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Singularity-locus expression of a class of parallel mechanisms

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SUMMARY

In parallel mechanisms, singular configurations (singularities) have to be avoided during motion. All the singularities should be located in order to avoid them. Hence, relationships involving all the singular platform poses (singularity locus) and the mechanism geometric parameters are useful in the design of parallel mechanisms. This paper presents a new expression of the singularity condition of the most general mechanism (6-6 FPM) of a class of parallel mechanisms usually named fully-parallel mechanisms (FPM). The presented expression uses the mixed products of vectors that are easy to be identified on the mechanism. This approach will permit some singularities to be geometrically found. A procedure, based on this new expression, is provided to transform the singularity condition into a ninth-degree polynomial equation whose unknowns are the platform pose parameters. This singularity polynomial equation is cubic in the platform position parameters and a sixth-degree one in the platform orientation parameters. Finally, how to derive the expression of the singularity condition of a specific FPM from the presented 6-6 FPM singularity condition will be shown along with an example.

KEYWORDS: Parallel manipulators; Kinematics; Mobility analysis; Singularity locus.

1. INTRODUCTION

Spatial parallel mechanisms (SPM) comprise two rigid bodies (platform and base) connected to one another by a number of kinematic chains (legs). The base is the frame and the platform is the end-effector, whereas only some leg kinematic pairs are actuated.

A large class of SPMs is the one collecting all those whose legs are either kinematic chains of the SPS type (S and P stand for spherical pair and prismatic pair respectively) with actuated prismatic pair, or kinematic chains equivalent to the SPS chain with actuated prismatic pair. The mechanisms of this class are usually named fully-parallel mechanisms[1] (FPM). The underlying architectures are called m-n FPM where m and n are numbers ranging from 1 to 6 and indicate the numbers of distinct spherical pairs in the base and in the platform respectively. The 6-6 FPM architecture is the most general one (Figure 1). All the m-n FPM can be derived from the 6-6 FPM making two or more spherical pairs coincide in the base and/or in the platform.

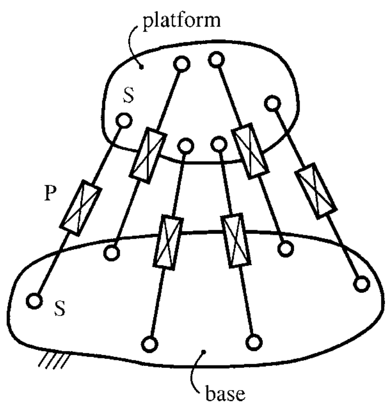

Fig. 1. The 6-6 fully-parallel mechanism.

Accordingly, the theoretical results, regarding the 6-6 FPM, are applicable to any m-n FPM, provided that the m-n FPM peculiar geometry is introduced in the general relationships obtained for the 6-6 FPM.

The direct position analysis of the 6-6 FPM has been discussed for a long time and just recently solved.[2,3]

Another theoretical problem, regarding the same class of parallel mechanisms, is the identification of all the singular configurations. Singular configurations (singularities) are those in which the relationship between the velocities of the end-effector points and the time derivatives of the active joint coordinates is not one-to-one. When a parallel mechanism reaches a singularity, the platform pose (position and orientation) cannot be controlled any longer and infinite active forces are required to balance external forces applied to the platform. Therefore, singularities have to be avoided during motion.

The first step to avoid singularities is to identify all of them. As a consequence, relationships, relating in explicit form all the singular platform poses (singularity locus) to the mechanism geometric parameters, are very useful in the design of parallel mechanisms.

Singularities have been classified[4] and physically interpreted.[5] Moreover, Merlet[6] characterized the FPM singularities by using geometric concepts (Grassmann geometry). Singularity-locus expressions (singularity equations) have been reported in the literature for some parallel mechanisms having three degrees of freedom (dof).[7–11]

Two procedures to obtain partial expressions of the 6-6 FPM singularity locus for given mechanism geometry and platform orientation were discussed by St-Onge and

Gosselin.[12] The first procedure is based on the linear decomposition of the determinant of the Jacobian matrix; the other is based on the systematic use of standard software packages implementing algebraic manipulations. The only relevant result of the approach used by St-Onge and Gosselin[12] is that the 6-6 FPM singularity equation is a cubic polynomial equation in the coordinates of one platform point, when the platform orientation and the mechanism geometry are given. No information is provided about how the 6-6 FPM singularity equation depends on the platform orientation parameters.

This paper shows that a general explicit-form singularity condition can be written for the 6-6 FPM. This condition has to be met by all 6-6 FPM platform singular poses. The same condition can be exploited in order to find geometric conditions to be satisfied by singularities, even if a systematic search of the geometric singular conditions is difficult to implement.

Moreover, using the Rodrigues parameters to parameterize the platform orientation, the singularity condition in question can be transformed into a ninth-degree polynomial equation which is cubic in the coordinates of one platform point and a sixth-degree one in the Rodrigues parameters.

Eventually, how to obtain the singularity condition of an m-n FPM from the 6-6 FPM singularity condition derived here will be shown along with an example.

2. SINGULARITY CONDITION

Figure 2 illustrates the notation that will be used. The B_i points for $i=1, \ldots, 6$ are the base spherical pair centers; the P_i points for $i=1, \ldots, 6$ are the platform spherical pair centers. The S_b and S_p coordinate frames are fixed to the base and to the platform, respectively. The O_b and P points are the origins of S_b and S_p, respectively. Moreover, the i-th leg is the one whose spherical pair centers are B_i and P_i; the i-th leg axis is the line through B_i and P_i; the i-th leg length is that of the B_iP_i segment and will be called d_i.

The 6-6 FPM closure equations can be written as

$$(\mathbf{P}_i - \mathbf{B}_i)^2 = d_i^2, \quad i=1, \ldots, 6 \tag{1}$$

where bold capital letters indicate position vectors measured in S_b.

If Equations (1) are dependent, the platform can accomplish finite displacements without changing the leg lengths

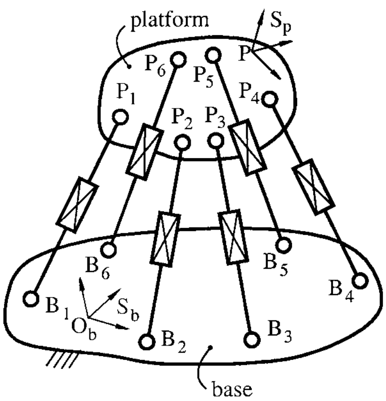

Fig. 2. Notations.

(global singularity). This condition can occur in (i) special mechanism configurations (kinematotropic mechanism) or in (ii) any mechanism configuration (architecture singularity). Henceforth, the mechanism geometry is supposed to be such that the global singularities are excluded.

Differentiation of Equations (1) gives the relationships

$$\dot{\mathbf{P}}_i \cdot (\mathbf{P}_i - \mathbf{B}_i) = d_i \dot{d}_i, \quad i=1, \ldots, 6 \tag{2}$$

where $\dot{\mathbf{P}}_i$ and $\dot{d}_i$ are the velocity of P_i and the time derivative of d_i, respectively.

The $\dot{\mathbf{P}}_i$ velocities for $i=1, \ldots, 6$ can be written as

$$\dot{\mathbf{P}}_i = \dot{\mathbf{P}} + \omega \times (\mathbf{P}_i - \mathbf{P}), \quad i=1, \ldots, 6 \tag{3}$$

where $\dot{\mathbf{P}}$ and ω are the velocity of point P and the platform angular velocity, respectively.

Taking into account relationships (3), Equations (2) become

$$(\mathbf{P}_i - \mathbf{B}_i) \cdot \dot{\mathbf{P}} + [(\mathbf{P}_i - \mathbf{P}) \times (\mathbf{P}_i - \mathbf{B}_i)] \cdot \omega = d_i \dot{d}_i, \quad i=1, \ldots, 6 \tag{4}$$

Equations (4) can be written in vector form as

$$\mathbf{J}^T \begin{Bmatrix} \dot{\mathbf{P}} \\ \omega \end{Bmatrix} = \mathbf{D}\dot{\mathbf{d}} \tag{5}$$

where

$$\mathbf{d} = [d_1, d_2, d_3, d_4, d_5, d_6]^T \tag{6.1}$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix} \tag{6.2}$$

$$\dot{\mathbf{d}} = [\dot{d}_1, \dot{d}_2, \dot{d}_3, \dot{d}_4, \dot{d}_5, \dot{d}_6]^Y \tag{6.3}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \tag{6.4}$$

with

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6] \tag{6.5}$$

$$\mathbf{u}_i = \mathbf{P}_i - \mathbf{B}_i, \quad i=1, \ldots, 6 \tag{6.6}$$

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6] \tag{6.7}$$

$$\mathbf{v}_i = (\mathbf{P}_i - \mathbf{P}) \times \mathbf{u}_i, \quad i=1, \ldots, 6 \tag{6.8}$$

(symbol $[.]^T$ denotes the transpose of [.]).

$\mathbf{J}^T$ and $\mathbf{D}$ are the 6-6 FPM Jacobian matrices. If the rank of any of $\mathbf{D}$ and $\mathbf{J}$ is smaller than six, system (5) is not a one-to-one mapping between the twist of the platform, $[\dot{\mathbf{P}}^T, \omega^T]^T$, and the joint rates, $\dot{d}_i$ for $i=1, \ldots, 6$, (local singularity).

If rank($\mathbf{D}$) is smaller than six, infinitesimal leg-length variations can occur with the platform at rest. Since $\mathbf{D}$ is a diagonal matrix, whose diagonal entries are the leg lengths, its rank will be smaller than six, if and only if one or more leg length vanishes. This condition is readily identified and, in practice, excluded by the mechanism physical limits.

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If rank(**J**) is smaller than six, infinitesimal platform displacements can occur without changing the leg lengths, i.e., with all the $\dot{d}_i$ for $i=1, \dots, 6$ equal to zero. This condition occurs when the platform assumes a pose (position and orientation) making **J** singular. Therefore, the singularity condition is

$$\det(\mathbf{J})=0 \tag{7}$$

The determinant of **J** can be written by using the third-order minors of the **U** and **V** submatrices of **J** by suitably exploiting the properties of determinants.[13] So doing, the following expression of det(**J**) results:

$$\begin{aligned}\det(\mathbf{J})=&u_{123}v_{456}-u_{124}v_{356}+u_{125}v_{346}-u_{126}v_{345}+u_{134}v_{256}\\ &-u_{135}v_{246}+u_{136}v_{245}+u_{145}v_{236}-u_{146}v_{235}\\ &+u_{156}v_{234}-u_{234}v_{156}+u_{235}v_{146}-u_{236}v_{145}\\ &-u_{245}v_{136}+u_{246}v_{135}-u_{256}v_{134}+u_{345}v_{126}\\ &-u_{346}v_{125}+u_{356}v_{124}-u_{456}v_{123}\end{aligned} \tag{8}$$

where

$$u_{ijk}=\det([\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k])=\mathbf{u}_i\cdot\mathbf{u}_j\times\mathbf{u}_k, \quad i, j, k=1, \dots, 6 \tag{9.1}$$

$$v_{ijk}=\det([\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k])=\mathbf{v}_i\cdot\mathbf{v}_j\times\mathbf{v}_k, \quad i, j, k=1, \dots, 6 \tag{9.2}$$

Since the S_p origin, P (Figure 2), can be arbitrarily chosen, it can be coincident with P_1 without losing generality. If P coincides with P_1, vector $\mathbf{v}_1$, defined by relationship (6.8), vanishes and all the v_{1jk} mixed products for $j, k=2, \dots, 6$ vanish. Therefore, expression (8) becomes

$$\begin{aligned}\det(\mathbf{J})=&u_{123}v_{456}-u_{124}v_{356}+u_{125}v_{346}-u_{126}v_{345}\\ &+u_{134}v_{256}-u_{135}v_{246}+u_{136}v_{245}+u_{145}v_{236}\\ &-u_{146}v_{235}+u_{156}v_{234}\end{aligned} \tag{10}$$

The singularity condition (7), featuring expression (8) or (10) instead of det(**J**), is the condition that vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ for $i=1, \dots, 6$ have to satisfy in any singular configuration.

2.1. Geometric interpretation

The geometric relationships among the leg axes in a singular configuration can be obtained by analyzing expression (8) or (10) of det(**J**). In this subsection, some geometric conditions, making each addend in expressions (8) and/or (10) vanish, will be presented. The zeroing of each addend is just one of the conditions making det(**J**) vanish. Therefore, the geometric conditions reported below are not exhaustive.

(a) All the leg axes are parallel to a single plane.
PROOF: If all the leg axes are parallel to a single plane, the u_{ijk} mixed products for $i, j, k=1, \dots, 6$ will be equal to zero. Therefore, each addend of expression (8) vanishes. Q.E.D.

(b) One leg length vanishes.
PROOF: Without loss of generality, the first leg will be assumed to have zero length. If d_1 is zero, $\mathbf{u}_1$ vanishes. Hence, all the u_{1jk} mixed products for $j, k=2, \dots, 6$, are zero. As a consequence, expression (10) vanishes. Q.E.D.

(c) One leg axis is perpendicular to all the straight lines normal to each couple of leg axes chosen among the remaining five leg axes.
PROOF: Without loss of generality, the reference leg will be assumed to be the first one. If the first leg axis is perpendicular to all the straight lines normal to each couple of leg axes chosen among the 2nd, 3rd, . . . , 6th leg axes, then all the u_{1jk} mixed products for $j, k=2, \dots, 6$ will be zero. As a result, expression (10) vanishes. Q.E.D.

(d) All the leg axes pass through a point, P′ (Figure 3a).
PROOF: Without loss of generality, the origin, P, of S_p can be chosen coincident with P′. If P coincides with P′ and all the leg axes pass through P′, definition (6.8) will lead to the conclusion that all the $\mathbf{v}_i$ vectors for $i=1, \dots, 6$ vanish. Therefore, all the v_{ijk} mixed products for $i, j, k=1, \dots, 6$ vanish and expression (8) also vanishes. Q.E.D.

(e) All the leg axes intersect a single straight line (Figure 3b).
PROOF: Without loss of generality, a point lying on the line intersecting all the leg axes will be chosen as the origin P of S_p. If P lies on this line, definition (6.8) will lead to the conclusion that all the $\mathbf{v}_i$ vectors for $i=1, \dots, 6$ are parallel to a plane perpendicular to this

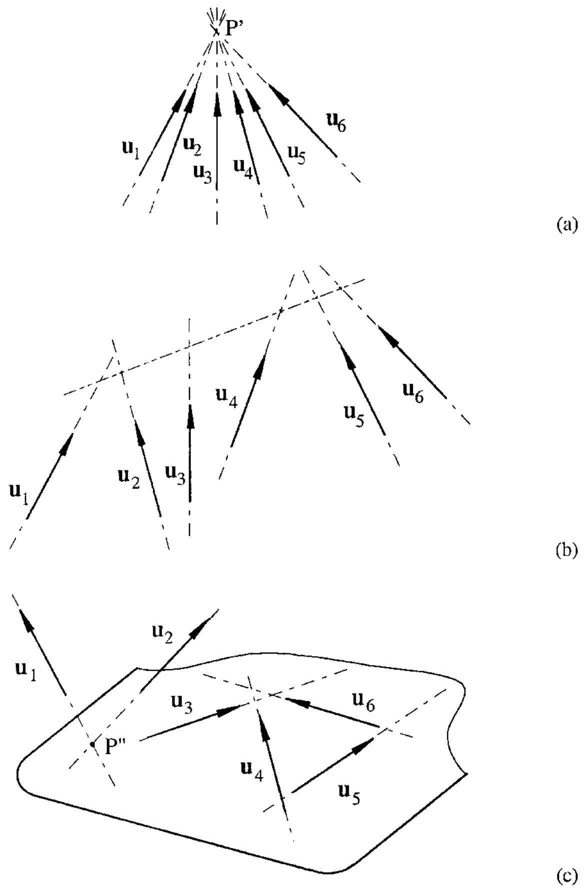

Fig. 3. Some singular configurations: (a) all the leg axes converge towards a point; (b) all the leg axes intersect a straight line; (c) four leg axes lie on a plane and the other two leg axes intersect one another in a point lying on the same plane.

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line. Therefore, all the v_{ijk} mixed products for $i, j, k=1, \ldots, 6$ are equal to zero and expression (8) vanishes, too. Q.E.D.

(f) Four leg axes lie on a plane and the other two intersect each other in a point, P″, lying on the same plane (Figure 3c).

PROOF: Without loss of generality, it will be assumed that the axes of the 3rd, 4th, 5th and 6th legs lie on a plane and the origin P of S_p coincides with P″. If $\mathbf{u}_3$, $\mathbf{u}_4$, $\mathbf{u}_5$ and $\mathbf{u}_6$ lie on a plane and P coincides with P″, definition (6.8) will lead to the conclusion that $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$ are perpendicular to the same plane and $\mathbf{v}_1$ and $\mathbf{v}_2$ are null vectors. Therefore, the v_{1jk} and v_{2jk} mixed products for $j, k=1, \ldots, 6$ are equal to zero because $\mathbf{v}_1$ and $\mathbf{v}_2$ are null vectors whilst v_{456}, v_{356}, v_{346} and v_{345} are equal to zero because $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$ are parallel vectors. As a consequence, all the addends of expression (8) vanish. Q.E.D.

3. SINGULARITY EQUATION

The expression (10) of det(**J**) is the sum of ten addends of type $u_{1ij}v_{kmn}$, where the i, j, k, m and n index values are obtained by permuting the 2-3-4-5-6 integer number sequence. Therefore, an explicit expression of det(**J**), containing all the mechanism geometric parameters and the platform pose parameters, can be obtained with the following procedure:

(i) the generic mixed product u_{1ij} is written in explicit form;

(ii) the generic mixed product v_{kmn} is written in explicit form;

(iii) the symbolic explicit expression of the generic addend $u_{1ij}v_{kmn}$ is obtained by multiplying the explicit expressions of u_{1ij} (step (i)) and v_{kmn} (step (ii));

(iv) the explicit expression of det(**J**) is obtained by calculating each expression (10) addend from the symbolic explicit expression of the generic addend (step (iii)) and by adding the computed addends.

This procedure can be implemented without systematically resorting to computer algebra, which makes it possible to find the type of explicit expression of det(**J**) after the first two steps have been implemented. In the following paragraphs, the explicit expressions of u_{1ij} and v_{kmn} will be derived, followed by the transformation of the singularity condition (7) into a ninth-degree polynomial equation, whose unknowns are the platform pose parameters.

The platform pose will be assigned by means of the P_1 coordinates, measured in S_b, and the rotation matrix $\mathbf{R}_{bp}$ transforming vector components measured in S_p into vector components measured in S_b.

3.1. *Explicit expression of u_{1ij}*

Definitions (6.6) and (9.1) yield the following expression of u_{1ij}:

$$u_{1ij}=(\mathbf{P}_1-\mathbf{B}_1)\cdot(\mathbf{P}_i-\mathbf{B}_i)\times(\mathbf{P}_j-\mathbf{B}_j) \tag{11}$$

The $\mathbf{B}_q$ position vectors, $q=1, i, j$, are constant vectors depending on the base geometry. On the contrary, the $\mathbf{P}_q$ position vectors, $q=1, i, j$, depend on the platform pose parameters and the platform geometry, according to the following relationship

$$\mathbf{P}_q=\mathbf{P}_1+\mathbf{R}_{bp}{}^{p}(\mathbf{P}_q-\mathbf{P}_1) \tag{12}$$

where the left superscript p indicates a vector expressed in S_p coordinates.

By taking into account relationship (12) and expanding the cross products, expression (11) becomes

$$\begin{aligned} u_{1ij}=(\mathbf{P}_1-\mathbf{B}_1)\cdot\{&\mathbf{R}_{bp}{}^{p}[(\mathbf{P}_i-\mathbf{P}_1)\times(\mathbf{P}_j-\mathbf{P}_1)] \\ &+\mathbf{B}_i\times\mathbf{B}_j-\mathbf{P}_1\times(\mathbf{B}_j-\mathbf{B}_i)+\mathbf{P}_1\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_j-\mathbf{P}_i)] \\ &+\mathbf{B}_j\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_i-\mathbf{P}_1)]-\mathbf{B}_i\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_j-\mathbf{P}_1)]\} \end{aligned} \tag{13}$$

Moreover, by expanding the dot product in expression (13) and simplifying the mixed products having two parallel vectors, expression (13) becomes

$$\begin{aligned} u_{1ij}=\mathbf{P}_1\cdot\{&\mathbf{R}_{bp}{}^{p}[(\mathbf{P}_i-\mathbf{P}_1)\times(\mathbf{P}_j-\mathbf{P}_1)]+\mathbf{B}_i\times\mathbf{B}_j \\ &+\mathbf{B}_j\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_i-\mathbf{P}_1)]-\mathbf{B}_i\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_j-\mathbf{P}_1)]\} \\ &-\mathbf{B}_1\cdot\{\mathbf{R}_{bp}{}^{p}[(\mathbf{P}_i-\mathbf{P}_1)\times(\mathbf{P}_j-\mathbf{P}_1)]+\mathbf{B}_i\times\mathbf{B}_j \\ &-\mathbf{P}_1\times(\mathbf{B}_j-\mathbf{B}_i)+\mathbf{P}_1\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_j-\mathbf{P}_i)]+\mathbf{B}_j \\ &\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_i-\mathbf{P}_1)]-\mathbf{B}_i\times[\mathbf{R}_{bp}{}^{p}(\mathbf{P}_j-\mathbf{P}_1)]\} \end{aligned} \tag{14}$$

The ${}^{p}(\mathbf{P}_i-\mathbf{P}_1)$, ${}^{p}(\mathbf{P}_j-\mathbf{P}_1)$ and ${}^{p}(\mathbf{P}_j-\mathbf{P}_i)$ vectors are constant vectors depending on the platform geometry.

Expression (14) is the u_{1ij} sought-after explicit form depending on the mechanism geometric parameters and the platform pose parameters. Expression (14) is linear in the P_1 coordinates and in the $\mathbf{R}_{bp}$ entries; moreover, it includes the products of the P_1 coordinates and the $\mathbf{R}_{bp}$ entries.

Eventually, it is noteworthy that expression (14) would be greatly simplified if the origin O_b of S_b (Fig. 2) coincided with B_1.

3.2. *Explicit expression of v_{kmn}*

Definitions (6.6), (6.8) ($P\equiv P_1$) and (9.2) yield the following expression for v_{kmn}:

$$\begin{aligned} v_{kmn}=[(\mathbf{P}_k-\mathbf{P}_1)\times(\mathbf{P}_k-\mathbf{B}_k)]\cdot[(\mathbf{P}_m-\mathbf{P}_1)\times(\mathbf{P}_m-\mathbf{B}_m)] \\ \times[(\mathbf{P}_n-\mathbf{P}_1)\times(\mathbf{P}_n-\mathbf{B}_n)] \end{aligned} \tag{15}$$

By using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, where **a**, **b** and **c** are any vectors, and the mixed product properties, the following identity can be obtained

$$\begin{aligned} &[(\mathbf{P}_m-\mathbf{P}_1)\times(\mathbf{P}_m-\mathbf{B}_m)]\times[(\mathbf{P}_n-\mathbf{P}_1)\times(\mathbf{P}_n-\mathbf{B}_n)] \\ &\quad=[(\mathbf{P}_m-\mathbf{B}_m)\times(\mathbf{P}_n-\mathbf{B}_n)\cdot(\mathbf{P}_m-\mathbf{P}_1)](\mathbf{P}_n-\mathbf{P}_1) \\ &\qquad+[(\mathbf{P}_m-\mathbf{P}_1)\times(\mathbf{P}_n-\mathbf{P}_1)\cdot(\mathbf{P}_m-\mathbf{B}_m)](\mathbf{P}_n-\mathbf{B}_n) \end{aligned} \tag{16}$$

By introducing expression (16) into (15) and expanding (15), expression (15) becomes

$$v_{kmn}=-s_{mn}t_{kn}+t_{mn}s_{kn} \tag{17}$$

where

$$s_{in}=(\mathbf{P}_i-\mathbf{B}_i)\times(\mathbf{P}_n-\mathbf{B}_n)\cdot(\mathbf{P}_i-\mathbf{P}_1),\quad i=m, k \tag{18.1}$$

$$t_{in}=(\mathbf{P}_i-\mathbf{P}_1)\times(\mathbf{P}_n-\mathbf{P}_1)\cdot(\mathbf{P}_i-\mathbf{B}_i),\quad i=m, k \tag{18.2}$$

By substituting expressions (12) with q=i and q=n, respectively, for $\mathbf{P}_i$ and $\mathbf{P}_n$ in definitions (18), expanding and

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simplifying those definitions, the following explicit expressions of s_{in} and t_{in} are obtained:

$$s_{in} = -t_{in} + (\mathbf{B}_i \times \mathbf{B}_n) \cdot [\mathbf{R}_{bp}{}^{p}(\mathbf{P}_i - \mathbf{P}_1)] + \mathbf{P}_1 \times (\mathbf{B}_i - \mathbf{B}_n) \cdot [\mathbf{R}_{bp}{}^{p}(\mathbf{P}_i - \mathbf{P}_1)] \quad (19.1)$$

$$t_{in} = \mathbf{P}_1 \cdot \{\mathbf{R}_{bp}{}^{p}[(\mathbf{P}_i - \mathbf{P}_1) \times (\mathbf{P}_n - \mathbf{P}_1)]\} - \mathbf{Bi} \cdot \{\mathbf{R}_{bp}{}^{p}[(\mathbf{P}_i - \mathbf{P}_1) \times (\mathbf{P}_n - \mathbf{P}_1)]\} \quad (19.2)$$

Expressions (19) are linear in the coordinates of P_1 and in the entries of $\mathbf{R}_{bp}$. Moreover, they contain products of the P_1 coordinates and the $\mathbf{R}_{bp}$ entries. Therefore, by substituting expressions (19) for s_{in} and t_{in}, $i = m, k$, into expression (17), the resultant expression of v_{kmn} is quadratic in the P_1 coordinates and the $\mathbf{R}_{bp}$ entries; besides, it contains products of quadratic terms of the P_1 coordinates and the quadratic terms of the $\mathbf{R}_{bp}$ entries.

*3.3. Explicit expressions of det(**J**) and the singularity condition*

The product of u_{1ij}, (Eq. (14)), and v_{kmn}, (Eq. (17)), gives the explicit expression of the generic addend $u_{1ij}v_{kmn}$ of det(**J**). Since u_{1ij} and v_{kmn} are linear and quadratic, respectively, in the P_1 coordinates and the $\mathbf{R}_{bp}$ entries, the product $u_{1ij}v_{kmn}$ is cubic in the P_1 coordinates and the $\mathbf{R}_{bp}$ entries; moreover, it contains the products of the monomials cubic in the P_1 coordinates and the monomials cubic in the $\mathbf{R}_{bp}$ entries. Therefore, the det(**J**) is at most a sixth-degree polynomial, which is cubic in the P_1 coordinates and the $\mathbf{R}_{bp}$ entries.

If the platform orientation is parameterized by using the Rodrigues parameters[14] the $\mathbf{R}_{bp}$ rotation matrix will have the following expression

$$\mathbf{R}_{bp} = \frac{1}{1+\mathbf{x}^2}[2(\mathbf{x}\mathbf{x}^T + \mathbf{x}^{sk}) + (1 - \mathbf{x}^2)\mathbf{I}_{3\times 3}] \quad (20)$$

where $\mathbf{x}$ is the vector $(x_1, x_2, x_3)^T$ of the three Rodrigues parameters x_i for $i = 1, 2, 3$; $\mathbf{I}_{3\times 3}$ is the 3×3 identity matrix and $\mathbf{x}^{sk}$ denotes the skew-symmetric cross-product matrix of vector $\mathbf{x}$. When the Rodrigues parameters become infinite, expression (20) of $\mathbf{R}_{bp}$ fails, even if $\mathbf{R}_{bp}$ is defined (representation singularity). Therefore, the only valid values of the Rodrigues parameters are the finite ones.

A rational expression of det(**J**) in the six platform pose parameters is obtained by substituting expression (20) for $\mathbf{R}_{bp}$ in the expression of det(**J**). The numerator of this rational expression is a polynomial, which is cubic in the P_1 coordinates and a sixth-degree one in the Rodrigues parameters, whereas its denominator is $(1+\mathbf{x}^2)^3$. The denominator does not make det(**J**) vanish for finite values of the Rodrigues parameters. Hence, the only platform pose parameters, making det(**J**) zero are the numerator roots.

If the rational expression of det(**J**) is introduced into the singularity condition (7) and the factor $(1+\mathbf{x}^2)^{-3}$ is cleared, a ninth-degree polynomial equation in the platform pose parameters will result. This polynomial equation is the singularity equation of the 6-6 FPM in explicit form.

4. SPECIAL CASES OF FPMs

All the m-n FPM can be derived from the 6-6 FPM, making two or more spherical pairs coincide at the base and/or at the platform[1]. The Jacobian matrices of these special cases can be derived from the **J** and **D** matrices of the 6-6 FPM by introducing their particular geometry into expressions (6.2) and (6.4).

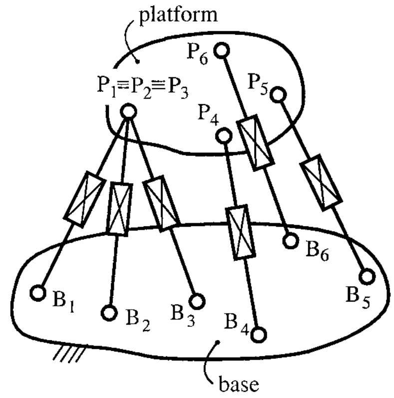

Fig. 4. A 6–4 fully parallel mechanism.

The singularity condition (7) and the det(**J**) expressions (8) and (10) still hold for the m-n FPMs, provided that the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ for $i = 1, \ldots, 6$ are computed by taking into account their particular geometry. In the following paragraphs, an example of how to derive the singularity equation of an m-n FPM from the singularity equation of the 6-6 FPM will be illustrated.

Figure 4 shows a 6-4 FPM. The mechanism of Fig. 4 is obtained from the 6-6 FPM by making the P_1, P_2 and P_3 platform points coincide. If the origin P of S_p is chosen coincident with P_1, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, defined by relationship (6.8), vanish. Hence, all the v_{kmn} mixed products containing at least one of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ vanish. As a consequence, expression (10) of det(**J**) becomes

$$\det(\mathbf{J}) = u_{123}v_{456} \quad (21)$$

and the singularity condition of the 6-4 FPM shown in Figure 4 is

$$u_{123}v_{456} = 0 \quad (22)$$

Singularity condition (22) was reported by Wohlhart[15] by directly addressing the mobility analysis of the 6-4 FPM shown in Figure 4.

5. CONCLUSIONS

A new expression of the singularity condition of the most general mechanism (6-6 FPM) of a class of parallel mechanisms usually named fully-parallel mechanisms (FPM) has been derived. The presented expression uses the mixed products of vectors that are easy to be identified on the mechanism.

This approach has allowed some singularities to be geometrically found.

The singularity condition derived here lead to a ninth-degree polynomial equation in the platform pose parameters by using the Rodrigues parameters to parameterize the platform orientation. This singularity polynomial equation

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is cubic in the platform position parameters and a sixth-degree one in the Rodrigues parameters.

The singularity condition of a specific m-n FPM can be easily derived from the 6-6 FPM singularity condition reported here. An example has been included.

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References

1. C. Innocenti and V. Parenti-Castelli, "Exhaustive enumeration of fully-parallel kinematic chains", *Proc. of the 1994 ASME Int. Winter Annual Meeting*, Chicago, USA (1994) **DSC-Vol. 55-2**, pp. 1135–1141.
2. M.L. Husty, "An algorithm for solving the direct kinematics of the Stewart-Gough platform", *Mechanism and Machine Theory* **31**(4), 365–380 (1996).
3. C. Innocenti, "Forward kinematics in polynomial form of the general Stewart platform", *Proc. of the 1998 ASME Design Engineering Technical Conference*, Atlanta (GA), USA (1998) Paper No. DETC98/MECH-5894.
4. C. Gosselin and J. Angeles, "Singularity analysis of closed-loop kinematic chains", *IEEE Transactions on Robotics and Automation* **6**(3), 281–290 (1990).
5. D. Zlatanov, R.G. Fenton and B. Benhabib, "Singularity analysis of mechanisms and robots via a motion-space model of the instantaneous kinematics", *Proc. of the 1994 IEEE Int. Conf. on Robotics and Automation*, San Diego (CA), USA (1994) **Vol. 2**, pp. 980–991.
6. J.-P. Merlet, "Singular configurations of parallel manipulators and Grassmann geometry", *Int. J. Robotics Research* **8**(5), 45–56 (1989).
7. C. Gosselin and J. Wang, "Singularity loci of planar parallel manipulators", *Proc. of the IXth World Congress on the Theory of Machines and Mechanisms*, Milano, Italy (1995) **Vol. 3**, pp. 1982–1986.
8. J. Sefrioui and C. Gosselin, "Étude et représentation des lieux de singularité des manipulateurs parallèles sphériques à trois degrés de liberté avec actioneurs prismatiques", *Mechanism and Machine Theory* **29**(4), 559–579 (1994).
9. J. Sefrioui and C. Gosselin, "On the quadratic nature of the singularity curves of planar three-degree-of-freedom parallel manipulators", *Mechanism and Machine Theory* **30**(4), 533–551 (1995).
10. R. Di Gregorio and V. Parenti-Castelli, "Influence of the geometric parameters of the 3-UPU parallel manipulator on the singularity loci", *Proc. of the 1999 Int. Workshop on Parallel Kinematic Machines*, Milano, Italy (1999) pp. 79–86.
11. R. Di Gregorio, "Statics and singularity loci of the 3-UPU wrist", *Proc. of the 2001 IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, Como, Italy (2001) pp. 470–475.
12. B.M. St-Onge and C. Gosselin, "Singularity analysis and representation of the general Gough-Stewart platform", *Int. J. Robotics Research* **19**(3), 271–288 (2000).
13. M. Villa, *Elementi di Algebra* (Patron Publisher, Italy 1965) p. 32.
14. R.E. Roberson and R. Schwertassek, *Dynamics of Multibody Systems* (Springer-Verlag Publishers, 1988) p. 77.
15. K. Wohlhart, "Displacement analysis of the general spherical Stewart platform", *Mechanism and Machine Theory* **29**(4), 581–589 (1994).