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ANOTHER WEAK STONE-WEIERSTRASS THEOREM FOR C^*-ALGEBRAS

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1. The purpose of this article is to present a new generalization of the classical Stone-Weierstrass theorem for commutative C^*-algebras.

Under the assumption that B is a sub-C^*-algebra of A separating the pure states of A and zero, Kaplansky has conjectured that $B=A$ [4, p. 246]. He gave a proof for the case that A is postliminary ([4, Theorem 7.2]; see also [2, 11.1.8]). Glimm, Akemann, and Sakai have established the conjecture in the presence of various other additional hypotheses, most of which hold in the commutative case ([3], [1], [7]).

We note that it involves no loss of generality to assume that A is generated by B together with a single self-adjoint element. Our additional assumption is that this element can be chosen so that the derivation it defines leaves B invariant.

THEOREM. *Let A be a C^*-algebra, and let B be a sub-C^*-algebra of A separating the pure states of A and zero. Suppose that A is generated by B together with a single self-adjoint element x_0 such that $x_0x-xx_0 \in B$ for all $x \in B$. Then $B=A$.*

The proof resembles Sakai's proof [6] that every derivation of a simple C^*-algebra with unit is inner.

2. **Proof of Theorem 1.** Suppose that $B\neq A$. Then there exists a nonzero self-adjoint bounded linear functional on A which is zero on B. Denote by K the set of all such functionals of norm ≤ 1, together with zero. K is convex and weak* compact, and so has a nonzero extreme point, say f. In what follows, we shall deduce the absurdity that $f=0$.

Let f^+ and f^- denote positive linear functionals on A such that $f=f^+-f^-$ and $\|f\|=\|f^+\|+\|f^-\|$ [2, 2.6.4]. Set $f^++f^-=g$, and let π be the representation of A defined by g [2, 2.4.4].

Let us first show that $\pi(B)''$ is a factor. To be able to use the argument of Lemma 1 of [6], we need only know that the centre of $\pi(B)''$ is contained in the centre of $\pi(A)''$. This follows easily from the fact that any derivation of $\pi(B)''$ (in particular that defined by $\pi(x_0)$) must be zero on the centre of $\pi(B)''$.

By [5], there exists an element T_0 of $\pi(B)''$ which defines the same derivation of $\pi(B)''$ as $\pi(x_0)$. Denote by C the C^*-algebra generated by $\pi(x_0)-T_0$ and 1. Since

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$\pi(B)''$ is a factor and C is in its commutant, the algebra generated by $\pi(B)''$ and C is their tensor product. Hence by Theorem 1 of [9], the C^*-algebra generated by $\pi(B)''$ and C is their C^*-algebra tensor product. Denote it by R.

We wish to show that C is the scalars. To do this it suffices to show that C has a unique character. If χ is a character of C then with R as above there exists a unique involutive algebra morphism from R to $\pi(B)''$ which fixes each element of $\pi(B)''$ and reduces to χ on C. Denote this morphism by φ_χ. We have

$$\chi(\pi(x_0)-T_0) = \varphi_\chi(\pi(x_0)-T_0) = \varphi_\chi(\pi(x_0))-T_0.$$

Hence

$$\chi(\pi(x_0)-T_0) + \mathrm{Sp}\ T_0 = \mathrm{Sp}\ \varphi_\chi(\pi(x_0)). \tag{1}$$

If we can show that $\varphi_\chi \mid \pi(A)$ is injective, then we shall have [2,1.3.10]

$$\mathrm{Sp}\ \phi_\chi(\pi(x_0)) = \mathrm{Sp}\ \pi(x_0). \tag{2}$$

From (1) and (2) it follows that χ is unique.

Denote $\varphi_\chi \mid \pi(A)$ by P. To show that P is injective it suffices to show that its transpose P' is surjective. Since P preserves adjoints it is enough that P' should take the self-adjoint elements of the dual of $\pi(B)''$ onto the self-adjoint elements of the dual of $\pi(A)$. Since the image by P' of the unit ball of the self-adjoint part of the dual of $\pi(B)''$ is compact, by the Krein-Milman theorem it is enough to prove that this image contains all extreme points of the unit ball of the self-adjoint part of the dual of $\pi(A)$. These are all either pure states of $\pi(A)$ or their negatives. If p is a pure state of $\pi(A)$, then $p_0=p \mid \pi(B)$ is a pure state of $\pi(B)$ [2, 11.1.7]; it follows that p is the unique state extension of p_0 to $\pi(A)$. Let p_1 be a state extension of p_0 to $\pi(B)''$. Then $P'p_1$ is a state of $\pi(A)$; since it extends p_0, it must be equal to p.

From the fact that C is the scalars we get $\pi(x_0)-T_0 \in \pi(B)''$. Hence $\pi(B)''=\pi(A)''$ (A is generated by B together with x_0). In view of [2, 2.5.1 (iii)], we can now conclude by a continuity argument that f_1 and f_2 (which agree on B) agree on all of A. Thus we have $f=f_1-f_2=0$, the promised absurdity.

3. **Problems.** 3.1. If A is a simple C^*-algebra and B is a nonzero sub-C^*-algebra such that A is generated by B together with a single self-adjoint element which defines a derivation of B, then it can be shown that B separates the pure states of A. Theorem 1 then is applicable, and Sakai's theorem [6] that every derivation of a simple C^*-algebra with unit is inner follows without difficulty. Is it possible to prove that B separates the pure states of A without using the fact that every derivation of a von Neumann algebra is inner?

3.2. If A is a C^*-algebra and B is a sub-C^*-algebra of A separating pure states of A and zero, and if π is a representation of A, must the centre of $\pi(B)''$ be contained in the centre of $\pi(A)''$? (If A is separable then an affirmative answer can be obtained using direct integral theory.) If so then a new proof of the Stone-Weierstrass theorem for type I C^*-algebras (Kaplansky's result, by virtue of the fact [8] that a type I C^*-algebra is postliminary) can be obtained by replacing paragraphs four to six of the proof of Theorem 1 with a result of Akemann [1, Lemma III.2]: if $\pi|B$ is a type I factor representation then $\pi(B)''=\pi(A)''$.

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