ON THE NØRLUND SUMMABILITY OF A CLASS OF FOURIER SERIES

BADRI N. SAHNEY

1. Our aim in this paper is to determine a necessary and sufficient condition for Nørlund summability of Fourier series and to include a wider class of classical results. A Fourier series, of a Lebesgue-integrable function, is said to be summable at a point by Nørlund method (N, p_n), as defined by Hardy [**1**], if $p_n \to 0$, $\sum p_n \to \infty$, and the point is in a certain subset of the Lebesgue set. The following main results are known.

THEOREM A. *Let $\phi(u)$ be even, $\phi(u) \in L(-\pi, \pi)$, and let S_n denote the $(n+1)$st partial sum of its Fourier series at the origin. Then the assumption*

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(\frac{t}{\log(1/t)}\right) \quad \text{as } t \to +0 \tag{1.1}$$

implies that S_n is summable $(N, (n+1)^{-1})$, or summable by harmonic means to the sum 0.

THEOREM A′. *If* (1.1) *is replaced by*

$$\Phi(t) = o\left(\frac{t}{\prod_{q=0}^{p-1} \{(\log)^{q+1}(1/t)\}}\right) \quad \text{as } t \to +0, \tag{1.1)$'$}$$

then $\sigma_n(p)$ is summable (H, p) to the sum 0 *and $\sigma_n(p)$ is defined as follows:*

$$\sigma_n(p) = \frac{\sum_{k=0}^{n} \left\{ S_{n-k}(t) \Big/ \prod_{q=0}^{p-1} (\log)^q (k+1) \right\}}{\{(\log)^p(n+1)\}},$$

for each positive integer p.

THEOREM B. *If $\phi(u)$ is defined as in Theorem* A, *then the assumption*

$$\Phi(t) = o(t) \quad \text{as } t \to +0 \tag{1.2}$$

implies that S_n is summable to 0 *by the Cesàro method (C, k), for any $k > 0$.*

---

*Received September 25, 1968. This work was carried out while the author was a Fellow at the Summer Research Institute at Queen's University and University of British Columbia in 1967 and 1968, respectively.

THEOREM C. *Let a function* $P(u)$, *tending to* ∞ *with* u, *and a sequence* $\{p_n\}$ *be defined as follows in terms of* $p(u)$, *monotonic decreasing and strictly positive for* $u \geqq 0$,

$$P(u) \equiv \int_0^u p(x)\,dx, \qquad p_n \equiv p(n). \tag{1.3}$$

Then (1.2) *and*

$$\int_1^u \frac{P(x)}{x}\,dx = O\{P(u)\} \quad \text{as } u \to \infty \tag{1.4}$$

ensure that either S_n *is summable* (N, p_n) *to* 0, *or*

$$t_n \equiv \left\{\sum_{r=0}^{n} p_r S_{n-r} \Big/ \sum_{r=0}^{n} p_r\right\} \to 0 \quad \text{as } n \to \infty. \tag{1.5}$$

THEOREM D. *Let* p_n *be defined as in* (1.3). *A necessary and sufficient condition that the Nørlund method* (N, p_n) *should sum the Fourier series of* $\phi(u)$ *to* 0 *such that*

$$\phi(t) = o\left(\frac{1}{\log|1/t|}\right) \tag{1.6}$$

is that the sequence

$$\frac{1}{P_n}\sum_{k=1}^{n} \frac{P_k}{(k+1)\log(k+1)} = O(1). \tag{1.7}$$

Theorem A has been proved by Hille and Tamarkin [**3**], Iyengar [**5**], and Siddiqi [**11**], while I proved Theorem A′ in [**9**]. Theorem B has been proved by Fejér, Lebesgue, and Hardy (see [**15**, p. 49]). Rajagopal [**8**] has proved Theorem C, and Theorem D is due to Varshney [**13**].

Rajagopal had to use the two alternative forms of Theorem C, from which he deduced Theorems A and B, respectively. He had indicated that condition (1.4) is violated in the application of Theorem C as Theorem A and Theorem A′. I have attempted to improve Theorems C and D in such a way that both Theorems A and B can be deduced from Theorem 1, which also generalizes Theorem D.

2. The following is our main result.

THEOREM 1. *Let the sequence* p_n *be defined as in* (1.3) *and let*

$$\Phi(t) = o(t/\psi(1/t)) \quad \text{as } t \to +0; \tag{2.1}$$

https://doi.org/10.4153/CJM-1970-011-3 Published online by Cambridge University Press

let $\psi(t)$ be positive, non-decreasing with t; then a necessary and sufficient condition to ensure (1.5) *is that*

$$\int_1^n \frac{P(x)}{x\psi(x)}\,dx = O(P(n)). \tag{2.2}$$

3. Proof of Sufficiency. We write the formula for the nth partial sum of the Fourier series as:

$$\begin{aligned} S_n &= \frac{1}{\pi}\int_0^\pi \phi(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du \\ &= \frac{1}{\pi}\int_0^\delta \phi(u)\,\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du + o(1). \end{aligned}$$

Using (1.5) and the last equation, we obtain:

$$\begin{aligned} t_n &= \frac{1}{\pi P_n}\int_0^\delta \phi(u)\sum_{r=0}^n p_{n-r}\,\frac{\sin(r+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du + o(1) \\ &= \frac{1}{\pi P_n}\left[\int_0^{1/n} + \int_{1/n}^\delta\right]\frac{\phi(u)}{\sin\frac{1}{2}u}\sum_{r=0}^n p_{n-r}\sin(r+\tfrac{1}{2})u\,du + o(1) \\ &= I_1 + I_2 + o(1), \quad \text{say,} \end{aligned} \tag{3.1}$$

by virtue of (1.3).

Now (1.3) implies that $mp_m < P_m$. If we choose m to be the integral part of $1/u$ and if we suppose that $1/n \leqq u \leqq \delta$, we obtain $m \sin\frac{1}{2}u > mu/\pi$. Now for $u > 0$ and $m \leqq n$ [**10**, Lemma] we have:

$$\left|\sum_{r=0}^n p_{n-r}\sin(r+\tfrac{1}{2})u\right| < P(m) + \frac{AP(m)}{m\sin\frac{1}{2}u} < cP(1/u). \tag{3.2}$$

Furthermore,

$$\left|\frac{\sum_{r=0}^n p_{n-r}\sin(r+\frac{1}{2})u}{\sin\frac{1}{2}u}\right| \leqq \frac{cP(1/u)}{\sin\frac{1}{2}u} < \frac{c\pi P(1/u)}{u} < \frac{cP(1/u)}{u}. \tag{3.3}$$

Considering I_1, we obtain:

$$\begin{aligned} I_1 &= \frac{1}{\pi P_n}\int_0^{1/n}\frac{\phi(u)}{\sin\frac{1}{2}u}\sum_{r=0}^n p_{n-r}\sin(r+\tfrac{1}{2})u\,du \\ &= O\left(\frac{1}{P_n}\right)\int_0^{1/n}|\phi(u)|(2n+1)P_n\,du \\ &= O(2n+1)o\left(\frac{t}{\psi(1/t)}\right)_0^{1/n} \\ &= o\left(\frac{1}{\psi(n)}\right) \to 0. \end{aligned} \tag{3.4}$$

https://doi.org/10.4153/CJM-1970-011-3 Published online by Cambridge University Press

Next, by (3.3), we have:

$$
\begin{aligned}
(3.5)\qquad I_2 &= O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} |\phi(u)| \left| \frac{\sum_{r=0}^{n} p_{n-r}\sin(r+\frac{1}{2})u}{\sin\frac{1}{2}u} \right| du \\
&= O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} |\phi(u)| \frac{P(1/u)}{u}\, du \\
&= O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} |\phi(u)| \left\{\frac{P(1/u)}{u\psi(u)}\,\psi(u)\right\} du \\
&= O\left(\frac{1}{P_n}\right)\left[\Phi(u)\,\frac{P(1/u)}{u}\right]_{1/n}^{\delta} \\
&\qquad + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} \Phi(u)d\left\{\frac{P(1/u)}{u\psi(1/u)}\,\psi(1/u)\right\} \\
&= o\left(\frac{1}{\psi(n)P(n)}\right) + o\left(\frac{1}{\psi(n)}\right) \\
&\qquad + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} o\left(\frac{u}{\psi(1/u)}\right)\frac{P(1/u)}{u\psi(1/u)}\, d\psi(1/u) \\
&\qquad + o\left(\frac{1}{P_n}\right) + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} o\left(\frac{u}{\psi(1/u)}\right)d\left\{\frac{P(1/u)}{u\psi(1/u)}\right\}\psi(1/u) \\
&= o\left(\frac{1}{\psi(n)}\right) + o\left(\frac{1}{\psi(n)P(n)}\right) + o(1)\int_{1/n}^{\delta} \frac{d\psi(1/u)}{\{\psi(1/u)\}^2} \\
&\qquad + o\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} u\, d\left\{\frac{P(1/u)}{u\psi(1/u)}\right\} \\
&= o\left(\frac{1}{\psi(n)}\right) + o\left(\frac{1}{\psi(n)P(n)}\right) + o(1)\left[\frac{1}{\psi(1/u)}\right]_{1/n}^{\delta} \\
&\qquad + o\left(\frac{1}{P_n}\right)\left\{\left[u\,\frac{P(1/u)}{u\psi(1/u)}\right]_{1/n}^{\delta} - \int_{1/n}^{\delta} 1\,\frac{P(1/u)}{u\psi(1/u)}\, du\right\} \\
&= o\left(\frac{1}{\psi(n)}\right) + o\left(\frac{1}{\psi(n)P(n)}\right) + o\left(\frac{1}{P_n}\right)O(P_n) \to 0,
\end{aligned}
$$

by virtue of (2.2).

Thus $t_n \to 0$, and this completes the proof.

4. Proof of Necessity. The proof of sufficiency shows that all we need to prove here is that

$$
(4.1)\qquad O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} \frac{u}{\psi(1/u)}\, d\left\{\frac{P(1/u)}{u}\right\} = O(1).
$$

https://doi.org/10.4153/CJM-1970-011-3 Published online by Cambridge University Press

Considering the left hand side, we have:

$$
\text{(4.2)}\quad O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} ud\left\{\frac{P(1/u)}{u\psi(1/u)}\,\psi(1/u)\right\}\frac{1}{\psi(1/u)}
$$

$$
= O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta} ud\left\{\frac{P(1/u)}{u\psi(1/u)}\right\} + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta}\frac{uP(1/u)}{u\psi(1/u)}\,\frac{d\psi(1/u)}{\psi(1/u)}
$$

$$
= O\left(\frac{1}{P_n}\right)\left[u\,\frac{P(1/u)}{u\psi(1/u)}\right]_{1/n}^{\delta} + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta}\frac{P(1/u)}{u\psi(1/u)}\,du + O(1)\int_{1/n}^{\delta}\frac{d\psi(1/u)}{\{\psi(1/u)\}^2}
$$

$$
= O\left(\frac{1}{\psi(n)}\right) + O\left(\frac{1}{P(n)}\right) + O\left(\frac{1}{P(n)}\right)\int_{1/n}^{\delta}\frac{P(1/u)}{u\psi(1/u)}\,du + O(1)\left[\frac{1}{\psi(1/u)}\right]_{1/n}^{\delta}
$$

$$
= O\left(\frac{1}{\psi(n)}\right) + O\left(\frac{1}{P(n)}\right) + O\left(\frac{1}{P(n)}\right)\int_{1/n}^{\delta}\frac{P(1/u)}{u\psi(1/u)}\,du.
$$

Hence from (4.1), we have:

$$
O\left(\frac{1}{\psi(n)}\right) + O\left(\frac{1}{P(n)}\right) + O\left(\frac{1}{P(n)}\right)\int_{1/n}^{\delta}\frac{P(1/u)}{u\psi(1/u)}\,du = O(1).
$$

But the first two terms tend to a constant with large n and the last equation, in that case, will reduce to (2.2), which proves the desired result.

5. Theorem 1 has the advantage over Rajagopal's result in two ways. First, it gives a set of necessary and sufficient conditions, while Rajagopal has proved only the sufficiency part. Secondly, Theorems A (Theorem A′ also) and B can be deduced directly from Theorem 1, which was not possible in his case.

If we consider the case $\psi(u) \equiv \log u$ and $p_n = 1/(n+1)$, we obtain Theorem A, while the case

$$
\psi(u) = \prod_{q=0}^{p-1}\log^{q+1}(u) \quad\text{and}\quad p_n = \left(\prod_{q=0}^{p-1}\log^{q}(n+1)\right)^{-1}
$$

is Theorem A′. By choosing $\psi(u) = 1$ and $p_n = \Gamma(n+\alpha)/\Gamma(n+1)\Gamma(\alpha)$ for $0 < \alpha < 1$, we obtain Theorem B.

The particular cases $\psi(u) \equiv \log u$ and $\psi(u) = 1$ are Theorems C and D, respectively.

6. Iyengar [**4**] has shown that harmonic summability of Fourier series implies Valiron summability of Fourier series, and Varshney [**12**] has shown that harmonic summability of Fourier series implies Riesz summability of first order and of type $\exp(n^{\alpha})$, $0 < \alpha < 1$. Hardy and Littlewood [**2**] proved

https://doi.org/10.4153/CJM-1970-011-3 Published online by Cambridge University Press

that the conditions $S_n - S_{n-1} = O(n^{\alpha-1})$ for $0 < \alpha < 1$ along with the Valiron summability imply convergence of S_n. Later on, Wang [**14**] and Iyengar [**5**] used the same conditions to prove the convergence of S_n, the former via Riesz summability of the type $\exp(n^\alpha)$ and the latter via harmonic summability. Jurkat [**7**] has discussed the advantage of Wang's method and proved that under a condition similar to (1.1)' (see [**7**]), the Riesz summability of S_n of any positive order and a certain type [**7**] along with the appropriate Tauberian condition implies the convergence of S_n. In the same way we can introduce in either Theorem A or Theorem A' the Tauberian condition appropriate to Nørlund summability of that theorem and establish the convergence of S_n. One such Tauberian condition, as given by Iyengar, is $S_n - S_{n-1} = O(n^\alpha)$ for $0 < \alpha < 1$.

References

1. G. H. Hardy, *Divergent series* (Oxford, at the Clarendon Press, 1949).

2. G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* XVIII: *On the convergence of Fourier series*, Proc. Cambridge Philos. Soc. *31* (1935), 317–323.

3. E. Hille and J. D. Tamarkin, *On the summability of Fourier series*, Trans. Amer. Math. Soc. *34* (1932), 757–783.

4. K. S. K. Iyengar, *A Tauberian theorem and its applications to convergence of Fourier series*, Proc. Indian Acad. Sci. Sect. A *18* (1943), 81–87.

5. ——— *New convergence and summability test for Fourier series*, Proc. Indian Acad. Sci. Sect. A *18* (1943), 113–120.

6. ——— *Notes on the summability*. II. *On the relation between summability by Nörlund means of a certain type and summability by Valiron means*, Half-Yearly J. Mysore Univ. Sect. B (N.S.) *4* (1944), 161–166.

7. W. Jurkat, *Zur Konvergenztheorie der Fourier-Reihen*, Math. Z. *53* (1950–51), 309–339.

8. C. T. Rajagopal, *Nörlund summability of Fourier series*, Proc. Cambridge Philos. Soc. *59* (1963), 47–53.

9. B. N. Sahney, *On the* (H, p) *summability of Fourier series*, Boll. Un. Mat. Ital. *16* (1961), 156–163.

10. ——— *On the Nörlund summability of Fourier series*, Pacific J. Math. *13* (1963), 251–262.

11. J. A. Siddiqi, *On the harmonic summability of Fourier series*, Proc. Indian Acad. Sci. Sect. A *28* (1948), 527–531.

12. O. P. Varshney, *On the relation between harmonic summability and summability by Riesz means of a certain type*, Tôhoku Math. J. (2) *11* (1959), 20–24.

13. ——— *On the Nörlund summability of Fourier series*, Acad. Roy. Belg. Bull. Cl. Sci. (5) *52* (1966), 1552–1558.

14. F. T. Wang, *On the Riesz summability of Fourier series*, Proc. London Math. Soc. (2) *47* (1942), 308–325.

15. A. Zygmund, *Trigonometrical series*, 2nd ed. (Chelsea, New York, 1952).

The University of Calgary,
Calgary, Alberta