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The convective viscous Cahn–Hilliard equation: Exact solutions

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In this paper, we give exact solutions for the convective viscous Cahn–Hilliard equation. This equation with a general symmetric double-well potential and Burgers-type convective term was introduced by T. P. Witelski (1996 *Studies in Applied Mathematics* **96**, 277–300) to study the joint effects of nonlinear convection and viscosity. We consider this equation with a polynomial, generally asymmetric potential. We also consider both Burgers-type and cubic convective terms. We obtained exact travelling-wave solutions for both cases. For the former case, with an additional constraint on nonlinearity and viscosity, we also obtained an exact two-wave solution.

Key words: Cahn–Hilliard equation, convective term, viscosity, travelling-wave solution

1 Introduction

The Cahn–Hilliard equation [5, 29] is now a well-established model of phase separation in binary systems. The basic underlying idea of this model is that for an inhomogeneous system, e.g. a system undergoing a phase transition, the thermodynamic potential (e.g. free energy) should depend not only on the order parameter (that is on concentration for the binary system), but also on its gradient as well. The idea of such dependence was introduced by Van der Waals [36] in his theory of capillarity. Due to this dependence, instead of the usual second order, the resulting diffusion equation for the concentration becomes a fourth-order PDE. It was introduced as early as 1958 [5]. The linearised version was treated and the corresponding instability of homogeneous state was identified. However, it was only much later on that an intensive study of the fully nonlinear form of this equation started [30]. An impressive amount of work has now been done on the *nonlinear Cahn–Hilliard equation*, see [29], as well as on its numerous modifications. The present paper is devoted to studying the following *convective-viscous Cahn–Hilliard equation*

$$u_t - \left(\alpha_1 u - 2\alpha_2 u^3\right) u_x = \left(u^3 - \delta u^2 - u - u_{xx} + \mu u_t\right)_{xx}. \tag{1.1}$$

To understand the meaning of different terms and corresponding special cases, we need to give some insight into the history of this equation. The classic nonlinear Cahn–Hilliard equation with cubic polynomial for the homogeneous part of the chemical potential, as was studied in the works of Novick–Cohen and Segel [30], corresponds to $\alpha_1 = \alpha_2 = \mu = 0$,

$$u_t = \left(u^3 - \delta u^2 - u - u_{xx}\right)_{xx} \tag{1.2}$$

(here and below our scaling and notation differ from those in the original papers). For this equation, an exact static kink solution was found. As usual, [8], we call 'kinks/anti-kinks' those solutions which approach different constant values at $x \to \pm\infty$ with a more or less localised transition region, typically of a tanh-like form. The solutions with $u_x > 0$ are called kinks and solutions with $u_x < 0$ anti-kinks. If $\delta \neq 0$, equation (1.2) represents the system with the asymmetric fourth-order polynomial potential and possesses an asymmetric kink solution. For the classic Cahn–Hilliard equation (1.2), the asymmetric potential and correspondingly the static asymmetric kink solution were usually discarded [30] because for this case the global conservation of the order parameter is violated. Later several authors introduced the nonlinear *convective* Cahn–Hilliard equation (CCHE) in one space dimension [8, 24, 39]. It corresponds to $\alpha_2 = \mu = \delta = 0$ in equation (1.1):

$$u_t - \alpha u u_x = \left(u^3 - u - u_{xx}\right)_{xx} \tag{1.3}$$

(for the equations with $\alpha_2 = 0$ we will drop the index at α_1). Leung [24] proposed this equation as a continual description of lattice gas phase separation under the influence of an external field. Similarly, Emmott and Bray [8] proposed this equation as a model for the spinodal decomposition of a binary alloy in an external field E. As they noticed, if the mobility (see the mobility definition, e.g., in [7]) is independent of the order parameter (concentration), the term involving E will drop out of the dynamics. To get nontrivial results, they presumed the simplest possible symmetric dependence of mobility on the order parameter, viz. $M \sim 1 - au^2$. Then, they obtained the Burgers-type convection term in equation (1.3) with the coefficient $\alpha = 2aE$. Thus, the sign of α depends both on the direction of the field and on the sign of a. Witelski [39] introduced the equation (1.3) as a generalisation of the classic Cahn–Hilliard equation or as a generalisation of the Kuramoto–Sivashinsky equation [23, 35] by including a nonlinear diffusion term. In [8], [24], [34], and [37]– [39], several approximate solutions and only two exact static kink and anti-kink solutions were obtained. The 'coarsening' of domains separated by kinks and anti-kinks was also discussed. The convective Cahn–Hilliard equation with cubic nonlinearity in the convective term was first derived in a model of kinetically controlled evolution of two-dimensional crystals [18, 19]. This equation corresponds to $\mu = 0$ in equation (1.1):

$$u_t - \alpha_1 u u_x + 2\alpha_2 u^3 u_x = \left(u^3 - \delta u^2 - u - u_{xx}\right)_{xx}. \tag{1.4}$$

Recently, a somewhat different equation with cubic nonlinearity in the convective term and containing additional derivatives terms was obtained in [17] in the completely different context of nonlinear optics, see Section 2 below. Equation (1.4) was approximately solved in [18]; in [19] the exact travelling-wave solution of this equation was obtained for the asymmetric potential, that is $\delta \neq 0$. Higher-order polynomials for both the convective terms and potential were also considered in [41]. On the other hand, Novick–Cohen [28] introduced the *viscous* Cahn–Hilliard equation (VCHE) to include some viscous effects which are neglected in the derivation of the classic Cahn–Hilliard equation [5]. It corresponds to $\alpha_1 = \alpha_2 = \delta = 0$ in equation (1.1):

$$u_t = \left(u^3 - u - u_{xx} + \mu u_t\right)_{xx}. \tag{1.5}$$

The VCHE could also be derived [1] as a certain limit of the classic phase-field model (see [4], [9], [32], and [33] and references therein for the derivation of phase-field model and its relations to other phase-separation models).

To study the joint effects of nonlinear convection and viscosity, Witelski [40] introduced the *convective-viscous-Cahn–Hilliard* equation (CVCHE) with a general symmetric double-well potential $F(u)$:

$$u_t - \alpha u u_x = \left(\frac{dF(u)}{du} - u_{xx} + \mu u_t\right)_{xx}. \tag{1.6}$$

It is worth noting that all results, including the stability of solutions, were obtained without specifying a particular functional form of the potential. Thus, they are valid both for the polynomial and logarithmic [29] potential. With an additional constraint imposed on nonlinearity and viscosity, the approximate travelling-wave solutions were obtained in [40]. In our previous communication [27], we obtained exact travelling-wave solutions for several special cases of equation (1.1): (a) $\alpha_2 = \mu = 0$; $\delta \neq 0$, which is the generalisation of (1.3); (b) $\alpha_2 = 0$; $\mu \neq 0$, both for $\delta = 0$ and $\delta \neq 0$, which is similar to equation (1.6), but with generally asymmetric polynomial potential; (c) $\mu = 0$; $\alpha_1 \neq 0$; $\alpha_2 \neq 0$, both for $\delta = 0$ and $\delta \neq 0$ (we were not aware of the existence of an exact solution for $\delta \neq 0$ in [19]). For case (a), depending on the signs of α and δ, there are four cases for the travelling-wave solutions as shown in Figure 1, which is essentially different from the simple symmetry observed for equation (1.3) [8,24]. For case (b), with $\delta = 0$, an additional exact two-wave solution was given. The distinction between the symmetric and asymmetric potential, i.e. zero and non-zero δ, is important and will be traced throughout the present paper because it reflects the physical difference between the absence and presence of thermodynamic preference for one of the stationary states.

Interestingly, equation (1.1) with both $\alpha_2 \neq 0$ and $\mu \neq 0$ exhibits some new features which are absent for any of above special cases. This equation is considered in Section 2 where we give exact travelling-wave solutions and discuss several special cases. We also give the exact solutions for the Gelens–Knobloch equation [17]. In Section 3, we consider in detail the convective-viscous Cahn–Hilliard equation with the Burgers-type convective term, i.e. equation (1.6), but with a quartic polynomial potential. We also give derivation of several results which were only listed in [27]. Both the single- and two-wave exact solutions are presented. For the latter, the time evolution is discussed in some detail. In Section 4, we discuss some limiting forms of the CVCH equation and their relations to some other well-known nonlinear equations. In Section 5, we consider the hyperbolic modification of the CVCHE (memory effects). In Section 6, we discuss the obtained results. Despite several hundreds of papers published on the subject of nonlinear Cahn–Hilliard equation and its modifications, the existing list of exact solutions is surprisingly short (see above). Our results add to this list one exact static kink/antikink solution, four exact travelling kink/antikink solutions, and an exact two-wave solution.

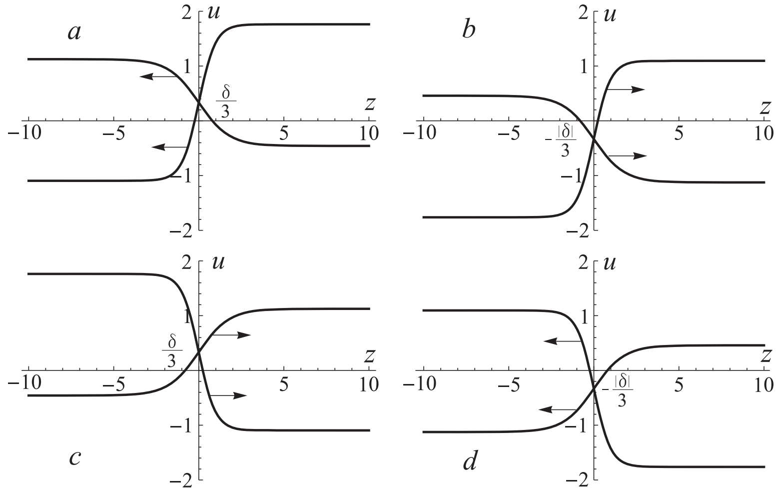

FIGURE 1. Kinks and anti-kinks for different values of α and δ: a) $\alpha = 1$ and $\delta = 1$, b) $\alpha = 1$ and $\delta = -1$, c) $\alpha = -1$ and $\delta = 1$, d) $\alpha = -1$ and $\delta = -1$.

2 Travelling wave solutions of the convective-viscous Cahn–Hilliard equation with a cubic convective term

Here we give exact travelling-wave solutions of (1.1). As usual, we call the solutions with $u_x > 0$ as 'kinks' , and solutions with $u_x < 0$ as 'anti-kinks'. Introducing the travelling-wave coordinate $z = x - vt$ and integrating once, we obtain the following nonlinear third-order ordinary differential equation

$$\frac{1}{2}\left(\alpha_2 u^4 - \alpha_1 u^2 - 2vu - c\right) = \left(u^3 - \delta u^2 - u - u_{zz} - \mu v u_z\right)_z, \tag{2.1}$$

where c is an arbitrary constant. At $z = \pm\infty$, all derivatives equal zero, i.e. the left-hand side also equals zero. For a large enough c (having the same sign as α_2), the fourth-order polynomial on the left-hand side has at least two real roots u_1, u_2 (for definiteness we take $u_1 < u_2$). They will be the stationary values at $z = \pm\infty$, where the right-hand side of equation (2.1) equals zero. Then, the polynomial on the left-hand side of (2.1) may be presented as

$$\alpha_2 u^4 - \alpha_1 u^2 - 2vu - c = (u - u_1)(u - u_2)\left(\alpha_2 u^2 + pu + g\right), \tag{2.2}$$

Here, we have only three constraints on five unknown constants u_1, u_2, p, g and v, because c is an arbitrary constant:

$$p = \alpha_2 (u_1 + u_2), \tag{2.3}$$

$$g = \left[(u_1 + u_2)^2 - u_1 u_2\right]\alpha_2 - \alpha_1, \tag{2.4}$$

$$v = \frac{u_1 + u_2}{2}\left[\left(u_1^2 + u_2^2\right)\alpha_2 - \alpha_1\right]. \tag{2.5}$$

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Looking for monotonic solutions, we introduce the ansatz:

$$\frac{du}{dz} = -\kappa\,(u - u_1)\,(u - u_2)\,. \tag{2.6}$$

If $\kappa > 0$, the solution we are looking for increases monotonically from the smaller stationary value u_1 at $z = -\infty$ to the larger stationary value u_2 at $z = +\infty$. According to the above definition, it is a kink. If $\kappa < 0$, the solution decreases from u_2 at $z = -\infty$ to u_1 at $z = +\infty$, i.e. it is an anti-kink. Actually, equation (2.6) is the simplest possible polynomial expression, yielding zero values of the derivative u_z at $z = \pm\infty$. Using the ansatz, we rewrite equation (2.2) as

$$\alpha_2 u^4 - \alpha_1 u^2 - 2vu - c = -\frac{1}{\kappa}\left(\alpha_2 u^2 + pu + g\right) u_z = -\frac{1}{\kappa}\left(\frac{1}{3}\alpha_2 u^3 + \frac{1}{2}pu^2 + gu\right)_z, \tag{2.7}$$

where p and g can be expressed through u_1, u_2 using equations (2.3) and (2.4), respectively. Using again the ansatz (2.6), we express the derivatives on the right-hand side of equation (2.1). Next, inserting equation (2.7) into the left-hand side of equation (2.1), rearranging the terms, and equating coefficients at the same powers of u, we finally obtain three constraints:

$$2\kappa^2 - 1 - \frac{\alpha_2}{6\kappa} = 0, \tag{2.8}$$

$$\left(3\kappa^2 + \frac{\alpha_2}{4\kappa}\right)(u_1 + u_2) - \delta + \kappa\mu v = 0, \tag{2.9}$$

$$\kappa^2\left[(u_1 + u_2)^2 + 2u_1 u_2\right] + 1 + \frac{\alpha_1}{2\kappa} - \frac{\alpha_2}{2\kappa}\left[(u_1 + u_2)^2 - u_1 u_2\right] + \kappa\mu v\,(u_1 + u_2) = 0. \tag{2.10}$$

Equations (2.8)–(2.10) together with (2.5) constitute the system of four algebraic equations for the four unknown constants κ, u_1, u_2, and v. If this system is satisfied, the corresponding solutions of the first-order equation (2.6) are simultaneously exact travelling-wave solutions of equation (1.1). Equation (2.6) is easily integrated yielding

$$u = \frac{u_1 + u_2 \exp\{\kappa\,(u_2 - u_1)\,(z + \phi)\}}{1 + \exp\{\kappa\,(u_2 - u_1)\,(z + \phi)\}}, \tag{2.11}$$

where ϕ is an arbitrary constant. It is natural to take the position of the maximal value of the derivative u_z (where $u_{zz} = 0$), as $z = 0$; then $\phi = 0$. The solution (2.11) could be rewritten in the form

$$u = \frac{u_2 + u_1}{2} + \frac{u_2 - u_1}{2}\tanh\left[\frac{1}{2}\kappa\,(u_2 - u_1)\,(x - vt)\right], \tag{2.12}$$

which is generally an asymmetric kink (for $\kappa > 0$), or anti-kink (for $\kappa < 0$) moving with the velocity v, equation (2.5). Contrary to the simple functional form of equation (2.12), the parametric dependence of κ, u_1, u_2, and v on α_1, α_2, δ, and μ is quite complicated, see equations (2.8)–(2.10) and (2.5) and exhibits different families of solutions.

For $\alpha_2 = 0$, equation (2.8) reduces to $2\kappa^2 = 1$. Results for this case were briefly listed in [27] and will be discussed in more detail in the next section. For an arbitrary non-zero α_2 equation (2.8) is a cubic equation. It has three different roots if $\alpha_2^2 < 8/3$, and only a single root if $\alpha_2^2 > 8/3$, see Figure 2. In [27] for $\alpha_2 \neq 0$ we considered only $\mu = 0$,

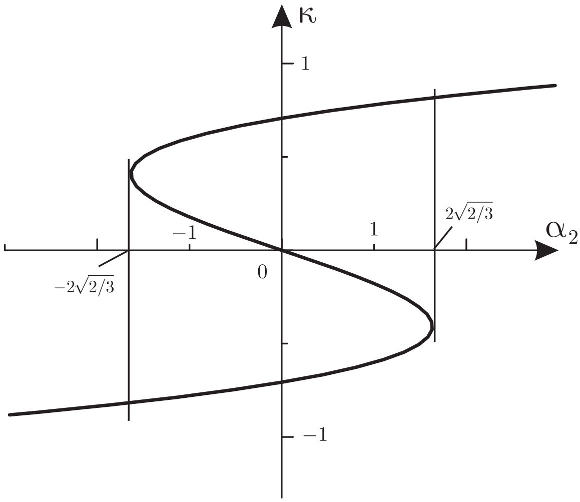

FIGURE 2. The roots κ of equation (2.8) as functions of α_2.

i.e. the zero viscosity case. In this case for arbitrary α_2, the non-zero velocity of the travelling-wave solution is possible if $\delta \neq 0$ only, which is evident from equations (2.5) and (2.9). The non-zero δ means that one of the stationary states is thermodynamically preferential. If, on the other hand,

$$3\kappa^2 + \frac{\alpha_2}{4\kappa} = 0, \tag{2.13}$$

i.e. an additional constraint is imposed on α_2 and κ, the non-zero velocity becomes possible for $\delta = 0$ as well. However, for this case, equation (2.13) together with equation (2.8) completely determines both α_2 and κ, i.e. $\alpha_2 = \mp\frac{3}{2}$ and $\kappa = \pm\frac{1}{2}$. This means that for $\mu = 0$ and symmetric potential the travelling-wave solutions are possible only for two special values of the cubic convective term coefficient. Therefore, without thermodynamic preference of any stationary state (and without dissipation) the dynamic transition between them is possible for a very special form of nonlinear ‘forcing’ only.

For the asymmetric potential in equation (1.1), the transition to the non-zero viscosity μ results in only quantitative changes. However, for the symmetric potential, $\delta = 0$, the combined action of the cubic nonlinearity and viscosity changes the situation drastically. For this case, introducing auxiliary variables $X = u_1 + u_2$, $Y = u_2 - u_1$, we can rewrite equations (2.9)–(2.10) and (2.5) as

$$\left(3\kappa^2 + \frac{\alpha_2}{4\kappa}\right) X + \kappa\mu v = 0, \tag{2.14}$$

$$\frac{3}{2}\left(\kappa^2 - \frac{\alpha_2}{4\kappa}\right) X^2 - \frac{1}{2}\left(\kappa^2 + \frac{\alpha_2}{4\kappa}\right) Y^2 + \kappa\mu v X + 1 + \frac{\alpha_1}{2\kappa} = 0, \tag{2.15}$$

$$v = \frac{1}{2} X \left[\frac{\alpha_2}{2}\left(X^2 + Y^2\right) - \alpha_1\right]. \tag{2.16}$$

To obtain the values of X, Y (i.e. u_1, u_2) and v, the system (2.14)–(2.16) should be solved for every real root of the cubic equation (2.8). The roots of equation (2.8) yielding the

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'possibilities' of nontrivial solutions are shown in Figure 2 as a function of α_2. However, this is only a *possibility*: for the existence of corresponding kinks and anti-kinks, the system (2.14)–(2.16) should have real solutions X, Y and v . Eliminating v from this system, we obtain

$$\left(3+\frac{\alpha_2}{\kappa}\right)X+\kappa\mu\left[\frac{\alpha_2}{2}\left(X^2+Y^2\right)-\alpha_1\right]X=0, \tag{2.17}$$

$$3\left(1+\frac{\alpha_2}{\kappa}\right)X^2+\left(1+\frac{2\alpha_2}{3\kappa}\right)Y^2=4\left(1+\frac{\alpha_1}{2\kappa}\right). \tag{2.18}$$

The root $X = 0$ corresponds to the static kink/anti-kink solutions. For $\alpha_2 = 0$, they coincide with the well-known static solutions [8]. For $X \neq 0$ equation, (2.17) reduces to

$$X^2+Y^2=\frac{2\alpha_1}{\alpha_2}-\frac{2}{\mu}\left(\frac{3}{\kappa\alpha_2}+2\right). \tag{2.19}$$

For given α_1, α_2, and μ any positive solution (X^2, Y^2) of the linear system (2.18)–(2.19) corresponds to the travelling-wave solution (2.12) with $u_1 + u_2 = X$ and $u_2 - u_1 = Y$. Thus, the simultaneous presence of the cubic convective term and viscosity enables the existence of the exact travelling-wave solutions without additional constraints imposed on the parameters, even if neither of the stationary states is thermodynamically preferential. Mostly, the presence of the higher-order convective term influences the properties of the nonlinear Cahn–Hilliard equation to a great extent. However, on the whole, it will not enable travelling-wave solutions in the case of a symmetric potential. Here, we consider another example of the modified nonlinear Cahn–Hilliard equation with the cubic nonlinearity in the convective term. It arises in a field quite remote from the applications mentioned above.

Recently, the convective Cahn–Hilliard equation with the cubic nonlinearity in the convective term and additional nonlinear odd-order derivative terms was derived by Gelens and Knobloch [17]. They studied the supercritical complex Swift–Hohenberg equation (CSHE) focusing on its applications in the field of nonlinear optics (for the details see [16] and [17]). The CSHE is one of universal, generic equations, which describe nonlinear systems near the threshold of stability [6, 31]. The general contents of [17] go far beyond the scope of the present paper. Here, we focus on a particular subject of the nonlinear phase equations (Sections V and VI of [17]) and, even more narrowly, on the modified convective Cahn–Hilliard equation which arises in this context. Rewriting the CSHE in terms of the amplitude and phase gradient, after a careful analysis of the relevant time, space, phase and amplitude scales, Gelens and Knobloch derived a single nonlinear equation for small perturbations of the phase gradient above the finite amplitude spatially homogeneous oscillations. From further on this equation will be called Cahn–Hilliard–Gelens–Knobloch equation (CHGKE):

$$u_t=\left[-2\left(1+\tilde{b}\tilde{\beta}\right)u-u_{xx}+2u^3\right]_{xx}+\left[2\left(\tilde{\beta}-\tilde{b}\right)u^2+\tilde{\beta}\left(3u_x^2-u^4+4uu_{xx}\right)\right]_x, \tag{2.20}$$

where u is the phase gradient deviation from zero value for the spatially homogeneous oscillations. In (2.20), $\tilde{\beta}$ and $\tilde{b}$ are imaginary parts of the constant coefficients at the fourth-order differential operator and the nonlinear term, respectively, in the CSHE [17].

To keep some resemblance to the notation in [17], we have used the same letters for these parameters. However, to avoid confusion with the notation in other sections, we supplied them with tildes. Looking for the travelling-wave solutions, viz. $u(z)$; $z = x - vt$, we have

$$\left[-2\left(1+\tilde{b}\tilde{\beta}\right)u - u_{zz} + 2u^3\right]_{zz} + \left[vu + 2\left(\tilde{\beta}-\tilde{b}\right)u^2 + \tilde{\beta}\left(3u_z^2 - u^4 + 4uu_{zz}\right)\right]_z = 0. \quad (2.21)$$

In the terminology of [17], we are looking for the 'single domain wall solution'. Denoting the stationary values at infinity as u_1 and u_2 (for definiteness presuming again $u_1 < u_2$), we again use the ansatz (2.6). We proceed along the same lines as above: substitute the ansatz into equation (2.21) and equate to zero coefficients of all powers of u. Denoting for the convenience $u_1 + u_2 = X$ and $u_1u_2 = r$, we get four algebraic constraints on four unknown parameters κ, r, X, and v:

$$\kappa^3 - \kappa + \frac{\tilde{\beta}}{6}\left(11\kappa^2 - 1\right) = 0, \quad (2.22)$$

$$X\left[\kappa^3 + \frac{\tilde{\beta}}{6}\left(7\kappa^2 + 1\right)\right] = 0, \quad (2.23)$$

$$r\left[2\kappa^3 + 3\tilde{\beta}\kappa + \tilde{\beta}\right] = X^2\left(\tilde{\beta} - \kappa^3\right) - 2\left(\tilde{\beta} - \tilde{b}\right) - 2\left(1 + \tilde{\beta}\tilde{b}\right)\kappa, \quad (2.24)$$

$$v = X\left[\tilde{\beta}X^2 - 2\tilde{\beta}r - 2\left(\tilde{\beta} - \tilde{b}\right)\right]. \quad (2.25)$$

If these constraints are satisfied, the solution of the first-order equation (2.6) is simultaneously a special solution of equation (2.20). For $\tilde{\beta} = 0$, equations (2.22) and (2.23) reduce to $\kappa^2 = 1$ and $X = 0$, respectively. Then, the equation (2.21) is reduced to the CCHE [8, 24]. Considering $\tilde{\beta} \neq 0$, let us assume $X \neq 0$; then equations (2.22) and (2.23) may be compatible only for special values of $\tilde{\beta}$ and κ. A simple check shows that the corresponding value of κ should be imaginary. Thus, even for $\tilde{\beta} \neq 0$, the only possibility is $X = 0$ and, consequently, $v = 0$. So only the static symmetric domain wall solutions are possible. The stationary states are $u_2 = -u_1 = \sqrt{|r|}$, and the solution (2.12) simplifies to

$$u = \sqrt{|r|}\tanh\left(\left(\kappa\sqrt{|r|}\right)x\right), \quad (2.26)$$

where the values for κ are given by the solutions of the cubic equation (2.22) and r is given by

$$r\left[2\kappa^3 + 3\tilde{\beta}\kappa + \tilde{\beta}\right] = -2\left(\tilde{\beta} - \tilde{b}\right) - 2\left(1 + \tilde{\beta}\tilde{b}\right)\kappa. \quad (2.27)$$

Thus, even for a non-zero $\tilde{\beta}$, the exact *single* domain wall solutions are only the static ones.

Of course, in complete analogy to the exact static solutions of the CCHE (where it was proven both numerically and by the approximate asymptotic analysis [8, 37, 38]), this does not prevent *movement* and *coarsening* for the *systems of multiple defects* in the CHGKE, as was observed numerically [17]. The values of κ as the function of $\tilde{\beta}$ are shown in Figure 3, where there are three monotonically descending branches of the $\kappa(\tilde{\beta})$ dependence. For $\tilde{\beta} < 0$, two sources (kinks) and one sink (anti-kink) solutions are possible, while for $\tilde{\beta} > 0$ one source (kink) and two sinks (anti-kinks) are possible. Again, these are only the possibilities: for each root κ the negativity of r should be checked. For the

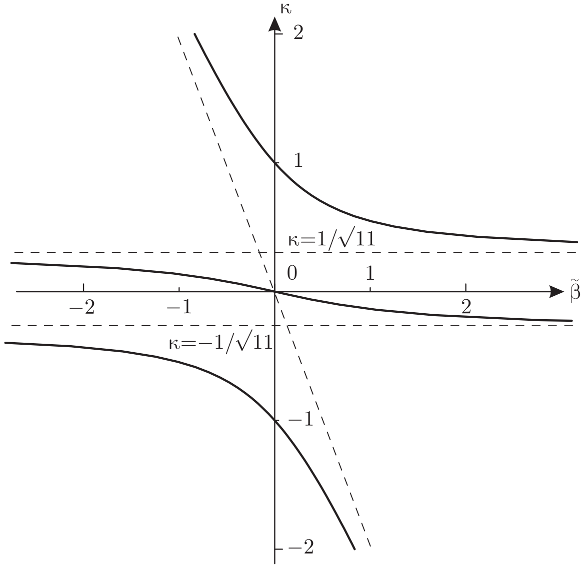

FIGURE 3. The roots κ of equation (2.22) as functions of $\tilde{\beta}$.

CHGKE, unlike the CCHE, the number of κ and (enlarging with $|\tilde{\beta}|$) asymmetry of their values are increased, see Figure 3. This may enhance, for the multiple defects system, the numerically observed irregular behaviour [17]. We see that despite the presence of the cubic convective term for the CHGKE all exact domain wall solutions are static. Thus, such a term does not guarantee, by itself, the existence of the exact travelling-wave solutions.

3 The convective-viscous Cahn–Hilliard equation with the Burgers-type convective term

The convective-viscous Cahn–Hilliard equation (CVCHE) with the Burgers-type convective term and with a general symmetric double-well potential was introduced by Witelski [40], see equation (1.6). This equation is interesting in several aspects. Firstly, it explicitly demonstrates that the combined action of the external forcing, represented by the convective term, and the dissipation, represented by the viscous term, enables stable travelling-wave solutions even if neither of the stationary states is thermodynamically preferential. Secondly, as was shown in [1], the viscous CH equation could be obtained as a limit of zero specific heat from the standard phase-field model [4]. Let us move 'backwards' to the non-zero specific heat case, now from the CVCHE. To this end, we introduce an auxiliary variable ω, which is equal to the expression in the brackets on the right-hand side of equation (1.6) and add the time derivative ω_t to the left-hand side of this equation. Thus, we arrive at a system of two nonlinear equations for u and ω. Remarkably, the travelling-wave solutions for this system are closely related to the travelling-wave solutions for the Penrose–Fife phase field model [9, 26, 31, 32]. If the potential is the asymmetric quartic polynomial, (1.6) reduces to

$$u_t - \alpha u u_x = \left(u^3 - \delta u^2 - u - u_{xx} + \mu u_t\right)_{xx}. \tag{3.1}$$

This is equation (1.1) with $\alpha_2 = 0$ (here and below we drop the lower index of α_1). First, we look for a single travelling-wave solution. The insertion of zero value of α_2 into the algebraic system (2.8)–(2.10) and (2.5) yields $2\kappa^2 = 1$, and

$$3\kappa^2 (u_1 + u_2) - \delta + \kappa\mu v = 0, \tag{3.2}$$

$$\frac{1}{2}\left[(u_1 + u_2)^2 + 2u_1u_2\right] + 1 + \frac{\alpha}{2\kappa} + \kappa\mu v (u_1 + u_2) = 0, \tag{3.3}$$

$$v = -\alpha\frac{u_1 + u_2}{2}. \tag{3.4}$$

For a non-zero δ, i.e. for an asymmetric potential, the solution of the system (3.2)–(3.4) yields the non-zero velocity v and stationary values u_1, u_2 for the solutions (2.12) without any additional limitations on the parameters α, μ, and δ [27]. For the single travelling wave we will be mainly interested, due to the reason mentioned above, in the case of symmetric potential $\delta = 0$. Then, neither of the stationary states has a thermodynamic preference. For this case, from equations (3.2) and (3.4), it follows that

$$v\left(\kappa\mu - \frac{3}{\alpha}\right) = 0. \tag{3.5}$$

It is now evident that for

$$\kappa\mu\alpha = 3, \tag{3.6}$$

i.e. for the special balance between the nonlinearity and dissipation, equation (3.5) is satisfied for an arbitrary v. This is evidently in accord with the above-mentioned result of Witelski [40]. As opposed to the $\delta \neq 0$ case, this is an essentially non-equilibrium situation, where the wave travels due to the precise balance between the external forcing and dissipation. On the other hand, if equation (3.6) is satisfied, the introduction of the thermodynamic preference, i.e. non-zero δ, will destroy this balance. Thus, this solution becomes impossible for the *special values* of α and μ which satisfy equation (3.6). As was shown in [40], such solutions are stable. The viscosity μ is positive, so there is always a kink, $\kappa > 0$, for a positive α, i.e. the positive direction of the applied field, see [8], and an anti-kink for a negative α. However, one has to remember that the sign of α depends both on the direction of the field and on the sign of the coefficient a in the expression for mobility [8], see Section 1. For this family of solutions there remains only one constraint imposed on the stationary values u_1, u_2, see equation (3.3):

$$3(u_1 + u_2)^2 + (u_2 - u_1)^2 = 4\left(1 + \frac{\alpha}{2\kappa}\right). \tag{3.7}$$

As is evident from the form of solution (2.12) and the expression for velocity (3.4), the absolute value of velocity and asymmetry of the kink are proportional to $u_1 + u_2$, while the amplitude and "steepness" are proportional to $u_2 - u_1$. This means that the trivial constant solution corresponds to $u_1 = u_2$. This yields simultaneously the strict upper limit on the absolute value of the velocity for this family of solutions:

$$|v| < v_m = \frac{|\alpha|}{\sqrt{3}}\left(1 + \frac{\alpha}{2\kappa}\right)^{\frac{1}{2}}. \tag{3.8}$$

Because $\alpha = \frac{3}{\kappa\mu}$, the upper limit of the value $\mu|v|$, as determined by equation (3.8), increases without limit for $\mu \to 0$ but approaches a constant value for increasing μ.

The CVCHE (3.1) possesses one more interesting feature: it admits an exact *two-wave* solution if an additional constraint is imposed on the parameters. In [27], such a solution was given for the case of $\delta = 0$. Here, we obtain a solution for the generally asymmetric potential $\delta \neq 0$ and afterwards the symmetric potential case will be discussed in some detail. To obtain this solution, we use Hirota's method [22]. The first step of this method is to cast equation (3.1) into a bilinear form using the Hopf–Cole substitution

$$u = \beta\frac{F_x}{F}. \tag{3.9}$$

To obtain the bilinear form, we use the ansatz

$$u^3 - u - u_{xx} + \mu u_t = \gamma u, \tag{3.10}$$

where γ is an unknown parameter. Then, it follows from equation (3.1) that

$$u_t - \alpha u u_x = \gamma u_{xx}. \tag{3.11}$$

Evidently, if u satisfies both equations (3.10) and (3.11), it satisfies equation (3.1) as well, but of course, not necessarily vice versa. Thus, our ansatz selects only a special set of solutions. As is well known, the substitution of equation (3.9) for u into equation (3.11) yields the linear diffusion equation for F; however only if $2\gamma = \alpha\beta$:

$$F_t = \gamma F_{xx}. \tag{3.12}$$

The substitution of equation (3.9) into equation (3.10) yields the following bilinear equation, if $\beta^2 = 2$:

$$F_x\left(3F_{xx} - \delta\beta F_x - \mu F_t\right) - F\left[F_{xxx} + (1+\gamma)F_x - \mu F_{xt}\right] = 0. \tag{3.13}$$

We look for solution of equation (3.13) in a form of a series

$$F = 1 + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + \varepsilon^3 F^{(3)} + \cdots \tag{3.14}$$

Terms of order ε are

$$\frac{\partial}{\partial x}\left[F^{(1)}_{xx} + (1+\gamma)F^{(1)} - \mu F^{(1)}_t\right] = 0, \tag{3.15}$$

or, integrating and setting integration constant equal to zero,

$$F^{(1)}_{xx} + (1+\gamma)F^{(1)} - \mu F^{(1)}_t = 0. \tag{3.16}$$

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Terms of order ε^2 are

$$\frac{\partial}{\partial x}\left[F^{(2)}_{xx} + (1+\gamma)\,F^{(2)} - \mu F^{(2)}_t\right] = F^{(1)}_x\left(3F^{(1)}_{xx} - \delta\beta F^{(1)}_x - \mu F^{(1)}_t\right). \qquad (3.17)$$

If equation (3.17) is homogeneous, $F^{(2)} = 0$ is a solution. Therefore, for $F^{(2)} = 0$ to be a solution, the equation

$$3F^{(1)}_{xx} - \delta\beta F^{(1)}_x - \mu F^{(1)}_t = 0 \qquad (3.18)$$

should be satisfied. If this is the case, for the terms of order ε^3 we also have the homogeneous equation,

$$\frac{\partial}{\partial x}\left[F^{(3)}_{xx} + (1+\gamma)\,F^{(3)} - \mu F^{(3)}_t\right] = 0, \qquad (3.19)$$

i.e. $F^{(3)} = 0$ is a solution of equation (3.19). This means that the series (3.14) could be terminated, if equations (3.16) and (3.18) are satisfied simultaneously. Substitution of equation (3.14) into the linear equation (3.12) yields, evidently,

$$F^{(1)}_t = \gamma F^{(1)}_{xx}. \qquad (3.20)$$

Thus, $F^{(1)}$ should satisfy simultaneously three linear equations (3.16), (3.18) and (3.20). Looking for solutions of equations (3.16), (3.18) and (3.20) in the form

$$F^{(1)} \sim \exp\left\{\rho\,(x - vt)\right\}, \qquad (3.21)$$

we get the following equations for ρ and v ($\gamma = \frac{1}{2}\alpha\beta$):

$$\rho^2 + \mu v\rho + (1+\gamma) = 0, \qquad (3.22)$$

$$3\rho^2 - \delta\beta\rho + \mu v\rho = 0, \qquad (3.23)$$

$$\gamma\rho^2 + v\rho = 0. \qquad (3.24)$$

The system (3.22)–(3.24) is overdetermined; again here there necessarily should be a constraint on α, μ and δ, reflecting the balance between external forcing, dissipation and (for $\delta \neq 0$) thermodynamic preference. Eliminating ρ and v from equations (3.22)–(3.24), we obtain the following constraint

$$(\alpha\beta\mu - 6)^2\,(\alpha + \beta) = 8\delta^2\,(\alpha\mu - \beta)\,. \qquad (3.25)$$

Remarkably, for $\delta = 0$ this constraint reduces to $\alpha\beta\mu = 6$; here $\beta^2 = 2 = 1/\kappa^2$, and we regain the crucial constraint (3.6), i.e. $\alpha\kappa\mu = 3$. If the condition (3.25) is fulfilled, the allowed values of ρ and v are

$$\rho_{1,2} = \frac{1}{4}\delta\beta \pm \frac{1}{2}\sqrt{\frac{1}{2}\delta^2 + 2 + \alpha\beta}, \qquad (3.26)$$

$$v_{1,2} = -\frac{1}{4}\alpha\delta \mp \frac{1}{4}\alpha\beta\sqrt{\frac{1}{2}\delta^2 + 2 + \alpha\beta}. \qquad (3.27)$$

Then, the solution F is

$$F = 1 + \exp\{\rho_1 (x - v_1 t) + \phi_1\} + \exp\{\rho_2 (x - v_2 t) + \phi_2\}, \tag{3.28}$$

where ϕ_1 and ϕ_2 are arbitrary constants. Then, the two-wave solution of equation (3.1) is

$$u = \beta \frac{\rho_1 \exp\{\rho_1 (x - v_1 t) + \phi_1\} + \rho_2 \exp\{\rho_2 (x - v_2 t) + \phi_2\}}{1 + \exp\{\rho_1 (x - v_1 t) + \phi_1\} + \exp\{\rho_2 (x - v_2 t) + \phi_2\}}. \tag{3.29}$$

It is instructive to consider in some detail the time evolution of equation (3.29). To avoid unnecessary complications, we do this for the physically important case of a symmetric potential, where all expressions are essentially simplified. For $\delta = 0$, equations (3.26) and (3.27) reduce to

$$\rho_{1,2} = \pm\sigma = \pm\frac{1}{2}\sqrt{2 + \alpha\beta}, \tag{3.30}$$

$$v_{1,2} = \mp v = \mp\frac{1}{2}\alpha\beta\sigma, \tag{3.31}$$

and solution (3.29) takes the form

$$u = \frac{\sigma}{\kappa} \frac{\exp\{\sigma (x + vt) + \phi_1\} - \exp\{-\sigma (x - vt) + \phi_2\}}{1 + \exp\{\sigma (x + vt) + \phi_1\} + \exp\{-\sigma (x - vt) + \phi_2\}}. \tag{3.32}$$

If we select $u = 0$ at $x = 0$, it follows that $\phi_1 = \phi_2 = \phi$. Let us look at the limits of this solution both for $t \to \infty$ and $t \to -\infty$. For definiteness, we take $\kappa > 0$; it follows from equation (3.6) that α should also be positive. For $t \to \infty$, two waves merge asymptotically into the well-known static kink [8, 24]

$$u = \frac{\sigma}{\kappa} \tanh(\sigma x). \tag{3.33}$$

Introducing the moving frame $z_1 = x + vt$, we can rewrite (3.29) as

$$u = \sqrt{1 + \frac{\alpha}{\sqrt{2}}} \frac{\exp(\sigma z_1) - \exp(-\sigma z_1 + 2vt)}{\exp(-\phi) + \exp(\sigma z_1) + \exp(-\sigma z_1 + 2vt)}. \tag{3.34}$$

Now, taking limit $t \to -\infty$, we obtain

$$u = \sqrt{1 + \frac{\alpha}{\sqrt{2}}} \frac{\exp(\sigma z_1)}{\exp(-\phi) + \exp(\sigma z_1)}, \tag{3.35}$$

i.e., a single kink moving in the negative direction and connecting $u = 0$, at $z_1 \to -\infty$ to $u = \sqrt{1 + \frac{\alpha}{\sqrt{2}}}$ at $z_1 \to \infty$.

On the other hand, introducing the moving frame $z_2 = x - vt$ and taking again the limit $t \to -\infty$ we get

$$u = -\sqrt{1 + \frac{\alpha}{\sqrt{2}}} \frac{\exp(-\sigma z_2)}{\exp(-\phi) + \exp(-\sigma z_2)}. \tag{3.36}$$

This is a single kink moving in the positive direction and connecting $u = -\sqrt{1+\frac{\alpha}{\sqrt{2}}}$ at $z_2 \to -\infty$ to zero value at $z_2 \to \infty$. So the solution (3.29) describes the kink initially

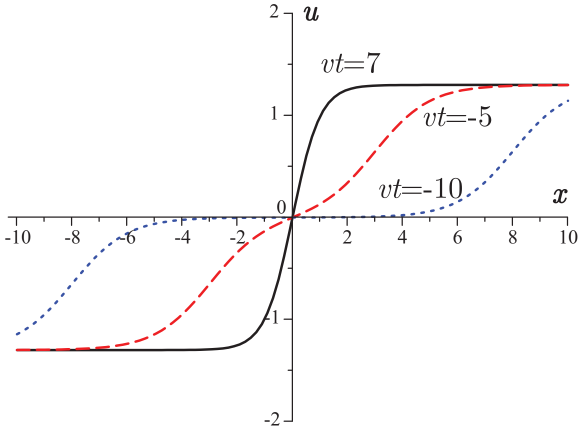

FIGURE 4. The two-wave solution (3.32) of equation (3.1) for $\sigma/\kappa = 1.3$ and $\phi = 2$.

"split" into two parts, moving towards each other and merging asymptotically in t to the well-known static kink (see Figure 4).

4 Small and large limits of α for CVCHE

As is evident from above, the presence of the convective term changes the properties of the nonlinear Cahn–Hilliard equation substantially. Hence, it makes sense to consider the limits of weak and strong external "forcing", i.e. small and large α limits in CVCHE.

For small α (for definiteness, we presume here $\alpha > 0$), it is convenient to introduce $W(y, \tau) = u(x, t)$,

$$y = \alpha x; \quad \tau = \alpha^2 t. \tag{4.1}$$

Then, equation (3.1) is rewritten as

$$W_\tau - W W_y = \left(W^3 - W - \alpha^2 W_{yy} + \alpha^2 \mu W_\tau\right)_{yy}. \tag{4.2}$$

It is natural to take $\mu = a/\alpha$ (also see [40], equation (6.1)); then, $a = 3\sqrt{2}$ corresponds to an exactly solvable case, see equation (3.6). Solving equations (3.3)–(3.4) for $u_1 + u_2$ and $u_2 - u_1$, we substitute the corresponding expressions (in terms of v and α) into equation (2.12). Introducing travelling-wave coordinate $\eta = y - \bar{v}\tau$, where $\bar{v} = v/\alpha$, we rewrite the stationary solution (2.12) as

$$w(\eta) = -\bar{v} + \left[1 + \frac{\alpha}{\sqrt{2}} - 3\bar{v}^2\right]^{\frac{1}{2}} \tanh\left\{\frac{1}{\sqrt{2}\alpha}\left[1 + \frac{\alpha}{\sqrt{2}} - 3\bar{v}^2\right]^{\frac{1}{2}} \eta\right\}. \tag{4.3}$$

For $\alpha \to 0$, this solution is close to w_1 for $\eta < 0$, close to w_2 for $\eta > 0$ and changes between these values very fast in the narrow region of the width $\sim \alpha$ at $\eta = 0$, where

$$w_{1,2} \approx -\bar{v} \mp \left[1 - 3\bar{v}^2\right]^{\frac{1}{2}}. \tag{4.4}$$

For small α, the solution (4.3) is just a special (exact) example of stable solutions obtained in [40] for the general symmetric double-well potential. To get some direct insight, we linearise (4.2) in $\hat{w}(y,\tau) = W - w(\eta)$ and consider the perturbations $\sim \exp(\theta\tau + iky)$ around the stationary states w_1 and w_2. This yields (in the limit $\alpha \to 0$) the dispersion relations

$$\theta_1 \approx -iw_1k - \left(3w_1^2 - 1\right)k^2, \tag{4.5}$$

$$\theta_2 \approx -iw_2k - \left(3w_2^2 - 1\right)k^2 \tag{4.6}$$

around stationary states w_1 and w_2 respectively. For $0 < \bar{v} < 1/\sqrt{3}$, the factor $(3w_1^2 - 1)$ is positive, i.e. θ_1 has a negative real part and the 'advancing' stationary state w_1 is stable. On the other hand, $(3w_2^2 - 1)$ is positive for $0 < \bar{v} < 1/2\sqrt{3}$, i.e. the "receding" state w_2 is linearly stable, while it is negative for $1/2\sqrt{3} < \bar{v} < 1/\sqrt{3}$, i.e. the "receding" state is linearly unstable. So this stable solution corresponds to the transition between 'metastable' (locally stable) and stable states for the former case and to the transition between unstable and stable states for the latter case. Another interesting issue would be the 'structural stability' of the solution with respect to a (slight) violation of the condition (3.6), i.e. $a = 2\sqrt{3} + \varepsilon$. It appears that for a small α the corresponding deviations $\sim O(\varepsilon)$ arise only in higher-order (in α) terms.

Let us now consider the case of a large α. It was first noticed in [20] that after rescaling $u = U/\alpha$ in the limit $\alpha \to \infty$, the convective Cahn–Hilliard equation (1.3) reduces to the famous Kuramoto–Sivashinsky equation [23, 35]:

$$U_t - UU_x = -U_{xx} - U_{xxxx}. \tag{4.7}$$

The latter equation is well known to possess quite complicated and even chaotic spatiotemporal dynamics. Golovin *et al.* [20] showed that the CCHE also exhibits, with the increase of α, a transition from coarsening to roughening. Unlike the CCHE, the CVCHE contains two generally independent parameters, α and μ. Let us first presume that the viscosity μ is fixed, while α is increasing. Rescaling $u = U/\alpha$ in equation (3.1) and taking the limit $\alpha \to \infty$, we obtain, instead of equation (4.7), the following equation:

$$U_t - UU_x = \mu U_{txx} - U_{xx} - U_{xxxx}. \tag{4.8}$$

The latter equation may be considered as 'compound' Kuramoto–Sivashinsky [23, 35] and Benjamin–Bona–Mahoney [2] equation. Here, we give three exact travelling-wave solutions for this equation, corresponding to three possible values of μv. To more closely resemble the original work [23], we change the sign of U. Introducing the travelling-wave coordinate $z = x - vt$, we have the following equation:

$$-vU_z + UU_z = -\mu v U_{zzz} - U_{zz} - U_{zzzz}. \tag{4.9}$$

For the latter equation, we obtained the travelling-wave solution

$$U = r_3(\tanh kz)^3 + r_2(\tanh kz)^2 + r_1 \tanh kz + r_0, \tag{4.10}$$

where

$$r_3 = 120k^3, \tag{4.11}$$

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$$r_2 = -15k^2 \mu v, \tag{4.12}$$

$$r_1 = 30k \left[-4k^2 + \frac{2}{19} - \frac{1}{152} (\mu v)^2 \right], \tag{4.13}$$

$$r_0 = v + \frac{1}{4} \mu v \left[40k^2 + \frac{7}{19} - \frac{13}{152} (\mu v)^2 \right]. \tag{4.14}$$

Additionally, we have the system of two nonlinear algebraic equations to determine the values of the parameters $q = k^2$ and $s = (\mu v)^2 / 152$,

$$s (19s - 2) (380q + 247s - 7) = 0, \tag{4.15}$$

$$19 \left[304q^2 + 20q (19s - 2) + 87s \right] = 47291s^2 + 11. \tag{4.16}$$

Evidently, U is an even function of k, so it is enough to consider only $\sqrt{q} = k > 0$. On the other hand, the sign of μv is determined by the sign of v, so $\mu v = \operatorname{sign}(v) \left| \sqrt{s} \right|$. The system (4.15)–(4.16) has four non-negative solutions:

$$s = 0; q = \frac{11}{76} = \frac{11}{4 \times 19}; \tag{4.17}$$

$$s = \frac{2}{19}; \ q = \frac{1}{4}; \tag{4.18}$$

$$s = \frac{18}{893} = \frac{18}{19 \times 47}; \ q = \frac{1}{188} = \frac{1}{4 \times 47}; \tag{4.19}$$

$$s = \frac{32}{1387} = \frac{32}{19 \times 73}; \ q = \frac{1}{292} = \frac{1}{4 \times 73}. \tag{4.20}$$

In terms of (μv) and k, the first solution (4.17) is $\mu = 0$ and $k^2 = 11/76$, which yields the classic Kuramoto solution for the Kuramoto–Sivashinski equation [23]. For this solution, the velocity v is a free parameter. So we consider here only three positive s solutions. The solution (4.18) corresponds to $k^2 = 1/4$, $(\mu v)^2 = 16$, and equation (4.10) takes the following form:

$$U = 15 \left[\left(\tanh \frac{z}{2} \right)^3 - \operatorname{sign}(v) \left(\tanh \frac{z}{2} \right)^2 - \tanh \frac{z}{2} \right] + \operatorname{sign}(v) \left(\frac{4}{\mu} + 9 \right). \tag{4.21}$$

This is an asymmetric solitary wave, approaching $\operatorname{sign}(v) \left(\frac{4}{\mu} - 6 \right)$ for $z \to \pm\infty$. It is a 'hump' for $v = 4/\mu$, and a 'gap' for $v = -4/\mu$. The solution (4.19) corresponds to $k^2 = 1/(4 \times 47)$, $(\mu v)^2 = (12)^2 / 47$, and (4.10) takes the form

$$U = \frac{15}{(47)^{\frac{3}{2}}} \left[\left(\tanh \frac{z}{2\sqrt{47}} \right)^3 - \operatorname{sign}(v)\, 3 \left(\tanh \frac{z}{2\sqrt{47}} \right)^2 + 3 \tanh \frac{z}{2\sqrt{47}} \right] + \operatorname{sign}(v) \frac{3}{\sqrt{47}} \left(\frac{4}{\mu} + \frac{15}{47} \right), \tag{4.22}$$

which is a monotonous asymmetric kink with $v = \pm 12 \Big/ \left(\mu \sqrt{47} \right)$, approaching asymptotic

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values

$$U^+ = \frac{60}{(47)^{\frac{3}{2}}} + \operatorname{sign}(v)\frac{12}{\mu\sqrt{47}}, \quad z \to \infty, \tag{4.23}$$

$$U^- = -\frac{60}{(47)^{\frac{3}{2}}} + \operatorname{sign}(v)\frac{12}{\mu\sqrt{47}}, \quad z \to -\infty. \tag{4.24}$$

The solution (4.20) corresponds to $k^2 = 1/(4 \times 73)$, $(\mu v)^2 = (16)^2/73$, and equation (4.10) takes the form

$$U = \frac{15}{(73)^{\frac{3}{2}}}\left[\left(\tanh\frac{z}{2\sqrt{73}}\right)^3 - \operatorname{sign}(v)\,4\left(\tanh\frac{z}{2\sqrt{73}}\right)^2 + 5\tanh\frac{z}{2\sqrt{73}}\right] + \operatorname{sign}(v)\frac{4}{\sqrt{73}}\left(\frac{4}{\mu} + \frac{15}{73}\right), \tag{4.25}$$

i.e. the kink with asymptotic values

$$U^+ = \frac{90}{(73)^{\frac{3}{2}}} + \operatorname{sign}(v)\frac{16}{\mu\sqrt{73}}, \quad z \to \infty, \tag{4.26}$$

$$U^- = -\frac{90}{(73)^{\frac{3}{2}}} + \operatorname{sign}(v)\frac{16}{\mu\sqrt{73}}, \quad z \to -\infty. \tag{4.27}$$

Remarkably, the transition from larger to smaller values of $|\mu v|$ results in a transition from the "quasi-BBM" solitary wave equation (4.21) to "quasi-KS" kinks, equations (4.22) and (4.25). Furthermore, for $|\mu v| = 16/\sqrt{73} \approx 1.873$, the difference between the stationary values is $(U^+ - U^-) \approx 0.228$, while for $|\mu v| = 12/\sqrt{47} \approx 1.750$ it is $(U^+ - U^-) \approx 0.372$. This means that if the effect of viscosity is diminished the amplitude of the kink increases. Thus, for the CVCHE in the limit of a large α (and fixed viscosity), a rather irregular, Kuramoto–Sivashinsky-like behaviour of solutions may be possible.

Now, let us presume that the viscosity is also scaled with α, $\mu = v\alpha^m$, with any positive m. Then, rescaling $u = U/\alpha$, $x = X\alpha^{\frac{m}{2}}$, $t = T\alpha^{\frac{m}{2}}$ and taking the limit $\alpha \to \infty$ yield the Benjamin–Bona–Mahoney [2] equation

$$U_T - UU_X = vU_{TXX}. \tag{4.28}$$

This equation is known to have unique and stable solutions. So, if the viscosity in the CVCHE increases with increasing "forcing", we may expect rather regular behaviour of solutions.

5 Memory effects in the convective-viscous Cahn–Hilliard equation

As a necessary step, the standard derivation of the Cahn–Hilliard equation [5, 9, 29] uses (generalised) Fick's law, i.e. the proportionality between the diffusional flux J and gradient of the chemical potential $\nabla\Phi$ [7]. Fick's law was often criticised for the infinite speed of the spread of diffusing substance. The most popular alternative is the Maxwell–Cattaneo approach, which introduces the time delay between the gradient and the flux, see e.g. [10] for a comprehensive discussion. Here, we give only the one-dimensional formulae, in accord with the spirit of the present paper. In the Maxwell–Cattaneo approach, the mass-conservation, or continuity equation, is not changed:

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}. \tag{5.1}$$

However, instead of Fick's law a more general relation is introduced:

$$\tau \frac{\partial J}{\partial t} + J = -D \frac{\partial}{\partial x} \Phi, \tag{5.2}$$

where τ is the 'delay' , or relaxation time. The integration of (5.2) yields

$$J = -\int_0^t \left(\frac{D}{\tau} \frac{\partial}{\partial x} \Phi \right) \exp\left(-\frac{t-t'}{\tau} \right) dt'. \tag{5.3}$$

Substituting the latter expression for J into (5.1), we get

$$\frac{\partial u}{\partial t} = \int_0^t \left(\frac{D}{\tau} \frac{\partial^2}{\partial x^2} \Phi \right) \exp\left(-\frac{t-t'}{\tau} \right) dt'. \tag{5.4}$$

The latter equation is an integro-differential equation with an exponentially decaying memory kernel. Depending on the form of Φ, it is the diffusion equation or Cahn–Hilliard equation with 'memory effects'. On the other hand, differentiation of equation (5.4) or, alternatively, elimination of J from equations (5.2)–(5.3) leads to the equation

$$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(D \frac{\partial}{\partial x} \Phi \right). \tag{5.5}$$

Hence, such modification is also called hyperbolic. The hyperbolic modification of the classic one-dimensional Cahn–Hilliard equation was proposed in [11] to model rapid spinodal decomposition in a binary alloy. This paper was followed by many others, both of physical and mathematical nature [12–14], to mention just a few. The hyperbolic modification of the viscous Cahn–Hilliard equation was also considered in [3], [15] and [21]. Here, we consider the hyperbolic modification of the CVCHE (3.1); for the sake of brevity we consider only $\delta = 0$, i.e. the symmetric potential case. Then, equation (5.5) takes the following form:

$$\tau u_{tt} + u_t - \alpha u u_x = \left(u^3 - u - u_{xx} + \mu u_t \right)_{xx}. \tag{5.6}$$

Introducing the travelling-wave coordinate $z = x - vt$, we get

$$-\left(\frac{1}{2} \alpha u^2 + vu \right)_z = \left[u^3 - \left(1 + \tau v^2 \right) u - u_{zz} - \mu v u_z \right]_{zz}. \tag{5.7}$$

Proceeding as above, we again obtain solution (2.12). Again we have $2\kappa^2 = 1$,

$$v = -\alpha \frac{u_1 + u_2}{2}, \tag{5.8}$$

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and the *crucial condition* (3.6), $\kappa\alpha\mu = 3$, is necessary. The only difference is that the single constraint on two stationary values u_1, u_2 is now

$$\left(3 - \tau\alpha^2\right)(u_1 + u_2)^2 + (u_2 - u_1)^2 = 4\left(1 + \frac{\alpha}{2\kappa}\right) \tag{5.9}$$

instead of equation (3.7). For $\tau\alpha^2 < 3$, this produces only quantitative changes, but for $\tau\alpha^2 > 3$ the absolute value of the velocity of the kink will increase with steepness. However, because of the condition (3.6), the latter inequality is equivalent to $\mu^2 < 6\tau$. Usually, τ is considered as a small parameter and the term τu_{tt} in equation (5.6) as a singular perturbation. So for the CVCHE with memory effects the increase of the velocity with the steepness may occur for extremely low viscosity only.

6 Discussion

The central aim of the present work is to study the joint effect of the nonlinear convective term and dissipation in the nonlinear Cahn–Hilliard equation. Such a study was pioneered by Witelski [40] for equation (1.6), i.e. for the case of a general symmetric double-well potential and Burgers-type convective term. To obtain the approximate travelling-wave solution, he used a singular asymptotic expansion. From the analysis of the internal shock layer, it was derived that such stable solutions exist only for a proper balance between nonlinear effects and dissipation. In the present paper, we have considered both the cubic and Burgers-type convective terms and the influence of dissipation. Generally, the presence of the higher-order convective term essentially influences the properties of the solution. As is shown both for the CCHE with cubic nonlinearity (1.4) and for the CHGKE (2.21), the number of exact static single-domain-wall solutions and their asymmetry increases. For the strong forcing, this may enhance quite irregular behaviour of the multiple-wave system [17]. However, comparing these two equations we can see that while the presence of the cubic nonlinearity will not generally guarantee the existence of exact travelling-wave solutions, the presence of dissipation may allow such solutions without any additional constraints on the parameters. In the case of the Burgers-type convection term (3.1), the situation is more subtle. Further on it is more convenient to rewrite the solution (2.12) in a somewhat different form. Solving equations (3.3)–(3.4) for $u_1 + u_2$ and $u_2 - u_1$, and substituting into (2.12) we get

$$\begin{aligned} u = -\frac{v}{\alpha} + \left[1 + \frac{\alpha}{2\kappa} + \left(\frac{v}{\alpha}\right)^2 (3 - 2\kappa\mu\alpha)\right]^{\frac{1}{2}} \\ \times \tanh\left\{\kappa\left[1 + \frac{\alpha}{2\kappa} + \left(\frac{v}{\alpha}\right)^2 (3 - 2\kappa\mu\alpha)\right]^{\frac{1}{2}} (x - vt)\right\}, \end{aligned} \tag{6.1}$$

and a constraint

$$v\left[\kappa\mu - \frac{3}{\alpha}\right] = \delta. \tag{6.2}$$

Setting $\mu = 0$ in equations (6.1) and (6.2), we return to the solution for the case of the

CCHE with asymmetric potential considered in [27]:

$$u = -\frac{v}{\alpha} + \left[1 + \frac{\alpha}{2\kappa} + 3\left(\frac{v}{\alpha}\right)^2\right]^{\frac{1}{2}} \tanh\left\{\kappa\left[1 + \frac{\alpha}{2\kappa} + 3\left(\frac{v}{\alpha}\right)^2\right]^{\frac{1}{2}}(x - vt)\right\}, \quad (6.3)$$

$$v = -\frac{1}{3}\alpha\delta. \quad (6.4)$$

The velocity of this travelling wave is proportional both to α, i.e. to the applied field, and to the asymmetry of potential δ, roughly speaking to the "thermodynamic driving force". So, unlike the travelling-wave solutions of more general form [39], the moving exact stationary kink/anti-kink solutions exist for $\delta \neq 0$ only. For the classic Cahn–Hilliard equation [5], the asymmetric potential, and correspondingly, the static asymmetric kink solution were usually discarded [29] because for this case the global conservation of the order parameter is violated. However, there is generally no global conservation for the CCHE. The very notion of coarsening or "Ostwald ripening", as considered in the theory of first-order phase transitions [25], relates to the competitive growth of stable-phase domains inserted into the metastable phase. In terms of quartic polynomial potential, this corresponds to unequal depths of two potential wells and $\delta \neq 0$.

On the other hand, if both nonlinear convective term and viscosity are present in equation (3.1), the moving exact stationary kink/anti-kink solutions, see equation (6.1), exist both for $\delta \neq 0$ and $\delta = 0$. For the former case, the velocity is again proportional to the asymmetry of the potential. For the latter case, i.e. for a symmetric potential the solution is

$$u = -\frac{v}{\alpha} + \left[1 + \frac{\alpha}{2\kappa} - 3\left(\frac{v}{\alpha}\right)^2\right]^{\frac{1}{2}} \tanh\left\{\kappa\left[1 + \frac{\alpha}{2\kappa} - 3\left(\frac{v}{\alpha}\right)^2\right]^{\frac{1}{2}}(x - vt)\right\}, \quad (6.5)$$

where v is now a free parameter. However, the existence of these travelling kink/anti-kink solutions requires the additional constraint on nonlinearity and viscosity, i.e. $\kappa\alpha\mu = 3$, see equations (3.6) and (6.2). Evidently, this is in accord with the result of Witelski [40]. It is essentially a non-equilibrium situation, where the wave travels due to precise balance between the external forcing and dissipation. Unlike equation (6.3), for the solution (6.5), the steepness of the kink is decreasing with velocity. The maximal steepness is achieved for $v = 0$, i.e. for a static symmetric kink. On the other hand, the strict upper limit on the allowed absolute value of the velocity corresponds to $u_1 = u_2$, i.e. to the trivial constant solution. Physically, the increasing absolute value of the velocity increases the role of the viscous term; additionally, to preserve the balance between the nonlinearity and dissipation, the steepness should be diminished.

Interestingly, the dependence of the asymmetry of the kink on the velocity is the same both for equations (6.3) and (6.5). However, for the former case, v is fixed by the asymmetry of potential, see equation (6.4), while for the latter case the velocity is a free parameter, $|v| < v_m$. Thus, the presence or absence of asymmetry in the potential fundamentally influences the existence and properties of exact solutions. As it follows from the results of [40], for a small α the solution (6.5) should be stable. The direct exploration of the stability of stationary states u_1, u_2 in Section 4 has shown that for

$0 < v < \alpha/\sqrt{3}$ the advancing stationary state u_1 is stable, while the receding state u_2 is linearly stable for $0 < v < \alpha/2\sqrt{3}$, and linearly unstable for $\alpha/2\sqrt{3} < v < \alpha/\sqrt{3}$. Thus, this solution corresponds to the transition between 'metastable' and stable states for the former case, and to the transition between unstable and stable for the latter case. Application of the same procedure to the solution (6.3) (for a small α) yields the linear stability of both stationary states, so the solution for $\mu = 0$, $\delta \neq 0$ (i.e. for the CCHE with asymmetric potential) corresponds to the transition between metastable and stable states. The process of transition between the two locally stable states is usually more stable than the transition between unstable and stable states, so the solution (6.3) should not be less stable than (6.5).

In contrast to the small α case, the case of a large α is much less conclusive, e.g., as was shown in [20] the CCHE exhibits the transition from coarsening to roughening in the $\alpha \to \infty$ limit. To get some heuristic arguments, we have considered the limit of a large α in Section 4. Unlike the CCHE, the CVCHE contains two generally independent parameters α and μ. Presuming the viscosity μ to be fixed and taking limit $\alpha \to \infty$, we obtain equation (4.8), which may be considered as the compound Kuramoto–Sivashinsky [23, 35] and Benjamin–Bona–Mahoney [2] equation. We have obtained exact travelling-wave solutions of this equation corresponding to three possible values of μv, see equations (4.18)–(4.20). For largest $|\mu v| = 4$, there is a single asymmetric 'hump' moving in the positive direction, and a single asymmetric 'gap' moving in the negative direction, see equation (4.21). This is reminiscent of the well-known (symmetric) solitary wave solution for the Benjamin–Bona–Mahoney [2] equation. For the smaller values $|\mu v| = 16/\sqrt{73} \approx 1.873$ and $|\mu v| = 12/\sqrt{47} \approx 1.750$, there are monotonic (asymmetric) kinks, solutions (4.25) and (4.22), respectively. So, for a large $|\mu v|$ the solutions of equation (4.8) look more like the solutions of Benjamin–Bona–Mahoney [2] equation, and for a small $|\mu v|$ it is more like the solutions of Kuramoto–Sivashinsky equation [23, 35]. However, generally it is not clear whether the BBM term may counteract the 'roughening', which is typical for the Kuramoto–Sivashinsky equation.

Now, let us presume that the viscosity is also scaled with α, $\mu = \nu\alpha^m$, with any positive m. Then, the limit $\alpha \to \infty$ yields the Benjamin–Bona–Mahoney [2] equation (4.28). This equation is known to have unique and stable solutions. Therefore, we speculate that if the viscosity increases with increasing 'forcing', the solutions of the convective-viscous CH equation may exhibit a quite regular behaviour for a large α.

For the CVCHE, we also obtained an exact two-wave solution. While the presence of the viscous term is necessary for the existence of such solution, the solution exists both for symmetric and asymmetric potentials. For the sake of brevity, only the solution for the symmetric potential is considered in some detail in Section 3. Physically, the exact two-wave solution corresponds to the (one-dimensional) space initially (for $t \to -\infty$) divided into three domains: the domains at $\pm\infty$, occupied by stable stationary states, and an intermediate domain in the unstable state $u = 0$. Thus, the merging of waves into the well-known static kink could be considered as a special case of (anomalous) coarsening. For $t \to -\infty$, i.e. when the 'half-waves' are widely separated, the intermediate (unstable) domain shrinks linearly with t. For $t \gg 1/v\sigma$, the deviation from the final static tanh-profile is localised in the transition region $|x| \sim 1/\sigma$ and decays exponentially with time,

$$u - \frac{\sigma}{\kappa}\tanh(\sigma x) \sim -\frac{\sinh(\sigma x)\exp(-\sigma v t)}{\cosh^2(\sigma x)}. \tag{6.6}$$

Finally, the memory effects (hyperbolic modification) in the convective-viscous equation (5.6) were briefly discussed. The exact travelling-wave solutions for this equation exist both for symmetric and asymmetric potential. However, in the present communication, we consider only the former case. For symmetric potential, the crucial condition (3.6) is again necessary. The only essential difference from the CVCHE is that due to the memory effect for a very low viscosity (very strong 'forcing'), or a very large 'delay time', the velocity may increase with the steepness.

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