Characterisations of the parabola

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1. *Introduction*

Three familiar properties of a parabola are that it is the locus of points that are equidistant from the focus and the directrix, that it can be created by an intersection of a plane and a cone, and that incoming rays parallel to the axis are reflected to a single point. The first two are often used as definitions, and the third may be used as an alternative definition or characterisation.

We present a set of eight diverse properties, including the focusing property, which are also sufficient conditions for a curve to be a parabola. It is startling that there are so many different characterisations of the parabola. The conditions have been selected for their varied mathematical nature and the assorted methods of proof that appear to be most informative or efficient. None are included that use three dimensions or require input of another conic section, except for circles. The conditions are proved to be sufficient by utilising algebra, geometry of triangles and circles, differential equations, functional equations, and judicious choices of coordinates.

Statements and proofs of necessary conditions or properties of parabolas abound in the literature and textbooks, unlike statements and proofs of sufficient conditions. For this reason, and since the proofs of necessity are generally straightforward, they are not presented.

Section 2 contains an introduction to parabolas. Section 3 presents the focusing property and the locus that uses the focus and directrix, which are among the best-known properties of the parabola and are characterisations. Section 4 contains six additional characterisations of the parabola and their proofs.

2. *The parabola*

There are many ways of defining a parabola. Geometrically, one often finds a parabola to be defined as the collection of all points in the Euclidean plane whose distance to a fixed point (the focus) is equal to its distance to a fixed line (the directrix). Another common definition from analytic geometry identifies a parabola as the intersection of a right circular cone and a plane that is parallel to one of the cone's generating lines. We define a parabola in the analytic sense that is described below. As a matter of convenience, occasionally, we refer to an equation $y = y(x)$ as a parabola, provided that the graph of all points satisfying the equation is a parabola.

Many individuals first encounter a parabola in an introductory algebra course as the graph of a quadratic equation of the form $y = ax^2 + bx + c$. A more general algebraic definition of a parabola is the graph of all points in the xy-plane satisfying a quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

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with $B^2 - 4AC = 0$. Degenerate cases, such as $x^2 - 2xy + y^2 = 0$, which is the line $y = x$ and has $B^2 - 4AC = 0$, are excluded. While the above equation is more general, a change of the coordinate system involving nothing more than a rotation allows the equation to be expressed in the form

$$y = ax^2 + bx + c. \tag{1}$$

Furthermore, if one changes the coordinate system using rotations and translations, that is, rigid motions, the equation can be written

$$y = \frac{x^2}{4p} \text{ with } p > 0, \tag{2}$$

which is a *standard form* [1, pp. 122-141], [2, pp. 666-673]. Rigid motions preserve geometric properties such as angles, area, distance, and tangency, which are the basis of our eight sufficient conditions for a curve to be a parabola. Therefore, as desired, we take the liberty of choosing a convenient coordinate system in our proofs, so as to result in (1) or (2), which define y as a function of x.

A contraction or expansion of at least one coordinate is required to simplify (2) further to $y = x^2$, for example [3, pp. 84-85]. Those transformations are not employed here, since unlike rigid motions they can alter the geometric properties that are in the eight characterisations of the parabola.

For uniformity, all figures, except Figures 4(a) and (c), display the parabola in (2) with $p = 1$.

3. *Two of the most common properties of parabolas*

The first condition that we present is the *reflection, focusing*, or *optical property*. It corresponds to an application where there is a location for a point source of light (the focus) that produces a beam of light that is parallel to the axis. In reverse, it says that the curve focuses a beam of light that is parallel to the axis to one point. The law of reflection says that on the side of the curve that contains the focus, the angle that an outgoing or incoming ray makes with the tangent line of the curve at the point of intersection is equal to the angle between the tangent line and the ray from or to the focus. Refer to Figure 1.

Rotating the parabola in (2) in three dimensions about the y-axis gives $y = \dfrac{x^2 + z^2}{4p}$, which is a surface called a *paraboloid of revolution*. Antennae and mirrors are often manufactured using this shape [4, p. 271], [5, p. 752], [6]. A significant implication of Theorem 1 is that antenna and mirror designers cannot find a simple, one-piece device other than a paraboloid for broadcasting or focusing parallel rays.

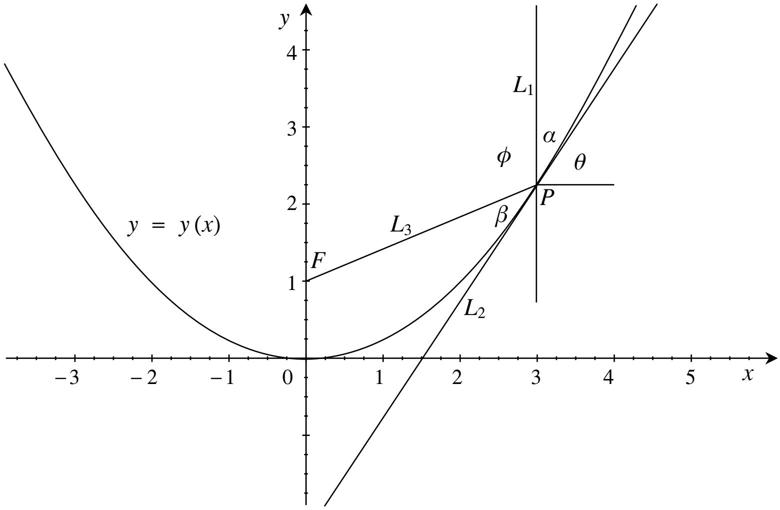

FIGURE 1: The reflection or focusing property, where $\alpha = \beta$ (Theorem 1)

Theorem 1: A sufficient condition for the differentiable function $y = y(x)$ to be a quadratic polynomial function (parabola) is that there is a line L and point F on the line, such that on the side of the curve containing F each line L_1 that is parallel to L intersects $y = y(x)$ in one point P and the angle between L_1 and the tangent line at P is equal to the angle between the tangent line at P and the line containing P and F.

Proof: Choose the coordinate system so that the line L is the y-axis and the point F has coordinates $(0, p)$ with $p > 0$. Take the equation of the line L_1 to be $x = x_1$, where for now suppose that $x_1 > 0$. This line intersects the graph of $y = y(x)$ at the point $P(x_1, y_1)$, where $y_1 = y(x_1)$. The slope of the tangent line L_2 at P is

$$y_1' = y'(x_1) = \tan\theta, \tag{3}$$

where θ is the positive angle from the x-axis to L_2. See Figure 1. The positive angle α from L_2 to L_1 is $\frac{1}{2}\pi - \theta$. Line L_3 contains P and F. The positive angle from L_3 to L_2 is given by

$$\beta = \alpha = \frac{\pi}{2} - \theta.$$

The positive angle ϕ from L_1 to L_3 is

$$\phi = \pi - 2\alpha = \pi - 2\left(\tfrac{1}{2}\pi - \theta\right) = 2\theta. \tag{4}$$

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Using (4), equation (3) can be written as

$$y_1' = \tan\left(\tfrac{1}{2}\phi\right) = \frac{1 - \cos\phi}{\sin\phi} = \frac{1 - \dfrac{p - y_1}{\sqrt{(p - y_1)^2 + x_1^2}}}{\dfrac{x_1}{\sqrt{(p - y_1)^2 + x_1^2}}}$$

$$= \frac{\sqrt{(p - y_1)^2 + x_1^2} - (p - y_1)}{x_1}. \tag{5}$$

By dropping the subscripts and letting

$$w = y - p, \tag{6}$$

(5) becomes

$$\frac{dw}{dx} = \frac{\sqrt{w^2 + x^2} + w}{x}. \tag{7}$$

Since (7) is a homogeneous differential equation, set

$$w = x\,u(x) \tag{8}$$

[7, pp. 13-14], [8, pp. 71-72]. Equation (7) becomes

$$\frac{1}{\sqrt{u^2 + 1}}\frac{du}{dx} = \frac{1}{x}.$$

Integrating results in

$$\ln\left(u + \sqrt{u^2 + 1}\right) = \ln x + \ln C$$

[9, pp. 63, 80]. Solving for u yields

$$u = \frac{C}{2}x - \frac{1}{2Cx}. \tag{9}$$

From (6), (8) and (9),

$$y = w + p = xu + p = \frac{C}{2}x^2 + p - \frac{1}{2C}, \tag{10}$$

which is an example of (1) for $x > 0$.

For $x_1 < 0$, lines L_1, L_2 and L_3 are replaced by their reflections in the y-axis. Therefore, for $x < 0$, $y = y(x)$ is a reflection in the y-axis of a curve in the family (10). From (10), the differentiability condition requires that the same C is used for $x < 0$. The point where $x = 0$ is determined by continuity. Thus (10) is the function for all real numbers and is a special case of (1). If $C = \frac{1}{2p}$, then (2) is obtained.

The criterion in Theorem 2 contains the standard locus definition. See Figure 2(a), where the points P of the parabola are equidistant from the focus F and directrix L. It supplies a way to construct or draw a parabola. Its statement does not require differentiability, and its proof is very elementary. This definition is ancient, since it can be traced back to at least Pappus of Alexandria (c. 290–c. 350) [10, pp. 8-10].

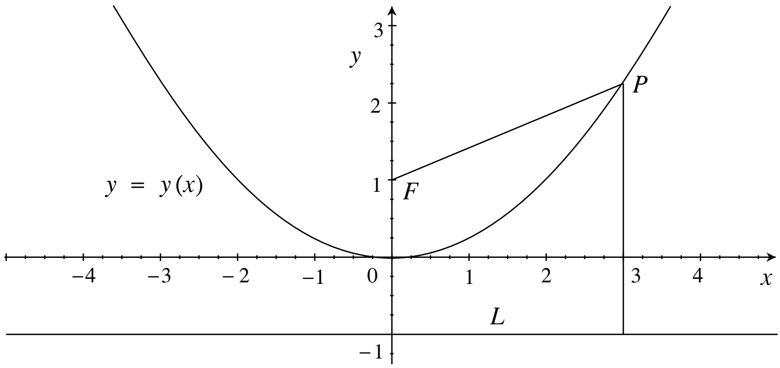

FIGURE 2(a): The common locus definition (Theorem 2)

Theorem 2: A sufficient condition for the function $y = y(x)$ to be a quadratic polynomial function (parabola) is that there exists a point F and a line L such that each point of $y = y(x)$ is equidistant from F and L.

Proof: Take F to be $(0, p)$ with $p > 0$ and L to be $y = -p$. Label an arbitrary point of $y = y(x)$ as P with coordinates $(x_1, y_1) = (x_1, y(x_1))$. From the construction, $y_1 \geqslant 0$. Setting the distance between F and P equal to the distance between P and L gives

$$\sqrt{(x_1 - 0)^2 + (y_1 - p)^2} = y_1 + p$$

or

$$x_1^2 + y_1^2 - 2py_1 + p^2 = y_1^2 + 2py_1 + p^2.$$

Omitting the subscripts and solving for y gives (2), that is, $y = \dfrac{x^2}{4p}$.

As Ogilvy [3, p. 76] points out, this characterisation is equivalent to the locus of the centres of all circles that pass through the point F and are tangent to line L. Figure 2(b) shows 17 of those circles, where the focus F is $(0, 1)$, the directrix L is $y = -1$, and the centres are on the parabola. The radius of each circle is equal to the two distances in Theorem 2.

Another point-by-point construction of a parabola is shown in Figure 2(c) [1, p. 92], [11, p. 220]. It uses a ruler and compasses. Select point $(0, p)$ with $p > 0$, which is the centre of all circles in a family of circles whose radii are $p + a$ with $a \geqslant 0$, and is the focus of the parabola. For each value of a, the points of intersection of the circle $(x - 0)^2 + (y - p)^2 = (p + a)^2$ and the horizontal line $y = a$ are on the parabola $y = \dfrac{x^2}{4p}$. This can be seen by substituting $y = a$ into the equation of the circle, which gives $x = \pm 2\sqrt{ap}$. The parameter a can be eliminated between $x = \pm 2\sqrt{ap}$ and $y = a$ to give

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$y = \dfrac{x^2}{4p}$, which is (2). For each point on the parabola, the radius of the circle and the vertical distance to the directrix L, which is $y = -p$, are $p + a$ and are the two distances in Theorem 2.

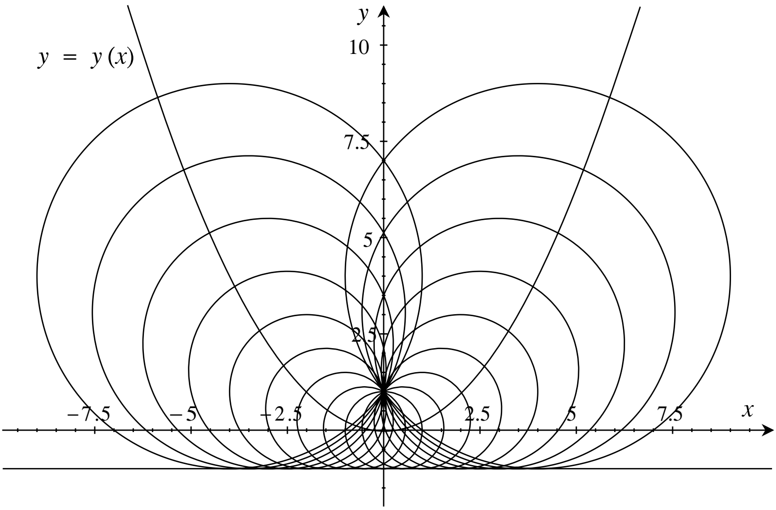

FIGURE 2(b): The common locus definition (Theorem 2)

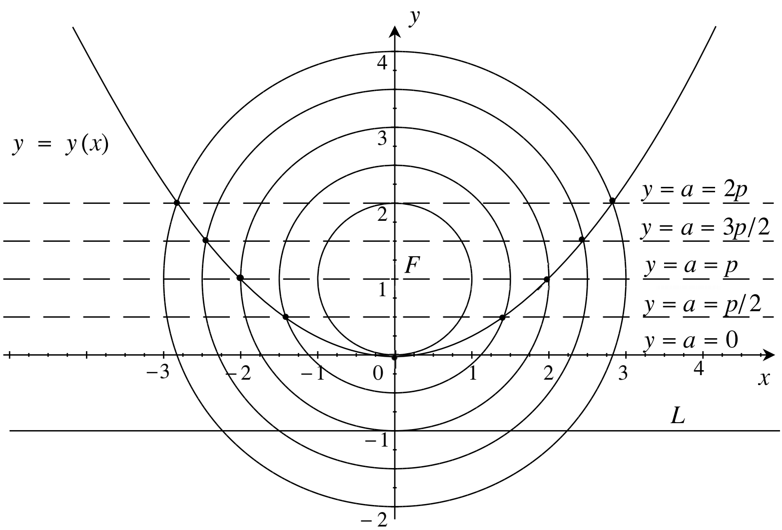

FIGURE 2(c): The common locus definition (Theorem 2)

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4. *Six characterisations of the parabola*

4.1. *Constructions based on a pedal curve, hypotenuses as tangent lines, and centres of circumcircles*

The line that is determined by the construction in Theorem 3 is called the *pedal curve* of the parabola with respect to the pedal point, which is the focus, since the points of the pedal curve are at the foot of perpendicular lines. Refer to Figure 3. Another way to express this condition is to say that the locus of the vertices of all right angles for which one side L_1 is tangent to the parabola and the other side L_2 contains the focus is a line, which is the pedal curve and is the x-axis in Figure 3. For more about pedal curves, see [11, pp. 227-229], [12, pp. 439-440], and [13, pp. 26-27]. Conversely, the pedal point F and the pedal curve can be used to construct tangent lines to the curve.

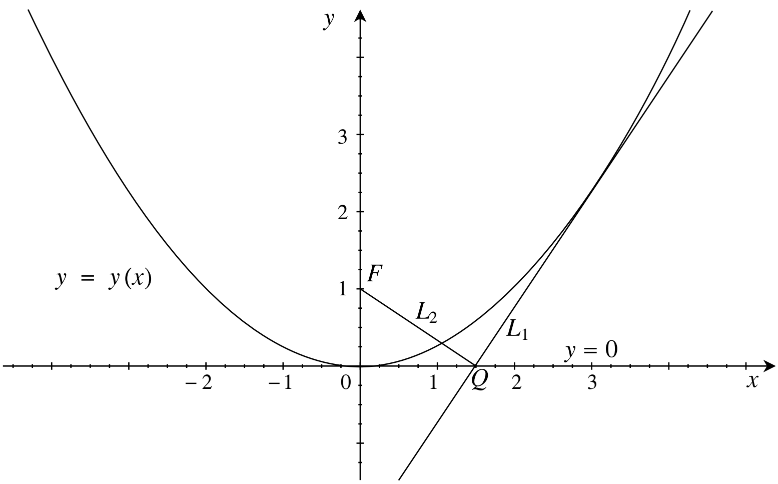

FIGURE 3: The pedal-curve construction (Theorem 3)

Theorem 3: A sufficient condition for the differentiable function $y = y(x)$ to be a quadratic polynomial function (parabola) is that the intersection points of all tangent lines to $y = y(x)$ with their perpendicular line through a fixed point are distinct and collinear.

Proof: Without loss of generality, take the point to be $F(0, p)$ with $p > 0$ and the line of intersections to be $y = 0$. The hypothesis says that the tangent line L_1 must meet its perpendicular line L_2 through F on the line $y = 0$. Let $Q(q,\ 0)$ be their point of intersection. Since L_2 passes through F and Q, its slope is $-\frac{p}{q}$, provided that $q \neq 0$. In order for q to be 0, L_2 must be vertical. For each value of q, including $q = 0$, the corresponding tangent line L_1 to the curve has the equation

$$y = \frac{q}{p}(x - q). \tag{11}$$

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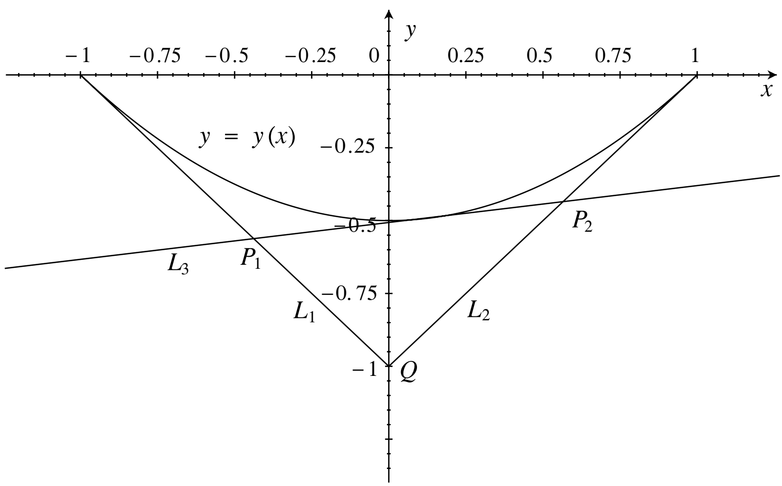
The curve $y = y(x)$ is the envelope of the family of its tangent lines. Recall that the *envelope* of a set of curves is a curve that intersects a member of the set in one point, and the envelope and the respective member share their tangent direction at the point of intersection [14, p. 273], [15, pp. 171-179]. To determine the envelope, differentiate (11) with respect to the parameter q, which gives $0 = \frac{x}{p} - 2\frac{q}{p}$ or

$$q = \frac{x}{2}. \tag{12}$$

Substituting (12) into (11) yields (2), that is, $y = \frac{x^2}{4p}$. The criterion that the points Q be distinct, precludes degenerate solutions, such as the line $y = x - p$ for which tangent lines to all points on the curve are the curve itself and there is only one point of intersection $(p,\ 0)$.

The next characterisation is that an arc of a parabola is created as the envelope of a set of hypotenuses. See Figure 4. A family of right triangles sharing the vertex Q at the right angle and directions for their legs along lines L_1 and L_2 produces the hypotenuses along L_3. The sum of the lengths of the legs is required to be a constant. A parabola may be defined by taking the domain in the formula for the arc to be the x-axis.



(a) $K = h$

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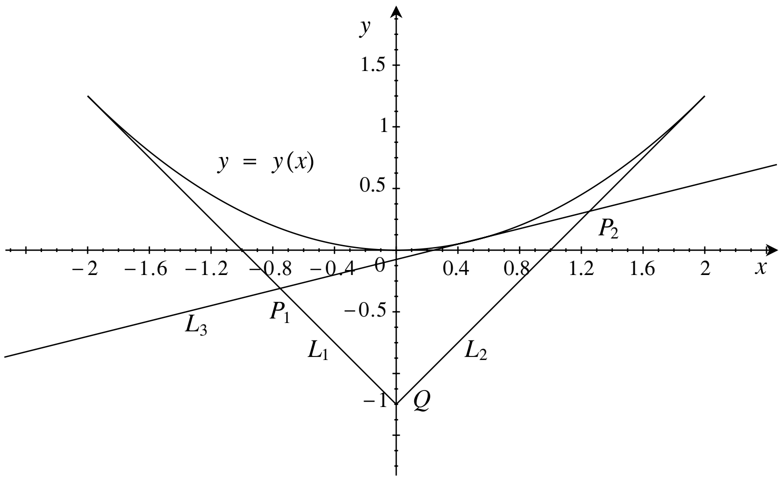
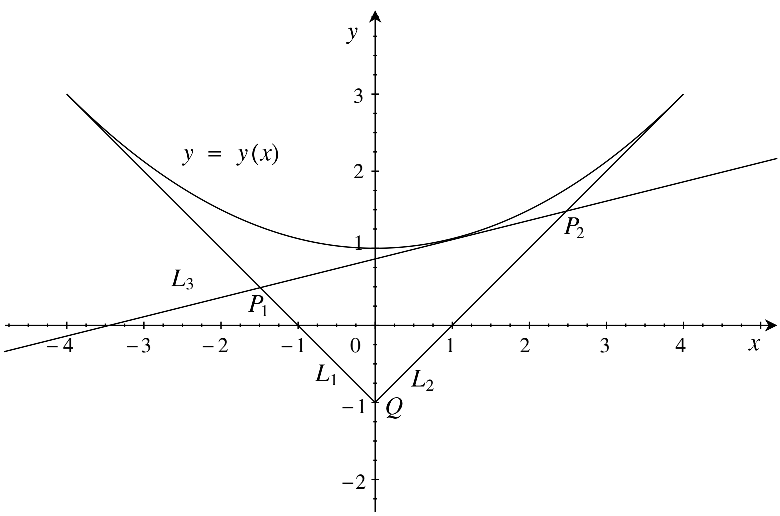


(b) $K = 2h$



(c) $K = 4h$

FIGURE 4: The envelope of the hypotenuses of a set of right triangles (Theorem 4)

Theorem 4: A sufficient condition for the differentiable function $y = y(x)$ to be an arc of a quadratic polynomial function (parabola) is that the arc is tangent to the hypotenuse of every right triangle in a family of right triangles that share a common vertex at the right angle, whose legs are in the same fixed directions, and the sum of the lengths of whose legs is a fixed constant.

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Proof: Place the right angle of the triangle at the point $Q(0, -h)$ with $h > 0$ and legs along L_1, which is $y = -x - h$, and L_2, which is $y = x - h$. The triangle's other two vertices are $P_1(x_1, -x_1 - h)$ with $x_1 < 0$ and $P_2(x_2, -x_2 - h)$ with $x_2 > 0$. The hypotenuse, which contains P_1 and P_2, lies on L_3, which has the equation

$$y = \frac{(x_2 - h) - (x_1 - h)}{x_2 - x_1}(x - x_1) + (-x_1 - h). \tag{13}$$

The lengths of the legs are $D(Q, P_1) = -\sqrt{2}x_1$ and $D(Q, P_2) = \sqrt{2}x_2$, where D is the distance function. The condition says that

$$-\sqrt{2}x_1 + \sqrt{2}x_2 = \sqrt{2}K \text{ or } x_2 = x_1 + K \tag{14}$$

for $K > 0$ a constant, which may be freely chosen. Equation (14) says that $-K < x_1 < 0$ and $0 < x_2 < K$. Substituting from (14) into (13) gives

$$y = \left(1 + \frac{2x_1}{K}\right)x - \left(2x_1 + h + \frac{2x_1^2}{K}\right), \tag{15}$$

which are the tangent lines to their envelope.

To find the envelope, eliminate the parameter x_1 in the tangent lines. Differentiating (15) with respect to x_1 gives $x_1 = \frac{1}{2}(x - K)$. Substituting this into (15) yields

$$y = \frac{x^2}{2K} + \frac{K}{2} - h \text{ for } -K < x < K,$$

which are graphed in Figure 4 with $K = h$, $2h$ and $4h$ for $h = 1$. Selecting $K = 2h$ gives $y = \dfrac{x^2}{4h}$ for $-2h < x < 2h$.

In the extreme cases where x_1 or x_2 approach zero, the hypotenuse collapses onto the leg which is tangent to the parabola at the end point of the leg. The legs and the hypotenuse are tangent to the arc. Thus, the arc is said to be *inscribed* in each triangle with sides along L_1, L_2 and L_3, even though the arc is exterior to the triangle.

The following characterisation involves the construction of a set of triangles. See Figure 5, which shows one of the triangles, with vertices A, B and C, whose circumcentre P is a point of the parabola. A *perpendicular bisector* of a triangle is a line that is perpendicular to a side at the midpoint of the side. The perpendicular bisectors of a triangle's sides meet in a point, which is called the *circumcentre* of the triangle and is the centre of the triangle's *circumscribed circle*. The circle is called the *circumcircle*, contains the vertices of the triangle, and always exists for any triangle. See [3, pp. 117-120], [16, pp. 502-506, 511-515, 521-531], and [17, pp. 10-18] about these geometric ideas.

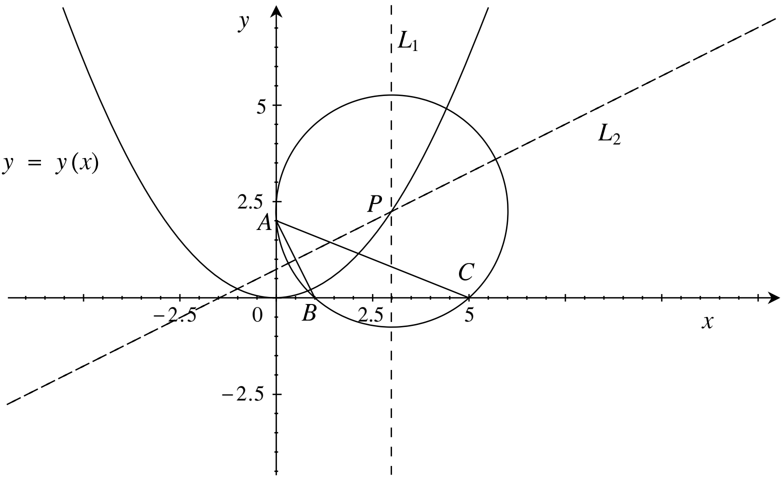

FIGURE 5: The locus of the centres of the circumcircles of triangles (Theorem 5)

Theorem 5: A sufficient condition for the function $y = y(x)$ to be a quadratic polynomial function (parabola) is that its curve is the locus of the circumcentres of a set of all triangles that have the same vertex in common and whose sides opposite the common vertex lie on the same line and have the same fixed length.

Proof: For specificity, coordinates are given for the triangles' vertices. The common vertex of all the triangles is $A(0,\ 2p)$ with $p > 0$. The opposite sides have constant length $2k > 0$ and are on the x-axis. The set of triangles is parameterised by t, which is the x-coordinate of the end-point of the opposite sides that is closer to the origin, so the other two vertices are $B(t,0)$ and $C\left(t + 2k\frac{t}{|t|},\ 0\right)$. By using symmetry, only positive values of t need to be considered.

Two perpendicular bisectors of each triangle meet at the circumcentre of the triangle. The perpendicular bisector of the side BC is the vertical line L_1, whose equation is $x = t + k$. The side that contains vertices A and B has slope $-2p/t$. Its perpendicular bisector is the line L_2, which has equation

$$y = p + \frac{t}{2p}\left(x - \frac{t}{2}\right). \tag{16}$$

The point P of intersection of L_1 and L_2 has x-coordinate

$$x = t + k \tag{17}$$

and y-coordinate as in (16). Eliminating the parameter t between (16) and (17) gives

$$y = \frac{x^2}{4p} + \frac{4p^2 - k^2}{4p},$$

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which is an example of (1). Setting $k = 2p$ gives (2). The circumcircles do not have to be constructed, since the intersections of lines L_1 and L_2 are the parabola's points.

4.2. *Two geometric properties*

Geometric characterisations, such as those in Theorems 6 and 7, are appealing. Their proofs are less intricate than those of some of the other characterisations. The apparent simplicity of the conditions does not mean that they call upon little information, since tangent lines at all points of the curve are involved. The trivial solution of a line is precluded by the requirement that the solution is a non-linear function. Refer to Figure 6.

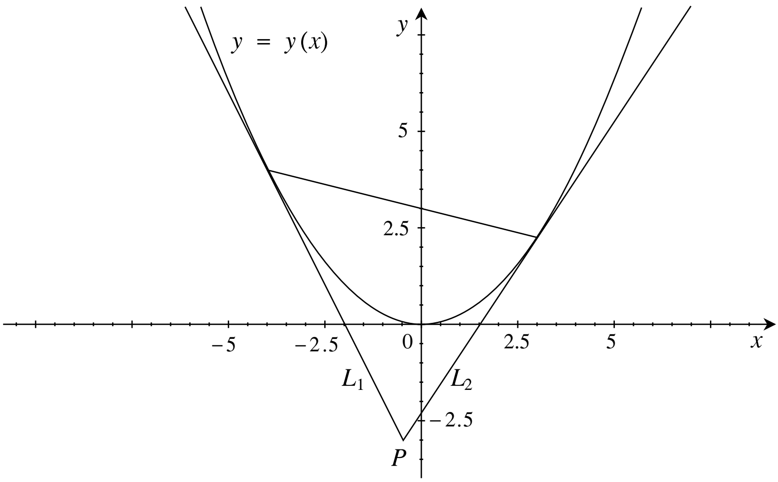

FIGURE 6: The slope of the chord is the mean of the slopes of the tangent lines at P_1 and P_2 (Theorem 6), and the x-coordinate of P is the mean of the x-coordinates of P_1 and P_2 (Theorem 7)

Theorem 6: A sufficient condition for the non-linear differentiable function $y = y(x)$ with no linear portions to be a quadratic polynomial function (parabola) is that the slope of the chord between arbitrarily selected distinct points $P_1(x_1, y_1) = P_1(x_1, y(x_1))$ and $P_2(x_2, y_2) = P_2(x_2, y(x_2))$ is the arithmetic mean of the slopes of the tangent lines at P_1 and P_2.

Proof: By a rigid motion of the coordinates, the origin (0, 0) can be moved to $P_1(x_1, y(x_1))$ with $y(x_1) = 0$. Then the slope of the chord is $\dfrac{y_2}{x_2}$ with $x_2 \neq 0$ and $y_2 \neq 0$, and the mean of the slopes of the tangent lines is $\frac{1}{2}(0 + y'(x_2))$. Dropping the subscript and equating the slope of the chord and the mean give the differential equation

$$y' = \frac{2y}{x}, \tag{18}$$

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whose solution is

$$y = Cx^2 \tag{19}$$

with $C \neq 0$, which is an example of (2). The requirement that $y = y(x)$ be non-linear rules out $C = 0$.

Theorem 7: A sufficient condition for the non-linear differentiable function $y = y(x)$ with no linear portions to be a quadratic polynomial function (parabola) is that the tangent lines at any pair of distinct points $P_1(x_1, x_2) = P_1(x_1, y(x_1))$ and $P_2(x_2, y_2) = P_2(x_2, y(x_2))$ meet in a point P whose x-coordinate is $\frac{1}{2}(x_1 + x_2)$, which is the arithmetic mean of the x-coordinates of P_1 and P_2.

Proof: Use the same coordinates as in the proof of Theorem 6. The criterion says that the coordinates of P are $\left(\frac{1}{2}x_2,\ 0\right)$. The tangent line at P_2 is

$$y = y_2 + y_2'(x - x_2).$$

where $y_2' = y'(x_2)$. Since P is on the tangent line, $0 = y_2 + y_2'\left(-\frac{1}{2}x_2\right)$, which gives (18) and hence (19).

4.3. *An area formula implies a parabola*

The final characterisation is at least unexpected, if not very surprising. Refer to Figure 7. The characterisation says that the validity of a formula for the area of an inscribed triangle implies that the curve is a parabola. An inscribed triangle is required to have its interior completely on one side of the curve. The formula contains only the x-coordinates of the three points on the curve.

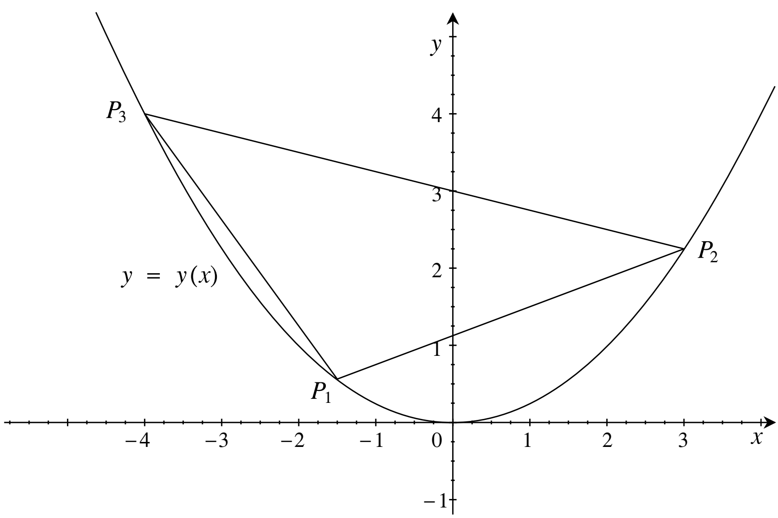

FIGURE 7: An area formula for all inscribed triangles implies that the curve is a parabola (Theorem 8)

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Theorem 8: A sufficient condition for the twice differentiable function $y(x)$ to be a quadratic polynomial function (parabola) is that any three distinct points (x_i, y_i), $i = 1, 2, 3$, that satisfy $y = y(x)$ with $x_1 < x_2 < x_3$, form an inscribed non-degenerate triangle and the formula for the area of the triangle with vertices at the points is

$$C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

for a single value of C for the curve.

Proof: Since $x_1 < x_2 < x_3$, $(x_3 - x_2)(x_3 - x_1)(x_2 - x_1) > 0$, and thus $C > 0$. Since all triangles using any three points are inscribed in the curve, the curve must be concave up or concave down. Without loss of generality, assume that it is concave up. The determinant in the expression below is positive, and the criterion can be expressed

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

where $y_i = y(x_i)$, $i = 1, 2, 3$. The left-hand side is a formula for the area of a triangle, given the coordinates of its vertices in a counterclockwise order [1, p. 28], [18, p. 202]. If the curve is concave down, the proof proceeds by interchanging two rows of the determinant. In the determinant, subtract row 1 from rows 2 and 3. By expanding the altered determinant, the criterion becomes the functional equation

$$(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 2C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Applying $\dfrac{d^2}{dx_1^2}$ gives

$$(x_3 - x_2)y_1'' = 4C(x_3 - x_2)$$

and, since $x_3 - x_2 \neq 0$,

$$y_1'' = 4C.$$

Omitting the subscript and integrating twice implies that

$$y = 2Cx^2 + C_1x + C_2,$$

that is (1), where C_1 and C_2 are constants of integration.

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