Glasgow Math. J. **62** (2020) S142–S164. © Glasgow Mathematical Journal Trust 2019.
doi:10.1017/S0017089519000314.

COALTERNATIVE COALGEBRAS

HELENA ALBUQUERQUE, ELISABETE BARREIRO, JOSÉ M. SÁNCHEZ
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal
e-mail: lena@mat.uc.pt, mefb@mat.uc.pt, txema.sanchez@mat.uc.pt

and CARLOS SONEIRA CALVO
Department of Pedagogy and Didactics A Coruña, University of A Coruña, Spain
e-mail: carlos.soneira@udc.gal

(Received 17 January 2019; revised 6 June 2019; accepted 18 June 2019; first published online 20 August 2019)

Abstract. In this paper, we define a Cayley–Dickson process for k-coalgebras proving some results that describe the properties of the final coalgebra, knowing the properties of the initial one. Namely, after applying the Cayley–Dickson process for k-coalgebras to a coassociative coalgebra, we obtain a coalternative one. Moreover, the first coalgebra is cocommutative if and only if the final coalgebra is coassociative. Finally we extend these results to a more general approach of D-coalgebras, where D is a k-coalgebra, presenting a class of examples of coalternative (non-coassociative) coalgebras obtained from group D-coalgebras.

2010 *Mathematics Subject Classification*. 16T15

1. Introduction and previous definitions. The first alternative (nonassociative) algebra which appeared in mathematical literature was the octonion algebra in the pioneer work of John Graves, in 1844. However, this structure was widely reported by Arthur Cayley as a result of a duplication process of the quaternion algebra: considering an algebra A with identity element endowed with an involutive antiautomorphism, the Cayley–Dickson process allows us to construct a new algebra $\bar{A}$ with double dimension that is an extension of the previous one, with same identity element and also endowed with an involutive antiautomorphism. This process is very useful for applications and also from a theoretical point of view, where we can predict some properties of $\bar{A}$ considering the known structure of A. In particular, in [3] it is proved that if we apply Cayley–Dickson process to an algebra $k_F G$, we obtain a deformed group algebra $k_{\bar{F}}(G \times \mathbb{Z}_2)$, where $\bar{F}$ is a cochain defined from the cochain F. Using this approach the octonions arise in [3] as an algebra in the monoidal category of graded vector spaces, that is, as an example of quasialgebra. Albuquerque, Elduque, and Pérez-Izquierdo in [2] went little further describing the octonions as an example of division alternative quasialgebra. Moreover, they classified all division alternative quasialgebras, proving that these algebras are obtained by applying three times a generalized Cayley–Dickson process to an associative and commutative algebra. Using a similar approach, in [5] Bulacu introduces the quasicoalgebras as coalgebras in the monoidal category of graded vector spaces, defining a Cayley–Dickson process for coalgebras starting with the trivial coalgebra (k, Id_k). In [5], the octonions are shown to be a natural example of quasicoalgebra obtained by this duplication process for coalgebras. Bulacu goes one step further studying the compatibility between the algebra and coalgebra structures for the Cayley algebras, proving that weak braided Hopf algebras are actually obtained.

In the algebra case, it was proved in [2] that if A is a strictly alternative division quasialgebra over a field k, then there are a field extension $K|k$, and an abelian group G with a symmetric 2-cochain $F : G \times G \to K \setminus \{0\}$, such that $A \simeq K_F G$. It is said, in the general algebra case, a quasialgebra over k is interpreted as a quasialgebra over K with $k|K$ a field extension. Moreover, the Cayley–Dickson process is reinterpreted in this situation by considering K as a new base field, in such a way that on $K_F[G]$ coexist two related structures, namely that of k-algebra and another one of K-algebra.

When turning to coalgebras, the Cayley–Dickson process defined in [5] considers the structure of k-coalgebra. It is enough in order to apply the construction to the particular case $k_F[G]$ tackled in [5], where the base field is always k. Nevertheless, in order to provide a general version of the Cayley–Dickson process for coalgebras, we must consider the coexistence of the starting k-coalgebra structure with other related but different coalgebra structure.

Notice that for a general algebra A, the construction considered in [2] is applied to the case where $K = A_e$, the subspace of A consisting on homogeneous elements of degree e the unit element of G, that in turn results to be a field extension of k, so it makes sense to tensorize over K. But for a general coalgebra C, we do not know whether the new base D is a field extension of k, because we do not have a division k-coalgebra structure on C. We just have that D is a k-coalgebra. In this situation, it makes no sense to tensorize over D and the suitable analogue to the tensor product is the cotensor product $-\Box_D-$. Actually, in the algebra case $-\otimes_K-$ is defined as a cokernel by using a K-bimodule structure of A, while in the coalgebra case $-\Box_D-$ is defined as a kernel by using a (D, D)-bicomodule structure of C. Notice however that if $D = k$, as actually happens in the case considered in [5], then $-\otimes_k - = -\Box_D-$, so the construction we will present in this work extends that of [5].

The paper is organized as follows. In this section, we present the preliminaries and basic definitions that can be seen in [1, 7, 8]. In Section 2, we construct a Cayley–Dickson process for k-coalgebras proving some theorems that describe the properties of the resulting coalgebra just from the properties of the initial one. Namely, after applying the Cayley–Dickson process for k-coalgebras to a coassociative coalgebra endowed with a strong anti-coinvolutive coalgebra map, we obtain a coalternative one. In this case, the first coalgebra is cocommutative if and only if the final coalgebra is coassociative. In Section 3, we show that all these results obtained for k-coalgebras can be extended to a more general structure of D-coalgebras, in such a way that we can accurately describe the relation between both coexisting structures. Finally, in Section 4, we present a class of examples of coalgebras, derived from group coalgebras, obtained from the generalized Cayley–Dickson process.

We recall that a *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*, an object $\underline{1} \in \mathcal{C}$ called the *unit object*, and natural isomorphisms $a : \otimes \circ (\otimes \times Id) \to \otimes \circ (Id \times \otimes)$ (the associativity constraint), $l : \otimes \circ (\underline{1} \times Id) \to Id$ (the left unit constraint) and $r : \otimes \circ (Id \times \underline{1}) \to Id$ (the right unit constraint). In addition, a satisfies the pentagon axiom, and l and r satisfy the triangle axiom. A monoidal category is called *strict* if the associativity, right unit, and left unit constraints are identities (see [7] for details). By Theorem XI.5.3 of [7], any monoidal category is monoidally equivalent to a strict one, and by Mac Lane's coherence theorem, in order to obtain general results we can proceed as if the constraints were all identities. A braiding for a strict monoidal category is a natural isomorphism $\Psi : \otimes \to \otimes \circ \tau$ satisfying some hexagon axioms, where

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

$\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the *switch functor* defined by $\tau(X, Y) := (Y, X)$, for X, Y in $\mathcal{C}$. If, moreover, $\Psi_{X,Y} = \Psi^{-1}_{Y,X}$ for any objects X, Y of $\mathcal{C}$, we say that $\mathcal{C}$ is *symmetric* (see [7] for details). From now on, we assume that our starting category $\mathcal{C}$ is strict, symmetric with braiding Ψ, and such that the base object is a field k, otherwise specified. For simplicity of notation, we will denote by unadorned $-\otimes-$, the tensor product $-\otimes_k-$ over k in $\mathcal{C}$.

A *k-algebra in $\mathcal{C}$* is a triple (A, m_A, η_A)—where A is an object of $\mathcal{C}$, $m_A : A \otimes A \to A$, a morphism called multiplication of A, and $\eta_A : \underline{1} \to A$, a morphism called unit morphism—such that $m_A \circ (\eta_A \otimes Id_A) = m_A \circ (Id_A \otimes \eta_A) = Id_A$.

A *k-coalgebra in $\mathcal{C}$* is a triple $(C, \Delta_C, \epsilon_C)$—where C is an object of $\mathcal{C}$, $\Delta_C : C \to C \otimes C$, a morphism called comultiplication of C, and $\epsilon_C : C \to \underline{1}$, a morphism called counit morphism—such that $(\epsilon_C \otimes Id_C) \circ \Delta_C = (Id_C \otimes \epsilon_C) \circ \Delta_C = Id_C$. Unless otherwise specified, we understand coalgebras as k-coalgebras, although in Section 3 we will explicitly define and actually work with a more general notion of coalgebra structure. Given two coalgebras $(C, \Delta_C, \epsilon_C)$, $(D, \Delta_D, \epsilon_D)$, a map $f : C \to D$ is said to be a *coalgebra morphism* if it satisfies that $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.

From now on, although all the definitions of this section as well as some results on Section 4 are valid in general, we will assume that the base field k is such that $char(k) \neq 2$ and we will consider $\mathcal{C}$ as the monoidal category of graded vector spaces.

In addition, if there is no risk of confusion, for simplicity of notation we will use Δ, ϵ, and Id instead of Δ_C, ϵ_C, and Id_C, respectively, when the referred coalgebra C is clearly understood.
We say that the coalgebra is *coassociative* if in addition

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta. \tag{1.1}$$

Notice that if $\mathcal{C}$ is not strict, then the coassociativity condition (1.1) is required up to the associativity constraint a. A coalgebra is said to be *cocommutative* provided that

$$\Delta = \Psi \circ \Delta. \tag{1.2}$$

Definition 1.1. A coalgebra (C, Δ, ϵ) is said to be *coalternative* if the coproduct satisfies the *coalternativity properties*:

$$(\Delta \otimes Id) \circ \Delta + ((\Psi \circ \Delta) \otimes Id) \circ \Delta = (\Psi \otimes Id) \circ (Id \otimes \Delta) \circ \Delta + (Id \otimes \Delta) \circ \Delta, \tag{1.3}$$

$$(Id \otimes \Delta) \circ \Delta + (Id \otimes (\Psi \circ \Delta)) \circ \Delta = (Id \otimes \Psi) \circ (\Delta \otimes Id) \circ \Delta + (\Delta \otimes Id) \circ \Delta. \tag{1.4}$$

It is clear that if a coalgebra (C, Δ, ϵ) is coassociative, then it is coalternative.

Remark 1.2. Let (C, Δ, ϵ) be a coalgebra. Then $(C, \Delta^{coop} = \Psi \circ \Delta, \epsilon)$ is also a coalgebra, called the *co-opposite coalgebra*. Furthermore, it holds that (C, Δ, ϵ) is coassociative if and only if so is $(C, \Delta^{coop}, \epsilon)$, and the same holds regarding to cocommutativity and coalternativity.

Definition 1.3. Let C, D be two coalgebras and $q : C \to D$ a coalgebra morphism.

1. The morphism q is said to be *coassociative* if it satisfies that

$$(q \otimes Id \otimes Id) \circ (Id \otimes \Delta) \circ \Delta = (q \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Delta, \tag{1.5}$$

$$(Id \otimes q \otimes Id) \circ (Id \otimes \Delta) \circ \Delta = (Id \otimes q \otimes Id) \circ (\Delta \otimes Id) \circ \Delta, \tag{1.6}$$

$$(Id \otimes Id \otimes q) \circ (Id \otimes \Delta) \circ \Delta = (Id \otimes Id \otimes q) \circ (\Delta \otimes Id) \circ \Delta. \tag{1.7}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

2. The morphism q is said to be *cocentral* if it satisfies that

$$(q \otimes Id) \circ \Delta = \Psi \circ (Id \otimes q) \circ \Delta. \tag{1.8}$$

Notice that the preceding definition of cocentral coalgebra morphism means a generalization of that introduced in [6].

Given a coalgebra D, a *right D-comodule* is a k-vector space C together with a k-linear map $\rho^C : C \to C \otimes D$ such that

$$(Id \otimes \Delta_D) \circ \rho^C = (\rho^C \otimes Id) \circ \rho^C, \tag{1.9}$$

$$(Id \otimes \epsilon) \circ \rho^C = Id. \tag{1.10}$$

Symmetrically, a *left D-comodule* is a k-vector space C together with a k-linear map ${}^C\rho : C \to D \otimes C$ such that

$$(\Delta_D \otimes Id) \circ {}^C\rho = (Id \otimes {}^C\rho) \circ {}^C\rho, \tag{1.11}$$

$$(\epsilon \otimes Id) \circ {}^C\rho = Id. \tag{1.12}$$

Finally, if C is a right and a left D-comodule with corresponding structure morphisms ρ^C and ${}^C\rho$, we say that it is a *(D, D)-bicomodule* if in addition it holds that

$$({}^C\rho \otimes Id) \circ \rho^C = (Id \otimes \rho^C) \circ {}^C\rho. \tag{1.13}$$

LEMMA 1.4. *Let C, D be two k-coalgebras and $q : C \to D$ a coassociative coalgebra morphism. Then C can be endowed with the (D, D)-bicomodule structure given by*

$$\rho^C := (Id \otimes q) \circ \Delta_C \quad \textit{and} \quad {}^C\rho := (q \otimes Id) \circ \Delta_C.$$

DEFINITION 1.5. A coalgebra (C, Δ, ϵ) is said to be *anti-coinvolutive* if it is endowed with an *anti-coinvolution* $\sigma : C \to C$, that is, a morphism satisfying

$$\sigma^2 = Id, \tag{1.14}$$

$$\epsilon = \epsilon \circ \sigma, \tag{1.15}$$

$$\Delta \circ \sigma = (\sigma \otimes \sigma) \circ \Psi \circ \Delta. \tag{1.16}$$

We say that C is *strongly anti-coinvolutive* if, in addition, the following conditions hold:

$$\Psi \circ \Delta + (\sigma \otimes Id) \circ \Psi \circ \Delta = \Delta + (\sigma \otimes Id) \circ \Delta, \tag{1.17}$$

$$\Psi \circ \Delta + (Id \otimes \sigma) \circ \Psi \circ \Delta = \Delta + (Id \otimes \sigma) \circ \Delta. \tag{1.18}$$

Obviously, if C is a cocommutative and anti-coinvolutive coalgebra, then the anti-coinvolution is necessarily strong.

2. The double process for k-coalgebras. From a non-necessarily coassociative k-coalgebra with anti-coinvolution, the Cayley–Dickson process provides an infinite chain of non-necessarily coassociative coalgebras over k. In each step of this chain, we have a k-coalgebra endowed with an anti-coinvolution map such that it contributes to the structure of the k-coalgebra in the next step. We proceed now to construct such a Cayley–Dickson process for k-coalgebras what means a generalization of that developed in [5]. In the next section, we will make use of this approach to extend the process to the more general structure of D-coalgebras.

Our goal is to obtain, from a finite dimensional anti-coinvolutive coalgebra $(C, \Delta, \epsilon, \sigma)$, a new coalgebra $C' = C \oplus uC$, of twice the dimension, with u just a formal symbol to make distinction between the two components of the direct sum. Notice that the symbol u can be interpreted as an isomorphism of k-vector spaces with inverse given by $u^{-1}(uc) = c$.

DEFINITION 2.1. For a fixed $\alpha \in k \setminus \{0\}$, we define the morphism $\Delta' : C' \to C' \otimes C'$ by

$$\Delta'|_C := \frac{1}{2}\big(\Delta + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\big), \tag{2.1}$$

$$\Delta'|_{uC} := \frac{1}{2}\big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\big) \circ u^{-1}, \tag{2.2}$$

the morphism $\epsilon' : C' \to k$ by

$$\epsilon'|C := 2\epsilon, \qquad \epsilon'|uC := 0, \tag{2.3}$$

and the morphism $\sigma' : C' \to C'$ by

$$\sigma'|C := \sigma, \qquad \sigma'|uC := -Id_{uC}. \tag{2.4}$$

We will say that $(C', \Delta', \epsilon', \sigma')$ is the result of applying the Cayley–Dickson process to the coalgebra $(C, \Delta, \epsilon, \sigma)$.

The following theorem shows the result of applying one step of the Cayley–Dickson process.

THEOREM 2.2. *Within the notations of Definition 2.1, if $(C, \Delta, \epsilon, \sigma)$ is a coassociative and strongly anti-coinvolutive coalgebra, then $(C', \Delta', \epsilon', \sigma')$ is a coalternative anti-coinvolutive coalgebra.*

Proof. First of all, $(C', \Delta', \epsilon', \sigma')$ is a coalgebra. Indeed,

$$\begin{aligned}(\epsilon' \otimes Id) \circ \Delta'|_C &= \frac{1}{2}(\epsilon' \otimes Id) \circ \Big(\Delta + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\Big)\\ &= (\epsilon \otimes Id) \circ \Delta = Id|_C,\end{aligned}$$

and since $\epsilon \circ \sigma = \epsilon$, we have

$$\begin{aligned}(\epsilon' \otimes Id) \circ \Delta'|_{uC} &= \frac{1}{2}(\epsilon' \otimes Id)\Big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\Big) \circ u^{-1}\\ &= (\epsilon \otimes Id) \circ (\sigma \otimes u) \circ \Delta \circ u^{-1}\\ &= (Id \otimes u) \circ (\epsilon \otimes Id) \circ \Delta \circ u^{-1} = Id_{uC}.\end{aligned}$$

Now, we prove that σ' is an anti-coinvolution for C'. It is easy to check that $\sigma'^2 = Id$ and $\epsilon' = \epsilon' \circ \sigma'$. Finally, in order to prove (1.16), when restricting to C on the one hand, we have

$$\Delta' \circ \sigma'|_C = \Delta'|_C \circ \sigma = \frac{1}{2}\Big(\Delta \circ \sigma + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta \circ \sigma\Big),$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

and on the other hand,

$$(\sigma' \otimes \sigma') \circ \Psi \circ \Delta'|_C = \frac{1}{2}\Big((\sigma' \otimes \sigma') \circ \Psi \circ \big(\Delta + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\big)\Big)$$
$$= \frac{1}{2}\Big((\sigma \otimes \sigma) \circ \Psi \circ \Delta + \alpha(u \otimes u) \circ \Psi \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\Big).$$

The required equality follows because by (1.16), for σ we get $\Delta(\sigma) = (\sigma \otimes \sigma) \circ \Psi \circ \Delta$, and using in addition (1.14) for σ, knowing that ψ is symmetric, we obtain that $(\sigma \otimes Id) \circ \Psi \circ \Delta(\sigma) = \Psi \circ (\sigma \otimes Id) \circ \Psi \circ \Delta$. When restricting to uC, by the same arguments we get that

$$\begin{aligned}
&(\sigma' \otimes \sigma') \circ \Psi \circ \Delta'|_{uC}\\
&= \frac{1}{2}(\sigma' \otimes \sigma') \circ \Big(\Psi \circ (\sigma \otimes u) \circ \Delta + \Psi \circ (u \otimes Id) \circ \Psi \circ \Delta\Big) \circ u^{-1}\\
&= -\frac{1}{2}\Big((Id \otimes \sigma) \circ \Psi \circ (\sigma \otimes u) \circ \Delta \circ u^{-1} + (\sigma \otimes Id) \circ \Psi \circ (u \otimes Id) \circ \Psi \circ \Delta \circ u^{-1}\Big)\\
&= -\frac{1}{2}\Big((Id \otimes \sigma) \circ (u \otimes \sigma) \circ \Psi \circ \Delta \circ u^{-1} + (\sigma \otimes u) \circ \Delta \circ u^{-1}\Big)\\
&= -\frac{1}{2}\Big((u \otimes Id) \circ \Psi \circ \Delta + (\sigma \otimes u) \circ \Delta\Big) \circ u^{-1} = -\Delta'|_{uC} = \Delta' \circ \sigma'|_{uC}.
\end{aligned}$$

Concerning coalternativity, we just describe the proof for condition (1.3) when restricting to C, because the proof for (1.4) is similar.

When restricting to C, the left-hand side of equality (1.3) is as follows:

$$\begin{aligned}
&(\Delta' \otimes Id) \circ \Delta' + ((\Psi \circ \Delta') \otimes Id) \circ \Delta'|_C\\
&\quad= \frac{1}{2}\Big((\Delta' \otimes Id)\big(\Delta + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\big)\\
&\quad\quad + ((\Psi \circ \Delta') \otimes Id)\big(\Delta + \alpha(u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\big)\Big)\\
&\quad= \frac{1}{4}\Big((\Delta \otimes Id) \circ \Delta + \alpha(u \otimes u \otimes Id) \circ (\sigma \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\
&\quad\quad + \alpha(Id \otimes u \otimes u) \circ (Id \otimes \sigma \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\\
&\quad\quad + \alpha(u \otimes Id \otimes u) \circ (\Psi \otimes Id) \circ (\sigma \otimes \sigma \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\\
&\quad\quad + (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\
&\quad\quad + \alpha(u \otimes u \otimes Id) \circ (Id \otimes \sigma \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\
&\quad\quad + \alpha(\Psi \otimes Id) \circ (Id \otimes u \otimes u) \circ (Id \otimes \sigma \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\\
&\quad\quad + \alpha(Id \otimes u \otimes u) \circ (\sigma \otimes \sigma \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\Big).
\end{aligned}$$

Moreover, if we reorder terms in the above sum and apply (1.14), (1.16), and the symmetry of the braiding, we get that this preceding sum can be expressed as

$$\begin{aligned}
&(\Delta' \otimes Id) \circ \Delta' + ((\Psi \circ \Delta') \otimes Id) \circ \Delta'|_C\\
&\quad= \frac{1}{4}\Big((\Delta \otimes Id) \circ \Delta + (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\
&\quad\quad + \alpha(u \otimes u \otimes Id) \circ (\sigma \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\
&\quad\quad + \alpha(u \otimes u \otimes Id) \circ (Id \otimes \sigma \otimes Id) \circ (\Delta \otimes Id) \circ \Delta
\end{aligned}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

$$+\alpha(u\otimes Id\otimes u)\circ(\sigma\otimes\sigma\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(u\otimes Id\otimes u)\circ(\sigma\otimes Id\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(Id\otimes u\otimes u)\circ(Id\otimes\sigma\otimes Id)\circ(\psi\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(Id\otimes u\otimes u)\circ(\sigma\otimes\sigma\otimes Id)\circ(\Psi\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta\Big).$$

For simplicity of notation, we define

$$X:=\Big(((\sigma\otimes Id)\circ\psi\circ\Delta)\otimes Id\Big)\circ\Delta+\Big(((Id\otimes\sigma)\circ\Delta)\otimes Id\Big)\circ\Delta,$$

taken from the partial sum of the second and third summands,

$$Y:=\Big(((\sigma\otimes\sigma)\circ\Delta)\otimes Id\Big)\circ\Psi\circ\Delta+\Big(((\sigma\otimes Id)\circ\Delta)\otimes Id\Big)\circ\Psi\circ\Delta,$$

taken from the partial sum of the fourth and fifth summands, and

$$Z:=\Big(((Id\otimes\sigma)\circ\psi\circ\Delta)\otimes Id\Big)\circ\Psi\circ\Delta+\Big(((\sigma\otimes\sigma)\circ\psi\circ\Delta)\otimes Id\Big)\circ\Psi\circ\Delta,$$

taken from the partial sum of the sixth and seventh summands.

In relation to the right side of equality (1.3), when restricting to C, we have

$$(Id\otimes\Delta')\circ\Delta'+(\Psi\otimes Id)\circ(Id\otimes\Delta')\circ\Delta'|_C$$
$$=\frac{1}{2}\Big((Id\otimes\Delta')\circ\big(\Delta+\alpha((u\otimes u)\otimes(\sigma\otimes Id)\circ\Psi\circ\Delta)\big)$$
$$+(\Psi\otimes Id)\circ(Id\otimes\Delta')\circ\big(\Delta+\alpha((u\otimes u)\otimes(\sigma\otimes Id)\circ\Psi\circ\Delta)\big)\Big)$$
$$=\frac{1}{4}\Big((Id\otimes\Delta)\circ\Delta+\alpha(Id\otimes u\otimes u)\circ(Id\otimes\sigma\otimes Id)\circ(Id\otimes\Psi)\circ(\Delta\otimes Id)\circ\Delta$$
$$+\alpha(u\otimes\sigma\otimes u)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(u\otimes u\otimes Id)\circ(Id\otimes\Psi)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta$$
$$+(\Psi\otimes Id)\circ(Id\otimes\Delta)\circ\Delta$$
$$+\alpha(\Psi\otimes Id)\circ(Id\otimes u\otimes u)\circ(Id\otimes\sigma\otimes Id)\circ(Id\otimes\Psi)\circ(\Delta\otimes Id)\circ\Delta$$
$$+\alpha(\Psi\otimes Id)\circ(u\otimes\sigma\otimes u)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(u\otimes u\otimes Id)\circ(Id\otimes\sigma\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta\Big).$$

Again by reordering the terms of the preceding sum and using that Δ and $\Psi\circ\Delta$ are coassociative, we can rewrite the sum in the following way:

$$(Id\otimes\Delta')\circ\Delta'+(\Psi\otimes Id)\circ(Id\otimes\Delta')\circ\Delta'$$
$$=\frac{1}{4}\Big((Id\otimes\Delta)\circ\Delta+(\Psi\otimes Id)\circ(Id\otimes\Delta)\circ\Delta$$
$$+\alpha(u\otimes u\otimes Id)\circ(Id\otimes\Psi)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(u\otimes u\otimes Id)\circ(Id\otimes\sigma\otimes Id)\circ(\Delta\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(u\otimes\sigma\otimes u)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta$$
$$+\alpha(\Psi\otimes Id)\circ(Id\otimes u\otimes u)\circ(Id\otimes\sigma\otimes Id)\circ(Id\otimes\Psi)\circ(\Delta\otimes Id)\circ\Delta$$
$$+\alpha(Id\otimes u\otimes u)\circ(Id\otimes\sigma\otimes Id)\circ(Id\otimes\Psi)\circ(\Delta\otimes Id)\circ\Delta$$
$$+\alpha(\Psi\otimes Id)\circ(u\otimes\sigma\otimes u)\circ(Id\otimes\Delta)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta\Big).$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

Proceeding as before, we define

$$X' := (\sigma \otimes (\Psi \circ \Delta)) \circ \Psi \circ \Delta + (Id \otimes \sigma \otimes Id) \circ (\Delta \otimes id) \circ \Psi \circ \Delta,$$

taken from the partial sum of the second and third summands,

$$Y' := \Big(\sigma \otimes ((\sigma \otimes Id) \circ \Delta)\Big) \circ \Psi \circ \Delta + \Big((\Psi \circ (Id \otimes \sigma)) \otimes Id\Big) \circ (Id \otimes \Psi) \circ (\Delta \otimes Id) \circ \Delta,$$

taken from the partial sum of the fourth and fifth summands, and

$$Z' := \Big((\Psi \circ (\sigma \otimes \sigma)) \otimes Id\Big) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta + (Id \otimes \sigma Id) \circ (Id \otimes \Psi) \circ (\Delta \otimes Id) \circ \Delta,$$

taken from the partial sum of the sixth and seventh summands.

We can now check that the left and right sides of (1.3) are equal in this case. Indeed, $X = X'$ holds because of Remark 1.2 and equation (1.17). As far as $Y = Y'$, first of all we observe that $Y = (\Psi \otimes Id) \circ Z$ and $Y' = (\Psi \otimes Id) \circ Z'$, so it is enough to prove that $Z = Z'$, that in turn follows applying equations (1.17) and (1.18) and Remark 1.2. Finally, the remaining corresponding summands are equal in virtue of the coassociativity of (C, Δ, ϵ). The proofs for conditions (1.3) and (1.4), when restricting to uC, follow similar arguments. □

THEOREM 2.3. *Within the notations of Theorem 2.2, if $(C, \Delta, \epsilon, \sigma)$ is a coassociative, cocommutative, and anti-coinvolutive coalgebra, then $(C', \Delta', \epsilon', \sigma')$ is a coassociative and anti-coinvolutive coalgebra.*

Proof. The proof of the fact that $(C', \Delta', \epsilon', \sigma')$ is a coalgebra with anti-coinvolution is the same as in Theorem 2.2.
We proceed now to check the coassociativity condition. As far as $\Delta'|_C$ is concerned, on the one hand, we have

$$\begin{aligned}
(\Delta' \otimes Id) \circ \Delta'|_C &= \frac{1}{2}(\Delta' \otimes Id) \circ \big(\Delta + \alpha((u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta)\big) \\
&= \frac{1}{4}\Big((\Delta \otimes Id) \circ \Delta \\
&\quad + \alpha(u \otimes u \otimes Id) \circ (\sigma \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta \\
&\quad + \alpha(Id \otimes u \otimes u) \circ (\sigma \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta \\
&\quad + \alpha(u \otimes Id \otimes u) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta\Big),
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
(Id \otimes \Delta') \circ \Delta'|_C &= \frac{1}{2}(Id \otimes \Delta') \circ \big(\Delta + \alpha((u \otimes u) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta)\big) \\
&= \frac{1}{4}\Big((Id \otimes \Delta) \circ \Delta \\
&\quad + \alpha(Id \otimes u \otimes u) \circ (Id \otimes \sigma \otimes Id) \circ (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ \Delta \\
&\quad + \alpha(u \otimes Id \otimes u) \circ (\sigma \otimes \sigma \otimes Id) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta \\
&\quad + \alpha(u \otimes u \otimes Id) \circ (\sigma \otimes Id \otimes Id) \circ (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta\Big).
\end{aligned}$$

But each summand in the former sum corresponds with another in the latter. Indeed, as C is coassociative we get $(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$. By cocommutativity and coassociativity on C, we obtain that

$$(\sigma \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta = (\sigma \otimes Id \otimes Id) \circ (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta,$$

so the second summand in the former sum corresponds to the fourth in the latter. By using the fact that σ is an anti-coinvolution and again cocommutativity and coassociativity on C, it holds that

$$(\sigma \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Psi \circ \Delta = (Id \otimes \sigma \otimes Id) \circ (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ \Delta,$$

so the second summand in the former sum corresponds to the first in the latter. For the remaining term, the proof is analogous.

As far as $\Delta'|_{uC}$ is concerned, on the one hand, we have

$$\begin{aligned}(\Delta' \otimes Id) \circ \Delta'|_{uC} &= \frac{1}{2}(\Delta' \otimes Id) \circ \big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\big) \circ u^{-1} \\ &= \frac{1}{4}\Big((Id \otimes Id \otimes u) \circ (\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Delta \\ &\quad + \alpha(u \otimes u \otimes u) \circ (\sigma \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Delta \\ &\quad + (Id \otimes u \otimes Id) \circ (\sigma \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta \\ &\quad + (u \otimes Id \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\Big) \circ u^{-1},\end{aligned}$$

and on the other hand, it happens that

$$\begin{aligned}(Id \otimes \Delta') \circ \Delta'|_{uC} &= \frac{1}{2}(Id \otimes \Delta') \circ \big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\big) \circ u^{-1} \\ &= \frac{1}{4}\Big((Id \otimes Id \otimes u) \circ (Id \otimes \sigma \otimes Id) \circ (Id \otimes \Delta) \circ (\sigma \otimes Id) \circ \Delta \\ &\quad + (Id \otimes u \otimes Id) \circ (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ (\sigma \otimes Id) \circ \Delta \\ &\quad + (u \otimes Id \otimes Id) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta \\ &\quad + \alpha(u \otimes u \otimes u) \circ (Id \otimes \sigma \otimes Id) \circ (Id \otimes (\Psi \circ \Delta)) \circ \Psi \circ \Delta\Big) \circ u^{-1}.\end{aligned}$$

The proofs of the equalities

$$\begin{aligned}&(\Delta \otimes Id) \circ (\sigma \otimes Id) \circ \Delta = (Id \otimes \sigma \otimes Id) \circ (Id \otimes \Delta) \circ (\sigma \otimes Id) \circ \Delta, \\ &(\sigma \otimes Id \otimes Id) \circ ((\Psi \circ \Delta \circ \sigma) \otimes Id) \circ \Delta = (Id \otimes \sigma \otimes Id) \circ (Id \otimes (\Psi \circ \Delta)) \circ \Psi \circ \Delta, \\ &(\sigma \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta = (Id \otimes \Psi) \circ (Id \otimes \Delta) \circ (\sigma \otimes Id) \circ \Delta, \\ &(\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta = (Id \otimes \Delta) \circ \Psi \circ \Delta\end{aligned}$$

use the same arguments to those applied to prove the corresponding equalities in the case of $\Delta|_C$, relying on the coassociativity and cocommutativity of C and the fact that σ is an anti-coinvolution for C. □

THEOREM 2.4. *Within the notations of Definition 2.1, if $(C, \Delta, \epsilon, \sigma)$ is a coassociative, non-cocommutative, and strongly anti-coinvolutive coalgebra, then $(C', \Delta', \epsilon', \sigma')$ is a coalternative non-coassociative, and anti-coinvolutive coalgebra.*

Proof. By Theorem 2.2, we know that $(C', \Delta', \epsilon', \sigma')$ is a coalternative anti-coinvolutive coalgebra. We suppose that C' is coassociative and try to reach a contradiction. Actually, if C' were coassociative, then $(\Delta' \otimes Id) \circ \Delta'|_C = (Id \otimes \Delta') \circ \Delta'|_C$, and in particular

$$((\Psi \circ \Delta) \otimes Id) \circ \Delta = (Id \otimes (\Psi \circ \Delta)) \circ \Psi \circ \Delta. \tag{2.5}$$

As C is coassociative by hypothesis, so is the co-opposite coalgebra and then we can rephrase equation (2.5) as

$$((\Psi \circ \Delta) \otimes Id) \circ \Delta = ((\Psi \circ \Delta) \otimes Id) \circ \Psi \circ \Delta.$$

Now, composing with $(\epsilon_C \otimes Id \otimes Id)$ on both sides of the last equation we get that $\Delta = \Psi \circ \Delta$, it is said, C is a cocommutative coalgebra. Thus, we reach a contradiction. □

3. The double process for *D*-coalgebras.

In this section, we show that all the results shown in Section 2 for k-coalgebras with tensor product $- \otimes_k -$ can be extended to a more general structure, such as the D-coalgebras, being D an arbitrary k-coalgebra (see [4] for details).

DEFINITION 3.1. Let D be a k-coalgebra. For C a right D-comodule and E a left D-comodule, the *cotensor product* $C\Box_D E$ is defined as the following equalizer in the category of vector spaces:

$$C\Box_D E \longrightarrow C \otimes E \underset{Id\otimes^E\rho}{\overset{\rho^C\otimes Id}{\rightrightarrows}} C \otimes D \otimes E.$$

If C, E are (D, D)-bicomodules, then $C\Box_D E$ can be endowed with a (D, D)-bicomodule structure as was shown in [4, 11.3-(3)]. Notice that $-\Box_D-$ preserves direct limits, so also direct sums [4, 10.5], and as we are dealing with vector spaces, it also preserves injections [4, 10.4]. Notice also that since $C\Box_D E$ is a kernel, for any morphism $f : P \to C \otimes E$ with image contained in $C\Box_D E$, there exists a unique $\tilde{f} : P \to C\Box_D E$ such that $i \circ \tilde{f} = f$.

DEFINITION 3.2. Let $(C, \Delta_C, \epsilon_C)$, $(D, \Delta_D, \epsilon_D)$ be two k-coalgebras and $q : C \to D$ a coassociative and cocentral coalgebra morphism. Under these conditions, we say that C is a D-coalgebra, with structure $(C, \Delta^q_C, \epsilon^q_C)$, where Δ^q_C is the factorization of Δ_C through $C\Box_D C$, and $\epsilon^q_C := q$.

These morphisms satisfy the coalgebra conditions, considering on C the D-bicomodule structure given by ${}^C\rho = (q \otimes Id_C) \circ \Delta_C$, $\rho^C = (Id_C \otimes q) \circ \Delta_C$. Indeed, using that q is coassociative and [4, 11.3], it follows that $D\Box_D C \simeq C \simeq C\Box_D D$ as D-comodules. Hence, two different coalgebra structures coexist on C.

REMARK 3.3. From now on, otherwise specified, we consider the (D, D)-bicomodule structure on $C \otimes C$ given by

$${}^{C\otimes C}\rho := ((q \otimes Id) \circ \Delta) \otimes Id \quad \text{and} \quad \rho^{C\otimes C} := Id \otimes ((Id \otimes q) \circ \Delta), \tag{3.1}$$

and on $C\Box_D C$ the one given by the factorization through $D \otimes (C\Box_D C)$, respectively, $(C\Box_D C) \otimes D$, of the morphims:

$${}^{C\otimes C}\rho \circ i_{C,C}, \quad \text{respectively,} \quad \rho^{C\otimes C} \circ i_{C,C}, \tag{3.2}$$

where $i_{C,C} : C\Box_D C \to C \otimes C$ denotes the canonical inclusion.

LEMMA 3.4. *Let C be a D-coalgebra. All compositions included as summands in the conditions* (1.3) *and* (1.4) *of coalternative coalgebra factorize through the corresponding cotensor products* $(C\Box_D C)\Box_D C$ *and* $C\Box_D(C\Box_D C)$.

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

Proof. First of all, it is obvious that $(\Delta \otimes Id) \circ \Delta$ and $(Id \otimes \Delta) \circ \Delta$ factorize through $(C\square_D C)\square_D C$ and $C\square_D(C\square_D C)$ because q is coassociative.

With concern to the composition $((\Psi \circ \Delta) \otimes Id) \circ \Delta$, we can proceed as follows:

$$
\begin{aligned}
&(Id \otimes q \otimes Id \otimes Id) \circ (Id \otimes \Delta \otimes Id) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta \\
&\quad = \Big(((\Psi \otimes Id) \circ (q \otimes \Psi) \circ (\Delta \otimes Id) \circ \Delta\big) \otimes Id\Big) \circ \Delta \\
&\quad = \Big(((\Psi \otimes Id) \circ (q \otimes \Psi) \circ (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\big) \otimes Id\Big) \circ \Delta \\
&\quad = \Big(((\Psi \otimes Id) \circ (Id \otimes \Psi) \circ ((\Psi \circ (Id \otimes q) \circ \Delta) \otimes Id) \circ \Delta\big) \otimes Id\Big) \circ \Delta \\
&\quad = \Big((\Psi \otimes Id) \circ (Id \otimes \Psi) \circ \big(((\Psi \circ (Id \otimes q)) \otimes Id) \circ ((Id \otimes \Delta) \circ \Delta)\big) \otimes Id\Big) \circ \Delta \\
&\quad = \Big((((Id \otimes q) \circ \Psi \circ \Delta) \otimes Id) \circ \Psi \circ \Delta\big) \otimes Id\Big) \circ \Delta \\
&\quad = \Big((((Id \otimes q) \circ \Delta) \otimes Id) \circ \Psi \circ \Delta\big) \otimes Id\Big) \circ \Delta,
\end{aligned}
$$

where the first, third, and fifth equalities hold by naturality of the braiding, the second and the sixth because q is cocentral, and the fourth one by coassociativity of q.

Hence, as $- \otimes_k C$ is left exact, we know that there exists a unique $f : C \to (C\square_D C) \otimes C$ such that

$$(i_{C,C} \otimes Id) \circ f = (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta.$$

Moreover,

$$
\begin{aligned}
&\Big(((i_{C,C} \otimes Id_D) \circ \rho^{C\square_D C}) \otimes Id\Big) \circ f \\
&\quad = \Big(((Id \otimes Id \otimes q) \circ (Id \otimes \Delta) \circ i_{C,C}) \otimes Id\Big) \circ f \\
&\quad = \Big(((Id \otimes Id \otimes q) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta) \otimes Id\Big) \circ \Delta \\
&\quad = \Big(((\Psi \otimes q) \circ (Id \otimes \Delta) \circ \Delta) \otimes Id\Big) \circ \Delta \\
&\quad = \Big(((\Psi \otimes Id) \circ (\Delta \otimes q) \circ \Delta) \otimes Id\Big) \circ \Delta \\
&\quad = (\Psi \otimes q \otimes Id) \circ (\Delta \otimes \Delta) \circ \Delta \\
&\quad = (Id \otimes q \otimes Id) \circ (i_{C,C} \otimes \Delta) \circ f.
\end{aligned}
$$

As $i_{C,C}$ is an equalizer, it is a monomorphism, and so is $i_{C,C} \otimes Id_D \otimes Id_C$, that in turns implies that

$$(\rho^{C\square_D C} \otimes Id) \circ f = (C\square_D C \otimes^C \rho) \circ f.$$

Henceforth, there exists a unique $g : C \to (C\square_D C)\square_D C$ such that $i_{C\square_D C,C} \circ g = f$. Summing up, we have

$$(i_{C,C} \otimes Id) \circ i_{C\square_D C,C} \circ g = (i_{C,C} \otimes Id) \circ f = (\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta,$$

so $(\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta$ factorizes through $(C\square_D C)\square_D C$.

Following a similar pattern, we could prove also that $(\Psi \otimes Id) \circ (\Delta \otimes Id) \circ \Delta$ factorizes through $C\square_D(C\square_D C)$. The corresponding proofs for the remaining compositions in (1.3) and (1.4) are analogous. □

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

We introduce now a generalization of the Cayley–Dickson process discussed in Section 2 by using an endomorphism instead of a scalar $\alpha \in k$.

DEFINITION 3.5. Let D be an arbitrary coalgebra and $(C, \Delta, \epsilon, \sigma)$ a strongly anti-coinvolutive coalgebra with a D-coalgebra structure $q : C \to D$ such that $q \circ \sigma = q$. Let $\gamma : D \to k$ be any nonzero coalgebra morphism and define $\tilde{\gamma} : C \to C$ as the composition

$$\tilde{\gamma} := ((\gamma \circ q) \otimes Id) \circ \Delta.$$

We take $C' = C \oplus uC$ as in Section 2, the morphisms ϵ' and σ' as in (2.3) and (2.4), respectively, and define $\Delta' : C' \to C' \otimes C'$ as follows:

$$\Delta'|_C := \tfrac{1}{2}\Big(\Delta + ((u \circ \sigma) \otimes u) \circ (Id \otimes \tilde{\gamma}) \circ \Psi \circ \Delta\Big),$$
$$\Delta'|_{uC} := \tfrac{1}{2}\Big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\Big) \circ u^{-1}.$$

We will say that $(C', \Delta', \epsilon', \sigma')$ is the result of applying the generalized Cayley–Dickson process to the coalgebra $(C, \Delta, \epsilon, \sigma)$.

REMARK 3.6. As q is cocentral and k the unit object of the category, it holds that

$$\tilde{\gamma} = ((\gamma \circ q) \otimes Id) \circ \Psi \circ \Delta = (Id \otimes (\gamma \circ q)) \circ \Delta. \tag{3.3}$$

Notice also that, if we take $D = k$ the trivial coalgebra and $q = \epsilon_C$, then it results that $-\square_D- = - \otimes_k -$ and the structure of D-coalgebra of C is just its k-structure. In addition, in this case $\tilde{\gamma} = \gamma \otimes Id_C$, it is said, multiplication by an scalar. As a consequence, we recover the Cayley–Dickson process discussed in Section 2.

PROPOSITION 3.7. *Within the conditions of Definition 3.5, the coproduct $\Delta' : C' \to C' \otimes C'$ factorizes through the cotensor*

$$C'\square_D C' = (C\square_D C) \oplus (C\square_D uC) \oplus (uC\square_D C) \oplus (uC\square_D uC).$$

Proof. First of all, as $-\square_D-$ preserves direct limits, we have

$$C'\square_D C' = (C\square_D C) \oplus (C\square_D uC) \oplus (uC\square_D C) \oplus (uC\square_D uC),$$

so we are reduced to check that each morphism acting as a summand in the definition of Δ' factorizes through the corresponding summand in the kernel. Notice also that as u is a (D, D)-bicomodule isomorphism, it is not necessary to consider it in this proof.

(i) For the summand Δ, it holds that

$$(Id \otimes (q \otimes Id) \circ \Delta) \circ \Delta = \big(((Id \otimes q) \circ \Delta) \otimes Id\big) \circ \Delta \tag{3.4}$$

just by condition (1.6) for q.

(ii) For the summand $\Psi \circ \Delta$, applying that q is cocentral, coassociative, and the naturality of the braiding, we have

$$\begin{aligned}(Id \otimes q \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta &= (\Psi \otimes Id) \circ (q \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Psi \circ \Delta\\ &= (\Psi \otimes Id) \circ (Id \otimes \Psi) \circ (\Psi \otimes Id) \circ (Id \otimes q \otimes Id) \circ (Id \otimes \Delta) \circ \Delta\\ &= (\Psi \otimes Id) \circ (Id \otimes \Psi) \circ (\Psi \otimes Id) \circ (Id \otimes q \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\ &= (\Psi \otimes Id) \circ (Id \otimes \Psi) \circ (q \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Delta\\ &= (Id \otimes q \otimes Id) \circ (Id \otimes \Delta) \circ \Psi \circ \Delta.\end{aligned}$$

(iii) For the summand $(\sigma \otimes Id) \circ \Delta$, we have

$$\begin{aligned}(\sigma \otimes ((q \otimes Id) \circ \Delta)) \circ \Delta &= (\sigma \otimes q \otimes Id) \circ (\Delta \otimes Id) \circ \Delta \\ &= \big(((\sigma \otimes q) \circ \Psi \circ \Delta) \otimes Id\big) \circ \Delta \\ &= \big(((\sigma \otimes (q \circ \sigma)) \circ \Psi \circ \Delta) \otimes Id\big) \circ \Delta \\ &= \big(((Id \otimes q) \circ \Delta_D \circ \sigma) \otimes Id_D\big) \circ \Delta .\end{aligned}$$

In the preceding computations, the first equality follows because q is coassociative, the second one because q is cocentral, the third one relies on $q \circ \sigma = q$, and the last one holds because σ is an anti-coinvolutive coalgebra morphism.

(iv) Finally, as far as the summand $(\sigma \otimes \tilde{\gamma}) \circ \Psi \circ \Delta$ is concerned, it holds that

$$\begin{aligned}&(Id \otimes q \otimes Id) \circ (\Delta \otimes Id) \circ (\sigma \otimes \tilde{\gamma}) \circ \Psi \circ \Delta \\ &\quad = (Id \otimes Id \otimes (\gamma \circ q) \otimes Id) \circ (Id \otimes q \otimes \Delta) \circ (\sigma \otimes \Delta) \circ \Psi \circ \Delta \\ &\quad = (Id \otimes Id \otimes \gamma \otimes Id) \circ (Id \otimes q \otimes q \otimes Id) \circ (Id \otimes \Delta \otimes Id) \circ (\sigma \otimes \Delta) \circ \Psi \circ \Delta \\ &\quad = (Id \otimes \gamma \otimes Id \otimes Id) \circ (Id \otimes q \otimes q \otimes Id) \circ (Id \otimes \Delta \otimes Id) \circ (\sigma \otimes \Delta) \circ \Psi \circ \Delta \\ &\quad = (Id \otimes Id \otimes q \otimes Id) \circ (Id \otimes (\gamma \circ q) \otimes \Delta) \circ (\sigma \otimes \Delta) \circ \Psi \circ \Delta .\end{aligned}$$

On the preceding calculations, the first equality follows by part (ii), the second and the fourth ones are true because q is coassociative, while on the third one we apply that $q \circ \sigma = q$. Summing up, we conclude that Δ' factorizes thorough $C' \square_D C'$. □

REMARK 3.8. Notice that the condition $q \circ \sigma = q$ is actually necessary in order to have Proposition 3.7, because the factorization of the third summand means that

$$(\sigma \otimes ((q \otimes Id) \circ \Delta)) \circ \Delta = (((Id \otimes q) \circ \Delta \circ \sigma) \otimes Id) \circ \Delta$$

and composing both sides of this equation with $\epsilon \otimes Id \otimes \epsilon$, it results that $q \circ \sigma = q$.

REMARK 3.9. The Cayley–Dickson process given in Definition 3.5 can be performed in an iterative way as that of Section 2. Indeed, given a D-coalgebra structure on C, we can define a D-coalgebra structure on C' in a natural way.

More precisely, let $q : C \to D$ be the given coalgebra morphism and denote by $p : C' \to C$ the natural projection of k-vector spaces. Then, the composition

$$q' : C' \xrightarrow{p} C \xrightarrow{q} D \tag{3.5}$$

satisfies, with respect to C', the same conditions than q with respect to C. Hence, we would be again in the conditions of Definition 3.5 and be able to carry out the next step of the Cayley–Dickson process. Actually, just by definition of Δ', ϵ', and σ', it follows that q' is a surjective k-coalgebra morphism such that $q' \circ \sigma' = q'$.

Concerning the cocentral character of q', we obtain

$$(q' \otimes Id_{C'}) \circ \Psi \circ \Delta'|_C = (q' \otimes Id_{C'}) \circ \Delta'|_C$$

because q is cocentral and $q' \circ u = 0$. The proof of the corresponding equality for the summand uC is similar, but using in addition that $q \circ \sigma = q$.

As far as the coassociative character of q', we have

$$\begin{aligned}
&(q'\otimes Id\otimes Id)\circ(\Delta'\otimes Id)\circ\Delta'|_C\\
&\quad=\frac{1}{2}\Big((q'\otimes Id\otimes Id)\circ(\Delta'\otimes Id)\circ\Delta\\
&\qquad+(q'\otimes Id\otimes Id)\circ(\Delta'\otimes Id)\circ(u\otimes u)\circ(\sigma\otimes\tilde{\gamma})\circ\Psi\circ\Delta\Big)\\
&\quad=\frac{1}{4}\Big((q\otimes Id\otimes Id)\circ(\Delta\otimes Id)\circ\Delta\\
&\qquad+((q'\circ u\sigma)\otimes(u\circ\tilde{\gamma})\otimes Id)\circ(\Psi\otimes Id)\circ(\Delta\otimes Id)\circ\Delta\\
&\qquad+((q'\circ u\sigma)\otimes u\otimes u)\circ(\Delta\otimes Id)\circ(\sigma\otimes\tilde{\gamma})\circ\Psi\circ\Delta\\
&\qquad+\big(((q'\circ u)\circ\Psi\circ\Delta)\otimes u\big)\circ(\sigma\otimes\tilde{\gamma})\circ\Psi\circ\Delta\Big)\\
&\quad=\frac{1}{4}\Big((q\otimes\Delta)\circ\Delta+(q\otimes(u\circ\sigma)\otimes Id)\circ(\Psi\otimes u)\circ(\Delta\otimes\tilde{\gamma})\circ\Psi\circ\Delta\Big)\\
&\quad=\frac{1}{4}\Big((q'\otimes\Delta)\circ\Delta\\
&\qquad+\big(((Id\otimes(u\circ\sigma))\circ\Psi)\otimes(u\circ\tilde{\gamma})\big)\circ(Id\otimes\Psi)\circ(\Psi\otimes q)\circ(Id\otimes\Delta)\circ\Delta\Big)\\
&\quad=\frac{1}{4}\Big((q'\otimes\Delta)\circ\Delta\\
&\qquad+(Id\otimes(u\circ\sigma)\otimes(u\circ\tilde{\gamma}))\circ(\Psi\otimes Id)\circ(Id\otimes\Psi)\circ(\Psi\otimes Id)\circ(\Delta\otimes q)\circ\Delta\Big)\\
&\quad=\frac{1}{4}\Big((q\otimes Id\otimes Id)\circ(p\otimes\Delta)\circ\Delta\\
&\qquad+(Id\otimes(u\circ\sigma)\otimes(u\circ\tilde{\gamma}))\circ(q'\otimes(\Psi\circ\Delta))\circ\Delta\Big)\\
&\quad=(q'\otimes\Delta')\circ\Delta'.
\end{aligned}$$

In the preceding computations, the third equality relies on the coassociativity of q and the fact that $q'\circ u=0$ and σ is an antimorphism of coalgebras, the fourth and the sixth ones on the naturality of the braiding, while the fifth one holds because q is coassociative and cocentral.

The corresponding proof for uC follows a similar pattern, as so do the proofs for the remaining equalities required for coassociativity.

As an application of Lemma 3.4 and Proposition 3.7, we state below the main result of this section. Specifically, the results about the Cayley–Dickson process developed in Section 2 for k-coalgebras and the tensor product $-\otimes_k-$ are extended to the additional structure of D-coalgebras and the cotensor product $-\square_D-$, in such a way that both structures are compatible.

THEOREM 3.10. *Within the hypothesis and the notation of Definitions 3.2 and 3.5, it holds that*

- *(i) if $(C,\Delta,\epsilon,\sigma)$ is a coassociative and strongly anti-coinvolutive D-coalgebra, then $(C',\Delta'^q,\epsilon'^q,\sigma')$ is a coalternative and anti-coinvolutive D-coalgebra;*
- *(ii) if $(C,\Delta,\epsilon,\sigma)$ is a coassociative, cocommutative, and anti-coinvolutive D-coalgebra, then $(C',\Delta'^q,\epsilon'^q,\sigma')$ is a coassociative and anti-coinvolutive D-coalgebra;*

(iii) if $(C, \Delta, \epsilon, \sigma)$ is coassociative, non-cocommutative, and strongly anti-coinvolutive D-coalgebra, then $(C', \Delta'^{q}, \epsilon'^{q}, \sigma')$ is a coaltenative, non-coassociative, and anti-coinvolutive D-coalgebra.

Proof. As the equalizer morphism is a monomorphism, taking into account Lemma 3.4 and Proposition 3.7, we are reduced to prove the results for $(C', \Delta', \epsilon', \sigma')$, it is said, just for k-algebras. But in this case, the arguments used in Theorems 2.2, 2.3, and 2.4 remain valid. This is because, in virtue of the coassociative and cocentral properties for q, and by (3.3), in the computations directed to prove the coalternativity condition of $(C', \Delta', \epsilon', \sigma')$, the morphism $\tilde{\gamma} : C \to C$ can be handled as if it were the multiplication by a scalar α in k. □

4. L^{FG} coalternative coalgebras.

We present in this section a class of examples of coalternative coalgebras inspired by Cayley algebras (quaternions and octonions) studied by Bulacu as quasicoalgebras in [**5**]. The following result is a generalization of Proposition 3.2 in [**5**].

PROPOSITION 4.1. *Let $L|k$ be a field extension such that L has a cocommutative and coassociative k-coalgebra structure, and let G be a finite abelian group such that $|G|$ is not divisible by $char(k)$ and provided with $F : G \times G \longrightarrow L \setminus \{0\}$ a 2-cochain. It holds the following:*

(i) The group algebra $L[G]$ with the coproduct and counit given, respectively, by

$$\Delta_F(x) := \frac{1}{|G|} \sum_{a \in G} F(a, a^{-1}x)^{-1} a \otimes_L a^{-1}x \quad \text{and} \quad \epsilon_F(x) := |G|\delta_{x,e}$$

for any $x \in G$, where $\delta_{x,e}$ is the Kronecker's delta, is an L-coalgebra denoted by $L^F G$.

(ii) Let $k^{F_0}G$ be the k-coalgebra defined as in (i) for the 2-cochain $F_0 := \epsilon_L \circ F : G \times G \longrightarrow k \setminus \{0\}$, and let σ_{F_0} be any strong coinvolution for $k^{F_0}G$. It holds that the product k-coalgebra $L \otimes k^{F_0}G$ together with the morphisms

$$q := Id_L \otimes \epsilon_{F_0}, \qquad \sigma_{L \otimes k^{F_0}G} := Id_L \otimes \sigma_{F_0}$$

falls under the conditions of Definition 3.5.

Proof. The demonstration of part (i) follows from the definitions and the fact that $F(a, e) = F(e, a) = 1_L$, for all $a, b \in G$.

As far as (ii) is concerned, it is well known (see [**7**]) that the product coalgebra $L \otimes k^{F_0}G$ is a k-coalgebra which structural morphisms are given by

$$\Delta_{L \otimes F_0} = (Id_L \otimes \Psi \otimes Id_{k^{F_0}G}) \circ (\Delta_L \otimes \Delta_{F_0}), \quad \epsilon_{L \otimes F_0} = \epsilon_L \otimes \epsilon_{F_0}. \tag{4.1}$$

The morphism q is a k-coalgebra morphism, because using that

$$(\epsilon_{F_0} \otimes L) \circ \Psi = L \otimes \epsilon_{F_0} \tag{4.2}$$

and the counit property for $k^{F_0}G$, we get

$$\begin{aligned}(q \otimes q) \circ \Delta_{L \otimes F_0} &= (Id_L \otimes \epsilon_{F_0} \otimes Id_L \otimes \epsilon_{F_0}) \circ (Id_L \otimes \Psi \otimes Id_{k^{F_0}G}) \\ &= \Delta_L \otimes \epsilon_{F_0} = \Delta_{L \otimes k^{F_0}G} \otimes q.\end{aligned}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

We prove now one of the coassociativity conditions for q, the others being analogous. Indeed, on the one hand, by the coassociative condition of L, (4.2) and the counit property for $k^{F_0}G$, we have

$$
\begin{aligned}
&(q \otimes Id_{L\otimes F_0}) \circ (\Delta_{L\otimes F_0} \otimes Id_{L\otimes k^{F_0}G}) \circ \Delta_{L\otimes F_0} \\
&\quad = \Big(Id_L \otimes ((\epsilon_{F_0} \otimes L) \circ \Psi) \otimes Id_{k^{F_0}G} \otimes Id_L \otimes Id_{k^{F_0}G}\Big) \\
&\qquad \circ \Big(\Delta_L \otimes ((\Delta_{F_0} \otimes L) \circ \Psi) \otimes Id_{k^{F_0}G}\Big) \circ (\Delta_L \otimes \Delta_{F_0}) \\
&\quad = (Id_L \otimes Id_L \otimes \Psi \otimes Id_{k^{F_0}G}) \circ \Big(((Id_L \otimes \Delta_L) \circ \Delta_L) \otimes \Delta_{F_0}\Big).
\end{aligned}
$$

On the other hand, by the same arguments, we have

$$
\begin{aligned}
&(q \otimes \Delta_{L\otimes k^{F_0}G}) \circ \Delta_{L\otimes F_0} \\
&\quad = (Id_L \otimes \epsilon_{F_0} \otimes Id_L \otimes \Psi_{Id_{k^{F_0}G}}) \circ (Id_L \otimes Id_L \otimes \Psi \otimes Id_{k^{F_0}G}) \\
&\qquad \circ \Big(((Id_L \otimes \Delta_L) \circ \Delta_L) \otimes \Delta_{F_0}\Big) \\
&\quad = (Id_L \otimes Id_L \otimes \Psi \otimes Id_{k^{F_0}G}) \circ \Big(((Id_L \otimes \Delta_L) \circ \Delta_L) \otimes \Delta_{F_0}\Big).
\end{aligned}
$$

The cocentral property of q follows by the cocommutativity of L, (4.2), the symmetry of Ψ, and the Hexagon Axiom in the definition of a braiding (see [7, XIII.1.1]). Finally, the fact that $Id_L \otimes \sigma_{F_0}$ is a coinvolution for the product coalgebra follows using the Hexagon Axiom and knowing that σ_{F_0} is a coinvolution for $k^{F_0}[G]$ and Ψ is symmetric and L is cocommutative. □

REMARK 4.2. Notice that the statement (ii) of Proposition 4.1 remains true if we take $\sigma_{L\otimes F_0} = \sigma_L \otimes \sigma_{F_0}$ with σ_L any strong coinvolution for L as a k-coalgebra.

Note that as L-vector spaces, we can identify $L \otimes_k kG$ and LG. Hence, we can summarize the results and main implications of Proposition 4.1 in the following statement.

COROLLARY 4.3. *Let $L|k$ be a field extension such that L has a cocommutative and coassociative k-coalgebra structure, and let G be a finite abelian group such that $|G|$ is not divisible by char(k) and provided with $F : G \times G \longrightarrow L \setminus \{0\}$ a 2-cochain. Under these conditions and keeping the notation of Proposition 4.1, it holds that three different coalgebra structures coexist in the group algebra LG with interrelations described in Proposition 4.1. Namely,*

(i) *the product k-coalgebra structure over k given by (4.1);*
(ii) *the L-coalgebra structure $(L^F G, \Delta_F, \epsilon_F)$; and*
(iii) *the L-coalgebra structure $(L^{F_0}G, \Delta^q_{L\otimes F_0}, Id_L \otimes \epsilon_{F_0})$ given by the factorization of $\Delta_{L\otimes F_0}$ through the cotensor $LG\Box_L LG$ as in Definition 3.2.*

In addition, the k-coalgebra structure of (i) and the L-coalgebra structure of (ii) fall under the conditions required to apply the Cayley–Dickson process of Definition 2.1 over k and L, respectively; while the L-coalgebra structure of (iii) satisfies the conditions of the Cayley–Dickson process described in Definition 3.5.

Let us consider the map $\phi : G \times G \times G \to k$ defined as $\phi(x, y, z) := \frac{F(x,y)F(xy,z)}{F(y,z)F(x,yz)}$, for all $x, y, z \in G$,. Next we generalize this result by presenting a coalgebra version of [3, Proposition 3.3], providing conditions, in terms of the map ϕ, that ensure the coalternativity of $k^F G$.

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

THEOREM 4.4. *Let k be a field and G be a finite abelian group such that $|G|$ is not divisible by $char(k)$ and provided with $F : G \times G \longrightarrow k \setminus \{0\}$ a 2-cochain. Consider the group coalgebra $k^F G$ with k-coalgebra structure given by (4.1). It holds the following:*

(i) If in the category of G-graded vector spaces the braiding Ψ is given by the usual flip functor, then $k^F G$ is a coalternative coalgebra if and only if

$$\phi(y, x, z) + R^{-1}(x, y)\phi(x, y, z) = 1 + R^{-1}(x, y)$$

and

$$\phi^{-1}(x, y, z) + R^{-1}(z, y)\phi^{-1}(x, z, y) = 1 + R^{-1}(z, y),$$

where $R(x, y) := \frac{F(x,y)}{F(y,x)}$, for all $x, y, z \in G$.

(ii) If in the category of G-graded vector spaces the braiding Ψ is given by $\Psi(x, y) = R(x, y)y \otimes x$, then $k^F G$ is coalternative if and only if $\phi(x, y, z) = 1$ for all $x, y, z \in G$.

Proof. With regard to part (i), we write the proof of condition (1.3), as condition (1.4) follows by similar arguments. Let $x \in G$. On the one hand, we have

$$\begin{aligned}
&\Big((\Delta \otimes Id) \circ \Delta + ((\Psi \circ \Delta) \otimes Id) \circ \Delta\Big)(x) \\
&= (\Delta \otimes Id)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big) \\
&\quad + ((\Psi \circ \Delta) \otimes Id)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big) \\
&= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z \otimes z^{-1}a \otimes a^{-1}x \\
&\quad + \sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z^{-1}a \otimes z \otimes a^{-1}x\Big) \\
&= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z \otimes z^{-1}a \otimes a^{-1}x \\
&\quad + \sum_{a\in G}\sum_{\xi\in G} F(a, a^{-1}x)^{-1}F(\xi^{-1}a, \xi)^{-1} \xi \otimes \xi^{-1}a \otimes a^{-1}x\Big),
\end{aligned}$$

being $\xi := z^{-1}a \in G$ in the last term. On the other hand, we have

$$\begin{aligned}
&\Big((\Psi \otimes Id) \circ (Id \otimes \Delta) \circ \Delta + (Id \otimes \Delta) \circ \Delta\Big)(x) = \\
&= (\Psi \otimes Id) \circ (Id \otimes \Delta)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big) \\
&\quad + (Id \otimes \Delta)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big) \\
&= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}(a^{-1}x))^{-1} z \otimes a \otimes z^{-1}(a^{-1}x) \\
&\quad + \sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}(a^{-1}x))^{-1} a \otimes z \otimes z^{-1}(a^{-1}x)\Big)
\end{aligned}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

$$= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(z^{-1}a, a^{-1}zx)^{-1}F(z, a^{-1}x)^{-1} z \otimes z^{-1}a \otimes a^{-1}x$$
$$+ \sum_{a\in G}\sum_{\xi\in G} F(\xi, \xi^{-1}x)^{-1}F(\xi^{-1}a, a^{-1}x)^{-1} \xi \otimes \xi^{-1}a \otimes a^{-1}x\Big),$$

because in the last equality we used the fact that a covers G if and only if so does $z^{-1}a$ and after have done this change we considered again the notation $\xi = z^{-1}a$.

Therefore, for $k^F G$, the coalternativity condition given by equation (1.3) remains

$$F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} + F(a, a^{-1}x)^{-1}F(z^{-1}a, z)^{-1} =$$
$$F(z^{-1}a, a^{-1}zx)^{-1}F(z, a^{-1}x)^{-1} + F(z, z^{-1}x)^{-1}F(z^{-1}a, a^{-1}x)^{-1}.$$

Doing the change of variable $v := a^{-1}x$ and $u := z^{-1}a$, the preceding equality can be written as

$$F(zu, v)^{-1}F(z, u)^{-1} + F(uz, v)^{-1}F(u, z)^{-1} = F(u, vz)^{-1}F(z, v)^{-1} + F(z, uv)^{-1}F(u, v)^{-1}$$

and replacing in turn u by x, z by y and v by z, we obtain that the condition given by equation (1.3) can be expressed in this case as the equality

$$F(x, y)^{-1}F(xy, z)^{-1} + F(y, x)^{-1}F(yx, z)^{-1} = F(y, z)^{-1}F(x, yz)^{-1} + F(x, z)^{-1}F(y, zx)^{-1},$$

for any $x, y, z \in G$.

Similarly, we can prove that

$$F(x, z)^{-1}F(xz, y)^{-1} + F(x, y)^{-1}F(xy, z)^{-1} = F(z, y)^{-1}F(x, zy)^{-1} + F(y, z)^{-1}F(x, yz)^{-1},$$

for any $x, y, z \in G$, which is equivalent to the second coalternativity condition given by equation (1.4).

With regard to part (ii), again we just write the proof of condition (1.3). On the one hand, we have

$$\Big((\Delta \otimes Id)\circ\Delta + ((\Psi\circ\Delta)\otimes Id)\circ\Delta\Big)(x)$$
$$= (\Delta \otimes Id)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big)$$
$$+ ((\Psi\circ\Delta)\otimes Id)\Big(\frac{1}{|G|}\sum_{a\in G} F(a, a^{-1}x)^{-1} a \otimes a^{-1}x\Big)$$
$$= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z \otimes z^{-1}a \otimes a^{-1}x$$
$$+ \sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1}F(z, z^{-1}a)F(z^{-1}a, z)^{-1} z^{-1}a \otimes z \otimes a^{-1}x\Big)$$
$$= \frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z \otimes z^{-1}a \otimes a^{-1}x$$
$$+ \sum_{a\in G}\sum_{z\in G} F(a, a^{-1}x)^{-1}F(z, z^{-1}a)^{-1} z \otimes z^{-1}a \otimes a^{-1}x\Big).$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

On the other hand, we have

$$\Big((\Psi\otimes Id)\circ(Id\otimes\Delta)\circ\Delta+(Id\otimes\Delta)\circ\Delta\Big)(x)=$$
$$=(\Psi\otimes Id)\circ(Id\otimes\Delta)\Big(\frac{1}{|G|}\sum_{a\in G}F(a,a^{-1}x)^{-1}a\otimes a^{-1}x\Big)$$
$$+(Id\otimes\Delta)\Big(\frac{1}{|G|}\sum_{a\in G}F(a,a^{-1}x)^{-1}a\otimes a^{-1}x\Big)=$$
$$=\frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G}F(a,a^{-1}x)^{-1}F(z,z^{-1}a^{-1}x)^{-1}F(a,z)F(z,a)^{-1}z\otimes a\otimes z^{-1}a^{-1}x$$
$$+\sum_{a\in G}\sum_{z\in G}F(a,a^{-1}x)^{-1}F(z,z^{-1}a^{-1}x)^{-1}a\otimes z\otimes z^{-1}a^{-1}x\Big)$$
$$=\frac{1}{|G|^2}\Big(\sum_{a\in G}\sum_{z\in G}F(z^{-1}a,za^{-1}x)^{-1}F(z,a^{-1}x)^{-1}F(z^{-1}a,z)F(z,z^{-1}a)^{-1}z\otimes z^{-1}a\otimes a^{-1}x$$
$$+\sum_{a\in G}\sum_{z\in G}F(z^{-1}a,za^{-1}x)^{-1}F(z,a^{-1}x)^{-1}z^{-1}a\otimes z\otimes a^{-1}x\Big),$$

where the last equation holds because a covers G if and only if so does $z^{-1}a$.

Then, in this case, condition (1.3) can be expressed by means of

$$F(a,a^{-1}x)^{-1}F(z,z^{-1}a)^{-1}=F(z^{-1}a,za^{-1}x)^{-1}F(z,a^{-1}x)^{-1}F(z^{-1}a,z)F(z,z^{-1}a)$$

and

$$F(a,a^{-1}x)^{-1}F(z^{-1}a,z)^{-1}=F(z^{-1}a,za^{-1}x)^{-1}F(z,a^{-1}x)^{-1}, \tag{4.3}$$

for all a, x, z in G. As these equalities are equivalent, it is enough to consider (4.3). Now, doing the changes of variables $v:=a^{-1}x$ and $u=z^{-1}a$, we can write (4.3) as

$$F(uz,v)^{-1}F(u,z)^{-1}=F(u,zv)^{-1}F(z,v)^{-1}. \tag{4.4}$$

Replacing again u by x, z by y and v by z we get, in turn, that (1.3) for $k^F(G)$ is equivalent to $\phi(x,y,z)=1$, with $x,y,z\in G$, as we wanted to prove. □

The next result shows the relationship between the coalgebra structures defined by means of 2-cochains and the Cayley–Dickson process.

PROPOSITION 4.5. *Let k be a field and G a finite abelian group such that char(k) does not divide $|G|$, and $N<H<G$ a chain of subgroups of G such that $G=H\cup uH$ with $u^2\in N$. Let also $\alpha\in k\setminus\{0\}$ be any scalar, $s:G\to k\setminus\{0\}$ a map such that $s(x)=1$ if $x\in N$ and $s(x)=-1$ if $x\in G\setminus N$, and $F:H\times H\to k\setminus\{0\}$ a 2-cochain satisfying that*

$$\frac{F(x,y)}{F(y,x)}=\frac{s(x)s(y)}{s(xy)},\quad \textit{for } x,y\in H. \tag{4.5}$$

Under this hypothesis, the following statements related to the k-coalgebras k^FH introduced in Proposition 4.1 hold:

(i) *The morphism $\sigma:k^FH\to k^FH$ defined by $\sigma(x):=s(x)x$ for all $x\in H$ is a strong anti-coinvolution for the coalgebra k^FH.*

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

(ii) *The map* $\overline{F}: G \times G \to k \setminus \{0\}$ *defined by*

$$\begin{aligned}\overline{F}(x, y) &:= F(x, y),\\ \overline{F}(x, uy) &:= s(x)^{-1}F(x, y),\\ \overline{F}(ux, y) &:= F(y, x),\\ \overline{F}(ux, uy) &:= \alpha^{-1}s(x)^{-1}F(y, x)\end{aligned}$$

is a 2*-cochain for the group* G *that satisfies the corresponding equality analogous to* (4.5) *with* $\overline{s}(x) = s(x)$ *and* $\overline{s}(ux) = -1$.

(iii) *The coalgebra* $k^{\overline{F}}G$ *with the strong anti-coinvolution given by* $\overline{\sigma}(x) = \overline{s}(x)x$ *is the* k*-coalgebra obtained by applying the Cayley–Dickson process of Definition 2.1 to* $k^F H$.

Proof. (i) By definition, σ trivially satisfies conditions (1.14) and (1.15). As far as condition (1.16), on the one hand, we have

$$\Delta \circ \sigma(x) = s(x)\Delta(x) = \frac{1}{|H|}\sum_{a\in H} s(x)F(a, a^{-1}x)^{-1} a \otimes a^{-1}x$$

and on the other hand,

$$(\sigma \otimes \sigma) \circ \Psi \circ \Delta(x) = \frac{1}{|H|}\sum_{a\in H} s(a^{-1}x)s(a)F(a, a^{-1}x)^{-1}(a^{-1}x \otimes a).$$

Now, if we define $b := a^{-1}x$, the preceding expression can be rewritten as

$$(\sigma \otimes \sigma) \circ \Psi \circ \Delta(x) = \frac{1}{|H|}\sum_{b\in h} s(b)s(b^{-1}x)F(b^{-1}x, b)^{-1}(b \otimes b^{-1}x),$$

that, in turn, can be expressed as

$$(\sigma \otimes \sigma) \circ \Psi \circ \Delta(x) = \frac{1}{|H|}s(x)\sum_{b\in H} F(b, b^{-1}x)^{-1}(b \otimes b^{-1}x)$$

because as F satisfies (4.5), we have $s(b)s(b^{-1}x)F(b^{-1}x, b)^{-1} = s(x)F(b, b^{-1}x)$. Thus, σ is an anti-coinvolution.

Now, in order to prove that σ is a strong anti-coinvolution, we just write the proof for (1.17), (1.18) being analogous. In the context of the present definition, the left-hand side of the equation (1.17) can be expressed as

$$\begin{aligned}&\frac{1}{|H|}\sum_{a\in H} F(a, a^{-1}x)^{-1}\big((a^{-1}x \otimes a) + s(a^{-1}x)(a^{-1}x \otimes a)\big)\\ &\quad= \frac{1}{|H|}\sum_{a\in H} F(a, a^{-1}x)^{-1}\big(1 + s(a^{-1}x)\big)(a^{-1}x \otimes a)\\ &\quad= \frac{1}{|H|}\sum_{b\in H} F(b^{-1}x, b)^{-1}\big(1 + s(b)\big)(b \otimes b^{-1}x)\end{aligned}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

$$= \frac{1}{|H|}\sum_{b\in N} 2F(b^{-1}x, b)^{-1}(b\otimes b^{-1}x)$$
$$= \frac{2}{|H|}\sum_{b\in N} F(b, b^{-1}, b)^{-1}(b\otimes b^{-1}x).$$

In the preceding computations, we used the change $b = a^{-1}x$, the definition of the morphism s, and the fact that $F(x, y) = F(y, x)$ if $y \in N$, that in turn follows from condition (4.5).

On the other hand, by the same arguments, the right side of (1.17) can be written as

$$\frac{1}{|H|}\sum_{b\in H} F(b, b^{-1}x)^{-1}(1+s(b))(b\otimes b^{-1}x) = \frac{2}{|H|}\sum_{b\in H} F(b, b^{-1}x)^{-1}(b\otimes b^{-1}x),$$

so σ actually satisfies equation (1.17).

(ii) It is clear that $\overline{F}$ is a 2-cochain because the unit element in $e_G = e$ and $\alpha \neq 0$. As far as condition (4.5), it is enough to note that

$$\frac{\overline{F}(ux, y)}{\overline{F}(y, ux)} = \frac{F(y, x)}{s(y)^{-1}F(y, x)} = s(y) = \frac{\overline{s}(ux)\overline{s}(y)}{\overline{s}(uxy)}$$

and

$$\frac{\overline{F}(ux, uy)}{\overline{F}(uy, ux)} = \frac{\alpha^{-1}s(x)^{-1}F(y, x)}{\alpha^{-1}s(y)^{-1}F(x, y)} = \frac{\overline{s}(ux)\overline{s}(uy)}{\overline{s}(xy)}.$$

(iii) Firstly, we recall that $k^{\overline{F}}G$ is a coalgebra with coproduct given by

$$\overline{\Delta}(x) = \frac{1}{2|H|}\Big(\sum_{a\in H}\overline{F}(a, a^{-1}x)^{-1}a\otimes a^{-1}x + \sum_{b\in H}\overline{F}(ub, ub^{-1}x)^{-1}ub\otimes ub^{-1}x\Big)$$

and

$$\overline{\Delta}(ux) = \frac{1}{2|H|}\Big(\sum_{a\in H}\overline{F}(a, ua^{-1}x)^{-1}a\otimes ua^{-1}x + \sum_{b\in H}\overline{F}(ub^{-1}x, b)^{-1}ub\otimes b^{-1}x\Big).$$

Moreover, the counity is defined as $\overline{\epsilon}(\zeta) := 2|G|\delta_{(\zeta,e)}$, for $\zeta \in G$.

On the other hand, the Cayley–Dickson process discussed in Definition 2.1 applied to $k^F H$ for a fixed $\alpha \in k\setminus\{0\}$ results in

$$\Delta'(x) = \frac{1}{2}\Big(\Delta + \alpha(u\otimes u)\circ(\sigma\otimes Id)\circ\Psi\circ\Delta\Big)(x)$$
$$= \frac{1}{2|H|}\Big(\sum_{a\in H}F(a, a^{-1}x)^{-1}\big(a\otimes a^{-1}x + \alpha s(a^{-1}x)u(a^{-1}x)\otimes ua\big)\Big)$$
$$= \frac{1}{2|H|}\Big(\sum_{a\in H}F(a, a^{-1}x)^{-1}(a\otimes a^{-1}x) + \alpha\sum_{b\in H}F(b^{-1}x, b)^{-1}s(b)ub\otimes ub^{-1}x\Big)$$
$$= \frac{1}{2|H|}\Big(\sum_{a\in H}\overline{F}(a, a^{-1}x)^{-1}(a\otimes a^{-1}x) + \sum_{b\in H}\overline{F}(ub, ub^{-1})^{-1}ub\otimes ub^{-1}x\Big),$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

so $\Delta'(x) = \overline{\Delta}(x)$ for all $x \in H$. Moreover, for the elements $ux \in G$, we have

$$\begin{aligned}\Delta'(ux) &= \frac{1}{2}\Big((\sigma \otimes u) \circ \Delta + (u \otimes Id) \circ \Psi \circ \Delta\Big)(x) \\ &= \frac{1}{2|H|}\Big(\sum_{a \in H} F(a, a^{-1}x)^{-1} s(a)(a \otimes u(a^{-1}x)) + \sum_{a \in H} F(a, a^{-1}x)^{-1}(u(a^{-1}x) \otimes a)\Big) \\ &= \frac{1}{2|H|}\Big(\sum_{a \in H} F(a, a^{-1}x)^{-1} s(a)(a \otimes ua^{-1}x) + \sum_{b \in H} F(b^{-1}x, b)^{-1}(ub \otimes b^{-1}x)\Big) \\ &= \frac{1}{2|H|}\Big(\sum_{a \in H} \overline{F}(a, ua^{-1}x)^{-1} a \otimes ua^{-1}x + \sum_{b \in H} \overline{F}(ub, b^{-1}x)^{-1} ub \otimes b^{-1}x\Big),\end{aligned}$$

so $\Delta'(ux) = \overline{\Delta}(ux)$. Finally, $\epsilon' = \epsilon_{\overline{F}}$ because $|G| = 2|H|$, and $\overline{\sigma} = \sigma'$ since $\overline{s}(ux) = -1$. □

REMARK 4.6. Note that the morphism σ defined in part (i) of Proposition 4.5 can be taken as an example of strong convolution σ_{F_0} required in part (ii) of Proposition 4.1.

COROLLARY 4.7. *Keeping the notation and within the same conditions as in Proposition 4.5, let us assume in addition that $L|k$ is a field extension with $(L, \Delta_L, \epsilon_L)$ a cocommutative coassociative k-coalgebra structure.*

Under these hypothesis, it holds that the product coalgebra

$$(L \otimes k^{\overline{F}}G, \Delta_{L\otimes\overline{F}}, \epsilon_L \otimes \epsilon_{\overline{F}}, Id_L \otimes \overline{\sigma})$$

is equal to

$$\Big((L \otimes k^F H)', \Delta'_{L\otimes F}, \epsilon'_{L\otimes F}, (Id_L \otimes \sigma)'\Big),$$

the coalgebra resulting of applying the Cayley–Dickson process to the product coalgebra $L \otimes k^F H$.

Proof. Firstly, as a consequence of part (ii) of Proposition 4.1, and taking into account the same considerations as in the proof of Theorem 3.10, we are reduced to prove the result for the Cayley–Dickson process, where $\tilde{\gamma}$ is the multiplication by an scalar $\alpha \in k$.

In order to fix the notation with respect to the formal label u, we establish that $(L \otimes k^F H)u = L \otimes k^F Hu$, interpreting it as the isomorphism

$$u\Big(\sum_{l \in L, x \in H} l \otimes x\Big) = \sum_{l \in L, x \in H} l \otimes xu$$

and identifying $l \otimes (x, \overline{0})$ and $l \otimes (x, \overline{1})$ with $l \otimes x$ and $l \otimes xu$ for all $l \in L$, $x \in H$, respectively.

When restricting to $L \otimes k^F H$, by Proposition 4.5, on the one hand, we have

$$\begin{aligned}\Delta_{L\otimes\overline{F}}|_{L\otimes k^F H} &= (Id \otimes \Psi \otimes Id) \circ \Delta_L \otimes \overline{\Delta}_{\overline{F}} \\ &= \frac{1}{2}\Big((Id \otimes \Psi \otimes Id) \circ (\Delta_L \otimes \Delta_F) \\ &\quad + \alpha(Id \otimes \Psi \otimes Id) \circ \big(\Delta_L \otimes (((u \circ \sigma) \otimes u) \circ \Psi \circ \Delta_F)\big)\Big) \\ &= \frac{1}{2}\Big((Id \otimes \Psi \otimes Id) \circ (\Delta_L \otimes \Delta_F) \\ &\quad + \alpha\big(Id \otimes (((u \circ \sigma) \otimes Id) \circ \Psi) \otimes u\big) \circ (\Delta_L \otimes (\Psi \circ \Delta_F))\Big).\end{aligned}$$

https://doi.org/10.1017/S0017089519000314 Published online by Cambridge University Press

On the other hand, we have

$$
\begin{aligned}
\Delta'_{L\otimes F}|_{L\otimes k^F H} &= \frac{1}{2}\Big((Id\otimes\Psi\otimes Id)\circ(\Delta_L\otimes\Delta_F)\\
&\quad +\alpha(Id\otimes(u\circ\sigma)\otimes Id\otimes u)\circ\Psi_{L\otimes k^F G, L\otimes k_F H}\\
&\qquad \circ(Id\otimes\Psi\otimes Id)\circ(\Delta_L\otimes\Delta_F)\Big)\\
&= \frac{1}{2}\Big((Id\otimes\Psi\otimes Id)\circ(\Delta_L\otimes\Delta_F)\\
&\quad +\alpha(Id\otimes(u\circ\sigma)\otimes Id\otimes u)\circ(Id\otimes\Psi\otimes Id)\\
&\qquad \circ(\Psi\otimes\Psi)\circ(Id\otimes(\Psi\circ\Psi)\otimes Id)\circ(\Delta_L\otimes\Delta_F)\Big)\\
&= \frac{1}{2}\Big((Id\otimes\Psi\otimes Id)\circ(\Delta_L\otimes\Delta_F)\\
&\quad +\alpha(Id\otimes(u\circ\sigma)\otimes Id\otimes u)\circ(Id\otimes\Psi\otimes Id)\circ(\Delta_L\otimes(\Psi\circ\Delta_F))\Big),
\end{aligned}
$$

where the second equality is true because of the Hexagon Axiom, and the last one because Ψ is symmetric and L cocommutative.

The proof for the corresponding equality when restricting to $L\otimes k^F Hu$ follows the same pattern and uses similar arguments, while the proofs for the counit and the coinvolution are straightforward. □

Acknowledgements. The first, second, and third authors acknowledge financial assistance by the Center for Mathematics of the University of Coimbra, UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. The third author is supported by the PCI of the UCA "Teoría de Lie y Teoría de Espacios de Banach," by the PAI with project numbers FQM298 and FQM7156, and by the project of the Spanish Ministerio de Educación y Ciencia MTM2013-41208P, and acknowledges the Fundação para a Ciência e a Tecnologia for the grant with reference SFRH/BPD/101675/2014.

REFERENCES

1. E. Abe, *Hopf algebras* (Cambridge University Press, Cambridge, New York, 1980).

2. H. Albuquerque, A. Elduque and J. M. Pérez-Izquierdo, Alternative quasialgebras, *Bull. Austral. Math. Soc.* **63** (2001), 257–268.

3. H. Albuquerque and S. Majid, Quasialgebra structure of the octonions, *J. Algebra* **220** (1999), 188–224.

4. T. Brzezinski and R. Wisbauer, *Corings and comodules*, London Mathematical Society Lecture Note Series, 309. (Cambridge University Press, Cambridge, 2003).

5. D. Bulacu, The weak braided Hopf algebra structure of some Cayley–Dickson algebras, *J. Algebra* **322** (2009), 2404–2427.

6. J. Cuadra, J. R. García Rozas and B. Torrecillas, Picard groups and strongly graded coalgebras, *J. Pure Appl. Algebra* **165**(30) (2001), 267–289.

7. C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155 (Springer-Verlag, Berlin, 1995).

8. M. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series (W.A. Benjamin, Inc., New York, 1969).