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Allen–Cahn equation with strong irreversibility†

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This paper is concerned with a fully non-linear variant of the Allen–Cahn equation with strong irreversibility, where each solution is constrained to be non-decreasing in time. The main purposes of this paper are to prove the well-posedness, smoothing effect and comparison principle, to provide an equivalent reformulation of the equation as a parabolic obstacle problem and to reveal long-time behaviours of solutions. More precisely, by deriving *partial* energy-dissipation estimates, a global attractor is constructed in a metric setting, and it is also proved that each solution $u(x,t)$ converges to a solution of an elliptic obstacle problem as $t \to +\infty$.

Key words: Strongly irreversible evolution equation, Allen–Cahn equation, obstacle parabolic problem, global attractor, ω-limit set, partial energy-dissipation

1 Introduction

Evolution equations along with *strong irreversibility* often appear in Damage Mechanics to describe a uni-directional evolution of damaging phenomena. For instance, damage accumulation and crack propagation exhibit strong irreversibility, since the degree of damage never decreases spontaneously. Therefore, to describe such phenomena as a phase field model, one may need to take into account the strong irreversibility (or *uni-directionality*) of the evolution. On the other hand, (spatial) *propagation* of damage is described in terms of diffusion (type) processes. However, these two effects, namely the uni-directionality of the evolution and the diffusive nature, often conflict each other. Such a conflict of two different effects may produce significant features of damaging phenomena. A few ways have been proposed to describe damaging phenomena in view of such two effects; above all, one often employs parabolic partial differential equations (PDEs) involving the *positive-part function* $(\,\cdot\,)_+ := \max\{\,\cdot\,,0\} \geqslant 0$ (or a *negative-part*

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one). The simplest example reads,

$$u_t = (\Delta u)_+ \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ and which is a classical problem (see, e.g., [39]) and also still revisited by many authors (see, e.g., [35, 36] and also [3, 45]). Furthermore, physical backgrounds of such irreversible models will be briefly reviewed in Section 2. From the viewpoints of mathematical analysis, equations such as (1.1) are classified as *fully non-linear* PDEs, and hence, the lack of gradient structure gives rise to difficulties and particularly prevents us to reveal *dissipation structure* driven by the diffusion term. Indeed, dissipative structures of parabolic equations are partially destroyed by applying the positive-part function. On the other hand, dissipative behaviours may also occur like a classical diffusion equation, unless they violate the strong irreversibility (see also Remark 4.2).

In this paper, we are concerned with the Allen–Cahn equation with strong irreversibility,

$$u_t = \Big(\Delta u - W'(u)\Big)_+ \quad \text{in } \ \Omega \times (0,\infty), \tag{1.2}$$

where $W'(u) = u^3 - \kappa u$ (with $\kappa > 0$) is the derivative of a double-well potential $W(u)$ simply given by

$$W(u) := \frac{1}{4}u^4 - \frac{\kappa}{2}u^2. \tag{1.3}$$

For simplicity, we shall use the form (1.3); however, most of the arguments throughout the present paper can be extended to more general (but still regular) double-well potential functions W defined on $\mathbb{R}$ (on the other hand, the dimensional restriction in (v) of Theorem 3.2 relies on the cubic growth of $W'(u)$. Moreover, it is more delicate to treat *singular* potentials, for example, logarithmic potential). Equation (1.2) is a strongly irreversible version of the celebrated Allen–Cahn equation,

$$u_t = \Delta u - W'(u) \quad \text{in } \ \Omega \times (0,\infty). \tag{1.4}$$

It is often employed to model phase-separation phenomena. Moreover, (1.2) also appears in a special setting of a phase field model describing crack propagation (see Remark 2.1).

As is already pointed out, (1.2) is classified as a fully non-linear parabolic equation, which is formulated in a general form $u_t = F(D^2u)$ with a non-linear function F and the Hessian matrix D^2u. Here, we shall reformulate the equation as a generalized gradient flow (of sub-differential type), which is fitter to distributional frameworks and energy techniques. By applying the (multivalued) inverse mapping $\alpha(\cdot)$ of $(\cdot)_+$ to both sides, (1.2) is reduced to

$$\alpha(u_t) \ni \Delta u - W'(u) \quad \text{in } \ \Omega \times (0,\infty).$$

The inverse mapping α of $(\cdot)_+$ can be decomposed as follows:

$$\alpha(s) = s + \partial I_{[0,\infty)}(s), \quad \partial I_{[0,\infty)}(s) = \begin{cases} 0 & \text{if } \ s > 0 \\ (-\infty, 0] & \text{if } \ s = 0 \\ \emptyset & \text{if } \ s < 0 \end{cases} \quad \text{for } \ s \in \mathbb{R}, \tag{1.5}$$

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where $\partial I_{[0,\infty)}$ stands for the sub-differential of the indicator function $I_{[0,\infty)}$ over the half-line $[0,+\infty)$. In the present paper, we shall particularly consider the Cauchy–Dirichlet problem for (1.2), which is hereafter denoted by (P) and equivalently given as

$$u_t + \eta - \Delta u + W'(u) = 0, \quad \eta \in \partial I_{[0,\infty)}(u_t) \quad \text{in } \Omega \times (0,\infty), \tag{1.6}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.7}$$

$$u = u_0 \quad \text{in } \Omega. \tag{1.8}$$

Furthermore, comparing (1.6) with (1.2), one can immediately find the relation,

$$\eta = -\left(\Delta u - W'(u)\right)_-, \tag{1.9}$$

where $(\,\cdot\,)_-$ stands for the negative part function, that is, $(s)_- := \max\{-s, 0\} \geqslant 0$. To be precise, such a doubly-non-linear reformulation including the relation (1.9) is justified in a strong formulation, for example, under the frame over $L^2(\Omega)$, where equations hold in a pointwise sense; on the other hand, in a weaker formulation, such as H^{-1}-framework, it is more delicate to verify the equivalence of two equations as well as (1.9).

Behaviours and properties of solutions to (P) can be imagined from the form of equations (1.2) and (1.6). For instance, each solution $u(x,t)$ of (P) behaves like that of the classical Allen–Cahn equation (1.4) at (x,t) where $\Delta u - W'(u)$ is positive. Otherwise, $u(x,t)$ never evolves. Therefore, one may expect that smoothing effect and energy-dissipation *partially* occur, but not everywhere. On the other hand, it is not easy to give a proof for such conjectures. Indeed, even existence and uniqueness of solutions have not yet been fully studied due to the severe non-linearity of (1.2) and (1.6). Moreover, to the best of authors' knowledge, such partial effects of smoothing and energy-dissipation have never been studied so far. Different from classical Allen–Cahn equations, such as (1.4), due to the defect of the (full) energy-dissipation structure, (P) has no absorbing set, and hence, no global attractor in any L^p-spaces. Indeed, from the non-decrease of $u(x,t)$ in time, that is, $u(x,t) \geqslant u(x,s)$ a.e. in Ω if $t \geqslant s$, one cannot expect any dissipation estimates for the L^p-norm $\|u(\cdot,t)\|_{L^p(\Omega)}$, provided that $u_0 \geqslant 0$. On the other hand, due to the presence of a gradient structure lying inside of $(\,\cdot\,)_+$ in (1.2), (P) shares a common Lyapunov energy with (1.4),

$$E(w) := \frac{1}{2}\int_\Omega |\nabla w(x)|^2 \,\mathrm{d}x + \int_\Omega W(w(x))\,\mathrm{d}x,$$

which decreases along the evolution of solutions $u = u(x,t)$ to (P) as well as of those to (1.4). So one may expect that a partial energy dissipation occurs (more precisely, a (quantitative) dissipative estimate for $E(u(t))$ holds in a proper sense) and it enables us to construct an absorbing set and a global attractor for (P) under a non-standard setting. However, it is unclear in which setting one can find out a partial energy-dissipation structure of (P) and establish quantitative dissipative estimates enough for a construction of a global attractor.

As we shall see in Section 6, the Cauchy–Dirichlet problem (P) (equivalently, (1.2), (1.7), (1.8)) can be rewritten as an obstacle problem of parabolic type,

$$\begin{aligned} u \geqslant u_0, \quad u_t - \Delta u + u^3 - \kappa u \geqslant 0 \quad &\text{in } \Omega \times (0,\infty),\\ (u-u_0)\left(u_t - \Delta u + u^3 - \kappa u\right) = 0 \quad &\text{in } \Omega \times (0,\infty),\\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0&, \end{aligned}$$

whose obstacle function coincides with the initial datum. Such parabolic obstacle problems whose obstacle functions coincide with initial data are also studied in the context of (American) option evaluation (see [23,43] and references therein). This reformulation will play a key role to discuss long-time behaviours of solutions as well as to investigate qualitative properties, for example, comparison principle and uniqueness (or selection principle), of solutions to (P) under milder assumptions.

The strongly irreversible evolution also exhibits a stronger dependence on initial state, compared to the classical Allen–Cahn equation. For example, solutions of (1.1) are constrained to be not less than initial data. Such a stronger initial-state-dependence of evolution can be found out more explicitly in the parabolic obstacle problem. Indeed, the evolution law (= the obstacle problem) explicitly depends on initial data. Furthermore, related issues (e.g., convergence to equilibria and Lyapunov stability of equilibria) of the dynamical system (DS) generated by (1.2) must be also affected by such a strong dependence of the DS on initial states. Therefore, it would be interesting to reveal the whole picture of such peculiar dynamics.

Main purposes of the present paper are to prove the well-posedness of (P) in an L^2-framework and to investigate qualitative and quantitative properties (e.g., comparison principle, smoothing effect, energy-dissipation estimates) and long-time behaviours of solutions. In particular, we shall focus on how to extract an energy-dissipation structure of (1.6) beyond the obstacle arising from the strong irreversibility, and moreover, we shall discuss in which setting (e.g., phase space) one can construct a global attractor for the DS generated by (P).

In Section 2, we briefly review several previous studies on strongly irreversible evolution equations (such as (1.2) and (1.6)) arising from Damage Mechanics and so on. Section 3 is devoted to discussing the well-posedness and a smoothing effect for (P) and providing a proof for the uniqueness and continuous dependence of solutions on initial data. In Section 4, we arrange energy inequalities which will be used to prove a smoothing effect for (P) as well as to reveal long-time behaviours of solutions. In this section, one may also find out energy-dissipation structures concealed in the equation. Finally, we also give a sketch of proof for the smoothing effect, that is, the existence of solutions to (P) for a wider class of initial data. A detailed proof will be given in Section 5 (and some part of it will be completed at the end of Section 6). In Section 6, we equivalently reformulate (P) as a parabolic variational inequality of obstacle type. This fact also indicates the lack of classical regularity of solutions to (P); indeed, it is well known that solutions to (elliptic) obstacle problems are at most of class $C^{1,1}$ (see, e.g., [22]). The argument for justifying the reformulation is somewhat delicate and deeply related to the construction of solutions to (P). Moreover, in Section 7, we shall discuss a comparison principle for the equation resulting from the reformulation. Furthermore, we shall obtain a uniform estimate for solutions to (P), and in particular, it will be verified that solutions of (P) enjoy a *range-preserving property*, that is, if u_0 takes a value within a certain range, then so does $u(\cdot, t)$ for any $t > 0$. This is a fundamental requirement for phase-field models. Here, the comparison principle is not directly proved for (P), since there arise some difficulties from the double non-linearity in the L^2-framework (on the other hand, it can be directly proved for (P) under some additional assumptions). Sections 8 and 9 are devoted to constructing a global attractor for a DS generated by (P) in a proper sense. As mentioned previously, no global attractor exists in any L^p-spaces, because of the strong irreversibility. Therefore, it is most crucial how to set up a phase set, which will be given by a metric space without linear and convex structures. In Section 10, we shall prove the convergence of each solution $u(x, t)$ for (P) as $t \to \infty$ and characterize the limit as a solution of an elliptic variational inequality of obstacle type. Also here, the reformulation exhibited in Section 6 will play a crucial role to characterize equilibria.

Notation We denote by $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$ the $L^p(\Omega)$-norm, that is, $\|f\|_p := (\int_\Omega |f(x)|^p \, dx)^{1/p}$ for $p \in [1, \infty)$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x \in \Omega} |f(x)|$. Denote also by $(\cdot, \cdot)$ the L^2-inner product, that is, $(u, v) := \int_\Omega u(x)v(x) \, dx$ for $u, v \in L^2(\Omega)$. For each normed space X and $T > 0$, $C_w([0, T]; X)$ denotes the space of weakly continuous functions on $[0, T]$ with values in X. We also simply write $u(t)$ instead of $u(\cdot, t)$, which is regarded as a function from Ω to $\mathbb{R}$, for each fixed $t \geqslant 0$. Here and henceforth, we use the same notation $I_{[0,\infty)}$ for the indicator function over the half-line $[0, \infty)$ as well as for that defined on $L^2(\Omega)$ over the closed convex set $K := \{u \in L^2(\Omega) : u \geqslant 0 \text{ a.e. in } \Omega\}$, namely,

$$I_{[0,\infty)}(u) = \begin{cases} 0 & \text{if } u \in K, \\ \infty & \text{otherwise} \end{cases} \quad \text{for } u \in L^2(\Omega),$$

if no confusion may arise. Moreover, let $\partial I_{[0,\infty)}$ also denote the sub-differential operator (precisely, $\partial_{\mathbb{R}} I_{[0,\infty)}$) in $\mathbb{R}$ (see (1.5)) as well as that (precisely, $\partial_{L^2} I_{[0,\infty)}$) in $L^2(\Omega)$ defined by

$$\partial_{L^2} I_{[0,\infty)}(u) = \left\{ \eta \in L^2(\Omega) : (\eta, u - v) \geqslant 0 \text{ for all } v \in K \right\} \quad \text{for } u \in K.$$

Here, we note that these two notions of sub-differentials are equivalent each other in the following sense: for $u, \eta \in L^2(\Omega)$,

$$\eta \in \partial_{L^2} I_{[0,\infty)}(u) \quad \text{if and only if} \quad \eta(x) \in \partial_{\mathbb{R}} I_{[0,\infty)}(u(x)) \text{ a.e. in } \Omega$$

(see, e.g., [20, 21]). We denote by C a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line.

2 Evolution equations with strong irreversibility

Evolution equations including the positive-part function, such as (1.1) and (1.2), have been studied in several papers and they play important roles particularly in Damage Mechanics. In this section, we briefly review some of those models and related non-linear PDEs including the positive-part function.

2.1 Quasi-static brittle fracture models

Francfort and Marigo [33] proposed a quasi-static evolution of brittle fractures in elastic bodies based on Griffith's criterion (see also [26] and [31]). Let $\Omega \subset \mathbb{R}^3$ be an elastic body and let $\Gamma_n \subset \overline{\Omega}$ be a crack at time t_n. Then, the crack $\Gamma_{n+1} \subset \overline{\Omega}$ and the displacement $\vec{u}_{n+1} : \Omega \setminus \Gamma_{n+1} \to \mathbb{R}^3$ at time t_{n+1} are obtained as a minimizer of the elastic energy,

$$\mathcal{F}(\vec{u}, \Gamma) = \underbrace{\int_{\Omega \setminus \Gamma} \mu |\varepsilon(\vec{u})|^2 + \lambda |\mathrm{tr}\, \varepsilon(\vec{u})|^2 \, \mathrm{d}x}_{\text{bulk energy}} + \underbrace{\mathcal{H}^2(\Gamma)}_{\text{surface energy}}$$

among $\Gamma \subset \overline{\Omega}$ including Γ_n and $\vec{u} : \Omega \setminus \Gamma \to \mathbb{R}^3$ satisfying a boundary condition $\vec{u}|_{\partial\Omega} = \vec{g}$ associated with the external load $\vec{g}$ on (some part of) the boundary. Here, $\varepsilon(\vec{u})$ is the symmetric part of the gradient matrix of $\vec{u}$, λ and μ are positive constants, and $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure. Furthermore, concerning the mode III (i.e., anti-planar shear) crack growth, the displacement vector $\vec{u} = \vec{u}(x)$ is reduced to a scalar-valued function $u = u(x)$ of class $SBV(\Omega)$ (see [32]). In order to perform numerical analysis of the mode III crack propagation, $\mathcal{F}$ is often regularized as the Ambrosio–Tortorelli energy (see [4, 5]),

$$\mathcal{F}_\varepsilon(u, z) = \frac{\mu}{2} \int_\Omega (1-z)^2 |\nabla u|^2 \, \mathrm{d}x + \int_\Omega f u \, \mathrm{d}x + \int_\Omega \gamma(x) \left(\frac{|\nabla z|^2}{2\varepsilon} + \varepsilon V(z) \right) \mathrm{d}x,$$

where u and z stand for the deformation of the material and a phase parameter describing the degree of crack (e.g., $z = 1$ means 'completely cracked' configuration), respectively, $V(\cdot)$ is a potential function, $\varepsilon > 0$ is a relaxation parameter (which is also related to the thickness of the diffuse interface), μ is a positive constant and $\gamma(x)$ denotes the *fracture toughness* of the material. It is proved in [4, 5] that $\mathcal{F}_\varepsilon$ converges to the Francfort–Marigo energy in the sense of Γ-convergence as $\varepsilon \to 0$. Quasi-static dynamics of the approximated brittle fracture model is also studied by introducing a constrained minimization scheme associated with $\mathcal{F}_\varepsilon$ (see [34]). Here, we stress again that the evolution of the phase parameter $z(x, t)$ is supposed to be monotone (i.e., non-decreasing in time).

A couple of non-linear evolution equations have been also proposed to describe (or approximate) quasi-static evolution of brittle fractures. Above all, Kimura and Takaishi [38, 48] developed a crack propagation model for numerical simulation. Their model is derived as a *double* gradient flow (i.e., in both variables (u, z)) for $\mathcal{F}_\varepsilon(u, z)$:

$$\alpha_1 u_t = \mu \mathrm{div}\left((1-z)^2 \nabla u\right) + f(x, t) \qquad \text{in } \Omega \times (0, \infty),$$

$$\alpha_2 z_t = \left(\varepsilon \mathrm{div}(\gamma(x) \nabla z) - \frac{\gamma(x)}{\varepsilon} V'(z) + \mu |\nabla u|^2 (1-z) \right)_+ \qquad \text{in } \Omega \times (0, \infty),$$

where α_1, α_2 are positive constants (related to numerical efficiency), together with boundary and initial conditions (see also [10], where mixed boundary conditions are imposed on u and z). Here, we remark that the second equation of the system includes the positive-part function in the right-hand side, in order to reproduce the non-decreasing (in time) evolution of the phase parameter $z(x, t)$.

Remark 2.1 *Equation* (1.2) *can be derived as an extreme case of the quasi-static model for the regularized energy. More precisely, let* $V(\cdot)$ *be a double-well potential,* $V(s) = s^4/4 - \kappa s^2/2$, *to confine the phase parameter into an interval (see [49]). Moreover, for (mathematical) simplicity, set* $\gamma(x) \equiv 1$, $f \equiv 0$, $\varepsilon = 0$ *and take* $\alpha_1 = 0$ *and* $\alpha_2 = 1$ *to the double-gradient flow model. Testing the first equation by u and integrating by parts, we see that*

$$(1-z)^2|\nabla u|^2 = 0 \quad \textit{for a.e. } (x,t) \in \Omega \times (0,\infty),$$

which means that either $(1-z)$ *or* $|\nabla u|$ *is zero a.e. in* $\Omega \times (0,\infty)$. *Hence, the system is reduced to the single equation* (1.2).

2.2 Damage accumulation models

Barenblatt and Prostokishin [14] proposed a damage accumulation model, which derives the following fully non-linear parabolic PDE including the positive-part function:

$$u_t = u^\alpha (u_{xx} + \kappa u)_+ \quad \text{in} \quad (a,b) \times (0,\infty)$$

with parameters $\alpha > 1$, $\kappa > 0$ to describe the evolution of damage factor $u(x,t)$, that is, an internal variable used in the Kachanov theory [37]. Their model was mathematically studied by Bertsch and Bisegna in [15], where the solvability of the initial-boundary value problem is proved in a classical framework and long-time behaviours of solutions are also investigated. In particular, it is proved that the *regional* blow-up phenomena occur (i.e., the blow-up set of a solution is a sub-interval of (a,b); however, it is neither a point set nor the whole of the interval) under suitable assumptions on λ, α and the interval (a,b) (see also [1]).

2.3 Irreversible evolution equations governed by sub-differentials

As is explained in Section 1, the strongly irreversible evolution can be also described in terms of the sub-differential operator $\partial I_{[0,\infty)}$ of the indicator function over the half-line. In what follows, we shall recall strongly irreversible evolution equations formulated in such a way. Let us start with a *rate-independent* uni-directional flow along with the Ambrosio–Tortorelli energy (see Knees, Rossi and Zanini [40, 41] and references therein, e.g., [29]). In [40, 41], they discussed the existence of solutions to the Cauchy problem for the rate-independent evolution equation,

$$\partial \mathcal{R}(z_t) + \mathrm{D}_z \mathcal{F}_\varepsilon(u,z) \ni 0, \quad 0 < t < T, \quad u = \arg\min_v \mathcal{F}_\varepsilon(v,z),$$

where D_z denotes a functional derivative (e.g., Fréchet derivative) of $\mathcal{F}_\varepsilon$ with respect to the second variable z, with a 1-positively homogeneous and uni-directional dissipation functional $\mathcal{R}$ given by

$$\mathcal{R}(\eta) = \begin{cases} \int_\Omega \kappa|\eta(x)|\,\mathrm{d}x & \text{if } \ \eta \geqslant 0 \ \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases} \qquad \text{for } \ \eta \in L^1(\Omega)$$

for some $\kappa > 0$ (more precisely, in [40], a modified Ambrosio–Tortorelli energy is treated).

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Strongly irreversible evolution equations also appear in other topics. For instance, the following *irreversible phase transition* model is proposed by Frémond and studied, for example, in [17, 18],

$$\begin{aligned} &\theta_t - \theta\chi_t - \Delta\theta = \chi_t^2 && \text{in } \Omega \times (0,\infty), \\ &\chi_t + \partial I_{[0,\infty)}(\chi_t) - \Delta\chi + \beta(\chi) \ni \theta - \theta_c && \text{in } \Omega \times (0,\infty), \end{aligned}$$

where θ and χ denote the absolute temperature (θ_c is a transition temperature) and a phase parameter, respectively, and moreover, β is a maximal monotone graph in $\mathbb{R}^2$. Due to the presence of the sub-differential term $\partial I_{[0,\infty)}(\chi_t)$, the evolution of χ is constrained to be non-decreasing. We refer the reader to [46] and references therein for mathematical analysis of the model. Moreover, Aso and Kenmochi [8] (see also [7]) studied the existence of solutions for a quasi-variational evolution inequality of reaction–diffusion type such as

$$\begin{aligned} &\theta_t - \Delta\theta + k(\theta, w) = h(t,x) && \text{in } \Omega \times (0,\infty), \\ &w_t + \partial I_{[g(\theta),\infty)}(w_t) - \Delta w + \ell(\theta, w) \ni q(t,x) && \text{in } \Omega \times (0,\infty), \end{aligned}$$

where k and ℓ are Lipschitz continuous functions in both variables, g is a smooth non-negative function and h and q are given functions in a suitable class. These systems are also reduced to (1.2) in an isothermal setting, that is, $\theta =$ constant (with suitable assumptions). Furthermore, we also refer the reader to references [16, 47] and references therein.

Equations reviewed in this section have been studied mostly in view of well-posedness. On the other hand, qualitative and quantitative analysis on behaviours of solutions is still open, since the equations have several different complexities. So, the study on intrinsic phenomena arising from the strong irreversibility has not yet been fully pursued. In the present paper, we shall treat a simpler equation, (1.2), but we shall investigate various properties and behaviours of solutions as well as the well-posedness of (P) in order to find out intrinsic features of parabolic PDEs involving the positive-part function.

3 Existence of L^2-solutions

The $L^2(\Omega)$-solvability of (P) ($= \{(1.2), (1.7), (1.8)\}$) can be ensured for smooth data by applying a general theory due to Barbu [13] and Arai [6]; more precisely, for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$, (P) possesses at least one $L^2(\Omega)$-*solution* $u = u(x,t)$ defined by

Definition 3.1 *A function* $u \in C([0,\infty); L^2(\Omega))$ *is said to be a* solution (*or an* $L^2(\Omega)$-*solution*) *of* (P), *if the following conditions are all satisfied*:

(i) u *belongs to* $W^{1,2}(\delta, T; L^2(\Omega))$, $C([\delta, T]; H_0^1(\Omega) \cap L^4(\Omega))$, $L^2(\delta, T; H^2(\Omega))$ *and* $L^6(\delta, T; L^6(\Omega))$ *for any* $0 < \delta < T < \infty$;

(ii) *there exists* $\eta \in L^\infty(0,\infty; L^2(\Omega))$ *such that*

$$u_t + \eta - \Delta u + u^3 - \kappa u = 0, \quad \eta \in \partial I_{[0,\infty)}(u_t) \text{ for a.e. } (x,t) \in \Omega \times (0,\infty) \tag{3.1}$$

and $\eta = -\left(\Delta u - u^3 + \kappa u\right)_-$ *for a.e.* $(x,t) \in \Omega \times (0,\infty)$. *Hence,* u *also solves* (1.2) *a.e. in* $\Omega \times (0,\infty)$;

(iii) $u(\cdot, 0) = u_0$ *a.e. in* Ω.

On the other hand, the uniqueness of solutions does not follow from general theories. Furthermore, by focusing on specific structures of the equation (1.6), we shall improve the result on the $L^2(\Omega)$-solvability. More precisely, we shall prove a *smoothing effect* for (P), that is, even if initial data belong to a closure of a set D (of more regular functions), corresponding solutions belong to the set D instantly. To state more details, let us introduce a set

$$D_r := \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega) : \|(\Delta u - u^3 + \kappa u)_-\|_2^2 \leqslant r \right\}$$

for each $r > 0$. Here, we stress that D_r is an unbounded set. Indeed, let $z \in C^2(\Omega) \cap C(\overline{\Omega})$ be the negative solution of the classical elliptic Allen–Cahn equation,

$$-\Delta z + z^3 - \kappa z = 0 \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Then, any multiple $w = cz$ of z satisfies $\Delta w - w^3 + \kappa w \geqslant 0$ a.e. in Ω, provided that $c \geqslant 1$. Then, w belongs to D_r, and therefore, D_r is unbounded.

Now, let us state a theorem on the well-posedness and smoothing effect.

Theorem 3.2 (Well-posedness and smoothing effect) *Let $r > 0$ be arbitrarily fixed.*

(i) *Let u_0 belong to the closure $\overline{D_r}^{L^2}$ of D_r in $L^2(\Omega)$. Then* (P) *admits a solution $u = u(x,t)$ satisfying*

$$\begin{aligned} u &\in L^2(0,T;H_0^1(\Omega)) \cap L^4(0,T;L^4(\Omega)),\\ t^{1/2}u_t \in L^2(0,T;L^2(\Omega)), \quad tu_t &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\ t^{1/2}u &\in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\\ t^{1/4}u \in L^\infty(0,T;L^4(\Omega)), \quad t^{1/6}u &\in L^6(0,T;L^6(\Omega)),\\ t^{1/3}u \in L^\infty(0,T;L^6(\Omega)), \quad tu &\in L^\infty(0,T;H^2(\Omega)),\\ u \in C_w((0,T];H^2(\Omega)\cap L^6(\Omega)) &\cap C((0,T];H_0^1(\Omega)\cap L^4(\Omega)),\\ u(t) &\in D_r \quad \textit{for all} \quad t \in (0,T] \end{aligned}$$

for any $0 < T < \infty$.

(ii) *If u_0 belongs to the closure $\overline{D_r}^{H_0^1 \cap L^4}$ of D_r in $H_0^1(\Omega) \cap L^4(\Omega)$, then it further holds that*

$$\begin{aligned} u &\in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^6(0,T;L^6(\Omega)),\\ u \in C([0,T];H_0^1(\Omega)\cap L^4(\Omega)), \quad t^{1/2}u_t &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\ t^{1/2}u \in L^\infty(0,T;H^2(\Omega)), \quad t^{1/6}u &\in L^\infty(0,T;L^6(\Omega)) \end{aligned}$$

for any $0 < T < \infty$.

(iii) *If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$, then $u \in C_w([0,T];H^2(\Omega) \cap L^6(\Omega))$ and $u_t \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ for any $0 < T < \infty$.*

(iv) *Let $T > 0$ be fixed. For $N \leqslant 3$, $L^2(\Omega)$-solutions u belonging to the class*

$$C([0,T];H_0^1(\Omega)) \tag{3.3}$$

are uniquely determined by initial data $u_0 \in H_0^1(\Omega)$ *and they continuously depend on initial data* u_0 *in the following sense: let* u_i *be the unique solution of* (P) *for the initial data* $u_{0,i} \in H_0^1(\Omega)$ *(for* $i = 1, 2$*) and set* $w := u_1 - u_2$*. Then, there exists a constant* $C > 0$*, which depends only on* $\sup_{t\in(0,T)} \|\nabla u_i(t)\|_2$ $(i = 1, 2)$ *such that*

$$\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 \leqslant \left(\|w(0)\|_2^2 + \|\nabla w(0)\|_2^2\right) e^{Ct} \tag{3.4}$$

for all $t \in [0, T]$.

(v) *For* $N \leqslant 4$*,* $L^2(\Omega)$*-solutions belonging to* (3.3) *and*

$$L^2(0, T; H^2(\Omega)) \cap L^6(0, T; L^6(\Omega)) \tag{3.5}$$

are uniquely determined by initial data $u_0 \in H_0^1(\Omega)$ *and* (3.4) *holds true with a constant* C *depending only on* $\|u_i\|_{L^2(0,T;H^2(\Omega))}$ *and* $\|u_i\|_{L^6(0,T;L^6(\Omega))}$ $(i = 1, 2)$.

(vi) *Furthermore, for general* N*,* $L^2(\Omega)$*-solutions belonging to* $L^\infty(\Omega\times(0, T))$ *as well as* (3.3) *are uniquely determined by initial data* $u_0 \in H_0^1(\Omega)$ *and they satisfy* (3.4) *with a constant* C *which depends only on uniform bounds* $\|u_i\|_{L^\infty(\Omega\times(0,T))}$ *of solutions* u_i $(i = 1, 2)$.

Remark 3.3 (Invariance of the set D_r) *Thanks to (i), the set* D_r *(and its closures) turns out to be invariant under the evolution generated by (P) (see also* (4.6)*). Hence,* D_r *will play a role of a phase space in order to investigate the dynamics of solutions to (P) (see Sections 8 and 9).*

Remark 3.4 (Difference between D_r and its closure) *To observe how smoothing effect occurs (in Theorem 3.2), let us consider the following two examples (with* $N = 1$*,* $\Omega = (-1, 1)$ *and* $\kappa = 1$ *for simplicity):*

(i) *Set* $u_0(x) = |x| - 1 \in H_0^1(-1, 1) \setminus H^2(-1, 1)$*. Then, define* $u_{0,\varepsilon} \in W^{2,\infty}(-1, 1)$ *by*

$$u_{0,\varepsilon}(x) = \begin{cases} |x| - 1 & \text{if } |x| > \varepsilon \\ \frac{1}{\varepsilon}\frac{x^2}{2} + \frac{\varepsilon}{2} - 1 & \text{if } |x| \leqslant \varepsilon \end{cases},$$

for $\varepsilon > 0$*. Then, one observes that*

$$u_{0,\varepsilon}'' - u_{0,\varepsilon}^3 + u_{0,\varepsilon} = \begin{cases} -u_{0,\varepsilon}^3 + u_{0,\varepsilon} < 0 & \text{if } |x| > \varepsilon \\ \frac{1}{\varepsilon} \underbrace{-u_{0,\varepsilon}^3 + u_{0,\varepsilon}}_{\text{close to zero}} > 0 & \text{if } |x| \leqslant \varepsilon \end{cases},$$

for $\varepsilon > 0$ *small enough. Therefore,*

$$\|(u_{0,\varepsilon}'' - u_{0,\varepsilon}^3 + u_{0,\varepsilon})_-\|_2^2 = \int_{|x|>\varepsilon} (u_{0,\varepsilon}^3 - u_{0,\varepsilon})^2 \, dx \leqslant \|u_0^3 - u_0\|_2^2 =: r < +\infty.$$

Moreover, one can check that $u_{0,\varepsilon} \to u_0$ *strongly in* $H_0^1(-1, 1)$*. Hence,* u_0 *belongs to the closure of* D_r *in* $H_0^1(-1, 1)$*. However,* u_0 *does not belong to* D_r *(*$\subset H^2(-1, 1)$*). On the other hand, by Theorem 3.2,* $u(x, t)$ *belongs to (at least)* $H^2(-1, 1) \subset C^{1+\alpha}([-1, 1])$ *at any* $t > 0$*. Therefore, the sharp edge of* $u_0(x)$ *at* $x = 0$ *instantly vanishes.*

(ii) *Set* $u_0(x) \equiv -1 \in L^2(-1,1) \setminus H^1_0(-1,1)$ *(hence,* u_0 *violates the homogeneous Dirichlet condition) and define approximated data by*

$$u_{0,\varepsilon}(x) = \begin{cases} -1 & \text{if } |x| < 1-\varepsilon, \\ -1 + \frac{1}{\varepsilon^2}(|x| - 1 + \varepsilon)^2 & \text{if } |x| \geqslant 1-\varepsilon. \end{cases}$$

Then, $u_{0,\varepsilon} \in H^2(-1,1) \cap H^1_0(-1,1)$ *and* $u_{0,\varepsilon} \to u_0$ *strongly in* $L^2(-1,1)$ *as* $\varepsilon \to 0$. *Moreover, we observe that*

$$u''_{0,\varepsilon} - u^3_{0,\varepsilon} + u_{0,\varepsilon} = \begin{cases} 0 & \text{if } |x| < 1-\varepsilon, \\ \frac{2}{\varepsilon^2} - u^3_{0,\varepsilon} + u_{0,\varepsilon} > 0 & \text{if } |x| \geqslant 1-\varepsilon, \end{cases}$$

which yields $\|(u''_{0,\varepsilon} - u^3_{0,\varepsilon} + u_{0,\varepsilon})_-\|_2^2 = 0$. *Hence,* u_0 *belongs to the closure of* D_r *in* $L^2(-1,1)$ *(but* $u_0 \notin D_r$*). Since the solution to (P) satisfies the boundary condition* $u(\pm 1, t) = 0$ *for any* $t > 0$ *by Theorem 3.2, the values of* $u(\pm 1, t)$ *jump to* 0 *from* -1 *at* $t = 0$.

Proof of (iv)–(vi) Before starting a proof for (iv), we remark that the uniqueness of solutions for (P) is not ensured by the abstract results (e.g., Arai [6], Colli-Visintin [25], Colli [24], Visintin [50]). For instance, in [24, 25], the uniqueness is proved for (abstract) doubly non-linear equations, $A(u_t) + B(u) \ni 0$, provided that either A or B is linear.

Fix $\delta > 0$ arbitrarily. Let u_i $(i = 1, 2)$ be two solutions for (P) belonging to (3.3) with initial data $u_{0,i} \in H^1_0(\Omega)$ $(i = 1, 2)$ and set $w := u_1 - u_2$. Then,

$$w_t + \eta_1 - \eta_2 - \Delta w + u_1^3 - u_2^3 = \kappa w,$$

where η_i is a section of $\partial I_{[0,\infty)}(\partial_t u_i)$ for $i = 1, 2$. Test both sides by w_t and employ the monotonicity of $\partial I_{[0,\infty)}$ to find that

$$\|w_t\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_2^2 \leqslant \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 - \left(u_1^3 - u_2^3, w_t\right) \leqslant \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 + C\|u_1^3 - u_2^3\|_2^2 + \frac{1}{2}\|w_t\|_2^2$$

for a.e. $t \in (\delta, T)$ (see Definition 3.1). Here, note that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 = (w_t, w) \leqslant \varepsilon\|w_t\|_2^2 + C_\varepsilon\|w\|_2^2.$$

Therefore, choosing $\varepsilon > 0$ small enough, one obtains

$$\alpha\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_2^2 \leqslant \frac{C_\varepsilon}{2\varepsilon}\|w\|_2^2 + C\|u_1^3 - u_2^3\|_2^2 \tag{3.6}$$

for some $\alpha > 0$. In case $N \leqslant 3$, thanks to the Mean-Value Theorem and Sobolev's embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$, it follows that

$$\|u_1^3 - u_2^3\|_2^2 \leqslant C\left(\|\nabla u_1\|_2^4 + \|\nabla u_2\|_2^4\right)\|\nabla w\|_2^2. \tag{3.7}$$

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Thus, Gronwall's inequality yields

$$\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 \leqslant \left(\|w(\delta)\|_2^2 + \|\nabla w(\delta)\|_2^2\right) e^{C_0(t-\delta)} \quad \text{for all } t \geqslant \delta, \tag{3.8}$$

where C_0 is a constant depending only on $\sup_{t\in(0,T)} \|\nabla u_i(t)\|_2$. From the fact that $u_i \in C([0,T];H_0^1(\Omega))$, one can pass to the limit as $\delta \to 0_+$ and obtain (3.8) with $\delta = 0$. If $w(0) = 0$, then we conclude that $w \equiv 0$, that is, $u_1 \equiv u_2$. This completes the proof of (iv).

Concerning (v), for any $3 \leqslant N \leqslant 5$ (then, $H^2(\Omega) \subset L^{2N}(\Omega)$), by Gagliardo–Nirenberg's inequality we infer that

$$\begin{aligned} \|u_1^3 - u_2^3\|_2^2 &\leqslant C\left(\|u_1\|_{2N}^4 + \|u_2\|_{2N}^4\right) \|\nabla w\|_2^2 \\ &\leqslant C\left(\|u_1\|_{H^2(\Omega)}^{4\theta} \|u_1\|_6^{4(1-\theta)} + \|u_2\|_{H^2(\Omega)}^{4\theta} \|u_2\|_6^{4(1-\theta)}\right) \|\nabla w\|_2^2, \end{aligned}$$

where θ is given by

$$\frac{1}{2N} = \theta\left(\frac{1}{2} - \frac{2}{N}\right) + \frac{1-\theta}{6}.$$

Furthermore, assuming $N \leqslant 4$, one finds that

$$2\theta + \frac{2(1-\theta)}{3} \leqslant 1,$$

which yields

$$\|u_i\|_{H^2(\Omega)}^{4\theta} \|u_i\|_6^{4(1-\theta)} \in L^1(0,T)$$

for $i = 1, 2$. Therefore, by Gronwall's inequality, we can obtain the desired conclusion.

To prove (vi), the argument presented can be also generalized for general dimension N by assuming the boundedness of solutions, that is, $u_i \in L^\infty(Q)$ ($i = 1, 2$) with $Q = \Omega \times (0,T)$, and by replacing (3.7) by

$$\|u_1^3 - u_2^3\|_2^2 \leqslant C\left(\|u_1\|_\infty^4 + \|u_2\|_\infty^4\right) \|w\|_2^2.$$

Therefore, for general N, bounded solutions are uniquely determined by initial data and a similar inequality to (3.8) holds with a constant C_0 depending on $\|u_i\|_{L^\infty(Q)}$. Thus, (vi) is proved. □

Before giving a (sketch of) proof for the existence part (i)–(iii) of Theorem 3.2, we shall (formally) derive energy estimates in the next section.

4 Energy inequalities and partial energy-dissipation estimates

In this section, we first collect key energy inequalities, which will play a crucial role later; in particular, we shall derive *partial energy-dissipation estimates* and apply them to construct a global attractor in a customized setting (cf. as we mentioned in Section 1, due to the strong irreversibility, no absorbing set and no global attractor exist in any L^p-spaces). In order to derive (some of) them in an intuitive way, we here carry out formal arguments only. Secondly, we shall give a sketch of proof for the existence part of

Theorem 3.2. In Section 5, we shall give detailed proofs for the existence part and energy inequalities.

Energy Inequality 1 Test (1.6) by u_t and employ the relation $(\eta, u_t) = 0$ for any $\eta \in \partial I_{[0,\infty)}(u_t)$ to see that

$$\|u_t\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = 0 \quad \text{a.e. in } (0,\infty), \tag{4.1}$$

where $E : H_0^1(\Omega) \cap L^4(\Omega) \to \mathbb{R}$ is an energy functional given by

$$E(w) := \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{4}\|w\|_4^4 - \frac{\kappa}{2}\|w\|_2^2 \quad \text{for } w \in H_0^1(\Omega) \cap L^4(\Omega).$$

Since E is coercive, one can observe that

$$\int_0^T \|u_t\|_2^2 \,\mathrm{d}t + \sup_{t\in[0,T]} \left(\|\nabla u\|_2^2 + \|u\|_4^4\right) \leqslant C\left(E(u_0) + 1\right) \tag{4.2}$$

for any $T > 0$. Hence, for $u_0 \in H_0^1(\Omega) \cap L^4(\Omega)$, (if a solution exists, then) one can expect that $u \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega) \cap L^4(\Omega))$. Multiplying (4.1) by t, we also have

$$t\|u_t\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(tE(u(t))\big) = E(u(t)) \quad \text{a.e. in } (0,\infty). \tag{4.3}$$

Energy Inequality 2 The following is a formal computation. Differentiate both sides of (1.6) in t and set $v = u_t$. Then, we have

$$v_t + \eta_t - \Delta v + 3u^2 v = \kappa v \quad \text{in } \Omega \times (0,\infty), \tag{4.4}$$

where η is a section of $\partial I_{[0,\infty)}(v)$. Test both sides by v. It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t} I^*_{[0,\infty)}(\eta) + \|\nabla v\|_2^2 + 3\int_\Omega u^2 v^2 \,\mathrm{d}x = \kappa\|v\|_2^2,$$

where $I^*_{[0,\infty)}$ stands for the convex conjugate of $I_{[0,\infty)}$, that is,

$$I^*_{[0,\infty)}(\sigma) = \sup_{s\in\mathbb{R}} \left(s\sigma - I_{[0,\infty)}(s)\right) = \sup_{s\geqslant 0} s\sigma = I_{(-\infty,0]}(\sigma).$$

Here, we used the fact $(\eta_t, v) = (\mathrm{d}/\mathrm{d}t) I^*_{[0,\infty)}(\eta)$ by the relation $v \in \partial I^*_{[0,\infty)}(\eta)$. Moreover, we note that $I^*_{[0,\infty)}(\eta) = 0$.

Now, for each potential function $V = V(x)$, let us denote by $\lambda_\Omega(V)$ the first eigenvalue of the Schrödinger operator $-\Delta + V(x)$ over Ω equipped with the homogeneous Dirichlet boundary condition. If $u_0 \geqslant 0$, then one observes that

$$\|\nabla v\|_2^2 + 3\int_\Omega u^2 v^2 \,\mathrm{d}x \geqslant \|\nabla v\|_2^2 + 3\int_\Omega u_0^2 v^2 \,\mathrm{d}x \geqslant \lambda_\Omega(3u_0^2)\|v\|_2^2.$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_2^2 + \left(\lambda_\Omega(3u_0^2) - \kappa\right)\|v\|_2^2 \leqslant 0.$$

In addition, assuming $\lambda_\Omega(3u_0^2) > \kappa$, one can obtain the exponential decay estimate for $v = u_t$,

$$\|v(t)\|_2^2 \leqslant \|v_0\|_2^2 \exp\left(-2(\lambda_\Omega(3u_0^2) - \kappa)t\right) \tag{4.5}$$

for all $t > 0$. Here, we note that v_0 corresponds to $(\Delta u_0 - u_0^3 + \kappa u_0)_+$. Exponential decay estimate (4.5) will be used in Corollary 10.3 (see also Remark 10.4) in Section 10.

Energy Inequality 3 The following argument will play a key role to overcome difficulties arising from the double non-linearity of (1.6) and enable us to establish *partial energy-dissipation estimates.* Formally, test (4.4) by η to find that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{[0,\infty)}(v) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_2^2 + (-\Delta v, \eta) + 3\int_\Omega u^2 v\eta \,\mathrm{d}x = \kappa(v, \eta).$$

Note that $v\eta \equiv 0$, $I_{[0,\infty)}(v) = 0$ a.e. in $\Omega \times (0, \infty)$ and $(-\Delta v, \eta) \geqslant 0$. It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_2^2 \leqslant 0,$$

which implies that

$$\|\eta(t)\|_2^2 \leqslant \|\eta(s)\|_2^2 \quad \text{for a.e. } \ 0 \leqslant s \leqslant t < \infty. \tag{4.6}$$

Here, we recall that $\eta(0) = \eta_0 := -(\Delta u_0 - u_0^3 + \kappa u_0)_-$. Likewise, multiplying (4.4) by $|\eta|^{p-2}\eta \in \partial I_{[0,\infty)}(v)$, one can also derive

$$\|\eta(t)\|_p \leqslant \|\eta(s)\|_p \quad \text{for a.e. } \ 0 \leqslant s \leqslant t < \infty,$$

when $\eta(0) \in L^p(\Omega)$, for any $p \in (1, \infty)$, and hence,

$$\|\eta(t)\|_\infty \leqslant \|\eta(s)\|_\infty \quad \text{for a.e. } \ 0 \leqslant s \leqslant t < \infty,$$

provided that $\eta(0) \in L^\infty(\Omega)$.

Energy Inequality 4 Testing (1.6) by u, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_4^4 = \kappa\|u\|_2^2 - (\eta, u) \leqslant \kappa\|u\|_2^2 + \|\eta\|_2\|u\|_2.$$

By Hölder and Young inequalities and (4.6) with $s = 0$ and $\eta(0) = \eta_0$, we further derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + \|\nabla u\|_2^2 + \frac{1}{2}\|u\|_4^4 \leqslant C_1(1 + \|\eta_0\|_2^{4/3}) \tag{4.7}$$

for some constant $C_1 > 0$ depending only on $|\Omega|$ and κ. Integration of both sides over $(0, T)$ yields

$$\frac{1}{2}\|u(T)\|_2^2 + \int_0^T \left(\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_4^4\right) \mathrm{d}t \leqslant C_1 T(1 + \|\eta_0\|_2^{4/3}) + \frac{1}{2}\|u_0\|_2^2 \tag{4.8}$$

for any $T > 0$. Thus, one expects that $u \in L^2(0,T;H_0^1(\Omega)) \cap L^4(0,T;L^4(\Omega))$ for $u_0 \in \overline{D_r}^{L^2}$ (then, $\|\eta_0\|_2 \leqslant r < \infty$). Moreover, it follows from (4.3) and (4.8) that

$$\int_0^T t\|u_t\|_2^2 \,\mathrm{d}t + TE(u(T)) \leqslant \frac{C_1}{2} T \left(1 + \|\eta_0\|_2^{4/3}\right) + \frac{1}{4}\|u_0\|_2^2 \tag{4.9}$$

for any $T > 0$. It also implies that $t^{1/2}u_t \in L^2(0,T;L^2(\Omega))$, $t^{1/2}u \in L^\infty(0,T;H_0^1(\Omega))$ and $t^{1/4}u \in L^\infty(0,T;L^4(\Omega))$ whenever $u_0 \in \overline{D_r}^{L^2}$.

We next derive a *partial energy-dissipation estimate*. Assume that $\|\eta_0\|_2^2 \leqslant r$ for some $r > 0$. Then, combining (4.7) with (4.1), one finds that

$$\|u_t\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) + 2\kappa\phi(t) \leqslant \kappa C_1(1 + r^{2/3}) =: C_r, \tag{4.10}$$

where $\phi : H_0^1(\Omega) \cap L^4(\Omega) \to \mathbb{R}$ is a functional given by

$$\phi(t) := \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{4}\|u(t)\|_4^4.$$

Therefore, we conclude that

$$\phi(t) \leqslant \frac{C_r}{2\kappa} + e^{-2\kappa t}\left[\phi(0) - \frac{C_r}{2\kappa}\right] \quad \text{for all } \ t \geqslant 0, \tag{4.11}$$

which will play an important role to construct an absorbing set in Section 8. Here, it is noteworthy that C_r is independent of u_0 belonging to $\overline{D_r}^{H_0^1 \cap L^4}$; however, C_r cannot be chosen uniformly for all $r > 0$. Hence, (4.11) can be regarded as a *partial* energy-dissipation estimate.

Energy Inequality 5 Test (1.6) by $-\Delta u + u^3 - \kappa u$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) - \|\eta\|_2^2 + \| - \Delta u + u^3 - \kappa u\|_2^2 = 0.$$

Here, we used the fact that $\eta = -(\Delta u - u^3 + \kappa u)_-$. Combining this with (4.6) where $s = 0$ and $\eta(0) = \eta_0$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) + \| - \Delta u + u^3 - \kappa u\|_2^2 \leqslant \|\eta_0\|_2^2 \quad \text{a.e. in } (0,\infty), \tag{4.12}$$

which implies

$$E(u(T)) + \int_0^T \| - \Delta u + u^3 - \kappa u\|_2^2 \,\mathrm{d}t \leqslant T\|\eta_0\|_2^2 + E(u_0) \tag{4.13}$$

for any $T > 0$. Thus, we infer that $u \in L^2(0,T;H^2(\Omega)) \cap L^6(0,T;L^6(\Omega))$, provided that $u_0 \in \overline{D_r}^{H_0^1 \cap L^4}$. Furthermore, it also follows from (4.12) that

$$TE(u(T)) + \int_0^T t\| - \Delta u + u^3 - \kappa u\|_2^2 \,\mathrm{d}t \leqslant \int_0^T E(u(t))\,\mathrm{d}t + \frac{T^2}{2}\|\eta_0\|_2^2, \tag{4.14}$$

which along with (4.8) implies $t^{1/2}u \in L^2(0,T;H^2(\Omega))$ and $t^{1/6}u \in L^6(0,T;L^6(\Omega))$ if $u_0 \in \overline{D_r}^{L^2}$.

Energy Inequality 6 The following argument is also formal; indeed, the differentiability (in t) of $-\Delta u + u^3 - \kappa u$ is not supposed in Definition 3.1. Test (1.6) by $(-\Delta u + u^3 - \kappa u)_t$. Then, we observe that

$$\big(u_t, (-\Delta u + u^3 - \kappa u)_t\big) = \|\nabla u_t\|_2^2 + 3\int_\Omega u^2 u_t^2 \,\mathrm{d}x - \kappa \|u_t\|_2^2$$

and

$$\big(\eta, (-\Delta u + u^3 - \kappa u)_t\big) \geqslant 0.$$

Here, we used the fact that $(\eta, -\Delta u_t) \geqslant 0$ and $\eta u_t \equiv 0$ a.e. in $\Omega \times (0,\infty)$. Therefore,

$$\|\nabla u_t\|_2^2 + 3\int_\Omega u^2 u_t^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u - u^3 + \kappa u\|_2^2 \leqslant \kappa \|u_t\|_2^2.$$

Furthermore, by (4.1),

$$\|\nabla u_t\|_2^2 + 3\int_\Omega u^2 u_t^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u - u^3 + \kappa u\|_2^2 \leqslant \kappa \|u_t\|_2^2 = -\kappa \frac{\mathrm{d}}{\mathrm{d}t} E(u(t)),$$

which can be rewritten as

$$\|\nabla u_t\|_2^2 + 3\int_\Omega u^2 u_t^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t))\right] \leqslant 0 \tag{4.15}$$

for a.e. $t > 0$. In particular,

$$\int_0^T \|\nabla u_t\|_2^2 \,\mathrm{d}t + 3\int_0^T \int_\Omega u^2 u_t^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\Delta u(T) - u^3(T) + \kappa u(T)\|_2^2 \\ + \kappa E(u(T)) \leqslant \frac{1}{2}\|\Delta u_0 - u_0^3 + \kappa u_0\|_2^2 + \kappa E(u_0) \tag{4.16}$$

for all $T > 0$. Hence, $u \in L^\infty(0,T;H^2(\Omega) \cap L^6(\Omega))$ and $u_t \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ (by (1.2)) if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$.

On the other hand, multiply (4.15) by t and compute as follows:

$$t\|\nabla u_t\|_2^2 + 3t\int_\Omega u^2 u_t^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(t\left[\frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t))\right]\right) \\ \leqslant \frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t)). \tag{4.17}$$

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Integrating both sides over $(0, T)$, we conclude that

$$\int_0^T t\|\nabla u_t\|_2^2\,\mathrm{d}t + 3\int_0^T t\left(\int_\Omega u^2 u_t^2\,\mathrm{d}x\right)\,\mathrm{d}t \\ + T\left[\frac{1}{2}\|\Delta u(T) - u^3(T) + \kappa u(T)\|_2^2 + \kappa E(u(T))\right] \\ \leqslant \frac{1}{2}\int_0^T \|\Delta u - u^3 + \kappa u\|_2^2\,\mathrm{d}t + \kappa\int_0^T E(u(t))\,\mathrm{d}t$$

for all $T > 0$. Combining it with (4.13), one can obtain an estimate exhibiting a smoothing effect,

$$\int_0^T t\|\nabla u_t\|_2^2\,\mathrm{d}t + 3\int_0^T t\left(\int_\Omega u^2 u_t^2\,\mathrm{d}x\right)\,\mathrm{d}t \\ + T\left[\frac{1}{2}\|\Delta u(T) - u^3(T) + \kappa u(T)\|_2^2 + \kappa E(u(T))\right] \\ \leqslant \frac{1}{2}\left(T\|\eta_0\|_2^2 + E(u_0) - E(u(T))\right) + \kappa\int_0^T E(u(t))\,\mathrm{d}t \tag{4.18}$$

for all $T > 0$. Hence, one expects that $t^{1/2}u_t \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ (by (1.2)), $t^{1/2}u \in L^\infty(0, T; H^2(\Omega))$ and $t^{1/6}u \in L^\infty(0, T; L^6(\Omega))$ for $u_0 \in \overline{D_r}^{H_0^1 \cap L^4}$. Moreover, multiply (4.17) by t again. Then,

$$t^2\|\nabla u_t\|_2^2 + 3t^2\int_\Omega u^2 u_t^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\left[\frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t))\right]\right) \\ \leqslant 2t\left(\frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t))\right).$$

Integrate both sides over $(0, T)$. Then, it follows that

$$\int_0^T t^2\|\nabla u_t\|_2^2\,\mathrm{d}t + 3\int_0^T t^2\left(\int_\Omega u^2 u_t^2\,\mathrm{d}x\right)\,\mathrm{d}t \\ + T^2\left[\frac{1}{2}\|\Delta u(T) - u^3(T) + \kappa u(T)\|_2^2 + \kappa E(u(T))\right] \\ \leqslant 2\int_0^T t\left(\frac{1}{2}\|\Delta u - u^3 + \kappa u\|_2^2 + \kappa E(u(t))\right)\,\mathrm{d}t \\ \overset{(4.14)}{\leqslant} \int_0^T E(u(t))\,\mathrm{d}t + 2\kappa\int_0^T tE(u(t))\,\mathrm{d}t + \frac{T^2}{2}\|\eta_0\|_2^2 + \frac{\kappa T}{2}\|u(T)\|_2^2. \tag{4.19}$$

By virtue of (4.8), we may obtain $tu_t \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ (by (1.2)), $tu \in L^\infty(0, T; H^2(\Omega))$ and $t^{1/3}u \in L^\infty(0, T; L^6(\Omega))$ for $u_0 \in \overline{D_r}^{L^2}$.

Let us also derive another partial energy-dissipation estimate. Inequality (4.18) yields

$$\frac{1}{2}\|\Delta u(T)-u^3(T)+\kappa u(T)\|_2^2+\kappa E(u(T)) \leqslant \frac{1}{2}\left(\|\eta_0\|_2^2+\frac{1}{T}E(u_0)-\frac{1}{T}E(u(T))\right)+\frac{\kappa}{T}\int_0^T E(u(t))\,\mathrm{d}t \tag{4.20}$$

for any $T>0$. Due to the decrease of the energy $t\mapsto E(u(t))$ and the fact that $E(\cdot)\geqslant -M_0 := \inf_{w\in H_0^1(\Omega)} E(w) > -\infty$, it follows that

$$\begin{aligned}\frac{1}{2}\|\Delta u(t)-u^3(t)+\kappa u(t)\|_2^2 &\leqslant \kappa M_0+\frac{1}{2}\left(\|\eta_0\|_2^2+\frac{1}{t}E(u_0)+\frac{1}{t}M_0\right)+\frac{\kappa}{t}\int_0^t \phi(u(\tau))\,\mathrm{d}\tau\\ &\overset{(4.11)}{\leqslant} \kappa M_0+\frac{1}{2}\left(r+\frac{1}{t}E(u_0)+\frac{1}{t}M_0\right)+\frac{C_r}{2}+\frac{\phi(0)}{2t} \quad \text{for all } t>0,\end{aligned} \tag{4.21}$$

which will be used to construct an absorbing set in Section 8. On the other hand, one can also exhibit a dissipation structure in a more quantitative way. Since $t\mapsto E(u(t))$ is non-increasing and $E(\cdot)$ is bounded from below, we deduce that

$$E_\infty := \lim_{t\to\infty} E(u(t)) \geqslant -M_0,$$

and therefore,

$$\frac{1}{T}\int_0^T E(u(t))\,\mathrm{d}t \searrow E_\infty \quad \text{as } T\to\infty.$$

Consequently, by (4.20), one has the following result:

Corollary 4.1 *In case $u_0\in H^2(\Omega)\cap H_0^1(\Omega)\cap L^6(\Omega)$, for any $\varepsilon>0$, there exists $T_\varepsilon>0$ (possibly depending on each solution u) such that, for all $T\geqslant T_\varepsilon$,*

$$\|\Delta u(T)-u^3(T)+\kappa u(T)\|_2^2 \leqslant \left\|(\Delta u_0-u_0^3+\kappa u_0)_-\right\|_2^2+\varepsilon. \tag{4.22}$$

In case $u_0\in \overline{D_r}^{H_0^1\cap L^4}$, for any $\varepsilon>0$, one can take $T_\varepsilon>0$ such that, for all $T\geqslant T_\varepsilon$,

$$\|\Delta u(T)-u^3(T)+\kappa u(T)\|_2^2 \leqslant r+\varepsilon.$$

Remark 4.2 *Due to the non-decreasing constraint on solutions, energy-dissipation cannot be observed in a usual way. Indeed, since $u(\cdot,t)\geqslant u_0$ a.e. in Ω for all $t\geqslant 0$, it follows that*

$$\|u(t)\|_p \geqslant \|u_0\|_p \quad \textit{for any } p\in[1,\infty],$$

provided that $u_0\geqslant 0$. Hence, the L^p norm of $u(t)$ never decays and no absorbing set in $L^p(\Omega)$ exists. Moreover, let z be the positive solution of the classical elliptic Allen–Cahn equation (3.2). *Then, any multiple $w=cz$ for $c\geqslant 1$ turns out to be an equilibrium for (P), since it holds that $\Delta w-w^3+\kappa w\leqslant 0$ a.e. in Ω. Therefore, the set of equilibria is unbounded in any*

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(linear) space including z. On the other hand, for any initial data $u_0 \in D_r$, one can observe partial energy-dissipation *in* (4.11), (4.21) *and* (4.22). *Indeed, the set D_r excludes a part of the unbounded set of equilibria (still, we emphasize again that D_r itself is unbounded). Hence, there arises a question: whether or not one can construct an "attractor" for the DS generated by (P) over the set D_r. An answer to this question will be provided in Section 8 and Section 9.*

We close this section by giving a sketch of proof for the existence part of Theorem 3.2 and by exhibiting an idea to justify the formal arguments given so far.

A sketch of proof for the existence part of Theorem 3.2 Set $H = L^2(\Omega)$ and define a functional $\psi : H \to [0, \infty]$ by

$$\psi(u) := \begin{cases} \phi(u) & \text{if } u \in H_0^1(\Omega) \cap L^4(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{4.23}$$

Then, the sub-differential $\partial\psi$ of ψ has the representation, $\partial\psi(v) = -\Delta v + v^3$ for $v \in D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$. Hence, (P) is reduced to an abstract Cauchy problem in the Hilbert space H,

$$u_t + \partial I_{[0,\infty)}(u_t) + \partial\psi(u) \ni \kappa u \quad \text{in } H, \quad 0 < t < T, \quad u(0) = u_0, \tag{4.24}$$

whose solvability (i.e., existence of solutions) has been studied by [13] and [6] for $u_0 \in D(\partial\psi)$. Then, (iii) can be proved by checking some structure conditions proposed in [6] (see Section 5 for more details). For later use, let us briefly recall a strategy (similar to [6]) to construct a solution of (4.24): We construct approximate solutions for (P) and denote by u_λ the (unique) solution of

$$u_t + \partial I_{[0,\infty)}(u_t) + \partial\psi_\lambda(u) \ni \kappa u \quad \text{in } H, \quad 0 < t < T, \quad u(0) = u_0 \in D(\partial\psi), \tag{4.25}$$

where $\partial\psi_\lambda$ is the sub-differential operator of the *Moreau–Yosida regularization* ψ_λ of ψ (equivalently, the *Yosida approximation* of $\partial\psi$) (see, e.g., [20]). Here, one can write

$$\partial\psi_\lambda(v) = \partial\psi(J_\lambda v) = -\Delta(J_\lambda v) + (J_\lambda v)^3 \quad \text{for all } v \in H, \tag{4.26}$$

where $J_\lambda : H \to D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$ stands for the *resolvent* of $\partial\psi$, that is, $J_\lambda := (I + \lambda\partial\psi)^{-1}$ (see [20]). Indeed, equation (4.25) can be also rewritten as an evolution equation governed by a Lipschitz continuous operator in H, that is,

$$u_t = \left(I + \partial I_{[0,\infty)}\right)^{-1} (-\partial\psi_\lambda(u) + \kappa u) \quad \text{in } H, \quad 0 < t < T, \quad u(0) = u_0,$$

since $\partial\psi_\lambda$ and $\left(I + \partial I_{[0,\infty)}\right)^{-1}$ are Lipschitz continuous in H. Therefore, the solution u_λ of (4.25) is uniquely determined (by u_0) and u_λ is of class $C^{1,1}$ in time. Furthermore, the section η_λ of $\partial I_{[0,\infty)}(\partial_t u_\lambda)$ as in (3.1) belongs to $C^{0,1}([0, T]; H)$ by means of the relation $\eta_\lambda = \kappa u_\lambda - \partial\psi_\lambda(u_\lambda) - \partial_t u_\lambda$. Moreover, (4.25) is also equivalent to

$$u_t = (-\partial\psi_\lambda(u) + \kappa u)_+ \quad \text{in } H, \quad 0 < t < T.$$

As in the formal computations given above, one can derive corresponding energy inequalities for u_λ with η_0 replaced by $-(\kappa u_0 - \partial\psi_\lambda(u_0))_-$. Therefore, passing to the limit as $\lambda \to 0_+$ (see Section 5), one can construct an L^2-solution u of (P) (for $u_0 \in D(\partial\psi)$) and reproduce all the energy inequalities obtained so far. Moreover, in order to prove smoothing effects (e.g., (ii)), we approximate initial data $u_0 \in \overline{D_r}^{H_0^1 \cap L^4}$ by $u_{0,n} \in D_r$ satisfying

$$u_{0,n} \to u_0 \quad \text{strongly in } H_0^1(\Omega) \cap L^4(\Omega) \quad \text{as } n \to \infty$$

and particularly employ the fact $\|(\Delta u_{0,n} - u_{0,n}^3 + \kappa u_{0,n})_-\|_2^2 \leqslant r$ (by $u_{0,n} \in D_r$) to reproduce the energy inequalities. For more precise arguments as well as a proof for (i), we refer the reader to Section 5. □

5 Proof of existence of solutions and energy inequalities

In this section, we give a proof for the existence part of Theorem 3.2 and a rigorous derivation of energy inequalities, which are derived in Section 4 by formal computations. More precisely, we shall prove

Theorem 5.1 *Let $r > 0$ be arbitrarily fixed.*

(i) *Let u_0 belong to the closure $\overline{D_r}^{L^2}$ of D_r in $L^2(\Omega)$. Then,* (P) *admits the unique solution $u = u(x,t)$ satisfying all the regularity conditions as in* (i) *of Theorem 3.2 such that* (4.1), (4.3), (4.7)–(4.10), (4.12), (4.14) *and* (4.19) *hold true with $\|\eta_0\|_2^2$ replaced by r. Moreover, it is also satisfied that*

$$\|\eta(t)\|_2^2 \leqslant r \quad \text{for a.e. } t > 0. \tag{5.1}$$

(ii) *If u_0 also belongs to the closure $\overline{D_r}^{H_0^1 \cap L^4}$ of D_r in $H_0^1(\Omega) \cap L^4(\Omega)$, then the solution $u = u(x,t)$ also satisfies all the regularity conditions as in* (ii) *of Theorem 3.2. Moreover,* (4.1)–(4.3), (4.7)–(4.14), (4.18)–(4.21) *and* (5.1) *are satisfied with $\|\eta_0\|_2^2$ replaced by r.*

(iii) *If $u_0 \in H^2(\Omega) \cap L^6(\Omega)$, then the solution $u = u(x,t)$ also fulfils all the regularity conditions as in* (iii) *of Theorem 3.2. Moreover,* (4.1)–(4.3), (4.7)–(4.14), (4.16), (4.18)–(4.21) *are satisfied with $\|\eta_0\|_2^2$ replaced by $\|(\Delta u_0 - u_0^3 + \kappa u_0)_-\|_2^2$. Furthermore, it holds that*

$$\|\eta(t)\|_2^2 \leqslant \|(\Delta u_0 - u_0^3 + \kappa u_0)_-\|_2^2 \quad \text{for a.e. } t > 0. \tag{5.2}$$

A rigorous proof for energy inequalities (4.5), (4.6), (4.15) and (4.17) will be postponed until the end of Section 6 (see Corollary 6.5). Indeed, we need uniqueness of solutions (for general initial data). Now, we give a proof of Theorem 5.1.

5.1 Reduction to an abstract Cauchy problem

Let $T > 0$ be arbitrarily fixed. Set $H = L^2(\Omega)$ (with $\|\cdot\|_H := \|\cdot\|_2$), $V = H_0^1(\Omega) \cap L^4(\Omega)$ (with $\|\cdot\|_V := \|\nabla\cdot\|_2 + \|\cdot\|_4$) and define a functional ψ on H as in (4.23). Moreover, set

$\varphi : H \to [0, \infty]$ by

$$\varphi(u) = \frac{1}{2}\|u\|_2^2 + I_{[0,\infty)}(u) \quad \text{for} \quad u \in H, \tag{5.3}$$

which is homogeneous of degree 2. Then, as in Section 4, (P) is reduced to the abstract Cauchy problem (4.24), which is also equivalent to

$$\partial\varphi(u_t) + \partial\psi(u) \ni \kappa u \;\; \text{in } H, \quad 0 < t < T, \quad u(0) = u_0. \tag{5.4}$$

In order to prove the existence of solutions to (5.4), it suffices to check all the assumptions of [6] (see also [13]). For the readers' convenience, let us recall them as follows:

(A1) there exists a reflexive Banach space $(V, \|\cdot\|_V)$ which is densely and compactly embedded in $(H, \|\cdot\|_H)$,

(A2) $D(\psi) \subset V$ and $\psi(u) + \|u\|_H^2 \to +\infty$ as $\|u\|_V \to +\infty$,

(A3) there exist $C > 0$ and $R > 0$ such that $\varphi(u) \geqslant C\|u\|_H^2$ for $u \in D(\varphi)$ satisfying $\|u\|_H \geqslant R$,

(A4) there exists $p > 1$ such that $\varphi(\mu u) = \mu^p \varphi(u)$ for $u \in D(\varphi)$ and $\mu > 0$,

(A5) $\partial\psi$ is $\partial\varphi$-monotone, that is,

$$\varphi(J_\lambda u - J_\lambda v) \leqslant \varphi(u - v) \;\; \text{for} \;\; u, v \in H \;\; \text{and} \;\; \lambda > 0, \tag{5.5}$$

where J_λ is the resolvent of $\partial\psi$, that is, $J_\lambda := (I + \lambda\partial\psi)^{-1}$.

Since (A1)–(A4) follow immediately from the setting of ψ and φ, we only give a proof for checking (A5).

Lemma 5.2 *It holds that*

$$I_{[0,\infty)}(J_\lambda u - J_\lambda v) \leqslant I_{[0,\infty)}(u - v), \quad \|J_\lambda u - J_\lambda v\|_2^2 \leqslant \|u - v\|_2^2$$

for $u, v \in H$. In particular, (5.5) *holds true with φ defined as in* (5.3).

Proof The second inequality follows from a well-known fact that resolvents of maximal monotone operators are non-expansive, that is, $\|J_\lambda u - J_\lambda v\|_H \leqslant \|u - v\|_H$ for $u, v \in H$ (see, e.g., [20]). So, it remains to prove the first inequality. In case $I_{[0,\infty)}(u - v) = \infty$, we have nothing to prove. In case $I_{[0,\infty)}(u - v) = 0$, that is, $u \geqslant v$ a.e. in Ω, by the definition of J_λ, we see that

$$J_\lambda u - J_\lambda v + \lambda\left[\partial\psi(J_\lambda u) - \partial\psi(J_\lambda v)\right] = u - v. \tag{5.6}$$

Test both sides by $-(J_\lambda u - J_\lambda v)_- \leqslant 0$ to get

$$\int_\Omega (J_\lambda u - J_\lambda v)_-^2 \, \mathrm{d}x \leqslant -\int_\Omega (u - v)\,(J_\lambda u - J_\lambda v)_- \, \mathrm{d}x \leqslant 0,$$

which implies $J_\lambda u \geqslant J_\lambda v$ a.e. in Ω. Here, we used the fact that

$$(\partial\psi(J_\lambda u) - \partial\psi(J_\lambda u), -(J_\lambda u - J_\lambda v)_-) = (-\Delta(J_\lambda u - J_\lambda v), -(J_\lambda u - J_\lambda v)_-) \\ + \left(|J_\lambda u|^2 J_\lambda u - |J_\lambda v|^2 J_\lambda v, -(J_\lambda u - J_\lambda v)_-\right) \geqslant 0$$

by monotonicity. Thus, $I_{[0,\infty)}(J_\lambda u - J_\lambda v) = 0$. □

5.2 Proof of (iii)

Let us prove (iii). To this end, suppose that

$$u_0 \in D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega). \tag{5.7}$$

Then, thanks to Arai [6, Theorem 3.3] (see also Barbu [13]), we assure that (5.4) admits a solution $u \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)$, such that the function $t \mapsto \varphi(u'(t))$ belongs to $L^\infty(0,T)$ and the function $t \mapsto \psi(u(t))$ is absolutely continuous on $[0,T]$. Concerning energy inequalities, one can rigorously derive (4.1)–(4.3) as in Section 4 under the frame of Definition 3.1. So, we shall verify the other energy inequalities. To this end, the rest of this sub-section is devoted to preparing auxiliary steps.

Recall approximate problems (4.25) for (P) and denote by u_λ the unique solution. Furthermore, let η_λ be the section of $\partial I_{[0,\infty)}(\partial_t u_\lambda)$ satisfying

$$\partial_t u_\lambda + \eta_\lambda + \partial\psi_\lambda(u_\lambda) = \kappa u_\lambda, \quad u_\lambda(0) = u_0. \tag{5.8}$$

Set $p_\lambda := \partial_t u_\lambda + \eta_\lambda$. Then, p_λ is a section of $\partial\varphi(\partial_t u_\lambda)$.

Remark 5.3 (Approximate equations in [6]) *Approximate problems used in [6] seem slightly different from* (5.8)*; indeed, they involve a liner relaxation term such as*

$$\lambda u_t + \partial\varphi(u_t) + \partial\psi_\lambda(u) \ni \kappa u,$$

since the quadratic coercivity of φ is not assumed. However, concerning (1.6)*, one can reproduce the same arguments as in [6] for* (5.8)*, since the original equation* (1.6) *already includes the linear relaxation term. On the other hand, the following arguments also work well for approximate equations with the additional relaxation term as in [6].*

As mentioned in Section 4, we assure that $\partial_t u_\lambda, \eta_\lambda \in C^{0,1}([0,T];H)$ and $\eta_\lambda = -(\kappa u_\lambda - \partial\psi_\lambda(u_\lambda))_-$ in H for each $t \in [0,T]$. In particular, one finds that

$$\eta_\lambda(0) := \lim_{t\to 0_+} \eta_\lambda(t) = -\left(\kappa u_0 - \partial\psi_\lambda(u_0)\right)_-.$$

Moreover, every assertion obtained by [6] for u_λ is valid (see proofs of Theorems 3.1 and 3.3 in [6] for details). In particular, let us recall that, up to a (not relabelled) subsequence

$\lambda \to 0$,

$$\begin{aligned} J_\lambda u_\lambda &\to u && \text{strongly in } C([0,T];H),\\ u_\lambda &\to u && \text{strongly in } C([0,T];H),\\ \partial_t u_\lambda &\to u_t && \text{weakly star in } L^\infty(0,T;H),\\ \partial\psi_\lambda(u_\lambda) &\to \partial\psi(u) && \text{weakly star in } L^\infty(0,T;H),\\ p_\lambda &\to p && \text{weakly star in } L^\infty(0,T;H), \end{aligned}$$

and moreover, the function $t \mapsto \psi(u(t))$ is (absolutely) continuous on $[0,T]$ (hence, $u \in C([0,T];H_0^1(\Omega)\cap L^4(\Omega)))$ and $p \in \partial\varphi(u_t)$. Since $\partial\psi(u) \in L^\infty(0,T;H)$ and $u(t) \in D(\partial\psi) = H^2(\Omega)\cap H_0^1(\Omega)\cap L^6(\Omega)$ for a.e. $t \in (0,T)$, it follows that $u \in L^\infty(0,T;H^2(\Omega)\cap L^6(\Omega))$ from the fact that $\|\Delta w\|_2^2 + \|w\|_6^6 \leqslant \|\partial\psi(w)\|_2^2$ for all $w \in D(\partial\psi)$ along with the elliptic estimate $\|w\|_{H^2(\Omega)} \leqslant C(\|\Delta w\|_2 + \|w\|_2)$ for $w \in H^2(\Omega)$. Thus, u solves (P). Here, we further observe that

$$J_\lambda u_\lambda \to u \quad \text{weakly star in } L^\infty(0,T;H_0^1(\Omega)\cap L^4(\Omega))$$

and (see [20], [19] and [12])

$$\lim_{\lambda\to 0}\int_0^T (\partial\psi_\lambda(u_\lambda), J_\lambda u_\lambda)\ \mathrm{d}t \to \int_0^T (\partial\psi(u),u)\ \mathrm{d}t.$$

One can also verify that

$$\begin{aligned} \limsup_{\lambda\to 0}\int_0^T \|\nabla J_\lambda u_\lambda(t)\|_2^2\,\mathrm{d}t &= \limsup_{\lambda\to 0}\int_0^T (-\Delta J_\lambda u_\lambda, J_\lambda u_\lambda)\ \mathrm{d}t\\ &\leqslant \limsup_{\lambda\to 0}\int_0^T \left(\partial\psi_\lambda(u_\lambda) - (J_\lambda u_\lambda)^3, J_\lambda u_\lambda\right)\ \mathrm{d}t\\ &\leqslant \int_0^T (-\Delta u,u)\ \mathrm{d}t = \int_0^T \|\nabla u(t)\|_2^2\,\mathrm{d}t, \end{aligned}$$

which implies

$$J_\lambda u_\lambda \to u \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)).$$

Similarly,

$$J_\lambda u_\lambda \to u \quad \text{strongly in } L^4(0,T;L^4(\Omega)).$$

Hence,

$$\int_0^T \phi(J_\lambda u_\lambda(t))\,\mathrm{d}t \to \int_0^T \phi(u(t))\,\mathrm{d}t. \tag{5.9}$$

Moreover, by $u \in C([0,T];H_0^1(\Omega)\cap L^4(\Omega))\cap L^\infty(0,T;H^2(\Omega)\cap L^6(\Omega))$, we deduce that $u \in C_w([0,T];H^2(\Omega)\cap L^6(\Omega))$ (see [44]). It follows that

$$J_\lambda u_\lambda(t) \to u(t) \quad \text{weakly in } H^2(\Omega)\cap L^6(\Omega) \quad \text{for any } \ t \in [0,T]. \tag{5.10}$$

On the other hand, there exists $\eta \in L^\infty(0,T;H)$, such that

$$\eta_\lambda \to \eta \quad \text{weakly star in } L^\infty(0,T;H)$$

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and $\eta = p - u_t \in \partial I_{[0,\infty)}(u_t)$. From the equivalence between (1.6) and (1.2), we also remark that

$$\eta = -(\kappa u - \partial\psi(u))_- = -\left(\Delta u - u^3 + \kappa u\right)_- \quad \text{a.e. in} \quad \Omega \times (0,T). \tag{5.11}$$

We next justify formal arguments in Section 4 to derive energy inequalities (except *Energy Inequality 1* in Section 4). To this end, we claim that

$$J_\lambda u, \ |J_\lambda u_\lambda| J_\lambda u_\lambda \in W^{1,2}(0,T; H_0^1(\Omega)). \tag{5.12}$$

Indeed, recalling (5.6) with u and v replaced by $u_\lambda(t+h)$ and $u_\lambda(t)$, respectively, and multiplying it by $J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)$, one can derive that

$$\frac{1}{2}\|J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)\|_2^2 + \lambda \|\nabla (J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t))\|_2^2 \\ + \frac{3}{4}\left\|(|J_\lambda u_\lambda| J_\lambda u_\lambda)(t+h) - (|J_\lambda u_\lambda| J_\lambda u_\lambda)(t)\right\|_2^2 \leqslant \frac{1}{2}\|u_\lambda(t+h) - u_\lambda(t)\|_2^2$$

for a.e. $t \in (0,T)$ and $h \in \mathbb{R}$ satisfying $t+h \in [0,T]$. Here, we also used the fundamental inequality,

$$\frac{3}{4}\Big||a|a - |b|b\Big|^2 \leqslant (a^3 - b^3)(a-b) \quad \text{for all} \quad a, b \in \mathbb{R}. \tag{5.13}$$

From the arbitrariness of h, we deduce that $J_\lambda u_\lambda \in W^{1,2}(0,T; H_0^1(\Omega))$ by $u_\lambda \in C^{1,1}([0,T]; L^2(\Omega)) \subset W^{1,2}(0,T; L^2(\Omega))$.

By [6, Lemma 3.10] and the monotonicity of $\partial\psi_\lambda$ along with Lemma 5.2, we obtain

Lemma 5.4 *For $u \in C^1([0,T]; H)$ satisfying $u_t \geqslant 0$ a.e. in $\Omega \times (0,T)$, it holds that*

(i) *$I_{[0,\infty)}((J_\lambda u)_t) \leqslant I_{[0,\infty)}(u_t)$ for a.e. $t \in (0,T)$, in particular, $(J_\lambda u)_t \geqslant 0$ a.e. in $\Omega \times (0,T)$,*

(ii) *for any $\eta \in \partial I_{[0,\infty)}(u_t)$, one has*

$$\left(\eta(t), \frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u(t))\right) \geqslant 0 \quad \textit{for a.e. } t \in (0,T).$$

5.3 Derivation of Energy Inequalities under (5.7)

We next derive energy inequalities.

Energy Inequalities 3 Differentiate both sides of (5.8) in t (indeed, it is rigorously possible, since both sides of (5.8) are smooth (in t) enough by approximation) and put $v_\lambda := \partial_t u_\lambda \in C^{0,1}([0,T]; H) \subset W^{1,\infty}(0,T; H)$. Then,

$$\partial_t v_\lambda + \partial_t \eta_\lambda + \frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda) = \kappa v_\lambda. \tag{5.14}$$

Multiplying both sides by η_λ and employing (ii) of Lemma 5.4, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{[0,\infty)}(v_\lambda) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_\lambda\|_2^2 \leqslant \kappa \int_\Omega v_\lambda \eta_\lambda \,\mathrm{d}x = 0,$$

which leads us to get

$$\|\eta_\lambda(t)\|_2^2 \leqslant \|\eta_\lambda(0)\|_2^2 = \|(\kappa u_0 - \partial\psi_\lambda(u_0))_-\|_2^2 \quad \text{for all} \quad t \in [0, T]. \tag{5.15}$$

Since $\partial\psi_\lambda(u_0) \to \partial\psi(u_0)$ strongly in H as $\lambda \to 0$ by $u_0 \in D(\partial\psi)$ (see [20]), one has

$$\|\eta(t)\|_2^2 \leqslant \|\eta\|^2_{L^\infty(0,T;H)} \leqslant \liminf_{\lambda\to 0} \|\eta_\lambda\|^2_{L^\infty(0,T;H)} \leqslant \|(\kappa u_0 - \partial\psi(u_0))_-\|_2^2$$

for a.e. $t \in (0, T)$. Hence, (5.2) follows.

Energy Inequalities 4–6 Thanks to (5.2), as in Section 4, one can derive (4.7)–(4.14) by replacing $\|\eta_0\|_2$ by $\|(\Delta u_0 - u_0^3 + \kappa u_0)_-\|_2$. As for *Energy Inequality 6*, test (5.8) by $(\partial\psi_\lambda(u_\lambda) - \kappa u_\lambda)_t$, which is well-defined due to the smoothness of u_λ and $\partial\psi_\lambda(u_\lambda)$ in t. Then, it follows that

$$\left(\partial_t u_\lambda + \eta_\lambda, (\partial\psi_\lambda(u_\lambda) - \kappa u_\lambda)_t\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial\psi_\lambda(u_\lambda) - \kappa u_\lambda\|_2^2 = 0.$$

Here, we also observe by (ii) of Lemma 5.4 that

$$\left(\eta_\lambda, (\partial\psi_\lambda(u_\lambda) - \kappa u_\lambda)_t\right) = (\eta_\lambda, (\partial\psi_\lambda(u_\lambda))_t) \geqslant 0,$$

and moreover,

$$\begin{aligned}\left(\partial_t u_\lambda, \frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda)\right) &= \left((J_\lambda u_\lambda)_t + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda), \frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda)\right)\\ &\geqslant \left((J_\lambda u_\lambda)_t, \frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda)\right)\\ &\overset{(4.26)}{\geqslant} \|\nabla(J_\lambda u_\lambda)_t\|_2^2 + \frac{3}{4}\left\|\partial_t\left(|J_\lambda u_\lambda|J_\lambda u_\lambda\right)\right\|_2^2.\end{aligned} \tag{5.16}$$

Here, we also used the fact that

$$\begin{aligned}&(J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t), \partial\psi_\lambda(u_\lambda(t+h)) - \partial\psi_\lambda(u_\lambda(t)))\\ &\quad\geqslant \left\|\nabla\left(J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)\right)\right\|_2^2 + \frac{3}{4}\left\|(|J_\lambda u_\lambda|J_\lambda u_\lambda)(t+h) - (|J_\lambda u_\lambda|J_\lambda u_\lambda)(t)\right\|_2^2\end{aligned}$$

by (5.13). By combining all these facts,

$$\begin{aligned}\|\nabla(J_\lambda u_\lambda)_t\|_2^2 + \frac{3}{4}\left\|\partial_t\left(|J_\lambda u_\lambda|J_\lambda u_\lambda\right)\right\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial\psi_\lambda(u_\lambda) - \kappa u_\lambda\|_2^2&\\ \leqslant \kappa\|\partial_t u_\lambda\|_2^2 = -\kappa\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u_\lambda(t)),&\end{aligned}$$

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where $E_\lambda(w) := \psi_\lambda(w) - (\kappa/2)\|w\|_2^2$. Integrate both sides over $(0,t)$ to see that

$$\int_0^t \left(\|\nabla(J_\lambda u_\lambda)_\tau\|_2^2 + \frac{3}{4}\left\|\partial_\tau\left(|J_\lambda u_\lambda| J_\lambda u_\lambda\right)\right\|_2^2 \right) d\tau + \frac{1}{2}\|\partial\psi_\lambda(u_\lambda(t)) - \kappa u_\lambda(t)\|_2^2 + \kappa E_\lambda(u_\lambda(t)) \leqslant \frac{1}{2}\|\partial\psi_\lambda(u_0) - \kappa u_0\|_2^2 + \kappa E_\lambda(u_0). \tag{5.17}$$

Thus,

$$(J_\lambda u_\lambda)_t \to u_t \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{5.18}$$

$$\partial_t\left(|J_\lambda u_\lambda| J_\lambda u_\lambda\right) \to \partial_t(|u|u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{5.19}$$

Passing to the limit in (5.17) as $\lambda \to 0$ and recalling that $u \in C_w([0,T];H^2(\Omega)\cap L^6(\Omega))$, we have

$$\int_0^t \left(\|\nabla u_\tau\|_2^2 + \frac{3}{4}\left\|\partial_\tau\left(|u|u\right)\right\|_2^2 \right) d\tau + \frac{1}{2}\|\partial\psi(u(t)) - \kappa u(t)\|_2^2 + \kappa E(u(t)) \leqslant \frac{1}{2}\|\partial\psi(u_0) - \kappa u_0\|_2^2 + \kappa E(u_0)$$

for all $t \in (0,T)$. Furthermore, one can also derive (4.18) (with $\|\eta_0\|_2$ replaced by $\|(\varDelta u_0 - u_0^3 + \kappa u_0)_-\|_2$). Then, (4.21) also follows immediately from (4.18) as in Section 4.

5.4 Proof of (ii)

We next prove (ii). Take an approximate sequence $(u_{0,n})$ satisfying

$$u_{0,n} \in D_r, \quad u_{0,n} \to u_0 \quad \text{strongly in } H_0^1(\Omega)\cap L^4(\Omega). \tag{5.20}$$

Since $u_{0,n}$ fulfils (5.7), the solution u_n of (P) with u_0 replaced by $u_{0,n}$ and the section $\eta_n \in \partial I_{[0,\infty)}(\partial_t u_n)$ as in (3.1) satisfy all energy inequalities that have been justified in the proof of (iii). Here, we mainly use (4.1)–(4.3) and (4.13) and note by (5.2) and (5.20) that

$$E(u_{0,n}) \to E(u_0), \quad \|\eta_n(t)\|_2^2 \leqslant \left\|\left(\varDelta u_{0,n} - u_{0,n}^3 + \kappa u_{0,n}\right)_-\right\|_2^2 \leqslant r \quad \text{for a.e. } t > 0.$$

Hence, by a priori estimates (4.1)–(4.3) and (4.13) for u_n, one can obtain, up to a (not relabelled) subsequence $n \to \infty$,

$$\begin{aligned} u_n \to u \quad & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{weakly star in } L^\infty(0,T;H_0^1(\Omega)\cap L^4(\Omega)), \\ & \text{strongly in } C([0,T];H), \\ -\varDelta u_n + u_n^3 \to -\varDelta u + u^3 \quad & \text{weakly in } L^2(0,T;H), \\ \eta_n \to \eta \quad & \text{weakly star in } L^\infty(0,T;H), \end{aligned}$$

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which also implies $u(t) \in D(\partial\psi)$ for a.e. $t \in (0,T)$ and $u_t + \eta - \Delta u + u^3 = \kappa u$ a.e. in $\Omega \times (0,T)$. Moreover, as in the proof of (5.9), one finds that

$$\int_0^T E(u_n(t))\,\mathrm{d}t \to \int_0^T E(u(t))\,\mathrm{d}t.$$

We next identify the limit η. We see that

$$\int_0^T (\eta_n, \partial_t u_n)\,\mathrm{d}t = -\int_0^T \|\partial_t u_n\|_2^2 - E(u_n(T)) + E(u_{0,n}),$$

which implies

$$\limsup_{n\to\infty} \int_0^T (\eta_n, \partial_t u_n)\,\mathrm{d}t \leqslant -\int_0^T \|u_t\|_2^2 - E(u(T)) + E(u_0) = \int_0^T (\eta, u_t)\,\mathrm{d}t.$$

Hence, by Minty's trick, we conclude that $u_t \geqslant 0$ and $\eta \in \partial I_{[0,\infty)}(u_t)$ a.e. in $\Omega\times(0,T)$. Since the function $t \mapsto u(t)$ is weakly continuous on $[0,T]$ with values in $H_0^1(\Omega)\cap L^4(\Omega)$ (see [44]) and the function $t \mapsto \phi(u(t))$ is (absolutely) continuous on $[0,T]$ (by $u_t \in L^2(0,T;H)$ and $-\Delta u + u^3 \in L^2(0,T;H)$), we also assure by the uniform convexity of $H_0^1(\Omega)\cap L^4(\Omega)$ that

$$u \in C([0,T]; H_0^1(\Omega)\cap L^4(\Omega)).$$

Concerning energy inequalities, (4.1)–(4.3) are (rigorously) derived as in Section 4. Moreover, (5.1) is proved as in the proof of (iii). Hence, (4.7)–(4.14) can be also rigorously derived with $\|\eta_0\|_2^2$ replaced by r. Moreover, combining (4.18) with (4.13) for u_n, one can verify

$$\begin{aligned}
t^{1/2}\partial_t u_n &\to t^{1/2}u_t && \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\
t^{1/2}\partial_t(|u_n|u_n) &\to t^{1/2}\partial_t(|u|u) && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
t^{1/2}\Delta u_n &\to t^{1/2}\Delta u && \text{weakly star in } L^\infty(0,T;L^2(\Omega)),\\
t^{1/2}u_n^3 &\to t^{1/2}u^3 && \text{weakly star in } L^\infty(0,T;L^2(\Omega)),
\end{aligned}$$

which also yields (4.18)–(4.21). Thus, (ii) has been proved.

5.5 Proof of (i)

Finally, let us prove (i). To this end, take $u_{0,n}$ satisfying

$$u_{0,n} \in D_r, \quad u_{0,n} \to u_0 \quad \text{strongly in } L^2(\Omega). \tag{5.21}$$

The solution u_n of (P) with u_0 replaced by $u_{0,n}$ and the section η_n of $\partial I_{[0,\infty)}(\partial_t u_n)$ satisfy all the energy inequalities that are justified in (iii). Here, we mainly use (5.2), (4.7)–(4.9), (4.14) and (4.19) (with $\|\eta_0\|_2$ replaced by $\|(\Delta u_{0,n} - u_{0,n}^3 + \kappa u_{0,n})_-\|_2$) for u_n along with the fact that

$$\|\eta_n(t)\|_2^2 \leqslant \|\left(\Delta u_{0,n} - u_{0,n}^3 + \kappa u_{0,n}\right)_-\|_2^2 \leqslant r \quad \text{for a.e. } t > 0.$$

Moreover, (4.9) yields

$$\int_0^t \tau\|\partial_\tau u_n\|_2^2 \, d\tau + tE(u_n(t)) \leqslant \frac{C_1}{2} t \left(1 + \|\eta_0\|_2^{4/3}\right) + \frac{1}{4}\|u_{0,n}\|_2^2,$$

which implies

$$E(u_n(t)) \leqslant \frac{C_1}{2}\left(1 + r^{2/3}\right) + \frac{1}{4t}\|u_{0,n}\|_2^2 \quad \text{for any} \quad t > 0. \tag{5.22}$$

Due to the lack of the convergence $E(u_{0,n}) \to E(u_0)$, we need an extra argument. One can obtain the following estimate for solutions u of (P) in the dual space $V^* = H^{-1}(\Omega) + L^{4/3}(\Omega)$ of $V = H_0^1(\Omega) \cap L^4(\Omega)$:

$$\begin{aligned}\int_0^T \|u_t\|_{V^*}^{4/3} \, dt &\leqslant C \int_0^T \left(\|\eta\|_2^{4/3} + \|\Delta u\|_{V^*}^{4/3} + \|u^3\|_{V^*}^{4/3} + \|u\|_2^{4/3}\right) dt \\ &\leqslant C \int_0^T \left(\|\eta\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{L^4(\Omega)}^4 + \|u\|_2^2 + 1\right) dt.\end{aligned}$$

By Aubin–Lions–Simon's compactness lemma along with the compact embeddings $V \hookrightarrow L^2(\Omega) \equiv (L^2(\Omega))^* \hookrightarrow V^*$, it follows that

$$\begin{aligned} u_n \to u \quad & \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \\ & \text{weakly in } W^{1,4/3}(0,T;V^*) \cap L^2(0,T;H_0^1(\Omega)) \cap L^4(0,T;L^4(\Omega)), \\ & \text{strongly in } L^2(0,T;L^2(\Omega)) \cap C([0,T];V^*), \\ \eta_n \to \eta \quad & \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Moreover, $u \in C_w([0,T];L^2(\Omega))$ and $u(0) = u_0$. Let $\delta \in (0,T)$ be arbitrarily fixed. Then, it follows from (4.9), (4.14) and (4.19) for u_n that

$$\begin{aligned} u_n &\to u && \text{strongly in } C([\delta,T];L^2(\Omega)), \\ t^{1/2}\partial_t u_n &\to t^{1/2}u_t && \text{weakly in } L^2(0,T;L^2(\Omega)), \\ t^{1/2}u_n &\to t^{1/2}u && \text{weakly star in } L^\infty(0,T;H_0^1(\Omega)), \\ t^{1/4}u_n &\to t^{1/4}u && \text{weakly in } L^\infty(0,T;L^4(\Omega)), \\ t\left(-\Delta u_n + u_n^3\right) &\to t\left(-\Delta u + u^3\right) && \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \end{aligned}$$

and hence, $u_t + \eta - \Delta u + u^3 = \kappa u$ a.e. in $\Omega \times (0,T)$. Here, we used the demi-closedness of maximal monotone operators to identify the limit. Moreover, from the arbitrariness of $\delta > 0$, we see that $u \in C((0,T];L^2(\Omega))$. We claim that $u(t) \to u_0$ strongly in $L^2(\Omega)$ as $t \to 0_+$, which also implies $u \in C([0,T];L^2(\Omega))$. Indeed, since $u(t) \to u_0$ weakly in $L^2(\Omega)$ as $t \to 0_+$, by (4.8) and (5.21),

$$\|u_0\|_2 \leqslant \liminf_{t\to 0_+} \|u(t)\|_2 \leqslant \limsup_{t\to 0_+} \|u(t)\|_2 \leqslant \|u_0\|_2,$$

which concludes that $u(t) \to u_0$ strongly in $L^2(\Omega)$ as $t \to 0_+$. Thus, we obtain $u \in C([0,T];L^2(\Omega))$.

Now, it remains to identify the limit η of $\eta_n \in \partial I_{[0,\infty)}(\partial_t u_n)$. To this end, let $\varepsilon \in (0, T)$ be a constant, and observe that

$$\limsup_{n\to\infty} \int_\varepsilon^T (\eta_n, \partial_t u_n)\, \mathrm{d}t \overset{(1.6)}{\leqslant} -\liminf_{n\to\infty} \int_\varepsilon^T \|\partial_t u_n\|_2^2\, \mathrm{d}t - \liminf_{n\to\infty} E(u_n(T)) + \limsup_{n\to\infty} E(u_n(\varepsilon)).$$

By Aubin–Lions–Simon's compactness lemma along with the compact embedding $H^2(\Omega)\cap L^6(\Omega) \hookrightarrow H^1(\Omega) \cap L^4(\Omega)$, for any $\delta > 0$, we see that

$$u_n \to u \quad \text{strongly in } C([\delta, T]; H^1_0(\Omega) \cap L^4(\Omega)),$$

which particularly implies

$$u_n(t) \to u(t) \quad \text{strongly in } H^1_0(\Omega) \cap L^4(\Omega)$$

for $t \in (0, \infty)$. Therefore, for any $\varepsilon > 0$, we conclude that

$$E(u_n(\varepsilon)) \to E(u(\varepsilon)).$$

Here, we also remark that due to (5.22), $E(u(\varepsilon))$ is estimated by

$$E(u(\varepsilon)) \leqslant \frac{C_1}{2}\left(1 + r^{2/3}\right) + \frac{1}{4\varepsilon}\|u_0\|_2^2 \quad \text{for any} \quad \varepsilon > 0.$$

It follows that

$$\limsup_{n\to\infty} \int_\varepsilon^T (\eta_n, \partial_t u_n)\, \mathrm{d}t \leqslant -\int_\varepsilon^T \|\partial_t u\|_2^2\, \mathrm{d}t - E(u(T)) + E(u(\varepsilon)) = \int_\varepsilon^T (\eta, u_t)\, \mathrm{d}t,$$

and therefore, due to Minty's trick (see [20]), we conclude that $\eta \in \partial I_{[0,\infty)}(u_t)$ a.e. in $\Omega \times (\varepsilon, T)$ (see Section 5.4). Since one can also take $\varepsilon > 0$ arbitrarily close to zero, the desired conclusion is obtained. As for the energy inequalities, the idea of derivation is basically same as the proof of (ii). This completes the proof of Theorem 5.1.

6 Reformulation of (P) as an obstacle problem

In this section we shall verify that (1.6) is equivalently rewritten as a parabolic variational inequality of obstacle type. Such a reformulation will not only shed light on a characteristic behavior of solutions but also play an important role to reveal the long-time behavior of each solution (see Section 10). Moreover, it will be also employed to discuss the uniqueness of solutions and a comparison principle for (P) (see Section 7) as well as to investigate Lyapunov stability of equilibria in a forthcoming paper (see [2]).

Our result of this section is stated in the following:

Theorem 6.1 (Reformulation of (P) as an obstacle problem) *For* $u_0 \in \overline{D_r}^{L^2}$, *the Cauchy–Dirichlet problem* (P) *admits a solution* $u = u(x,t)$ *which also solves*

$$u_t + \partial I_{[u_0(x),\infty)}(u) - \Delta u + u^3 - \kappa u \ni 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{6.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{6.2}$$

$$u = u_0 \quad \text{in} \quad \Omega, \tag{6.3}$$

where $\partial I_{[u_0(x),\infty)}$ *is the sub-differential operator of the indicator function* $I_{[u_0(x),\infty)}$ *over* $[u_0(x),\infty)$. *Hence, the section* η *of* $\partial I_{[0,\infty)}(u_t)$ *as in* (3.1) *also belongs to* $\partial I_{[u_0(x),\infty)}(u)$ *for a.e. in* $\Omega \times (0,\infty)$. *Such a solution to* (P) *is uniquely determined by the initial datum* u_0. *Furthermore,* (P) *is equivalently rewritten as* (6.1)–(6.3), *provided that the solution of* (P) *is unique.*

Remark 6.2

(i) *In this paper, it is not proved that* all *solutions to (P) solve* (6.1)–(6.3), *unless solutions to (P) are uniquely determined by initial data. Since Theorem 6.1 will be proved through the approximation* (4.25) *of* (1.6), *the equivalence of two problems will be ensured only for the solutions constructed by the approximation as in Section 4 (see also Section 5).*

(ii) *The theorem stated previously also provides a* selection principle *for (P). Indeed, for* $u_0 \in \overline{D_r}^{L^2}$, *the uniqueness of solutions is not generally ensured. However, according to Theorem 6.1, (P) always possesses one and only one solution which also solves* (6.1)–(6.3). *Moreover, as discussed in Section 5, selected solutions fulfil energy inequalities derived in Section 4. Such a selection principle will be used to consider the DS generated by (P) and to prove the convergence of solutions as* $t \to +\infty$.

(iii) *It is noteworthy that the fully non-linear problem* (1.2) *is now converted to a* semi-linear *obstacle problem* (6.1). *However, such a semi-linear problem still involves another difficulty, since the obstacle function* u_0 *is supposed to lie on the* L^2 *closure of* D_r *and the problem is posed on the* L^2 *(i.e., strong) framework. On the other hand, it is also known (see [28]) that uniformly elliptic fully non-linear equations of the form* $f(D^2u) = 0$ *can be reduced to a quasi-linear one, provided that* f *is smooth enough (e.g., of class* $C^{3,\alpha}$*). However, it is not applicable to* (1.2), *for the corresponding* f *is not so smooth and not uniformly elliptic.*

Remark 6.3 (Parabolic obstacle problem) *Problem* (6.1)–(6.3) *can be equivalently rewritten as follows:*

$$u \geqslant u_0, \quad u_t - \Delta u + u^3 - \kappa u \geqslant 0 \quad \text{in} \quad \Omega \times (0,\infty),$$

$$(u - u_0)\left(u_t - \Delta u + u^3 - \kappa u\right) = 0 \quad \text{in} \quad \Omega \times (0,\infty),$$

$$u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0,$$

which is an obstacle problem of parabolic type and where the initial datum u_0 *also plays a role of the obstacle function from below (see [23,43]). One may no longer expect classical* *regularity of solutions to (P). Indeed, let us consider a simpler elliptic obstacle problem, e.g.,*

$$-\Delta\phi(x) \geqslant f(x), \quad \phi(x) \geqslant g(x), \quad (-\Delta\phi(x) - f(x))(\phi(x) - g(x)) = 0 \ \ \textit{in } \Omega$$

along with the homogeneous Dirichlet condition. It is well known that the optimal regularity of solution is $C^{1,1}(\Omega)$ (unless the contact set is non-empty), even though the obstacle function g is sufficiently smooth (e.g., $g \in C^\infty(\overline{\Omega})$) (see [22]).

Proof Let us recall again the approximate problems (5.8) whose solutions are sufficiently smooth in time. Throughout this proof, let $((0,T), \mathfrak{M}_t, \mu_t)$, $(\Omega, \mathfrak{M}_x, \mu_x)$ and $(Q, \mathfrak{M}_{x,t}, \mu_{x,t})$ be the measure spaces of Lebesgue measures with respect to t, x and (x,t), respectively. Moreover, for any $A \in \mathfrak{M}_{x,t}$, we write

$$A_x := \{t \in (0,T) : (x,t) \in Q\} \ \text{ for each } x \in \Omega,$$
$$A_t := \{x \in \Omega : (x,t) \in Q\} \ \text{ for each } t \in (0,T).$$

Then, $A_x \in \mathfrak{M}_t$ for μ_x-a.e. $x \in \Omega$ and $A_t \in \mathfrak{M}_x$ and for μ_t-a.e. $t \in (0,T)$, by Fubini–Tonelli's lemma.

Let u_λ be the solution of (5.8) for $u_0 \in D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$ and let η_λ be the section of $\partial I_{[0,\infty)}(\partial_t u_\lambda)$. We recall that $u_\lambda \in C^{1,1}([0,T]; L^2(\Omega))$ and $\eta_\lambda, \partial\psi_\lambda(u_\lambda) \in C^{0,1}([0,T]; L^2(\Omega))$. Hence, η_λ and $\partial\psi_\lambda(u_\lambda)$ are differentiable μ_t-a.e. in $(0,T)$ with values in $L^2(\Omega)$. Moreover, (by taking a continuous representation of η_λ) it holds that

$$u_\lambda(t) = u_\lambda(s) + \int_s^t \partial_\tau u_\lambda(\tau)\,\mathrm{d}\tau, \quad \eta_\lambda(t) = \eta_\lambda(s) + \int_s^t \partial_\tau \eta_\lambda(\tau)\,\mathrm{d}\tau \ \text{ in } L^2(\Omega)$$

for any $t, s \in [0,T]$. Moreover, recall that u_λ and η_λ satisfy (5.8) in $L^2(\Omega)$ for all $t \in [0,T]$.

Since both sides of (5.8) are differentiable μ_t-a.e. in $(0,T)$, for any $r \in (1,2)$ and $\zeta \in L^q(\Omega)$, $\zeta \geqslant 0$ with $q \in (1,\infty)$ satisfying $1/q + r/2 = 1$, one observes that

$$\int_\Omega \zeta(x)|\eta_\lambda(x,t)|^r\,\mathrm{d}x \leqslant \int_\Omega \zeta(x)|\eta_\lambda(x,s)|^r\,\mathrm{d}x \quad \text{if } \ t \geqslant s \tag{6.4}$$

for all $t, s \in [0,T]$. Indeed, let $\zeta \in C_0^\infty(\Omega)$ be such that $\zeta \geqslant 0$ in Ω and test (5.14) by $\zeta\eta_\lambda$. Then, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{[0,\infty)}(\partial_t u_\lambda) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \zeta\eta_\lambda^2\,\mathrm{d}x + \left(\frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda), \zeta\eta_\lambda\right) = \kappa\int_\Omega \partial_t u_\lambda \eta_\lambda \zeta\,\mathrm{d}x.$$

Here, we notice that $\zeta\eta_\lambda$ also belongs to $\partial I_{[0,\infty)}(\partial_t u_\lambda)$ by $\zeta \geqslant 0$ and $\eta_\lambda \in \partial I_{[0,\infty)}(\partial_t u_\lambda)$, and therefore, by (ii) of Lemma 5.4

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\partial\psi_\lambda(u_\lambda), \zeta\eta_\lambda\right) \geqslant 0.$$

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Then, integrate both sides over (s,t) and employ the facts that $\partial_t u_\lambda \eta_\lambda = 0$ and $I_{[0,\infty)}(\partial_t u_\lambda) = 0$ a.e. in $\Omega \times (0,\infty)$ to obtain

$$\int_\Omega \zeta \eta_\lambda^2(t)\,\mathrm{d}x \leqslant \int_\Omega \zeta \eta_\lambda^2(s)\,\mathrm{d}x \quad \text{for} \quad t \geqslant s \geqslant 0.$$

Likewise, noting $|\eta_\lambda|^{r-2}\eta_\lambda \in \partial I_{[0,\infty)}(\partial_t u_\lambda)$ for any $r \in (1,\infty)$, one can also obtain

$$\int_\Omega \zeta |\eta_\lambda(t)|^r\,\mathrm{d}x \leqslant \int_\Omega \zeta |\eta_\lambda(s)|^r\,\mathrm{d}x \quad \text{for} \quad t \geqslant s \geqslant 0 \quad \text{and} \quad 1 < r < \infty \tag{6.5}$$

for any $\zeta \in C_0^\infty(\Omega)$ satisfying $\zeta \geqslant 0$. In particular, let $r > 1$ be less than 2 and let $q \in (1,\infty)$ and $\zeta \in L^q(\Omega)$ be such that $\zeta \geqslant 0$ and $1/q + r/2 = 1$. Then, one can take $\zeta_n \in C_0^\infty(\Omega)$ such that $\zeta_n \geqslant 0$ and $\zeta_n \to \zeta$ strongly in $L^q(\Omega)$. Moreover, passing to the limit in (6.5) with $1 < r < 2$ and ζ replaced by ζ_n as $n \to \infty$, we deduce that

$$\int_\Omega \zeta |\eta_\lambda(t)|^r\,\mathrm{d}x \leqslant \int_\Omega \zeta |\eta_\lambda(s)|^r\,\mathrm{d}x \quad \text{for} \quad t \geqslant s \geqslant 0 \quad \text{and} \quad 1 < r < 2$$

for any $\zeta \in L^q(\Omega)$, $\zeta \geqslant 0$, $1/q + r/2 = 1$. Thus, (6.4) is proved.

By (6.4), we assure that $|\eta_\lambda(x,t)|$ is non-increasing in t for a.e. $x \in \Omega$. Indeed, suppose on the contrary that $|\eta_\lambda(x,t)| > |\eta_\lambda(x,s)|$ for all $x \in \Omega_0$ and for some $t > s$ and $\Omega_0 \subset \Omega$ satisfying $|\Omega_0| > 0$. Substitute $\zeta = \chi_{\Omega_0}$, which is the characteristics function over Ω_0, into (6.4) to obtain

$$\int_{\Omega_0} |\eta_\lambda(t)|^r\,\mathrm{d}x \leqslant \int_{\Omega_0} |\eta_\lambda(s)|^r\,\mathrm{d}x.$$

However, this fact contradicts the assumption. Hence, $|\eta_\lambda(x,\cdot)|$ is non-increasing in time for a.e. $x \in \Omega$, and furthermore, $\eta_\lambda(x,\cdot)$ is non-decreasing by $\eta_\lambda \leqslant 0$. Since $\eta_\lambda \in C^{0,1}([0,T];L^2(\Omega))$, one can verify that $\partial_t \eta_\lambda(t) \geqslant 0$ μ_x-a.e. in Ω for μ_t-a.e. $t \in (0,T)$.

For each $t \in (0,T)$, define the set $\Omega_t \in \mathfrak{M}_x$ of $x \in \Omega$ satisfying

(i) u_λ and η_λ satisfy (5.8) at (x,t),

(ii) u_λ, η_λ and $\partial\psi_\lambda(u_\lambda)$ are partially differentiable in t at (x,t),

(iii) the following identities hold at (x,t):

$$u_\lambda(x,t) = u_\lambda(x,0) + \int_0^t \partial_\tau u_\lambda(x,\tau)\,\mathrm{d}\tau,$$
$$\eta_\lambda(x,t) = \eta_\lambda(x,0) + \int_0^t \partial_\tau \eta_\lambda(x,\tau)\,\mathrm{d}\tau,$$

(iv) $\partial_t \eta_\lambda(x,t) \geqslant 0$ at (x,t).

Then, $\mu_x(\Omega \setminus \Omega_t) = 0$ for μ_t-a.e. $t \in (0,T)$. Define the set $Q_1 \in \mathfrak{M}_{x,t}$ of $(x,t) \in Q$ satisfying (i)–(iv). Then noting that $\Omega_t = (Q_1)_t$, we find by Fubini–Tonelli's lemma that Q_1 has full measure, that is, $\mu_{x,t}(Q \setminus Q_1) = 0$.

Now, set

$$\Omega_1 := \{x \in \Omega : \mu_t((0,T) \setminus (Q_1)_x) = 0\}.$$

First, we claim that $\Omega_1 \in \mathfrak{M}_x$. Indeed, by Fubini–Tonelli's lemma, we see that $(Q_1)_x \in \mathfrak{M}_t$ for μ_x-a.e. $x \in \Omega$, and moreover, the function $x \mapsto \mu_t((Q_1)_x)$ is $\mathfrak{M}_x$-measurable. Since the function

$$x \mapsto \mu_t((0,T) \setminus (Q_1)_x) = T - \mu_t((Q_1)_x)$$

is also $\mathfrak{M}_x$-measurable, the level set Ω_1 of the $\mathfrak{M}_x$-measurable function also belongs to $\mathfrak{M}_x$.

Next, we claim that $\mu_x(\Omega \setminus \Omega_1) = 0$. Indeed, note that

$$N_1 := \{(x,t) \in Q : x \in \Omega \setminus \Omega_1,\ t \in (0,T) \setminus (Q_1)_x\} \subset Q \setminus Q_1,$$
$$N_2 := \{(x,t) \in Q : x \in \Omega_1,\ t \in (0,T) \setminus (Q_1)_x\} \subset Q \setminus Q_1.$$

Since the measure space $(Q, \mathfrak{M}_{x,t}, \mu_{x,t})$ is complete, the sets N_1 and N_2 also belong to $\mathfrak{M}_{x,t}$. In particular, we obtain $\mu_{x,t}(N_1) = \mu_{x,t}(N_2) = 0$. By Fubini–Tonelli's lemma,

$$\int_{\Omega \setminus \Omega_1} \mu_t((0,T) \setminus (Q_1)_x)\,\mathrm{d}x = \mu_{x,t}(N_1) = 0,$$

which implies $\mu_x(\Omega \setminus \Omega_1) = 0$ by $\mu_t((0,T) \setminus (Q_1)_x) > 0$ for a.e. $x \in \Omega \setminus \Omega_1$.

Furthermore, the set

$$Q_2 := \{(x,t) \in Q : x \in \Omega_1,\ t \in (Q_1)_x\} = (\Omega_1 \times (0,T)) \setminus N_2$$

is $\mathfrak{M}_{x,t}$-measurable and has full measure, that is, $\mu_{x,t}(Q \setminus Q_2) = 0$; indeed, applying Fubini–Tonelli's lemma and combining all the facts obtained so far, we conclude that

$$0 \leqslant \mu_{x,t}(Q \setminus Q_2) \leqslant \mu_{x,t}(N_2) + \mu_x(\Omega \setminus \Omega_1)T = 0.$$

Moreover, Q_2 is a subset of Q_1. Now, we are ready to prove Theorem 6.1. Let $(x_0, t_0) \in Q_2$ be fixed. In case $u_\lambda(x_0,t_0) = u_0(x_0)$, by $\partial I_{[u_0(x),\infty)}(u_\lambda(x_0,t_0)) = (-\infty, 0]$, the relation

$$\partial_t u_\lambda + \partial I_{[u_0(x),\infty)}(u_\lambda) + \partial\psi_\lambda(u_\lambda) - \kappa u_\lambda \ni 0 \tag{6.6}$$

holds true at (x_0,t_0). In case $u_\lambda(x_0,t_0) > u_0(x_0)$ (then, $\partial I_{[u_0(x_0),\infty)}(u_\lambda(x_0,t_0)) = \{0\}$), since $(x_0,t_0), (x_0,t) \in Q_1$ for μ_t-a.e. $t \in (0,T)$, there exists $t_1 \in (0,t_0) \cap (Q_1)_{x_0}$ such that $\partial_t u_\lambda(x_0,t_1) > 0$, which implies $\eta_\lambda(x_0,t_1) = 0$. Moreover, it follows that

$$\begin{aligned} 0 \geqslant \eta_\lambda(x_0,t_0) &= \int_{t_1}^{t_0} \partial_\tau \eta_\lambda(x_0,\tau)\,\mathrm{d}\tau + \eta_\lambda(x_0,t_1) \\ &= \int_{(t_1,t_0)\cap(Q_1)_{x_0}} \partial_\tau \eta_\lambda(x_0,\tau)\,\mathrm{d}\tau + \eta_\lambda(x_0,t_1) \geqslant 0, \end{aligned}$$

which implies $\eta_\lambda(x_0,t_0) = 0$. Thus, (6.6) is satisfied at (x_0,t_0). In particular, the section η_λ of $\partial I_{[0,\infty)}(\partial_t u_\lambda)$ also belongs to the set $\partial I_{[u_0(x_0),\infty)}(u_\lambda)$ for $\mu_{x,t}$-a.e. in Q.

Recalling the convergence as $\lambda \to 0$ of solutions u_λ for (5.8) obtained in Section 5, one can deduce that

$$\eta_\lambda \to \eta \in \partial I_{[u_0(x),\infty)}(u) \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega))$$

by the demi-closedness of maximal monotone operators. Hence, the limit u of u_λ also solves (6.1)–(6.3).

We next consider the case that $u_0 \in \overline{D_r}^{L^2}$. Then, let us take $u_{0,n} \in D_r$ such that

$$u_{0,n} \to u_0 \quad \text{strongly in } L^2(\Omega). \tag{6.7}$$

Let u_n be the solution of (P) with the initial data $u_{0,n}$ such that u_n also solves (6.1)–(6.3) with u_0 replaced by $u_{0,n}$. In particular, the section η_n of $\partial I_{[0,\infty)}(\partial_t u_n)$ as in (3.1) also belongs to the set $\partial I_{[u_{0,n}(x),\infty)}(u_n)$. On the other hand, by (6.7), it holds that $I_{[u_{0,n}(x),\infty)} \to I_{[u_0(x),\infty)}$ on $L^2(\Omega)$ in the sense of Mosco (see Lemma 6.4 and [9]). Therefore, from the convergence of u_n obtained in Section 5.4, we deduce that the limit η of η_n fulfils

$$\eta \in \partial I_{[u_0(x),\infty)}(u) \quad \text{for a.e. } t \in (0, T).$$

Lemma 6.4 *Let $u_{0,n}, u_0 \in L^2(\Omega)$ be such that $u_{0,n} \to u_0$ strongly in $L^2(\Omega)$. Then, $I_{[u_{0,n}(x),\infty)} \to I_{[u_0(x),\infty)}$ on $L^2(\Omega)$ in the sense of Mosco.*

Proof *Existence of recovery sequences*: For each $w \in D(I_{[u_0(x),\infty)})$, define a recovery sequence $w_n := w - u_0 + u_{0,n} \in L^2(\Omega)$. Then, $w_n \geqslant u_{0,n}$, which gives $w_n \in D(I_{[u_{0,n}(x),\infty)})$. Moreover, $w_n \to w$ strongly in $L^2(\Omega)$ by assumption.
Weak lim inf convergence: Let $w_n, w \in L^2(\Omega)$ be such that $w_n \to w$ weakly in $L^2(\Omega)$. We shall check that

$$\liminf_{n\to\infty} I_{[u_{0,n}(x),\infty)}(w_n) \geqslant I_{[u_0(x),\infty)}(w).$$

In case $\liminf_{n\to\infty} I_{[u_{0,n}(x),\infty)}(w_n) = \infty$, the assertion follows immediately. In case $\liminf_{n\to\infty} I_{[u_{0,n}(x),\infty)}(w_n) < \infty$, up to a (not relabelled) subsequence, $I_{[u_{0,n}(x),\infty)}(w_n)$ is bounded. Hence, $w_n \geqslant u_{0,n}$ a.e. in Ω. For each $z \in C_0^\infty(\Omega)$ satisfying $z \geqslant 0$, it follows that

$$\int_\Omega w_n z \, dx \geqslant \int_\Omega u_{0,n} z \, dx.$$

Letting $n \to \infty$ and using the arbitrariness of z, we conclude that $w \geqslant u_0$ a.e. in Ω. Thus, $I_{[u_0(x),\infty)}(w) = 0$, and hence, the assertion follows. Consequently, $I_{[u_{0,n}(x),\infty)} \to I_{[u_0(x),\infty)}$ on $L^2(\Omega)$ in the sense of Mosco. □

One can prove in a standard way that the solution of (6.1)–(6.3) is uniquely determined by the initial datum u_0 (cf. Theorem 7.2). Hence, it turns out that (P) and (6.1)–(6.3) are equivalent to each other, provided that the solution to (P) is unique. This completes the proof of Theorem 6.1. □

We close this section by giving a proof for the energy inequalities (4.5), (4.6), (4.15) and (4.17), which have not yet been rigorously proved.

Corollary 6.5 *If $u = u(x,t)$ also solves* (P) *as well as the obstacle problem* (6.1)–(6.3) (*see Theorem 6.1*), *then* (4.6), (4.15) *and* (4.17) *hold. In addition, if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$, then* (4.5) *is also satisfied with $\|v_0\|_2^2$ replaced by $\|(\Delta u_0 - u_0^3 + \kappa u_0)_+\|_2^2$.*

Proof Let $u = u(x,t)$ be a solution to (P) which also solves (6.1)–(6.3). By Theorem 6.1, it is uniquely determined by u_0 (and actually exists). Therefore, by the proofs of (i)–(iii) of Theorem 5.1, $u(x,t)$ satisfies the energy inequalities which have already been verified in Section 5.1–5.5 and is obtained as a limit of unique solutions u_λ to (5.8) as $\lambda \to 0$.

Energy Inequality (4.6) By (5.11), for each $s \in [0,T)$ at which $\eta(s)$ satisfies (1.6), we can construct a solution to (P) with the initial datum $u(s)$ and deduce by (5.2) and the uniqueness of solutions that

$$\|\eta(t)\|_2^2 \leqslant \|\eta(s)\|_2^2 \quad \text{for a.e. } \ t \in (s,T). \tag{6.8}$$

We remark that the set of $t \in (s,T)$ at which (6.8) is satisfied may depend on the choice of s. We further claim that

$$\|\eta(t)\|_2^2 \leqslant \|\eta(s)\|_2^2 \quad \text{for a.e. } (s,t) \in \{(\sigma,\tau) \in [0,T]^2 : \sigma \leqslant \tau\} \tag{6.9}$$

(hereafter, we also simply write (4.6) instead of (6.9)). Indeed, the subset $I = \{(\sigma,\tau) \in [0,T]^2 : \sigma \leqslant \tau,\ \|\eta(\tau)\|_2 > \|\eta(\sigma)\|_2\}$ is (Lebesgue) measurable due to the measurability of $t \mapsto \|\eta(t)\|_2$. Hence, since $I_\sigma := \{\tau \in [\sigma,T] : (\sigma,\tau) \in I\}$ has Lebesgue measure zero for a.e. $\sigma \in (0,T)$, so is I by Fubin–Tonelli's lemma. Thus, (6.9) follows.

Energy Inequalities (4.15) and (4.17) Similarly, we can also prove by uniqueness that

$$\begin{aligned}
&\int_s^t \left(\|\nabla u_\tau\|_2^2 + \frac{3}{4} \left\|\partial_\tau \left(|u|u\right)\right\|_2^2 \right) \mathrm{d}\tau \\
&+ \frac{1}{2} \|\partial\psi(u(t)) - \kappa u(t)\|_2^2 + \kappa E(u(t)) \leqslant \frac{1}{2} \|\partial\psi(u(s)) - \kappa u(s)\|_2^2 + \kappa E(u(s)) \\
&\qquad \text{for a.e. } \ 0 < s < t < T.
\end{aligned}$$

In particular, the function $t \mapsto (1/2)\,\|\partial\psi(u(t)) - \kappa u(t)\|_2^2 + \kappa E(u(t))$ is non-increasing, and hence, it is differentiable a.e. in $(0,T)$. Dividing both sides by $t - s$ and taking a limit as $s \nearrow t$, we obtain (4.15). Furthermore, (4.17) also follows in a similar way.

Energy Inequality 2 Here, we suppose that u_0 satisfies (5.7). Multiplying (5.14) by v_λ, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\lambda\|_2^2 + \int_\Omega (\partial_t \eta_\lambda) v_\lambda \,\mathrm{d}x + \left(\frac{\mathrm{d}}{\mathrm{d}t} \partial\psi_\lambda(u_\lambda), v_\lambda \right) = \kappa \|v_\lambda\|_2^2.$$

Here, we remark that

$$\int_\Omega (\partial_t \eta_\lambda) v_\lambda \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} I^*_{[0,\infty)}(\eta_\lambda) = 0$$

by $v_\lambda \in \partial I^*_{[0,\infty)}(\eta_\lambda)$. Therefore, we find by (5.16) that

$$\frac{1}{2}\partial_t \|v_\lambda\|_2^2 + \|\nabla (J_\lambda u_\lambda)_t\|_2^2 + \frac{3}{4} \left\|\partial_t \left(|J_\lambda u_\lambda| J_\lambda u_\lambda\right)\right\|_2^2 \,\mathrm{d}x \leqslant \kappa \|v_\lambda\|_2^2.$$

To apply the convergence obtained so far (e.g., (5.18) and (5.19)) and employ the weak lower semi-continuity of norms,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\kappa t}\|v_\lambda\|_2^2\right) + e^{-2\kappa t}\|\nabla(J_\lambda u_\lambda)_t\|_2^2 \leqslant 0.$$

Integrate both sides over (s,t), pass to the limit as $\lambda \to 0$, divide both sides of the resulting inequality by $t-s$ and take the limit as $s \nearrow t$. Then, one can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\kappa t}\|u_t\|_2^2\right) + e^{-2\kappa t}\|\nabla u_t\|_2^2 \leqslant 0 \quad \text{for a.e. } 0 < t < T,$$

which implies (4.5). Thus, all energy inequalities (for u_0 satisfying (5.7)) obtained in Section 4 along with (iii) of Theorem 3.2 have been rigorously reproduced. This completes the proof. □

7 Comparison principle

This section is devoted to proving a comparison principle for the obstacle problem (6.1) as well as the strongly irreversible Allen–Cahn equation (1.2) (or equivalently, (1.6)). Let us begin with the definition of L^2 sub- and super-solutions of (6.1) (and (1.6)).

Definition 7.1 *Let $T \in (0,\infty)$ be fixed. A function $u \in C([0,T];L^2(\Omega))$ is said to be an L^2* sub-solution (*or sub-L^2-solution*) *of* (6.1) *on $Q_T = \Omega \times (0,T)$, if the following conditions are all satisfied:*

(i) *u belongs to the same class as in* (i) *of Definition 3.1,*

(ii) *there exists $\eta \in L^\infty(0,T;L^2(\Omega))$ such that*

$$u_t + \eta - \Delta u + u^3 - \kappa u \leqslant 0, \quad \eta \in \partial I_{[u_0(x),\infty)}(u) \ \text{ for a.e. } (x,t) \in \Omega \times (0,T). \tag{7.1}$$

A function $u \in C([0,T];L^2(\Omega))$ is said to be an L^2 super-solution (*super-L^2-solution*) *of* (6.1) *on $Q_T = \Omega \times (0,T)$, if* (i) *and* (ii) *are satisfied with the inverse inequality of* (7.1). *Furthermore, a sub- and a super-L^2-solution of* (1.6) *are also defined by replacing the inclusion of* (7.1) *with $\eta \in \partial I_{[0,\infty)}(u_t)$.*

Our result reads,

Theorem 7.2 (Comparison principle for (6.1)) *Let u and v be a sub- and a super-L^2-solution for* (6.1) *with the obstacle function replaced by $u_0 = u(0)$ and $v_0 = v(0)$, respectively, in $Q_T = \Omega \times (0,T)$ for some $T > 0$. Suppose that $u \leqslant v$ a.e. on the parabolic boundary $\partial_p Q_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Then, it holds that*

$$u \leqslant v \quad \text{a.e. in } Q_T.$$

In particular, the solution of (6.1)–(6.3) *is unique.*

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Proof By subtracting inequalities (see (7.1)) and by setting $w := u - v$, we see that

$$w_t - \Delta w + u^3 - v^3 \leqslant \kappa w + \nu - \mu \ \text{ in } Q_T,$$

where μ and ν are sections of $\partial I_{[u_0(x),\infty)}(u)$ and $\partial I_{[v_0(x),\infty)}(v)$, respectively. Test both sides by w_+. Then, we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_+\|_2^2 \leqslant \kappa\|w_+\|_2^2 + \int_\Omega (\nu - \mu)w_+ \,\mathrm{d}x \quad \text{for a.e. } 0 < t < T.$$

Here, we observe by $\nu \leqslant 0$ and $\mu \in \partial I_{[u_0(x),\infty)}(u)$ that

$$\begin{aligned}\int_\Omega (\nu - \mu)w_+ \,\mathrm{d}x &= \int_\Omega \nu w_+ \,\mathrm{d}x - \int_\Omega \mu w_+ \,\mathrm{d}x \\ &\leqslant - \int_{\{u=u_0\}\cap\{u\geqslant v\}} \mu w \,\mathrm{d}x.\end{aligned}$$

Due to the fact that $v \geqslant v_0 \geqslant u_0$ a.e. in Ω, one of the following (i) and (ii) holds: (i) the set $\{u = u_0\} \cap \{u \geqslant v\}$ has Lebesgue measure zero; (ii) $w = 0$ for a.e. $x \in \{u = u_0\} \cap \{u \geqslant v\}$. Hence, it follows that

$$\int_{\{u=u_0\}\cap\{u\geqslant v\}} \mu w \,\mathrm{d}x = 0.$$

Combining all these facts, we deduce that

$$\int_\Omega (\nu - \mu)w_+ \,\mathrm{d}x \leqslant 0.$$

Therefore, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_+\|_2^2 \leqslant \kappa\|w_+\|_2^2 \quad \text{for a.e. } 0 < t < T,$$

and hence, applying Gronwall's inequality, we conclude that $w_+ \equiv 0$ a.e. in Q_T, which completes the proof. □

Now, we exhibit a range-preserving property of solutions to (P) in the following:

Corollary 7.3 *Let u be the* unique *solution of* (P) *such that u also solves* (6.1)–(6.3) (*see Theorem 6.1*). *Assume $u_0 \in L^\infty(\Omega)$. Then, it holds that*

$$u_0(x) \leqslant u(x,t) \leqslant \max\left\{\sqrt{\kappa}, \|u_0\|_{L^\infty(\Omega)}\right\} \quad \text{a.e. in} \quad \Omega \times (0,\infty),$$

and hence $u \in L^\infty(\Omega \times (0,\infty))$.

Proof Due to the non-decrease of $u(x,t)$, it follows immediately that

$$u_0(x) \leqslant u(x,t) \ \text{ a.e. in } \Omega \times (0,\infty).$$

On the other hand, by assumption, u is also a solution of (6.1). Moreover, the constant function $U(x,t) \equiv \max\left\{\sqrt{\kappa}, \|u_0\|_{L^\infty(\Omega)}\right\} \geqslant \sqrt{\kappa}$ turns out to be a super-solution of (6.1), and furthermore, one can observe that

$$u(x,t) \leqslant U(x,t) \quad \text{a.e. on } \partial_p Q_T \quad \text{for any } T > 0.$$

Thus, by Theorem 7.2, we deduce that $u(x,t) \leqslant \max\left\{\sqrt{\kappa}, \|u_0\|_{L^\infty(\Omega)}\right\}$ a.e. in Q_T for any $T > 0$. □

As for (1.2) (or equivalently (1.6)), we shall exhibit two comparison principles under different additional assumptions. The following theorem provides a comparison principle for classical solutions of (1.2):

Theorem 7.4 (Comparison principle for classical solutions to (1.2)) *Let u and v be a sub- and a super-$C^{2,1}$-solution for* (1.2) *in $Q_T = \Omega \times (0,T)$ for some $T > 0$, respectively. Suppose, that $u \leqslant v$ a.e. on the parabolic boundary $\partial_p Q_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T)$. Then, it holds that*

$$u \leqslant v \quad \text{a.e. in } Q_T.$$

Proof Let u and v be a sub- and a super-solution for (P), respectively. Then, it holds that

$$\begin{aligned}\partial_t(u - v) &\leqslant \left(\Delta u - u^3 + \kappa u\right)_+ - \left(\Delta v - v^3 + \kappa v\right)_+ \\ &\leqslant \left(\Delta w - u^3 + v^3 + \kappa w\right)_+,\end{aligned}$$

where we set $w := u - v$. Let $\alpha > 0$ be fixed so that $r \mapsto \kappa r_+ - \alpha r$ is strictly decreasing (e.g., $\alpha > \kappa$). Subtracting αw from both sides, one has

$$w_t - \alpha w \leqslant \left(\Delta w - u^3 + v^3 + \kappa w\right)_+ - \alpha w.$$

Multiply both sides by $e^{-\alpha t}$ and set $z := e^{-\alpha t} w$. It then follows that

$$z_t \leqslant \left(\Delta z - e^{-\alpha t}(u^3 - v^3) + \kappa z\right)_+ - \alpha z.$$

We claim that

$$z \leqslant 0 \quad \text{in } Q := \Omega \times (0,T],$$

which also implies

$$u \leqslant v \quad \text{in } Q.$$

Indeed, assume on the contrary that

$$z(x_0, t_0) > 0$$

at some $(x_0, t_0) \in Q$. Then

$$\sup_{(x,t)\in Q} z(x,t) > 0,$$

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where the supremum is achieved by some $(x_1, t_1) \in \Omega \times (0, T]$. Then by Taylor's expansion,

$$z_t \geqslant 0, \quad \nabla z = 0, \quad \Delta z \leqslant 0 \quad \text{at} \ \ (x_1, t_1).$$

Hence

$$0 \leqslant z_t \leqslant \left(\Delta z - e^{-\alpha t}(u^3 - v^3) + \kappa z\right)_+ - \alpha z \leqslant \kappa z_+ - \alpha z < 0 \quad \text{at} \ \ (x_1, t_1).$$

This yields a contradiction. Thus $z \geqslant 0$ on Q. □

One can also prove a comparison principle for *strictly increasing* L^2-sub-solutions for (1.6).

Proposition 7.5 *Let u be an L^2-sub-solution of* (1.6) *in Q_T satisfying $u_t > 0$ and let v be an L^2-super-solution of* (1.6) *in Q_T. Suppose that $u \leqslant v$ a.e. on $\partial_p Q_T$. Then, it holds that*

$$u \leqslant v \quad \text{a.e. in } Q_T.$$

Proof By assumption, we find that $\partial I_{[0,\infty)}(u_t) = \{0\}$, and therefore, (1.6) holds with $\eta \equiv 0$. Subtract inequalities to see that

$$w_t - \nu - \Delta w + u^3 - v^3 \leqslant \kappa w \ \ \text{in } Q_T,$$

where $w := u - v$ and ν is a section of $\partial I_{[0,\infty)}(v_t)$. The multiplication of the both sides and w_+ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_+\|_2^2 - \int_\Omega \nu w_+ \, \mathrm{d}x + \|\nabla w_+\|_2^2 \leqslant \kappa \|w_+\|_2^2.$$

Here, recall that $\nu \leqslant 0$, and therefore, by Gronwall's inequality, we conclude that $w_+ \equiv 0$, that is, $u \leqslant v$ in Q_T. □

8 Phase set, semi-group and compact absorbing set

The following two sections are devoted to constructing a global attractor of the DS generated by (P) as well as (6.1)–(6.3). We emphasize again that due to the strong irreversibility, global attractor does not exist in any L^p-spaces. So, we need a customized setting to extract energy-dissipation structures of the equation and to construct a global attractor. We start with setting up a (non-linear) phase set and a metric on it.

Let $r > 0$ be arbitrarily fixed and set a phase set $D = D_r$ (see Section 3 for the definition of D_r). Thanks to Theorem 3.2, we see that D is invariant under the evolution of solutions to (P). Furthermore, let us define a metric $\mathrm{d}(\cdot,\cdot)$ over the set D by

$$\mathrm{d}(u, v) := \|u - v\|_{H_0^1(\Omega)} + \|u - v\|_{L^4(\Omega)} \quad \text{for} \ \ u, v \in D.$$

Moreover, denote by $S_t : D \to D$ the semi-group associated with (P) and (6.1)–(6.3), that is,

$$S_t u_0 := u(t) \quad \text{for} \ \ t \geqslant 0, \ u_0 \in D,$$

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where u is the (unique) solution of (P) which also solves (6.1)–(6.3) and whose initial datum is u_0. By Theorems 3.2 and 6.1, one can assure that S_t is a continuous semi-group.

We next set a subset of D by

$$B_0 := \left\{ u \in D : \|\Delta u - u^3 + \kappa u\|_2^2 \leqslant c_r + 1,\ \phi(u) \leqslant C_r + 1 \right\}$$

with $c_r = 2\kappa M_0 + r + C_r$ and $C_r := C_r/(2\kappa)$ (see (4.11) and (4.21)). Then, the partial energy-dissipation estimates (4.11) and (4.21) immediately ensure the following:

Lemma 8.1 *The set B_0 is D-*absorbing*, that is, for any bounded subsets B of (D, d), one can take $\tau_B \geqslant 0$ such that $S_t B \subset B_0$ for all $t \geqslant \tau_B$.*

We next prove the compactness of B_0 in (D, d).

Lemma 8.2 *The set B_0 is compact in (D, d).*

Proof To prove this lemma, let us define a functional $C : L^2(\Omega) \to [0, \infty)$ by

$$C(f) := \int_\Omega \left(f(x)\right)_-^2 \mathrm{d}x \quad \text{for } f \in L^2(\Omega).$$

We then observe that $C(\cdot)$ is (strongly) continuous in $L^2(\Omega)$ and convex. Hence, $C(\cdot)$ is also weakly lower semi-continuous in $L^2(\Omega)$ by the convexity.

Let (u_n) be a sequence in B_0. Then, obviously, (u_n) is bounded in $H_0^1(\Omega) \cap L^4(\Omega)$, and moreover,

$$\| -\Delta u_n + u_n^3\|_2 \leqslant \sqrt{c_r + 1} + \kappa \|u_n\|_2 \leqslant C.$$

Noting that

$$\| -\Delta v\|_2^2 + \|v^3\|_2^2 \leqslant \| -\Delta v + v^3\|_2^2 \quad \text{for all } v \in H^2(\Omega) \cap L^6(\Omega),$$

we deduce that (u_n) is bounded in $H^2(\Omega) \cap L^6(\Omega)$. Then,

$$u_n \to u \quad \text{weakly in } H^2(\Omega) \cap L^6(\Omega)$$

for some $u \in H^2(\Omega) \cap L^6(\Omega)$. Moreover, by the compact embedding $H^2(\Omega) \cap L^6(\Omega) \hookrightarrow H_0^1(\Omega) \cap L^4(\Omega)$, one can take a subsequence of (n) without relabelling such that

$$u_n \to u \quad \text{in } (D, \mathrm{d}), \quad \text{that is, strongly in } H_0^1(\Omega) \cap L^4(\Omega),$$

which also yields

$$\begin{aligned} \Delta u_n &\to \Delta u && \text{strongly in } H^{-1}(\Omega) \text{ and weakly in } L^2(\Omega), \\ u_n^3 &\to u^3 && \text{strongly in } L^q(\Omega) \text{ and weakly in } L^2(\Omega),\ 1 \leqslant q < 2. \end{aligned}$$

Hence, we see that

$$\Delta u_n - u_n^3 + \kappa u_n \to \Delta u - u^3 + \kappa u \quad \text{weakly in } L^2(\Omega).$$

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Since $C(\cdot)$ is weakly lower semi-continuous in $L^2(\Omega)$ and $u_n \in D$, it follows that

$$C(\Delta u - u^3 + \kappa u) \leqslant \liminf_{n\to\infty} C(\Delta u_n - u_n^3 + \kappa u_n) \leqslant r,$$

which implies $u \in D$. Moreover, the weak lower semi-continuity of $\|\cdot\|_2$ and ϕ leads us to obtain $u \in B_0$. □

Remark 8.3 (Set up of the phase set)

(i) *The phase space D assigned here is non-linear and non-convex (cf. see also Proposition 9.5). Furthermore, we stress that D is unbounded. The metric* d *is chosen such that B_0 becomes compact in (D, d).*

(ii) *One may replace the phase set D by its closure in $H_0^1(\Omega) \cap L^4(\Omega)$. Then, the compactness of B_0 follows in a simpler way; indeed, it suffices to prove the pre-compactness of B_0 in (D, d). However, we address ourselves to the phase set D instead of its closure.*

9 Construction of a (D, d)-global attractor

In this section, we shall construct a *global attractor* defined in the following sense for the DS generated by (P) on the phase set (D, d):

Definition 9.1 ((D, d)-global attractor) *A subset $\mathcal{U}$ of D is called a* (D, d)-global attractor *associated with the DS $(S_t, (D, \mathrm{d}))$ if the following conditions hold true:*

(i) *$\mathcal{U}$ is compact in (D, d);*

(ii) *$\mathcal{U}$ satisfies an* attraction property *in (D, d), that is, let $B \subset D$ be a* d*-bounded subset of D (i.e., the diameter* $\mathrm{diam}(B) := \sup\{\mathrm{d}(u, v) : u, v \in B\}$ *is finite). Then, for any neighbourhood $\mathcal{O}$ of $\mathcal{U}$ in (D, d), there exists $\tau_{\mathcal{O}} \geqslant 0$, such that $S_t B \subset \mathcal{O}$ for all $t \geqslant \tau_{\mathcal{O}}$;*

(iii) *$\mathcal{U}$ is strictly invariant, that is, for any $t \geqslant 0$, it holds that $S_t \mathcal{U} = \mathcal{U}$.*

Our result reads,

Theorem 9.2 (Existence of (D, d)-global attractor) *The DS $(S_t, (D, \mathrm{d}))$ admits the (D, d)-global attractor $\mathcal{U}$, which is given by*

$$\mathcal{U} := \bigcap_{\tau \geqslant \tau_0} \overline{F_\tau}, \quad F_\tau := \bigcup_{t \geqslant \tau} S_t B_0, \tag{9.1}$$

where τ_0 is a positive constant and $\overline{F_\tau}$ stands for the closure of F_τ in (D, d). Moreover, $\mathcal{U}$ is the maximal bounded strictly invariant set, and therefore, the (D, d)-global attractor is unique.

Proof For the convenience of the reader and self-containment, we give a proof specific to our setting instead of checking conditions and applying a ready-made theorem, for the argument below is not lengthy (e.g., in [42], one may find a general theory in a metric setting). The following argument is essentially based on a standard theory (see [11, Chap. 2, §2] and also [27], where a metric setting is not treated). In what follows, we shall directly check three conditions (i)–(iii) of Definition 9.1 for the set $\mathcal{U}$ given by (9.1).

Compactness in (D, d). Here, we note that $F_\tau \subset B_0$ for any $\tau \geqslant t_0$ and some $t_0 \geqslant 0$. Hence, $\overline{F_\tau}$ is compact in (D, d) and included in B_0. Therefore, $\mathcal{U}$ is included in B_0 and compact in (D, d).

Attraction property in (D, d). To prove this, let $B \subset D$ be a d-bounded set and suppose on the contrary that there exist a neighbourhood $\mathcal{O}_0$ of $\mathcal{U}$ in (D, d) and a sequence $t_n \to \infty$ such that $S_{t_n} B \cap (D \setminus \mathcal{O}_0) \neq \emptyset$. Let us take $y_n \in S_{t_n} B \cap (D \setminus \mathcal{O}_0)$. Since B_0 is D-absorbing, one can take $\tau_B \geqslant 0$ such that $S_t B \subset B_0$ for all $t \geqslant \tau_B$. Hence, for $n \gg 1$ satisfying $t_n \geqslant \tau_B$, one observes that $y_n \in S_{t_n} B \subset B_0$. Therefore, up to a subsequence, y_n converges to an element y of B_0 in (D, d). Moreover, let $n_0 \in \mathbb{N}$ be such that $S_{t_{n_0}} B \subset B_0$. Then, since $y_n \in S_{t_n - t_{n_0}} \circ S_{t_{n_0}} B \subset S_{t_n - t_{n_0}} B_0 \subset \overline{F_{t_n - t_{n_0}}}$ for all $n \gg 1$, the limit y belongs to $\mathcal{U}$ (see Lemma 9.3). On the other hand, by $y_n \in D \setminus \mathcal{O}_0$, the limit y never belongs to $\mathcal{U}$. This is a contradiction. Therefore, $\mathcal{U}$ enjoys the attraction property in (D, d).

Lemma 9.3 *Let* (X_n) *be a sequence of closed subsets of a metric space* (D, d)*. Let* $y_n \in X_n$ *be such that* $y_n \to y$ *in* (D, d)*. In addition, suppose that* $X_n \subset X_m$ *if* $n \geqslant m$*. Then, it holds that*

$$y \in \bigcap_{k \in \mathbb{N}} X_k.$$

Proof Let $k \in \mathbb{N}$ be arbitrarily fixed. For any $n \geqslant k$, we recall $y_n \in X_n \subset X_k$. Hence, the closedness of X_k implies $y \in X_k$. From the arbitrariness of k, we conclude that $y \in \cap_{k \in \mathbb{N}} X_k$. □

Strict invariance. We claim that $S_t \mathcal{U} \subset \mathcal{U}$. Let $y \in S_t \mathcal{U}$. Then, there exists $u \in \mathcal{U}$ such that $y = S_t u$. Moreover, u belongs to $\overline{F_\tau}$ for any $\tau \geqslant \tau_0$. Hence, in particular, by a diagonal argument, one can take a sequence $u_n \in F_n$ such that $u_n \to u$ in (D, d). Indeed, for each $m \in \mathbb{N}$, since u belongs to $\overline{F_m}$, one can take a sequence $(u_n^{(m)})_{n \in \mathbb{N}}$ in F_m such that $u_n^{(m)} \to u$ in (D, d) as $n \to \infty$. Now let $u_n := u_n^{(n)} \in F_n$ and observe that $u_n \to u$ in (D, d). Furthermore, there exist sequences $t_n \geqslant n$ and $b_n \in B_0$ such that $u_n = S_{t_n} b_n$. Here, one can suppose that t_n is increasing without any loss of generality. Thus, one can write

$$y = S_t u = S_t(\lim_{n \to \infty} u_n) = \lim_{n \to \infty} S_t u_n = \lim_{n \to \infty} S_t \circ S_{t_n} b_n = \lim_{n \to \infty} S_{t + t_n} b_n.$$

Here, we used the continuity of S_t in (D, d), which follows from the continuous dependence of solutions for (1.6)–(1.8) on initial data (see Theorem 3.2), to verify the third equality. Noting that $b_n \in B_0$, we deduce that $S_{t + t_n} b_n \in F_{t + t_n}$. Therefore, y belongs to $\mathcal{U}$ (see Lemma 9.3), which also implies the relation $S_t \mathcal{U} \subset \mathcal{U}$. We next show $\mathcal{U} \subset S_t \mathcal{U}$. Let $y \in \mathcal{U}$ be fixed. Then, y belongs to $\overline{F_\tau}$ for all $\tau \geqslant \tau_0$. Hence, one can particularly take a sequence $t_n \nearrow \infty$ and $b_n \in B_0$ such that $y = \lim_{n \to \infty} S_{t_n} b_n$. Note that

$$y = \lim_{n \to \infty} S_{t_n} b_n = \lim_{n \to \infty} S_t \circ S_{t_n - t} b_n = S_t \left(\lim_{n \to \infty} S_{t_n - t} b_n \right)$$

for each $t > 0$. Here, we used the continuity of S_t again and further noticed that

$$S_{t_n - t} b_n \in S_{t_n - t} B_0 \subset B_0$$

for $n \gg 1$, since B_0 is D-absorbing, and therefore, the compactness of B_0 implies that $S_{t_n-t} b_n$ converges to an element $u_1 \in B_0$ in (D, d), up to a subsequence, as $n \to \infty$. Recall that $S_{t_n-t} b_n \in F_{t_n-t}$ for $n \gg 1$ to obtain

$$u_1 = \lim_{n\to\infty} S_{t_n-t} b_n \in \mathcal{U}$$

(see also Lemma 9.3). Thus, y belongs to $S_t\mathcal{U}$. Consequently, we conclude that $S_t\mathcal{U} = \mathcal{U}$.

Finally, let us prove the maximality of $\mathcal{U}$ among bounded strictly invariant sets. Indeed, let $\mathcal{V}$ be a bounded strictly invariant set in (D, d). Then, since $\mathcal{V}$ is a bounded set in (D, d), one can take $\tau \geqslant 0$ such that $S_t\mathcal{V} \subset B_0$ for all $t \geqslant \tau$. From (9.1) along with the strict invariance of $\mathcal{V}$, it follows that $\mathcal{V} \subset \mathcal{U}$. Thus, $\mathcal{U}$ is maximal. □

Not surprisingly, we observe that

Proposition 9.4 *Let $r > 0$ and let $\psi \in D_r$ be a solution of the inclusion,*

$$\partial I_{[0,\infty)}(0) - \Delta\psi + \psi^3 - \kappa\psi \ni 0 \quad \text{in } L^2(\Omega) \tag{9.2}$$

(*hence, ψ is a super-solution to the elliptic Allen–Cahn equation* (3.2)).*Then, ψ belongs to the global attractor $\mathcal{U}$ constructed in Theorem 9.2 under the phase set $D = D_r$.*

Proof We note that (9.2) corresponds to a stationary equation for (P). More precisely, $u(x, t) \equiv \psi(x)$ is a solution for (P) with $u_0 = \psi$. Hence, ψ must belong to the absorbing set B_0 (see Lemma 8.1). Therefore, by means of (9.1) along with the fact $S_\tau\psi = \psi$, one can conclude that $\psi \in \mathcal{U}$. □

The connectedness of the global attractor $\mathcal{U}$ (with $D = D_r$) is not proved due to the peculiar setting of the phase set D_r. However, we can prove it by assigning the following set D_r^+ to the phase set D instead of D_r:

$$D_r^+ := \left\{ u \in D_r : u \geqslant 0 \ \text{ a.e. in } \Omega \right\}.$$

Here, we remark that D_r^+ is still non-compact in $H_0^1(\Omega)$ and unbounded in $H^2(\Omega)$ (cf. see (i) of Remark 3.4). Then, the preceding argument still runs as previously mentioned. Indeed, the non-negativity of initial data is inherited to solutions of (P).

Proposition 9.5 *It holds that*

(i) *D_r^+ is convex,*

(ii) *$\mathcal{U}$ is connected if $D = D_r^+$.*

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Proof We first prove (i). Let $u, v \in D_r^+$ and $\theta \in (0,1)$ and set $u_\theta := (1-\theta)u + \theta v$. Note that

$$\begin{aligned}\Delta u_\theta + \kappa u_\theta - u_\theta^3 &= (1-\theta)\left[\Delta u + \kappa u\right] + \theta\left[\Delta v + \kappa v\right] - ((1-\theta)u + \theta v)^3 \\ &\geqslant (1-\theta)\left[\Delta u - u^3 + \kappa u\right] + \theta\left[\Delta v - v^3 + \kappa v\right]\end{aligned}$$

by the convexity of the cubic function x^3 on $[0, \infty)$. Hence, the decrease as well as the convexity of the function $x \mapsto (x)_-^2$ lead us to observe that

$$\begin{aligned}\left(\Delta u_\theta - u_\theta^3 + \kappa u_\theta\right)_-^2 &\leqslant \left((1-\theta)\left[\Delta u - u^3 + \kappa u\right] + \theta\left[\Delta v - v^3 + \kappa v\right]\right)_-^2 \\ &\leqslant (1-\theta)\left(\Delta u - u^3 + \kappa u\right)_-^2 + \theta\left(\Delta v - v^3 + \kappa v\right)_-^2 .\end{aligned}$$

Thus, integrating both sides over Ω and recalling the fact that $u, v \in D_r^+$, we obtain

$$C(\Delta u_\theta - u_\theta^3 + \kappa u_\theta) \leqslant r,$$

which implies $u_\theta \in D_r^+$. Therefore, D_r^+ is convex.

We next prove (ii). In the proof of Lemma 8.2, we have shown that B_0 is bounded in $H^2(\Omega) \cap L^6(\Omega)$. Hence, one can take $R > 0$ such that

$$B_0 \subset B_1 := \left\{u \in D : \|u\|_{H^2(\Omega)} + \|u\|_{L^6(\Omega)} \leqslant R\right\}.$$

Then, since $D = D_r^+$ is convex, so is B_1, and hence, B_1 is connected in (X, d). Moreover, we can verify that B_1 is compact in (D, d). Furthermore, by Lemma 8.1, we can take $t_0 > 0$ such that $S_t B_1 \subset B_0$ for all $t \geqslant t_0$. Hence, $S_t B_1 \subset B_0 \subset B_1$ for all $t \geqslant t_0$, and therefore, it holds that

$$\mathcal{U} = \bigcap_{\tau \geqslant \tau_0} \overline{E_\tau}, \quad E_\tau := \bigcup_{t \geqslant \tau} S_t B_1.$$

Moreover, due to the continuity of S_t in (D, d), the set $S_t B_1$ is also connected for each $t \geqslant 0$. Furthermore, since the family $\{S_t B_1\}_{t \geqslant 0}$ has a non-empty intersection (indeed, every stationary point in B_1 (e.g., $0 \in B_1$) belongs to the intersection), the union $E_\tau = \cup_{t \geqslant \tau} S_t B_1$ is connected as well. Therefore, the closure $\overline{E_\tau}$ is also connected. Finally, Lemma 9.6 ensures the connectedness of $\mathcal{U} = \cap_{\tau \geqslant \tau_0} \overline{E_\tau}$, since $\overline{E_\tau}$ is included in the compact set B_1 for $\tau \geqslant t_0$. □

Lemma 9.6 (see e.g. [30, p.437]) *Let X be a compact Hausdorff space. Let $\mathcal{P}$ be a family of non-empty, closed and connected subsets of X such that either $A \subset B$ or $B \subset A$ holds true for any $A, B \in \mathcal{P}$. Then, the intersection*

$$\bigcap \mathcal{P} := \bigcap_{A \in \mathcal{P}} A$$

is also connected.

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10 Convergence to equilibria

We finally discuss the convergence of each solution $u = u(x,t)$ for (P) as t goes to ∞. We shall prove the ω-limit set is non-empty and a singleton. Moreover, the limit is characterized as a solution of an elliptic variational inequality of obstacle type.

Theorem 10.1 *Let* $u_0 \in \overline{D_r}^{L^2}$ *with an arbitrary* $r > 0$ *and let* u *be the solution of* (P) *as well as* (6.1)–(6.3) (*see Theorem 6.1*). *Then, it holds that*

$$\begin{aligned} u(t) \to \phi \quad & \text{strongly in } H^1_0(\Omega) \cap L^4(\Omega), \\ & \text{weakly in } H^2(\Omega) \cap L^6(\Omega) \quad \text{as} \quad t \to \infty \end{aligned}$$

for some $\phi \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^6(\Omega)$. *Hence, the* ω*-limit set* $\omega(u)$ *of* u *is non-empty and a singleton. Moreover, the limit* ϕ *is a solution of the following elliptic variational inequality of obstacle type*:

$$\partial I_{[u_0(x),\infty)}(\phi) - \Delta\phi + \phi^3 \ni \kappa\phi \quad \text{in } L^2(\Omega), \qquad \phi \in H^1_0(\Omega), \tag{10.1}$$

which is rewritten as

$$\begin{aligned} \phi \geqslant u_0, \quad -\Delta\phi + \phi^3 - \kappa\phi \geqslant 0 \quad & \text{in} \quad \Omega, \\ (\phi - u_0)\left(-\Delta\phi + \phi^3 - \kappa\phi\right) = 0 \quad & \text{in} \quad \Omega, \\ \phi|_{\partial\Omega} = 0. & \end{aligned}$$

Proof Even though the uniqueness of solutions to (P) is not guaranteed, the solution of (P) as well as of (6.1)–(6.3) is unique (see Theorem 6.1). Hence, all energy inequalities are valid (see Section 5). By (4.2), there is a sequence $\tau_n \in [n, n+1]$ such that

$$u_t(\tau_n) \to 0 \quad \text{strongly in } L^2(\Omega).$$

Furthermore, since $u(t)$ is bounded in $H^1_0(\Omega) \cap L^4(\Omega)$ for $t \geqslant 0$, up to a (not relabelled) subsequence, there exists $\phi \in H^1_0(\Omega) \cap L^4(\Omega)$ such that

$$u(\tau_n) \to \phi \quad \text{weakly in } H^1_0(\Omega) \cap L^4(\Omega) \text{ and strongly in } L^2(\Omega).$$

We also further derive from (4.6) (with s and $\|\eta(s)\|_2^2$ replaced by 0 and r, respectively) and Corollary 4.1 (hence $(u(\tau_n))$ is bounded in $H^2(\Omega) \cap L^6(\Omega)$) that

$$\begin{aligned} \eta(\tau_n) &\to \eta_\infty & & \text{weakly in } L^2(\Omega), \\ -\Delta u(\tau_n) + u(\tau_n)^3 &\to -\Delta\phi + \phi^3 & & \text{weakly in } L^2(\Omega), \\ u(\tau_n) &\to \phi & & \text{weakly in } H^2(\Omega) \cap L^6(\Omega), \end{aligned}$$

which along with the demi-closedness of $\partial I_{[0,\infty)}$ gives $\eta_\infty \in \partial I_{[0,\infty)}(0)$. It also particularly implies

$$u(\tau_n) \to \phi \quad \text{strongly in } H^1_0(\Omega) \cap L^4(\Omega).$$

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Therefore, we assure that

$$\eta_\infty - \Delta\phi + \phi^3 - \kappa\phi = 0, \quad \eta_\infty \in \partial I_{[0,\infty)}(0), \tag{10.2}$$

which is a necessary condition for (10.1).

Thus, ϕ is an element of the ω-limit set $\omega(u)$ of u. Furthermore, from the non-decrease of $t \mapsto u(x,t)$ for a.e. $x \in \Omega$, we conclude that

$$u(x,t) \nearrow \phi(x) \quad \text{for a.e.} \ \ x \in \Omega \ \ \text{as} \ \ t \to \infty.$$

Hence, $\omega(u) = \{\phi\}$.

Now, recall that u also solves (6.1)–(6.3) and η belongs to $\partial I_{[u_0(x),\infty)}(u)$. By the demi-closedness of $\partial I_{[u_0(x),\infty)}$ in $L^2(\Omega) \times L^2(\Omega)$, we conclude that η_∞ is a section of $\partial I_{[u_0(x),\infty)}(\phi)$ a.e. in Ω. Thus, ϕ turns out to be a solution of (10.1). This completes the proof. □

Remark 10.2

(i) *To prove that the ω-limit set is a singleton, Łojasiewicz-Simon type inequalities are often used. However, it seems difficult to apply them to (P), since* (1.6) *is not a gradient flow but a generalized one, which can be written in the form,*

$$u_t + \partial I_{[0,\infty)}(u_t) \ni -E'(u),$$

where E' stands for a functional derivative of E (i.e., Fréchet derivative). On the other hand, this point was proved more easily since solutions of (P) are non-decreasing in time.

(ii) *As for the parabolic obstacle problem* (6.1)–(6.3)*, it also seems difficult to apply a Łojasiewicz–Simon type inequality due to the presence of the non-smooth potential $I_{[u_0(x),\infty)}$; however, by reducing the obstacle problem to (P), one can prove that the ω-limit set of each solution for the obstacle problem is a singleton and consists of a single solution to* (10.1).

In Theorem 10.1, the rate of convergence is not estimated. Under a suitable assumption on initial data, by employing (4.5), one can verify an exponential convergence of $u(t)$ as $t \to \infty$.

Corollary 10.3 *In addition to the same assumptions as in Theorem 10.1, suppose that*

$$u_0 \geqslant 0 \quad \textit{and} \quad \lambda_\Omega(3u_0^2) > \kappa. \tag{10.3}$$

Set $\sigma := \lambda_\Omega(3u_0^2) - \kappa > 0$ and $C = \|(\Delta u_0 - u_0^3 + \kappa u_0)_+\|_2$. Then, it holds that

$$\|u(t) - \phi\|_2 \leqslant \frac{C}{\sigma} e^{-\sigma t} \quad \textit{for all} \ \ t \geqslant 0.$$

Proof By Theorem 10.1, it is already known that $u(t)$ converges to some equilibrium ϕ strongly in $H_0^1(\Omega) \cap L^4(\Omega)$ as $t \to \infty$. Moreover, setting $\sigma := \lambda_\Omega(3u_0^2) - \kappa > 0$ and letting

$s_0 > 0$, we observe that

$$\|u(t) - u(s)\|_2 \leqslant \int_s^t \|\partial_\tau u(\tau)\|_2 \, d\tau \overset{(4.5)}{\leqslant} C \int_s^t e^{-\sigma\tau} \, d\tau \leqslant \frac{C}{\sigma}\left(e^{-\sigma s} - e^{-\sigma t}\right) \quad \text{for} \quad s_0 \leqslant s \leqslant t < \infty$$

for some constant $C \geqslant 0$. Letting $t \to \infty$, we deduce that

$$\|\phi - u(s)\|_2 \leqslant \frac{C}{\sigma} e^{-\sigma s} \quad \text{for all} \quad s \geqslant s_0.$$

This completes the proof. □

Remark 10.4 (On assumption (10.3)) *Note that $\lambda_\Omega(3u_0^2) > \mu(\Omega) > 0$ by $u_0 \not\equiv 0$, where $\mu(\Omega)$ stands for the first eigenvalue of the Dirichlet Laplacian $-\Delta$ posed in Ω. Hence, the second inequality of* (10.3) *holds true if $\mu(\Omega) \geqslant \kappa$ (e.g., the diameter of Ω is small enough). On the other hand, even if $\mu(\Omega) < \kappa$, the second condition of* (10.3) *is also satisfied under an appropriate assumption on the initial datum u_0, for instance,*

$$3u_0^2 \geqslant U_\lambda \quad \text{a.e. in } \Omega,$$

where $U_\lambda = U_\lambda(x)$ is the (unique) positive solution of the elliptic equation for any $\lambda > \kappa$,

$$-\Delta U_\lambda + U_\lambda^2 = \lambda U_\lambda, \; U_\lambda > 0 \; \text{ in } \Omega, \quad U_\lambda = 0 \; \text{ on } \partial\Omega. \tag{10.4}$$

Indeed, for each $\lambda > \kappa$ (hence, $\lambda > \mu(\Omega)$), (10.4) *admits the unique positive solution $U_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $0 < U_\lambda \leqslant \lambda$ in Ω, and moreover, (λ, U_λ) turns out to be a principal eigenpair of the Schrödinger operator $v \mapsto -\Delta v + U_\lambda v$. Hence, if $3u_0^2 \geqslant U_\lambda$ a.e. in Ω, then, $\lambda_\Omega(3u_0^2) \geqslant \lambda_\Omega(U_\lambda) = \lambda > \kappa$.*

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