4.18.

(1) Indexed preorders (in particular triposes, defined below) can be used as categorical models of predicate logic. With this in mind, we often call their elements *predicates* – more precisely, if $\mathcal{D}$ is an indexed preorder, I is a set, and $\varphi \in \mathcal{D}(I)$, we say that φ is a *predicate on* I.
(2) If $\mathcal{D}$ is an indexed preorder and $f : J \to I$ is a function, applying the functor to f gives us a monotonic map $\mathcal{D}(f) : \mathcal{D}(I) \to \mathcal{D}(J)$. We call this function *reindexing along* f, and usually abbreviate it by f^*. Thus, if φ is a predicate on I, then *its reindexing* $f^*(\varphi)$ *along* f is a predicate on J. Semantically, reindexing corresponds to *substitution and context extension.*
(3) There are more general concepts of indexed preorder, one is that of a *pseudofunctor* of type $\mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$. Another generalization of indexed preorders is to replace $\mathbf{Set}$ by another category. We do not need these levels of generality.
(4) Preorders are a special case of *indexed categories*, which are functors $\mathcal{C} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Cat}$. The link between indexed categories and logic was discovered by Lawvere in the 60ies Lawvere (1969, 1970) ('quantifiers as adjoints'), and is at the heart of *categorical logic*.

Definition 4.19. Given indexed preorders $\mathcal{D}, \mathcal{E} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$, an *indexed monotonic map* $\sigma : \mathcal{D} \to \mathcal{E}$ is a family

$$\sigma_I : \mathcal{D}(I) \to \mathcal{E}(I) \qquad (I \in \mathbf{Set})$$

of monotonic functions, such that we have

$$\sigma_J(f^*(\varphi)) \cong f^*(\sigma_I(\varphi)) \tag{5}$$

for all functions $f : J \to I$ and predicates $\varphi \in \mathcal{D}(I)$.

Remarks 4.20.

(1) Indexed monotonic maps are special cases of *pseudo-natural transformations* Lack (2010). If we have equality in (5), we speak of a *strict* indexed monotonic map, which is an instance of a *2-natural transformation.*

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(2) Indexed preorders and indexed monotonic maps form a category, which wedenote by **IOrd**. Composition of indexed monotonic maps $\mathcal{C} \xrightarrow{\sigma} \mathcal{D} \xrightarrow{\tau} \mathcal{E}$ is defined by $(\tau \circ \sigma)_I(\varphi) = \tau_I(\sigma_I(\varphi))$ for $\varphi \in \mathcal{C}(I)$. The identity $\mathrm{id}_\mathcal{D}$ of an indexed preorder $\mathcal{D}$ is defined by $\mathrm{id}_{\mathcal{D},I}(\varphi) = \varphi$ for all $\varphi \in \mathcal{D}(I)$.

Definition 4.21. Let $\mathcal{D}, \mathcal{E}$ be indexed preorders.

(1) For indexed monotonic maps $\sigma, \tau : \mathcal{D} \to \mathcal{E}$, we define

$$\sigma \leqslant \tau :\Leftrightarrow \forall I \in \mathbf{Set}\,.\, \sigma_I \leqslant \tau_I.$$

We say that σ and τ are *isomorphic*, and write $\sigma \cong \tau$, if $\sigma \leqslant \tau$ and $\tau \leqslant \sigma$.

(2) An indexed monotonic map $\sigma : \mathcal{D} \to \mathcal{E}$ is called an *equivalence*, if there exists a indexed monotonic map $\tau : \mathcal{E} \to \mathcal{D}$ such that $\tau \circ \sigma \cong \mathrm{id}_\mathcal{D}$, and $\sigma \circ \tau \cong \mathrm{id}_\mathcal{E}$. In this case, τ is called an *(indexed) weak inverse* of σ.

(3) We say that $\mathcal{D}$ and $\mathcal{E}$ are *equivalent*, and write $\mathcal{D} \simeq \mathcal{E}$, if there exists an equivalence $\sigma : \mathcal{D} \to \mathcal{E}$.

Lemma 4.22. An indexed monotonic map $\sigma : \mathcal{D} \to \mathcal{E}$ is an equivalence, if and only if for every set I, the monotonic map $\sigma_I : \mathcal{D}(I) \to \mathcal{E}(I)$ is order reflecting and essentially surjective.

Proof. By Remark 4.3, (1), every σ_I has a weak inverse $\tau_I : \mathcal{E}(I) \to \mathcal{D}(I)$. Together these τ_I give rise to an indexed weak inverse of σ. □

4.4. *Triposes*

Next we consider a special kind of indexed Heyting preorders, called *triposes*, see Hyland *et al.* (1980).

Definition 4.23. A *tripos* is a functor $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HPO}$ such that

(1) For every function $f : J \to I$, the reindexing map $f^* : \mathcal{P}(I) \to \mathcal{P}(J)$ has a right adjoint $\forall_f : \mathcal{P}(J) \to \mathcal{P}(I)$.

(2) If

$$\begin{array}{ccc} P & \xrightarrow{q} & K \\ \downarrow_p & & \downarrow_g \\ J & \xrightarrow{f} & I \end{array} \qquad (6)$$

is a pullback square of sets and functions, then $\forall_q(p^*(\varphi)) \cong g^*(\forall_f(\varphi))$ for all $\varphi \in \mathcal{P}(J)$ (this is the *Beck-Chevalley condition*).

(3) $\mathcal{P}$ has a *generic predicate*, i.e. there exists a set Prop, and a $\mathsf{tr} \in \mathcal{P}(\mathsf{Prop})$ such that for every set I and $\varphi \in \mathcal{P}(I)$ there exists a (not necessarily unique) function $\chi_\varphi : I \to \mathsf{Prop}$ with $\varphi \cong \chi_\varphi^*(\mathsf{tr})$.

Remark 4.24.

(1) $\forall_f : \mathcal{P}(J) \to \mathcal{P}(I)$ is not required to preserve meets or implication.

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(2) The statement that the above square is a pullback, means explicitly that

$$\forall j \in J, k \in K \,.\, f(j) = g(k) \Leftrightarrow \Big(\exists! x \in P \,.\, p(x) = j \wedge q(x) = k\Big).$$

(3) To interpret disjunction we want joins in triposes, but we don't have to postulate disjunction (nor $\exists$), since they can be encoded in terms of the other connectives in second order logic.

(4) A simple example of the functor $\forall_f$ is furnished by considering a Heyting algebra H that is complete with respect to the order relation. In this case, the tripos is defined as a functor $\mathcal{T} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HPO}$ where over objects $\mathcal{T}$ is defined by $\mathcal{T}(X) := [X, \mathrm{H}] = \mathrm{H}^X$. Thus, given an arrow $f : A \to B$, we have a reindexing map $f^* : [B, \mathrm{H}] \to [A, \mathrm{H}]$ whose right adjoint is $\forall_f : [A, \mathrm{H}] \to [B, \mathrm{H}]$, given by the expression

$$\forall_f(\phi)(y) = \bigwedge\nolimits_{f(x)=y} \phi(x)$$

Intuitively, $\forall_f(\phi)(y)$ is the meaning in H of the universal quantification of $\phi(x)$, where x ranges over the f-preimages of y.

Lemma 4.25. Let $\mathcal{D}$ and $\mathcal{P}$ be indexed preorders, and assume that $\sigma : \mathcal{D} \to \mathcal{P}$ and $\tau : \mathcal{P} \to \mathcal{D}$ form an equivalence. If $\mathcal{P}$ is a tripos, then so is $\mathcal{D}$.

Proof. This is because all the defining properties of a tripos are stable under equivalence, and can be transported along σ and τ. In particular:

(1) for any set I, $\tau_I(\top)$ is a greatest element in $\mathcal{D}(I)$;
(2) meets in $\mathcal{D}(I)$ are given by $\varphi \wedge \psi = \tau_I(\sigma_I(\varphi) \wedge \sigma_I(\psi))$;
(3) Heyting implication in $\mathcal{D}(I)$ is given by $\varphi \to \psi \;=\; \tau_I(\sigma_I(\varphi) \to \sigma_I(\psi))$;
(4) universal quantification in $\mathcal{D}$ is can be defined by $\forall_f(\varphi) = \tau_I(\forall_f(\sigma_J(\varphi)))$ for $f : J \to I$ and $\varphi \in \mathcal{D}(J)$;
(5) a generic predicate for $\mathcal{D}$ is given by $\tau_{\mathsf{Prop}}(\mathsf{tr})$ where $\mathsf{tr} \in \mathcal{P}(\mathsf{Prop})$ is the generic predicate of $\mathcal{P}$. □

5. Constructing triposes from ordered structures

In this section we show how to construct triposes –or weaker structures such as indexed meet-semilattices or indexed preorders– from ordered combinatory algebras or abstract Krivine structures. We also consider the relations between the different constructions.

5.1. *From $\mathcal{AKS}$s to indexed preorders*

Definition 5.1. Let $\mathcal{K} = (\Lambda, \Pi, \ldots)$ be an abstract Krivine structure, and let I be any set. The *entailment relation* $\vdash$ in $\mathcal{P}(\Pi)^I$ is defined by

$$\varphi, \psi \in \mathcal{P}(\Pi)^I\,, \quad \varphi \vdash \psi \quad :\Leftrightarrow \quad \exists t \in \mathrm{QP}\; \forall i \in I \,.\, t \perp \varphi(i) \Rightarrow \psi(i) \tag{7}$$

for $\varphi, \psi : I \to \mathcal{P}(\Pi)$. An element $t \in \Phi$ as above is said to be "a realizer of the entailment $\varphi \vdash \psi$".

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Remark 5.2. Notice that the entailment relation above, could have been defined using the arrow $\Rightarrow_{\perp}$, because $t \perp \varphi(i) \Rightarrow \psi(i)$ if and only if $t \perp \varphi(i) \Rightarrow_{\perp} \psi(i)$, compare with Remark 4.10.

Lemma 5.3. Let $\mathcal{K}$ be an abstract Krivine structure.

(1) For any set I, the entailment relation $\vdash$ is a preorder on $\mathcal{P}(\Pi)^I$.
(2) For any function $f : J \to I$, precomposition defines a monotonic map

$$f^* : (\mathcal{P}(\Pi)^I, \vdash) \to (\mathcal{P}(\Pi)^J, \vdash), \qquad \varphi \mapsto \varphi \circ f.$$

(3) The preceding constructions give an indexed preorder

$$\mathcal{P}(\mathcal{K}) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}, \qquad I \mapsto (\mathcal{P}(\Pi)^I, \vdash), \qquad f \mapsto f^*.$$

Proof. (1) – This is proved in the same way as Lemma 4.11.

(2) – Let $\varphi, \psi : I \to \mathcal{P}(\Pi)$. If $t \in \Phi$ is a realizer of $\varphi \vdash \psi$, then it is also a realizer of $\varphi \circ f \vdash \psi \circ f$, thus f^* is monotonic.

(3) – We check the functoriality condition, i.e. $g^* \circ f^* = (f \circ g)^*$ and $\mathrm{id}_I^* = \mathrm{id}_{A^I}$ for $K \xrightarrow{g} J \xrightarrow{f} I$. This follows from associativity and unit laws for composition. □

Definition 5.4. The indexed preorder $\mathcal{P}_{\perp}(\mathcal{K}) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ is defined by

$$\mathcal{P}_{\perp}(\mathcal{K})(I) = (\mathcal{P}_{\perp}(\Pi)^I, \vdash), \qquad f \mapsto f^*$$

where the order on $\mathcal{P}_{\perp}(\Pi)^I$ is the restriction of the entailment order on $\mathcal{P}(\Pi)^I$ to predicates with values in $\mathcal{P}_{\perp}(\Pi)$.

Lemma 5.5. The canonical inclusion $\mathcal{P}_{\perp}(\mathcal{K}) \hookrightarrow \mathcal{P}(\mathcal{K})$ is an equivalence of indexed preorders.

Proof. By Lemma 4.22 it is suffices to show that the inclusion

$$(\mathcal{P}_{\perp}(\Pi)^I, \vdash) \hookrightarrow (\mathcal{P}(\Pi)^I, \vdash)$$

is an equivalence for all sets I and this is proved in the same way as Lemma 4.12. □

5.2. *From $\mathcal{OCA}$s to indexed meet-semilattices*

We now give the construction of indexed meet-semilattices out of $\mathcal{OCA}$s, which will turn out to produce triposes in case the $\mathcal{OCA}$ is an ${}^{\mathcal{I}}\mathcal{OCA}$.

Definition 5.6. Let $(\mathcal{A}, \Phi)$ be a filtered $\mathcal{OCA}$. The *entailment relation* $\vdash \;\subseteq A^I \times A^I$ is defined by

$$\varphi \vdash \psi \quad :\Leftrightarrow \quad \exists r \in \Phi \; \forall i \in I \,.\, r(\varphi(i)) \leqslant \psi(i) \tag{8}$$

for $\varphi, \psi : I \to A$. An element $r \in \Phi$ as above is said to be "a realizer of the entailment $\varphi \vdash \psi$".

Lemma 5.7. Let $(\mathcal{A}, \Phi)$ be a filtered $\mathcal{OCA}$.

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(1) For any set I, the entailment relation $\vdash$ is a preorder on A^I, and $(A^I, \vdash)$ is a meet-semi-lattice with the following definitions: $\top : I \to A$; $\top(i) = \top = \mathsf{k}$ and $\varphi \wedge \psi$ of two functions $\varphi, \psi : I \to A$ is $(\varphi \wedge \psi)(i) = \varphi(i) \wedge \psi(i)$.

(2) For any function $f : J \to I$, precomposition with f defines a meet preserving monotonic map

$$f^* : (A^I, \vdash) \to (A^J, \vdash), \qquad \varphi \mapsto \varphi \circ f.$$

(3) The preceding constructions define an indexed meet-semi-lattice

$$\mathcal{P}(\mathcal{A}) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{SLat}, \qquad I \mapsto (A^I, \vdash), \qquad f \mapsto f^*.$$

Proof. (1) – We use in the proof of this assertion the realizers exhibited in Lemma 4.14.

(2) – Let $\varphi, \psi : I \to A$. If $r \in \Phi$ is a realizer of $\varphi \vdash \psi$, then it is also a realizer of $\varphi \circ f \vdash \psi \circ f$, thus f^* is monotonic. For meets, we have

$$((\varphi \wedge \psi) \circ f)(j) = \mathsf{p}(\varphi(fj)\psi(fj)) = ((\varphi \circ f) \wedge (\psi \circ f))(j)$$

for all $j \in J$, which means $f^*(\varphi \wedge \psi) = f^*(\varphi) \wedge f^*(\psi)$. Preservation of $\top$ is shown in the same way.

(3) – It remains to check functoriality, i.e. $g^* \circ f^* = (f \circ g)^*$ and $\mathrm{id}_I^* = \mathrm{id}_{A^I}$ for $K \xrightarrow{g} J \xrightarrow{f} I$. This follows from associativity and unit laws for composition. □

5.3. *From ${}^{\mathcal{I}}\mathcal{OCA}$s to triposes*

Next we show that if the $\mathcal{OCA}$ considered above has the necessary additional structure to make it a ${}^{\mathcal{I}}\mathcal{OCA}$, the indexed meet semi-lattice just constructed is in fact a tripos.

For any ${}^{\mathcal{I}}\mathcal{OCA}$, $\mathcal{A} = (A, \leqslant, \mathrm{app}, \mathrm{imp}, \Phi, \mathsf{k}, \mathsf{s}, \mathsf{e})$, the quintuple $(A, \leqslant, \mathrm{app}, \mathsf{k}, \mathsf{s})$ is an $\mathcal{OCA}$ –that we also call $\mathcal{A}$–, and Φ is a filter on it. Thus, we can construct the indexed meet-semi-lattice $\mathcal{P}(\mathcal{A})$ from Definition 5.6.

Theorem 5.8. If $\mathcal{A} = (A, \leqslant, \mathrm{app}, \mathrm{imp}, \Phi, \mathsf{k}, \mathsf{s}, \mathsf{e})$ is a ${}^{\mathcal{I}}\mathcal{OCA}$, then $\mathcal{P}(\mathcal{A})$ is a tripos. Moreover, if the ${}^{\mathcal{I}}\mathcal{OCA}$ is a ${}^{\mathcal{K}}\mathcal{OCA}$ –with combinator $\mathsf{c} \in \Phi$–then $\neg\neg\varphi \vdash \varphi$, with $\neg\varphi := \varphi \to \bot$.

Proof. We know that $\mathcal{P}(\mathcal{A})$ is an indexed meet-semi-lattice, and it remains to be shown that it has implication, universal quantification, and a generic predicate.

For $\varphi, \psi : I \to A$, we define $\varphi \to \psi$ by

$$(\varphi \to \psi)(i) = \varphi(i) \to \psi(i)$$

To see that this gives a Heyting implication on $(A^I, \vdash)$, we have to check that

$$\varphi \vdash \psi \to \theta \quad \Leftrightarrow \quad \varphi \wedge \psi \vdash \theta$$

where $(\varphi \wedge \psi)(i) = \mathsf{p}\varphi(i)\psi(i)$. In a similar manner as before, the assertion can be proved along the lines of reasoning of Theorem 4.15.

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The universal quantification of a predicate $\psi : J \to A$ along a function $f : J \to I$ is defined by:

$$\forall_f(\psi)(i) = \inf_{f(j)=i}\psi(j)$$

With this definition it follows directly that for any $\varphi : I \to A$ and $r \in \Phi$ we have

$$\begin{array}{rl} & \forall j \in J \,.\, r\,\varphi(f(j)) \leqslant \psi(j) \\ \Leftrightarrow & \forall i \in I \,.\, r\,\varphi(i) \leqslant \forall_f(\psi)(i), \end{array}$$

which means that $f^*\varphi \vdash \psi$ if and only if $\varphi \vdash \forall_f\psi$ (with the same realizer), hence Remark 4.3, (2) implies that $\forall_f$ is right adjoint to f^*.

For the Beck-Chevalley condition, consider the pullback square (6) in Definition 4.23, and let $\varphi : J \to I$. For $k \in K$ we have:

$$g^*(\forall_f(\varphi))(k) = \inf\{\varphi(j) : j \in J,\, fj = gk\} \text{ and } \forall_q(p^*(\varphi))(k) = \inf\{\varphi(p(x)) : x \in P,\, qx = k\}.$$

In the first case, the infimum is taken over the set $\{j \in J \mid f(j) = g(k)\}$, and in the second case over the set $\{j \in J \mid \exists x \in P \,.\, p(x) = j, q(x) = k\}$. These two sets are equal since the square is a pullback (thus the Beck Chevalley condition holds even up to equality).

Finally, a generic predicate for $\mathcal{P}(\mathcal{A})$ is given by $\mathrm{id}_A \in \mathcal{P}(\mathcal{A})(A)$.

The fact that $\neg\neg\varphi \vdash \varphi$ for all predicates φ, follows directly from Lemma 4.16. □

Remark 5.9. For a ${}^{\mathcal{K}}\mathcal{OCA}$ named $\mathcal{A}$ constructed from complete Boolean algebras $(B, \leqslant)$ as in Example 3.9, the entailment relation (Definition 5.6) on predicates $\varphi, \psi : I \to B$ on a set I reduces to

$$\varphi \vdash \psi \quad \Leftrightarrow \quad \forall i \in I \,.\, \varphi(i) \leqslant \psi(i),$$

i.e. the ordering on predicates is simply the *pointwise ordering*. This implies that the induced triposes $\mathcal{P}(\mathcal{A})$ are $\forall$-*standard* in the terminology of Hyland *et al.* (1980), which means essentially that the associated realizability model is equivalent to a *Boolean valued model* Bell (1977), which substantiates our claim from the introduction that realizability in ${}^{\mathcal{K}}\mathcal{OCA}$'s subsumes Boolean valued models.

This remarkable generality of realizability models over ${}^{\mathcal{K}}\mathcal{OCA}$'s means that we cannot expect the *existence* and *disjunction properties* Troelstra (1973)(1.11.32) to hold in general . To see how the disjunction property (which says that validity of a disjunctive formula $\varphi \vee \psi$ implies validity of either φ or ψ) can fail, consider the Boolean algebra $B = \{\bot, \ell, r, \top\}$ with ℓ and r incomparable, and least and greatest elements $\bot$ and $\top$. If L and R are two propositional constants denoting ℓ and r respectively, then $L \vee R$ is valid, but neither of L and R is valid.

The previous argument also shows that the existence (or 'witness') property does not hold in general, since it would imply the disjunction property as is shown in loc. cit.

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5.4. *From $\mathcal{AKS}$s to ${}^{\mathcal{K}}\mathcal{OCA}$s*

Next, we recall the construction due to Streicher (see Streicher (2013)) that starting from an $\mathcal{AKS}$ abbreviated as $\mathcal{K}$ produces a ${}^{\mathcal{K}}\mathcal{OCA}$ –that we call $\mathcal{A}_{\mathcal{K}}$– and show that they induce isomorphic indexed preorders –in fact triposes–.

Definition 5.10. Given an $\mathcal{AKS}$:

$$\mathcal{K} = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push}, \mathrm{app}, \mathrm{store}, \mathsf{K}, \mathsf{S}, \mathsf{CC}, \mathrm{QP})$$

define

$$\mathcal{A}_{\mathcal{K}} = (A, \leqslant, \mathrm{app}, \mathrm{imp}, \mathsf{k}, \mathsf{s}, \mathsf{c}, \mathsf{e}, \Phi)$$

as follows.

(1) $(A, \leqslant) = (\mathcal{P}_{\perp}(\Pi), \supseteq)$;

(2) $\mathrm{app}(P,Q) = P \circ_{\perp} Q = ({}^{\perp}({}^{\perp}Q \rightsquigarrow P))^{\perp}$, $\mathrm{imp}(P,Q) = P \Rightarrow_{\perp} Q = ({}^{\perp}({}^{\perp}P \cdot Q))^{\perp}$;

(3) $\mathsf{k} = \{\mathsf{K}\}^{\perp}$, $\mathsf{s} = \{\mathsf{S}\}^{\perp}$, $\mathsf{c} = \{\mathsf{CC}\}^{\perp}$, $\mathsf{e} = \{\mathsf{E}\}^{\perp}$, where $\mathsf{E} = \mathsf{S}(\mathsf{K}(\mathsf{SKK}))$;

(4) $\Phi = \{P \in \mathcal{P}_{\perp}(\Pi) \mid \exists t \in \mathrm{QP} \,.\, t \perp P\}$.

If $a, b \in A$ we write $ab := \mathrm{app}(a,b)$ and $a \to b := \mathrm{imp}(a,b)$. See Definitions 2.5, 2.8 and 2.9.

We recall the following important theorem from Streicher (2013) and write down a short proof for later use.

Theorem 5.11. Let $\mathcal{K}$ be an $\mathcal{AKS}$ and consider the structure $\mathcal{A}_{\mathcal{K}}$ presented in Definition 5.10.

(1) Then, $\mathcal{A}_{\mathcal{K}}$ is a ${}^{\mathcal{K}}\mathcal{OCA}$.

(2) The associated indexed preorders $\mathcal{P}_{\perp}(\mathcal{K})$ and $\mathcal{P}(\mathcal{A}_{\mathcal{K}})$ are isomorphic.

Proof.

(1) The order is clearly inf-complete as we observed in Remark 2.2. The fact that the implication and application satisfy the monotonicity properties, is clear. The implication $\to$ satisfies the *half adjunction property*: if $a \leqslant (b \to c)$ then $ab \leqslant c$ as was established in Theorem 2.7.

Next we prove that $\mathsf{k}ab \leqslant a$. Lemma 2.12 (2.12) guarantees that for all $a, b \in A$, $\mathsf{K} \in {}^{\perp}\big({}^{\perp}a \cdot ({}^{\perp}b \cdot a)\big)$. This assertion means that $\{\mathsf{K}\} \subseteq {}^{\perp}\big({}^{\perp}a \cdot ({}^{\perp}b \cdot a)\big)$ and then $\mathsf{k} \supseteq \big({}^{\perp}\big({}^{\perp}a \cdot ({}^{\perp}b \cdot a)\big)\big)^{\perp} \supseteq {}^{\perp}a \cdot ({}^{\perp}b \cdot a)$ that can be written as ${}^{\perp}a \rightsquigarrow \mathsf{k} \supseteq {}^{\perp}b \cdot a$. Moreover, from Definition 2.5, (1) we deduce that $\mathsf{k} \circ_{\perp} a \supseteq {}^{\perp}a \rightsquigarrow \mathsf{k} \supseteq {}^{\perp}b \cdot a$, i.e. $\mathsf{k} \circ_{\perp} a \leqslant (b \to a)$ –compare with Definition 2.5–. Using the half adjunction property (Theorem 2.7), we deduce that $\mathsf{k}ab \leqslant a$.

The condition $\mathsf{s}abc \leqslant (ac)(bc)$ can be proved as follows. Take $t \perp a$, $s \perp b$, $u \perp c$. Using Lemma 2.10 we deduce that $(su) \perp bc$ and $(tu) \perp ac$ and also that $(tu)(su) \perp (ac)(bc)$ and if $\pi \in (ac)(bc)$ is an arbitrary element we have that $(tu)(su) \perp \pi$.

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Then by the Definition 2.8, (S3) we conclude that $\mathsf{s} \perp t \cdot s \cdot u \cdot \pi$. Hence we have proved that $\mathsf{s} \perp {}^{\perp}a \cdot {}^{\perp}b \cdot {}^{\perp}c \cdot (ac)(bc)$ or $\mathsf{s} \in {}^{\perp}({}^{\perp}a \cdot {}^{\perp}b \cdot {}^{\perp}c \cdot (ac)(bc))$ or equivalently that $\mathsf{s} \supseteq {}^{\perp}a \cdot {}^{\perp}b \cdot {}^{\perp}c \cdot (ac)(bc)$.

Assume now that we have a situation as follows: $x, y \in A$, $z \subseteq \Pi$ with ${}^{\perp}x{\cdot}z \subseteq y$, clearly it follows from Remark 2.4 that $z \subseteq {}^{\perp}x \rightsquigarrow y$

If we apply repeatedly the above observation to ${}^{\perp}a \cdot {}^{\perp}b \cdot {}^{\perp}c \cdot (ac)(bc) \subseteq \mathsf{s}$ we deduce that $(ac)(bc) \subseteq \mathsf{s}\, abc$, and the proof of this part is finished.

The proof that e as introduced in Definition 5.10, is an adjunctor is the content of Theorem 2.13.

The proof that $\Phi \subseteq A$ is a filter that contains $\mathsf{k}, \mathsf{s}, \mathsf{e}$ is the following. The subset Φ is closed under application because if $f, g \in \Phi$, i.e. if we have $t_f \in {}^{\perp}f \cap \mathrm{QP}$ and $t_g \in {}^{\perp}g \cap \mathrm{QP}$ then $t_f t_g \in {}^{\perp}f^{\perp}g \cap \mathrm{QP} \subseteq {}^{\perp}(f \circ_{\perp} g) \cap \mathrm{QP}$ (Lemma 2.10). Moreover, $\mathsf{k}, \mathsf{s}, \mathsf{e} \in \Phi$ because $\mathsf{K} \in {}^{\perp}\mathsf{k} \cap \mathrm{QP}$, $\mathsf{S} \in {}^{\perp}\mathsf{s} \cap \mathrm{QP}$ and $\mathsf{E} \in {}^{\perp}\mathsf{e} \cap \mathrm{QP}$.

Finally, as we took $\mathsf{c} = \{\mathsf{cc}\}^{\perp}$, it is clear that: $\mathsf{cc} \in {}^{\perp}\mathsf{c} \cap \mathrm{QP}$. Moreover, we proved in Lemma 2.12 that $\mathsf{cc} \in {}^{\perp}(((a \to b) \to a) \to a)$, that implies that $\mathsf{c} \supseteq ({}^{\perp}(((a \to b) \to a) \to a))^{\perp} = (((a \to b) \to a) \to a)$, i.e. $\mathsf{c} \leqslant (((a \to b) \to a) \to a)$.

(2) In both cases the predicates on a set I are functions $\varphi, \psi : I \to \mathcal{P}_{\perp}(\Pi)$, so we only have to check that the two definitions of entailment coincide. The entailment in $\mathcal{P}(\mathcal{A}_{\mathcal{K}})$ is given by: $\exists P \in \Phi\ \forall i \in I\,.\, P\varphi(i) \leqslant \psi(i)$, which using the adjunctor and substituting P by $\mathsf{e}P$ can be formulated equivalently as: $\exists P \in \Phi\ \forall i \in I\,.\, P \leqslant \varphi(i) \to \psi(i)$.
As to the equivalence we have that:

$$\begin{array}{cc} & \exists P \in \Phi\ \forall i \in I\,.\, P \leqslant \varphi(i) \to \psi(i) \\ \Leftrightarrow & \exists t \in \mathrm{QP}\ \forall i \in I\,.\, \{t\}^{\perp} \supseteq ({}^{\perp}({}^{\perp}\varphi(i){\cdot}\psi(i)))^{\perp} \\ \Leftrightarrow & \exists t \in \mathrm{QP}\ \forall i \in I\,.\, t \perp ({}^{\perp}({}^{\perp}\varphi(i){\cdot}\psi(i)))^{\perp} \\ \Leftrightarrow & \exists t \in \mathrm{QP}\ \forall i \in I\,.\, t \perp {}^{\perp}\varphi(i){\cdot}\psi(i) \\ \Leftrightarrow & \exists t \in \mathrm{QP}\ \forall i \in I\,.\, t \perp \varphi(i) \Rightarrow \psi(i) \end{array}$$

and the last line is the definition of entailment in $\mathcal{P}_{\perp}(\mathcal{K})$. □

5.5. *From ${}^{\mathcal{K}}\mathcal{OCA}$s to $\mathcal{AKS}$s*

In order to complete our program to set up the foundations of realizability in terms of ${}^{\mathcal{K}}\mathcal{OCA}$s, we reverse the construction presented in Subsection 5.4 and show how to construct from a ${}^{\mathcal{K}}\mathcal{OCA}$ called $\mathcal{A}$, an $\mathcal{AKS}$ named as $\mathcal{K}_{\mathcal{A}}$. Then, we prove that the corresponding triposes are equivalent.

Definition 5.12. Given a ${}^{\mathcal{K}}\mathcal{OCA}$

$$\mathcal{A} = (A, \leqslant, \mathrm{app}_{\mathcal{A}}, \mathrm{imp}, \mathsf{k}, \mathsf{s}, \mathsf{c}, \mathsf{e}, \Phi)$$

we define the structure:

$$\mathcal{K}_{\mathcal{A}} = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push}, \mathrm{app}, \mathrm{store}, \mathsf{K}, \mathsf{S}, \mathsf{cc}, \mathrm{QP})$$

as follows.

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(1) $\Lambda = \Pi := A$
(2) $\perp\!\!\!\perp := \leqslant$, i.e. $s \perp \pi :\Leftrightarrow s \leqslant \pi$
(3) $\mathrm{push}(s, \pi) := \mathrm{imp}(s, \pi) = s \to \pi$, $\mathrm{app}(s, t) := \mathrm{app}_{\mathcal{A}}(s, t) = st$, $\mathrm{store}(\pi) := \neg\pi$
(3) $\mathsf{K} := \mathsf{e}(\mathsf{bek})$, $\mathsf{S} := \mathsf{e}(\mathsf{b}(\mathsf{be}(\mathsf{be}))\mathsf{s})$, $\mathsf{CC} := \mathsf{ec}$
(4) $\mathrm{QP} := \Phi$

Here, b is an abbreviation for $\mathsf{s}(\mathsf{ks})\mathsf{k}$, which has the property that $\mathsf{b}abc \leqslant a(bc)$ for all $a, b, c \in A$, and $\neg\pi$ is a shorthand for $\pi \to \bot$ and $\bot := \inf(A)$.

Theorem 5.13. In the notations of Definition 5.12 the structure $\mathcal{K}_{\mathcal{A}}$ is an $\mathcal{AKS}$.

Proof. It is clear that QP is closed under application and contains $\mathsf{K}, \mathsf{S}, \mathsf{CC}$, and it remains to check the axioms about the orthogonality relation (see Definition 2.8). Substituting the above definitions, these axioms become:

(S1)	$t \leqslant u \to \pi$	$\Rightarrow$	$tu \leqslant \pi$
(S2)	$t \leqslant \pi$	$\Rightarrow$	$\mathsf{e}(\mathsf{bek}) \leqslant t \to u \to \pi$
(S3)	$tv(uv) \leqslant \pi$	$\Rightarrow$	$\mathsf{e}(\mathsf{b}(\mathsf{be}(\mathsf{be}))\mathsf{s}) \leqslant t \to u \to v \to \pi$
(S4)	$t \leqslant \neg\pi \to \pi$	$\Rightarrow$	$\mathsf{ec} \leqslant t \to \pi$
(S5)	$t \leqslant \pi$	$\Rightarrow$	$\neg\pi \leqslant t \to \pi', \quad \forall \pi'$

(S1) follows from Definition 3.8, (PA), and (S5) follows from monotonicity of the arrow in its second argument and the antitonicity in the first.

(S2) is shown by the following derivation:

$$t \leqslant \pi \Rightarrow \mathsf{k}tu \leqslant \pi \Rightarrow \mathsf{e}(\mathsf{k}t) \leqslant u \to \pi \Rightarrow \mathsf{bek}t \leqslant u \to \pi \Rightarrow \mathsf{e}(\mathsf{bek}) \leqslant t \to u \to \pi.$$

(S3) is proved using repeatedly the basic properties of b and e as follows:

$$tv(uv) \leqslant \pi \Rightarrow \mathsf{s}tuv \leqslant \pi \Rightarrow \mathsf{e}(\mathsf{s}tu) \leqslant v \to \pi \Rightarrow \mathsf{be}(\mathsf{s}t)u \leqslant v \to \pi \Rightarrow \mathsf{e}(\mathsf{be}(\mathsf{s}t))$$

$$\leqslant u \to v \to \pi \Rightarrow$$

$$\Rightarrow \mathsf{be}(\mathsf{be})(\mathsf{s}t) \leqslant u \to v \to \pi \Rightarrow \mathsf{b}(\mathsf{be}(\mathsf{be}))\mathsf{s}t \leqslant u \to v \to \pi \Rightarrow \mathsf{e}(\mathsf{b}(\mathsf{be}(\mathsf{be}))\mathsf{s})$$

$$\leqslant t \to u \to v \to \pi$$

Finally, (S4) is proved using the basic property of c –Definition 3.8, (PC), the monotony of the application and the definition of e– as follows. Applying (PC) for $a = \pi$ and $b = \bot$ we obtain that $\mathsf{c} \leqslant (\neg\pi \to \pi) \to \pi$. Moreover, for all t we have that $t \leqslant (\neg\pi \to \pi)$ and then by the monotony of the application we deduce that: $\mathsf{c}t \leqslant ((\neg\pi \to \pi) \to \pi)(\neg\pi \to \pi)$. Moreover, from the following implication $(\neg\pi \to \pi) \to \pi \leqslant (\neg\pi \to \pi) \to \pi \Rightarrow ((\neg\pi \to \pi) \to \pi)(\neg\pi \to \pi) \leqslant \pi$ we obtain by transitivity that $\mathsf{c}t \leqslant \pi$ and then that $\mathsf{ec} \leqslant t \to \pi$. □

Definition 5.14. Let $(D, \leqslant)$ be a preorder.

(1) A *principal filter* in D is a subset of D of the form

$$\uparrow d_0 := \{d \in D \mid d_0 \leqslant d\}.$$

for some $d_0 \in D$.

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(2) Dually, a *principal ideal* in D is a subset of the form

$$\downarrow d_0 := \{d \in D \mid d \leqslant d_0\}.$$

for $d_0 \in D$.

Lemma 5.15. Let $\mathcal{A}$ be a ${}^{\mathcal{K}}\mathcal{OCA}$ structure, and $\mathcal{K}_{\mathcal{A}}$ the $\mathcal{AKS}$ induced via the construction in Definition 5.12.

(1) For $U \subseteq A$ we have ${}^{\perp}U = \downarrow(\inf U)$, and $U^{\perp} = \uparrow(\sup U)$.
(2) For $a \in A$ we have $\inf(\uparrow a) = a = \sup(\downarrow a)$
(3) The set $\mathcal{P}_{\perp}(\Pi)$ consists precisely of the principal filters in A, and the maps

$$f : A \to \mathcal{P}_{\perp}(\Pi),\ a \mapsto \uparrow a \quad \text{and} \quad g : \mathcal{P}_{\perp}(\Pi) \to A,\ P \mapsto \inf P,$$

are mutually inverse and establish a bijection between A and $\mathcal{P}_{\perp}(\Pi)$.
(4) For $P, Q \in \mathcal{P}_{\perp}(\Pi)$ we have $\inf(P \Rightarrow_{\perp} Q) = \inf(P \Rightarrow Q) = \inf P \to \inf Q$.

Proof. (1) – ${}^{\perp}U$ is the set of lower bounds of U, and $\inf U$ is the greatest lower bound. An element $a \in A$ is a lower bound of U if and only if it is smaller than the greatest lower bound. The second claim is just the dual (recall that this duality is valid in a lattice).

(2) – a is a lower bound of $\uparrow a$, and since $a \in \uparrow a$ any other lower bound must be smaller. Thus a is the greatest lower bound. The second part is symmetric.

(3) – For $P \subseteq A$ we have $({}^{\perp}P)^{\perp} = (\downarrow(\inf P))^{\perp} = \uparrow(\inf P)$, thus all $({}^{\perp}(-))^{\perp}$-stable sets are principal filters.

Conversely, for a principal filter of the form $\uparrow a$ and using the previous parts of this Lemma, we have that $({}^{\perp}\uparrow a)^{\perp} = (\downarrow(\inf(\uparrow a)))^{\perp} = \uparrow(\sup(\downarrow(\inf(\uparrow a)))) = \uparrow(\sup(\downarrow a)) = \uparrow a$.

To see that f and g are mutually inverse, take first $a \in A$. Then $g(f(a)) = \inf(\uparrow a) = a$. In the other direction, let $P \in \mathcal{P}_{\perp}(\Pi)$. We know that P is a principal filter, thus $P = \uparrow a$ for some $a \in A$ and we have $f(g(P)) = \uparrow(\inf P) = \uparrow(\inf(\uparrow a)) = \uparrow a = P$.

(4) – The fact that $\inf(P \Rightarrow_{\perp} Q) = \inf(P \Rightarrow Q)$ follows also from the previous results. Indeed, we have that $\inf(P \Rightarrow_{\perp} Q) = \inf\big(({}^{\perp}(P \Rightarrow Q))^{\perp}\big) = \inf((\downarrow(\inf(P \Rightarrow Q)))^{\perp}) = \inf(\downarrow(\inf P \to \inf Q)^{\perp}) = \inf((\downarrow(a \to b))^{\perp}) = \inf(\uparrow(\sup(\downarrow(a \to b)))) = a \to b = \inf P \to \inf Q = \inf(P \Rightarrow Q)$. In the above computations we used that: $P = \uparrow a$, $Q = \uparrow b$ and the parts (1), (2) and (3) already proved. The last equality is proved below.

From the preceding claim we know that given P, Q as above, there are elements $a, b \in A$ such that $P = \uparrow a$ and $Q = \uparrow b$. We have

$$\uparrow a \Rightarrow \uparrow b \;=\; {}^{\perp}(\uparrow a)\cdot\uparrow b \;=\; \downarrow(\inf(\uparrow a))\cdot\uparrow b \;=\; \downarrow a\cdot\uparrow b \;=\; \{c \to d \mid c \leqslant a, b \leqslant d\}$$

and thus $\inf(\uparrow a \Rightarrow \uparrow b) \;=\; a \to b$ by monotonicity of the arrow. □

Theorem 5.16. The associated indexed triposes $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}_{\perp}(\mathcal{K}_{\mathcal{A}})$ are equivalent (see Definitions 5.7-5.7 and in 5.3-5.3 respectively).

Proof. Let I be a set. The elements of $\mathcal{P}(\mathcal{A})(I)$ are functions $\varphi : I \to A$, and the elements of $\mathcal{P}_{\perp}(\mathcal{K}_{\mathcal{A}})(I)$ are functions $\widehat{\varphi} : I \to \mathcal{P}_{\perp}(\Pi)$.

Post-composition with f and g from Lemma 5.15-(3) induces a bijection between $\mathcal{P}(\mathcal{A})(I)$ and $\mathcal{P}_{\perp}(\mathcal{K}_{\mathcal{A}})(I)$, and it remains to show that this bijection is compatible with the entailment orderings.

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Let $\varphi, \psi : I \to A$ be two predicates in $\mathcal{P}(\mathcal{A})(I)$, with corresponding predicates $f \circ \varphi, f \circ \psi$ in $\mathcal{P}_\perp(\mathcal{K}_\mathcal{A})(I)$. Then we can reformulate the entailment $f \circ \varphi \vdash f \circ \psi$ in $\mathcal{P}_\perp(\mathcal{K}_\mathcal{A})(I)$ as follows:

$$\begin{aligned} f \circ \varphi \vdash f \circ \psi \quad & \Leftrightarrow \exists a \in \Phi\ \forall i \in I\ .\ a \perp \uparrow(\varphi i) \Rightarrow \uparrow(\psi i) \\ & \Leftrightarrow \exists a \in \Phi\ \forall i \in I\ \forall b \in [\uparrow(\varphi i) \Rightarrow \uparrow(\psi i)].\ a \leqslant b \\ & \Leftrightarrow \exists a \in \Phi\ \forall i \in I\ .\ a \leqslant \inf[\uparrow(\varphi i) \Rightarrow \uparrow(\psi i)] \\ & \Leftrightarrow \exists a \in \Phi\ \forall i \in I\ .\ a \leqslant \varphi(i) \to \psi(i), \end{aligned}$$

and this is equivalent to the entailment $\varphi \vdash \psi$ in $\mathcal{P}(\mathcal{A})(I)$:

$$\exists a \in \Phi\ \forall i \in I\ .\ a\varphi(i) \leqslant \psi(i)$$

by axioms (PA) and (PE) in Definition 3.8. □

6. Internal realizability in ${}^{\mathcal{K}}\mathcal{OCA}$s

We have shown that the class of ordered combinatory algebras that, besides a filter of distinguished truth values are equipped with an implication, an adjunctor and satisfy a completeness condition with respect to the infimum over arbitrary subsets – i.e.: ${}^{\mathcal{K}}\mathcal{OCA}$ s– is rich enough as to allow the tripos construction and as such its objects can be taken as the basis of the categorical perspective on classical realizability –à la Streicher–. In this section we show that we can define realizability for this type of combinatory algebras, and thus, to define realizability in higher-order arithmetic.

Definition 6.1. The language of kinds is defined by the grammar:

$$\textbf{Kinds:} \qquad \sigma, \tau ::= c \quad | \quad \sigma \to \tau,$$

where c ranges over a fixed set of constants (base kinds) that contains at least a symbol o representing the kind of propositions. Consider an infinite set of variables labeled by kinds x^τ. Suppose that we have infinitely many variables labeled of the kind τ for each kind τ. Consider also a set of constants $a^\tau, b^\sigma, \ldots$ labeled with a kind. The language $\mathcal{L}^\omega$ of order ω is defined by the following grammar:

$$M^\sigma, N^{\sigma\to\tau}, A^o, B^o ::= x^\sigma \ |\ a^\sigma \ |\ (\lambda x^\sigma.M^\tau)^{\sigma\to\tau} \ |\ (N^{\sigma\to\tau}M^\sigma)^\tau \ |\ (A^o \Rightarrow B^o)^o \ |\ (\forall x^\tau.A^o)^o$$

o represents the type of truth values. The expressions labeled by o are called "formulæ". The symbols $\to$ and $\Rightarrow$, when iterated, are associated on the right side. On the other hand, the application, when iterated, are associated on the left side.

Definition 6.2. Let A be a ${}^{\mathcal{K}}\mathcal{OCA}$ and consider a set of variables $\mathcal{V} = \{x_1, x_2, \ldots\}$. A declaration is a string of the shape $x_i : A^o$. A context is a string of the shape $x_1 : A_1^o, \ldots, x_k : A_k^o$, i.e.: contexts are finite sequences of declarations. The contexts will be often denoted by capital Greek letters: Δ, Γ, Σ. A sequent is a string of the shape $x_1 : A_1^o, \ldots, x_k : A_k^o \vdash p : B^o$ where p is a polynomial of $A[x_1, \ldots, x_k]$. The left side of a sequent is a context. When we do not make the declarations of the context of a sequent

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explicit, we will write it as $\Gamma \vdash p : B^o$. Typing rules are trees with leaves of the shape:

$$\frac{S_1 \cdot \cdots \cdot S_h}{S_{h+1}}\ \text{(Rule)}$$

where $h \geqslant 0$ and $S_1, \ldots, S_{h+1}$ are sequents.

The typing rules for $\mathcal{L}^\omega$ are the following:

$$\text{(where } x_i : A_i^o \text{ appears in } \Gamma\text{)}\ \frac{}{\Gamma \vdash x_i : A_i^o}\ \text{(ax)}$$

$$\frac{\Gamma, x : A^o \vdash p : B^o}{\Gamma \vdash \mathsf{e}(\lambda^* x\ p) : (A^o \Rightarrow B^o)^o}\ (\rightarrow_i)$$

$$\frac{\Gamma \vdash p : (A^o \Rightarrow B^o)^o \quad \Gamma \vdash q : A^o}{\Gamma \vdash pq : B^o}\ (\rightarrow_e)$$

$$\text{(where } x^\sigma \text{ does not appears free in } \Gamma\text{)}\ \frac{\Gamma \vdash p : A^o}{\Gamma \vdash p : (\forall x^\sigma A^o)^o}\ (\forall_i)$$

$$\frac{\Gamma \vdash p : (\forall x^\sigma A^o)^o}{\Gamma \vdash p : (A^o\{x^\sigma := M^\sigma\})}\ (\forall_e)$$

Definition 6.3. Let us consider $\mathcal{A} = (A, \leqslant, \text{app}, \rightarrow, \Phi, \mathsf{k}, \mathsf{s}, \mathsf{e}, \mathsf{c})$ a ${}^{\mathcal{K}}\mathcal{OCA}$. We define the interpretation of $\mathcal{L}^\omega$ as follows:

(1) For *kinds*: The interpretation of a constant c is a set $[\![c]\!]$. In particular, the constant o is interpreted as the underlying set of $\mathcal{A}$, i.e.: $[\![o]\!] = A$. Given two kinds σ, τ, the interpretation $[\![\sigma \rightarrow \tau]\!]$ is the set of functions $[\![\tau]\!]^{[\![\sigma]\!]}$.

(2) For *expressions*: In order to interpret expressions, we start choosing an assignment a for the variables x^σ such that $\mathsf{a}(x^\sigma) \in [\![\sigma]\!]$. As it is usual in semantics, the substitution-like notation $\{x^\sigma := s\}$ affecting an assignment a –or an interpretation using a–, modifies it by redefining a over x^σ as the statement $\mathsf{a}\{x^\sigma := s\}(x^\sigma) := s$. We proceed similarly for interpretations.

— For an expression of the shape x^σ, its interpretation is $[\![x^\sigma]\!] = \mathsf{a}(x^\sigma)$.

— For an expression of the shape $\lambda x^\sigma M^\tau$, its interpretation is the function $[\![\lambda x^\sigma M^\tau]\!] \in [\![\sigma \rightarrow \tau]\!]$ defined as $[\![\lambda x^\sigma M^\tau]\!](s) := [\![M^\tau]\!]\{x^\sigma := s\}$ for all $s \in [\![\sigma]\!]$.

— For an expression of the shape $(N^{\sigma\rightarrow\tau} M^\sigma)^\tau$ its interpretation is $[\![(N^{\sigma\rightarrow\tau} M^\sigma)^\tau]\!] := [\![N^{\sigma\rightarrow\tau}]\!]\big([\![M^\sigma]\!]\big)$.

— For an expression of the shape $(A^o \Rightarrow B^o)^o$ its interpretation is $[\![(A^o \Rightarrow B^o)^o]\!] := [\![A^o]\!] \rightarrow [\![B^o]\!]$.

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— For an expression of the shape $(\forall x^\sigma A^o)^o$ its interpretation is $[\![(\forall x^\sigma A^o)^o]\!] := \inf\{[\![A^o]\!]\{x^\sigma := s\} \mid s \in [\![\sigma]\!]\}$.

We say that $\mathcal{A}$ satisfies a sequent $x_1 : A_1^o, \ldots, x_k : A^k \vdash p : B^o$ if and only if for all assignment $\mathtt{a}$ and for all $b_1, \ldots, b_k \in A$, if $b_1 \leqslant [\![A_1^o]\!], \ldots, b_k \leqslant [\![A_k^o]\!]$ then $p\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [\![B^o]\!]$. In this case we write that: $\mathcal{A} \models x_1{:}A_1^o, \ldots, x_k{:}A^k \vdash p{:}B^o$.

A rule:

$$\frac{S_1 \cdot \cdots \cdot S_h}{S_{h+1}}\ (\text{Rule})$$

is said to be *adequate* if and only if for every $\mathcal{A} \in {}^{\mathcal{K}}\mathcal{OCA}$, if $\mathcal{A} \models S_1, \ldots, S_h$ then $\mathcal{A} \models S_{h+1}$.

Theorem 6.4. The rules of the typing system appearing in Definition 6.2, are adequate.

Proof. For (ax) is evident. For the implication rules:

$(\rightarrow)_i$ Assume $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$ where $\Gamma = x_1 : A_1^o, \ldots, x_k : A_k^o$. Consider an assignment $\mathtt{a}$ and $b_1, \ldots, b_k \in A$ such that $b_i \leqslant [\![A_i^o]\!]$. We get:

$$(\lambda^* xp)\{x_1 := b_1, \ldots, x_k := b_k\}[\![A^o]\!] = (\lambda^* xp\{x_1 := b_1, \ldots, x_k := b_k\})[\![A^o]\!] \leqslant p\{x_1 := b_1, \ldots, x_k := b_k, x := [\![A^o]\!]\} \leqslant [\![B^o]\!]$$

the last inequality by the assumption $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$.
Applying the adjunction property we deduce that $\mathtt{e}(\lambda^* xp)\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [\![(A^o \Rightarrow B^o)^o]\!]$. Since the above is valid for all the assignments, we conclude that $\mathcal{A} \models \Gamma \vdash \mathtt{e}(\lambda^* x\, p) : (A^o \Rightarrow B^o)^o$.

$(\rightarrow)_e$ Assume $\mathcal{A} \models \Gamma \vdash p : (A^o \Rightarrow B^o)^o$ and $\mathcal{A} \models \Gamma \vdash q : A^o$ where $\Gamma = x_1 : A_1^o, \ldots, x_k : A_k^o$. Consider an assignment $\mathtt{a}$ and $b_1, \ldots, b_k \in A$ such that $b_i \leqslant [\![A_i^o]\!]$. By hypothesis we get:

$$p\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [\![A^o]\!] \rightarrow [\![B^o]\!] \quad \text{and} \quad q\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [\![A^o]\!],$$

and by the monotony of the application in $\mathcal{A}$ we deduce that:

$$pq\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant ([\![A^o]\!] \rightarrow [\![B^o]\!])\ [\![A^o]\!] \leqslant [\![B^o]\!].$$

Since the above is valid for all the assignments, we conclude that $\mathcal{A} \models \Gamma \vdash pq : [\![B^o]\!]$.

For the quantifiers:

$(\forall)_i$ Assume $\mathcal{A} \models \Gamma \vdash p : A^o$ and that x^σ does not appear free in Γ, where $\Gamma = x_1 : A_1^o, \ldots, x_k : A_k^o$.
Consider an assignment $\mathtt{a}$ and $b_1, \ldots, b_k \in A$ such that $b_i \leqslant [\![A_i^o]\!]$.
Since $A_1^o, \ldots, A_k^o$ does not depend upon x^σ, by the assumption $\mathcal{A} \models \Gamma \vdash p : A^o$, we get:

$$p\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [\![A^o]\!]\{x^\sigma := s\} \text{ for all } s \in [\![\sigma]\!].$$

Then $p\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant \inf\{[\![A^o]\!]\{x^\sigma := s\} \mid s \in [\![\sigma]\!]\} = [\![(\forall x^\sigma A^o)^o]\!]$. We conclude as before that $\mathcal{A} \models \Gamma \vdash p : (\forall x^\sigma A^o)^o$.

$(\forall)_e$ Assume $\mathcal{A} \models \Gamma \vdash p : (\forall x^\sigma A^o)^o$, where $\Gamma = x_1 : A_1^o, \ldots, x_k : A_k^o$. Consider an assignment $\mathtt{a}$ and $b_1, \ldots, b_k \in A$ such that $b_i \leqslant [\![A_i^o]\!]$. By the assumption $\mathcal{A} \models \Gamma \vdash p$:

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$(\forall x^\sigma A^o)^o$ we deduce that:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \leqslant [\![A^o]\!]\{x^\sigma := s\} \text{ for all } s \in [\![\sigma]\!]$$

Since $[\![M^\sigma]\!] \in [\![\sigma]\!]$ we obtain:

$$p\{x_1 := b_1, \dots, x_k := b_k\} \leqslant [\![A^o]\!]\{x^\sigma := [\![M^\sigma]\!]\} = [\![A^o\{x^\sigma := M^\sigma\}]\!]$$

We conclude as before that $\mathcal{A} \models \Gamma \vdash p : A^o\{x^\sigma := M^\sigma\}$. □

The language of higher-order Peano arithmetic –$(\mathrm{PA})^\omega$–is an instance of $\mathcal{L}^\omega$ where we distinguish a constant of kind I and two constants of expression 0^I and $\mathrm{succ}^{I\to I}$.

Definition 6.5. For each kind σ we define the Leibniz equality $=_\sigma$ as follows:

$$x_1^\sigma =_\sigma x_2^\sigma :\equiv \forall y^{\sigma\to o}\big((y^{\sigma\to o}x_1^\sigma)^o \Rightarrow (y^{\sigma\to o}x_2^\sigma)^o\big)^o$$

The axioms of Peano arithmetic are equalities over the kind I, except for $\forall x^I((\mathrm{succ}^{I\to I}x^I =_I 0^I) \Rightarrow \bot)^o$ –which we abbreviate $\forall x^I(\mathrm{succ}^{I\to I}x^I \neq 0^I)^o$– and for the induction principle.

Definition 6.6. Fixed $\mathcal{A} \in {}^{\mathcal{K}}\mathcal{OCA}$, we say that $a \in A$ realizes a formula F^o if $a \leqslant [\![F^o]\!]$. We write $a \Vdash_{\mathcal{A}} F^o$ for "a realizes F^o", or simply as $a \Vdash F^o$, whenever it does not cause confusion.

The theory of $\mathcal{A}$ is the set of closed formulæ F^o such that there is an $a \in \Phi$ which realizes F^o. The theory of $\mathcal{A}$ is denoted by $\mathrm{th}(\mathcal{A})$.

In this presentation of Krivine's realizability, the orthogonality is implicit in the implication $\to$ that is part of the structure of the ${}^{\mathcal{K}}\mathcal{OCA}$.

Lemma 6.7. Let us consider an equality $M^\sigma =_\sigma N^\sigma$ such that $[\![M^\sigma]\!] = [\![N^\sigma]\!]$. Then the equality $M^\sigma =_\sigma N^\sigma$ is realized by $\mathsf{e}(\lambda^* x.x)$.

Proof. Consider an $f \in [\![\sigma \to o]\!] = A^{[\![\sigma]\!]}$, since $[\![M^\sigma]\!] = [\![N^\sigma]\!]$ we have $f([\![M^\sigma]\!]) = f([\![N^\sigma]\!])$. We conclude that $(\lambda^* x.x)[\![y^{\sigma\to o}M^\sigma]\!] \leqslant [\![y^{\sigma\to o}M^\sigma]\!] = [\![y^{\sigma\to o}N^\sigma]\!]$ and $\mathsf{e}(\lambda^* x.x) \leqslant [\![y^{\sigma\to o}M^\sigma \Rightarrow y^{\sigma\to o}N^\sigma]\!]$ for every assignment of $y^{\sigma\to o}$. Hence $\mathsf{e}(\lambda^* x.x) \Vdash M^\sigma =_\sigma N^\sigma$. □

Proposition 6.8. In every ${}^{\mathcal{K}}\mathcal{OCA}$ $\mathcal{A}$ all axioms of Peano arithmetic but the induction principle are in $\mathrm{th}(\mathcal{A})$.

Proof. By 6.7 all the axioms which are equalities are realized by $\mathsf{e}(\lambda^* x.x)$. Moreover, the axiom which say that 0 is not a successor is also realized: It is easy verify that $[\![\forall x^I[\mathrm{succ}^{I\to I}x^I =_I 0^I \Rightarrow \bot]]\!] =$

$$[\![\top \Rightarrow \bot]\!] \Rightarrow [\![\bot]\!].$$

By monotonicity $[\![\top \Rightarrow \bot]\!]\mathsf{s} \leqslant [\![\top \Rightarrow \bot]\!][\![\top]\!] \leqslant [\![\bot]\!]$. Thus $[\![\top \Rightarrow \bot]\!]\mathsf{s} \leqslant [\![\bot]\!]$ and hence $\mathsf{e}(\lambda^* x.x\mathsf{s}) \Vdash [\![\forall x^I[\mathrm{succ}^{I\to I}x^I =_I 0^I \Rightarrow \bot]]\!]$ □

Definition 6.9. The formula $\mathsf{N}(z^I)$ is defined as:

$$\forall x^{I\to o}(\forall y^I((x^{I\to o}y^I)^o \Rightarrow (x^{I\to o}(\mathrm{succ}^{I\to I}y^I))^o \Rightarrow ((x^{I\to o}0^I)^o \Rightarrow (x^{I\to o}z^I)^o)^o$$

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Remark 6.10. Since the equational axioms of Peano arithmetic and the axiom $\forall x^I[\mathrm{succ}^{I\to I}x^I =_I 0^I \Rightarrow \bot]$ are universal formulæ, therefore imply their relativization to N. The relativization of the induction principle to N is $\forall x^I(\mathsf{N}(x^I) \Rightarrow \mathsf{N}(x^I))$, which is realized by means of $\mathsf{e}(\lambda^* x.x)$. Thus, relativizing to N all proofs of higher-order arithmetic, we find realizers in Φ for their theorems by means of adequacy 6.4. In other words, th($\mathcal{A}$) contains th($(PA)^\omega$).

7. Conclusion and further work

In Section 5.5, particularly in Theorem 5.16, we completed our program to set up the foundations of realizability. This is attained by showing that every $\mathcal{AKS}$ induces a $^{\mathcal{K}}\mathcal{OCA}$ and proving that the induced tripos is isomorphic to the one constructed by Streicher. Conversely to every $^{\mathcal{K}}\mathcal{OCA}$, named as $\mathcal{A}$, we associate an $\mathcal{AKS}$, called $\mathcal{K}_{\mathcal{A}}$, such that the associated triposes are equivalent. It is interesting to notice that the terms and stacks of $\mathcal{K}_{\mathcal{A}}$ are both given by the elements of $\mathcal{A}$ and that the realizability relation is given by the partial order of $\mathcal{A}$.

A natural development of the above work is to extend the theory of $^{\mathcal{K}}\mathcal{OCA}$s in order to model quantum computing; in particular one of the authors already has some work on the categorical models of quantum lambda calculus.

Moreover, the closure relation given by double perpendicularity, that is basic in Streicher's construction, has the cost to introduce the adjunctor e. We believe that this could be modifed, without trivializing the theory, in order to avoid the need for the adjunctor.

Another interesting line of work is to extend our considerations to $^{\mathcal{K}}\mathcal{OCA}$'s with the operation defined partially.

References

Bell, J.L. (1977) *Boolean-valued models and independence proofs in set theory* Oxford Logic Guides n. 4, Clarendon Press, Oxford.

Birkhoff, G. (1955) *Lattice theory* Colloquium Publications, Vol 25, American Mathematical Society, Third edition.

Friedman, H. (1973) *Some applications of Kleene's methods for intuitionistic system*, Cambridge summer school in mathematics, 337, pp 113–170.

Gierz,G., Hofmann,K. H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D. S. (2003) *Continuous Lattices and Domains*. Cambridge University Press, 2003.

Griffin, T.G. (1990) *A Formulæ-as-Types Notion of Control*. In Conference Record of the Seventeenth Annual ACM Symposium on Principles of Programming Languages.

Hofstra, P. (2006) *All realizability is relative*, Math. Proc. Cambridge Philos. Soc. 141, no. 2, pp. 239–264.

Hofstra, P. (2008) *Iterated realizability as a comma construction*, Math. Proc. Cambridge Philos. Soc. 144, no. 1, pp. 39–51.

Hyland, J.M.E. (1982) *The effective topos*, Proc. of The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout 1981) pp. 165-216, North Holland.

Hyland, J.M.E., Johnstone, P.T. and Pitts, A.M. (1980)*Tripos theory*, Math. Proc. Cambridge Phil. Soc. 88, pp 205–232.

Johnstone, P.T. (2002) *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York.

Hofstra, P. and van Oosten, J. (2004) *Ordered partial combinatory algebras*, Math. Proc. Cambridge Philos. Soc. 134, no. 3, pp. 445–463.

Kleene, S.C. (1945) *On the interpretation of intuitionistic number theory.*, Journal of Symbolic Logic, 10:109–124.

Krivine, J.L. (1994), *A general storage theorem for integers in call-by-name lambda-calculus.*, Th. Comp. Sc., 129, p. 79–94.

Krivine, J.L. (2001), *Realizability algebras: a program to well order* $\mathbb{R}$, Logical methods in computer science, vol 7, (3: 02), pp 1–47.

Krivine, J.L.(2001) *Types lambda-calculus in classical Zermelo-Fraenkel set theory*, Arch. Math. Log. 40, no. 3, pp. 189–205.

Krivine, J.L. (2003), *Dependent choice, 'quote' and the clock*, Th. Comp. Sc. 308, pp. 259–276.

Krivine, J.L. (2008) *Structures de réalisabilité, RAM et ultrafiltre sur* $\mathbb{N}$. http : //www.pps.jussieu.fr/ krivine/Ultrafiltre.pdf.

Krivine, J.L. (2009) *Realizability in classical logic in Interactive models of computation and program behaviour*, Panoramas et synthèses 27 (2009), SMF.

Lack, S. (2010) A 2-categories companion. In *Towards higher categories*, pages 105–191. Springer.

Lawvere, F.W. (1969) Adjointness in foundations. *Dialectica*, 23(3-4):281–296.

Lawvere, F. W. (1970) Equality in hyperdoctrines and the comprehension schema as an adjoint functor. *Applications of Categorical Algebra*, 17:1–14, 1970.

McCarty, D. (1983) *Realizability and recursive mathematics*, Technical Report CMU-CS-84-131. Departament of Computer Science, Carnegie-Mellon University, 1984. Report version of the author's PhD thesis, Oxford University.

Myhill, J.(1973) *Some properties of intuitionistic Zermelo-Fraenkel set theory*, Lecture notes in mathematics 337, pp 206-231.

Sørensen, M.H. and Urzyczyn,P. *Lectures in the Curry–Howard isomorphism* Elsvier, Studies in Logic and the foundations of mathematics, vol 149, 2006.

Streicher, T. (2013)*Krivine's Classical Realizability from a Categorical Perspective*, Math. Struct. in Comp. Science. vol. 23, n 6.

A.S. Troelstra (1973) *Metamathematical investigation of intuitionistic arithmetic and analysis*, volume 344. Springer Science & Business Media.

van Oosten, J. (2008) *Realizability, an Introduction to its Categorical Side*, Elsevier.