Zero-dimensional isomorphic dynamical models

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Abstract. By an *assignment* we mean a mapping from a Choquet simplex K to probability
measure-preserving systems obeying some natural restrictions. We prove that if Φ is an
aperiodic assignment on a Choquet simplex K such that the set of extreme points $\mathsf{ex}\,K$
is a countable union $\bigcup_n E_n$, where each set E_n is compact, zero-dimensional and the
restriction of Φ to the Bauer simplex K_n spanned by E_n can be 'embedded' in some
topological dynamical system, then Φ can be 'realized' in a zero-dimensional system.

1. *Introduction*

For a convex set K we shall denote by $\mathsf{ex}\,K$ the set of its extreme points. A *metrizable
Choquet simplex* (which in this note will be briefly called just a *simplex*) is a compact
convex subset K of a metric-linear space endowed with a convex metric such that every
point $p \in K$ is the barycenter of a unique probability distribution supported by $\mathsf{ex}\,K$ (we
are using the Choquet–Meyer characterization of simplices; see e.g. [**Ph01**]). A simplex
K is called *Bauer* if $\mathsf{ex}\,K$ is closed (and hence compact). Any simplex is a compact
connected space, but its set of extreme points may have various topological attributes: it

may but need not be compact, sigma-compact or zero-dimensional, etc. We will shortly say that the simplex has any of the above topological properties, meaning that its set of extreme points has it. For instance, we will speak about zero-dimensional, sigma-compact simplices. Exception: since Bauer simplices already have their name, we will not use the confusing term 'compact simplex'.

A *face* of a simplex K is a compact convex subset $F \subset K$ such that $\mathrm{ex}\,F \subset \mathrm{ex}\,K$. A face of a simplex is a simplex.

By a *topological dynamical system* we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. It is well known that the set $\mathcal{M}_T(X)$ of all T-invariant Borel probability measures (in the following we will skip the adjectives 'Borel' and 'probability') on X, equipped with the weak-star topology, is a simplex and we shall call it *the simplex of invariant measures*. The standard metric on measures

$$\mathsf{dist}(\mu, \nu) = \sum_{n \geq 1} 2^{-n} \left| \int f_n \, d\mu - \int f_n \, d\nu \right|,$$

where $(f_n)_{n \geq 1}$ is some fixed sequence of normed continuous functions linearly dense in $C(X)$ (the space of continuous real functions on X with the uniform norm), is well known to be convex and it induces the weak-star topology. We denote by $\mathcal{M}_T^e(X)$ the collection of all ergodic T-invariant measures on X, which coincides with the collection of extreme points of $\mathcal{M}_T(X)$. By a *zero-dimensional system* we understand a topological dynamical system (X, T), where the space X is zero-dimensional.

By an *assignment* we will understand a mapping Φ defined on a simplex K, whose 'values' are measure-preserving systems (by a measure-preserving system we mean a standard probability space (X, Σ, μ) together with a measure-automorphism $T : X \to X$). We also require that the assignment obeys the following rules:

(1) extreme points are assigned ergodic measure-preserving systems,
(2) if $p \in K$ is the barycenter of a probability distribution ξ on $\mathrm{ex}\,K$, then

$$\Phi(p) \approx \int \Phi(e) \, d\xi(e),$$

where '$\approx$' denotes measure-theoretic isomorphism and the integral is realized on a disjoint union of the spaces realizing the assignments $\Phi(e)$ for $e \in \mathrm{ex}\,K$ (we will say that Φ is *harmonic*).

Two assignments, Φ and Φ' defined on K and K', respectively, are said to be *equivalent* if there exists an affine homeomorphism $\pi : K \to K'$ such that $\Phi(p) \approx \Phi'(\pi(p))$ for every $p \in K$. Since Φ and Φ' are both harmonic, it suffices to check that $\Phi(e) \approx \Phi'(\pi(e))$ for $e \in \mathrm{ex}\,K$. By a *face of an assignment* Φ on a simplex K we mean the restriction of Φ to a face of K.

An assignment Φ on K is called *aperiodic* if $\Phi(e)$ is aperiodic for each $e \in \mathrm{ex}\,K$.

For a topological dynamical system (X, T), the assignment Φ on $\mathcal{M}_T(X)$ defined by $\Phi(\mu) = (X, \mathsf{Borel}(X), \mu, T)$ is called the *natural* assignment of (X, T). We say that an assignment can be *realized* (*embedded*) in a topological dynamical system (X, T) if it is equivalent to (a face of) the natural assignment of (X, T).

The general question about a characterization of assignments realizable in topological dynamical systems is wide open and seems to be hopelessly difficult. It is not only the

question about a possible affine-topological shape of the set $\mathcal{M}_T(X)$ in a topological dynamical system (X, T) but also about the possible configuration of ergodic systems placed over the extreme points of this set. For example, one can ask whether it is possible to have, in one topological dynamical system (perhaps minimal), a sequence of measures isomorphic to, say, irrational rotations, converging to, say, a Bernoulli measure. Or is it possible to have a closed arc of Bernoulli measures parametrized continuously and increasingly by their entropies, and no other ergodic measures. The variety of imaginable questions of this kind is endless. So far, there exist only partial results and most of them concern zero-dimensional systems. Let us review briefly some of them.

(1)   On any simplex K there exists an assignment realizable in a minimal zero-dimensional system (more precisely, in a Toeplitz subshift [**Dow91**]).

(2)   On any simplex K, given any non-negative affine function h on K, of the class LU (increasing limit of a sequence of upper semicontinuous functions), there exists an assignment realizable in a zero-dimensional minimal system such that the resulting entropy function on invariant measures coincides with h. If h is upper semicontinuous, the minimal system can be a subshift [**DS03**].

(3)   The natural assignment of any aperiodic zero-dimensional system can be realized in a minimal zero-dimensional system [**Dow06**].

(4)   Any aperiodic assignment that can be embedded in a zero-dimensional system can also be realized in a zero-dimensional system [**Dow08**].

(5)   If K is a simplex such that $\mathsf{ex}\,K$ is countable, then any aperiodic assignment on K can be realized in a Cantor minimal system [**KO06**].

The result (5) allows us to answer positively all questions of the kind 'can there be a sequence of such and such measures converging to such and such measure'. The result (4) has many applications; in particular, it allows us to settle the above-mentioned question about an arc of Bernoulli measures; it also allows us to construct universal topological systems (even minimal) which contain (up to isomorphism) every possible aperiodic ergodic system (both invertible and non-invertible). The result (4) will be heavily used also in this note.

  Since all the above results concern zero-dimensional systems, the following question seems to be of crucial importance toward understanding the assignments realizable in all topological dynamical systems.

(*)   Is every natural assignment arising from an aperiodic topological dynamical system equivalent to a natural assignment arising from a zero-dimensional dynamical system?

Aperiodicity restriction is added in order to avoid some trivial counterexamples, such as the identity map on a connected space. There is a large class of systems which have the so-called *small boundary property* (SBP), for instance all invertible finite entropy systems which possess an aperiodic minimal factor (see [**Lin99**]). It is fairly obvious that systems with SBP have so-called *isomorphic* zero-dimensional extensions (see e.g. [**BD04**]) and, since isomorphic extensions preserve natural assignments (up to equivalence), for such systems the answer to the question (*) is positive. It is worth mentioning that it remains an open problem whether possessing an isomorphic zero-dimensional extension is equivalent to SBP. But the property asked for in (*) is much weaker than possessing an isomorphic zero-dimensional extension, and the answer to that question beyond systems with SBP is

unknown and seems hopelessly difficult. Any progress in this direction is valuable. In this paper we will provide the following partial answer.

THEOREM 1.1. *Let (X, T) be an aperiodic topological dynamical system such that $\mathcal{M}_T(X)$ is a zero-dimensional Bauer simplex or a zero-dimensional sigma-compact simplex†. Then the natural assignment arising from the system (X, T) can be realized in a zero-dimensional system.*

In fact, our main achievement is the following, slightly more general, theorem (of which Theorem 1.1 is an obvious particular case).

THEOREM 1.2. *Suppose that Φ is an aperiodic assignment on a simplex K such that $\operatorname{ex} K$ is a countable union $\bigcup_{n \geq 1} E_n$, where every set E_n is zero-dimensional and compact. Assume that for each n the restriction of Φ to the Bauer simplex K_n spanned by E_n can be embedded in a topological dynamical system. Then Φ can be realized in a zero-dimensional system.*

The difference between the above two theorems is that in the latter we demand *separate* embeddings for the restrictions $\Phi|_{K_n}$. This does not directly imply the existence of a joined realization for Φ, which is assumed in the former theorem (obviously, the converse implication holds).

2. *Preliminaries*

In this section, we summarize necessary notions; in particular, we discuss array systems and markers in aperiodic systems. For details we refer to [**Boy83**, **DK16**]. We let (X, T) and (Y, S) be topological dynamical systems.

For $\mu \in \mathcal{M}_T(X)$, a point $x \in X$ is said to be *generic* for μ if the sequence of measures $(1/n) \sum_{i=0}^{n-1} \delta_{T^i(x)}$ tends to the measure μ as $n \to +\infty$, in the weak-star topology, where δ_x denotes the Dirac measure at the point x. It is well known that for any $\mu \in \mathcal{M}_T^e(X)$, the set of generic points in X has full μ-measure. We say that $\mathcal{M}_S(Y)$ is a *copy* of $\mathcal{M}_T(X)$ if (Y, S) and (X, T) are topologically conjugate. It is clear that then the natural assignments on $\mathcal{M}_S(Y)$ and $\mathcal{M}_T(X)$ are equivalent via a mapping π induced by the topological conjugacy.

Let $\Lambda_1, \Lambda_2, \ldots$ be finite sets each containing at least two elements (called *alphabets*, endowed with the discrete topology, the cardinalities need not be bounded). By an *array system* (over $\Lambda_1, \Lambda_2, \ldots$) we mean any closed, shift-invariant subset of the Cartesian product $\prod_{k \geq 1} \Lambda_k^{\mathbb{Z}}$ (endowed with the product topology). Each element of the array system can be pictured as an array $x = (x_{k,n})_{k \geq 1, n \in \mathbb{Z}}$ such that each *symbol* $x_{k,n}$ belongs to Λ_k. Any finite array of the form

$$a = a_{[1,k] \times [0,n-1]} = (a_{i,j})_{1 \leq i \leq k, \, 0 \leq j \leq n-1}$$

with each entry $a_{i,j}$ belonging to Λ_i will be called a $(k \times n)$-*rectangle*. The product topology is generated by the collection of all cylinder sets corresponding to centered

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† Recall that we consider properties of $\mathcal{M}_T^e(X)$. We remark that a set is zero-dimensional and sigma-compact if and only if it is a countable union of zero-dimensional compact sets.

$(k \times (2n + 1))$-rectangles a (with k and n ranging over $\mathbb{N}$), defined as follows:

$$[a] = \{x : x_{[1,k] \times [-n,n]} = a_{[1,k] \times [0,2n]}\}.$$

The array system is by default regarded with the action of the horizontal shift σ given by

$$(\sigma(x))_{k,n} = x_{k,n+1}, \quad x = (x_{k,n})_{k \geq 1, n \in \mathbb{Z}}.$$

Speaking about an array, we will refer to the indices $k \geq 1$ and $n \in \mathbb{Z}$ as *vertical* and *horizontal* coordinates (positions), respectively.

Notice that regardless of the cardinalities of the alphabets Λ_k (as long as each of these cardinalities is at least two), the product $\prod_k \Lambda_k$ is homeomorphic to the Cantor set $\mathfrak{C}$. Thus, the system $\prod_k \Lambda_k^{\mathbb{Z}}$ (with the horizontal shift σ) is conjugate to $\mathfrak{C}^{\mathbb{Z}}$ (with the shift). The latter is the *universal zero-dimensional system* in the sense that any zero-dimensional system is topologically conjugate to a subsystem of $(\mathfrak{C}^{\mathbb{Z}}, \sigma)$. We shall call the simplex of invariant measures of the universal system $(\mathfrak{C}^{\mathbb{Z}}, \sigma)$ the *universal simplex*. Recall that it has the affine-topological structure of the *Poulsen simplex*, i.e., its extreme points form a dense subset.

Let us return to the case of a general topological dynamical system (X, T). Let $(\mathcal{P}^{(k)})_{k \in \mathbb{N}}$ be a sequence of finite Borel-measurable partitions of X. For each k, let $\lambda \mapsto P_\lambda$ be a bijection from a finite alphabet Λ_k onto $\mathcal{P}^{(k)}$. By the *array-name* of a point $x \in X$ under the action of T with respect to $(\mathcal{P}^{(k)})_{k \in \mathbb{N}}$ we shall mean the array $(x_{k,n})_{k \geq 1, n \in \mathbb{Z}}$ obtained by the rule:

- the value $x_{k,n}$ is the symbol $\lambda \in \Lambda_k$ if and only if the point $T^n(x)$ is in the subset $P_\lambda \in \mathcal{P}^{(k)}$ of X.

Notice that the closure of all array-names (with respect to $(\mathcal{P}^{(k)})_{k \in \mathbb{N}}$) is an array system. Note that if X is zero-dimensional then there exists a sequence of clopen partitions $(\mathcal{P}^{(k)})_k$ which separate points. It is not hard to see that then (X, T) is topologically conjugate to the array system obtained as the collection of all corresponding array-names (which in this case is already closed).

Let (X, T) be a zero-dimensional system. By an *n-marker* we mean a *clopen* set $F \subset X$ such that:

(1)   no orbit visits F twice in n steps (i.e., $F, T^{-1}F, \ldots, T^{-(n-1)}F$ are disjoint);
(2)   every orbit visits F at least once (by compactness, this implies that for some $N \in \mathbb{N}$, we have $F \cup T^{-1}F \cup \cdots \cup T^{-(N-1)}F = X$).

We have the following key fact.

THEOREM 2.1. (Krieger's marker lemma, aperiodic case) *If (X, T) is an aperiodic zero-dimensional system, then for every $n \in \mathbb{N}$ there exists an n-marker. The parameter N in (2) above can be selected equal to $2n - 1$.*

3. *Special case*

To prove Theorem 1.2, we first deal with the following special case, which will be used in the main proof.

THEOREM 3.1. *Assume that Φ is an aperiodic assignment on a zero-dimensional Bauer simplex K and that Φ can be embedded in a topological dynamical system. Then Φ can be realized in a zero-dimensional system.*

Theorem 3.1 will be proved in two major steps. At first, in Theorem 3.2 below, we give a slightly weaker statement, saying that the assignment Φ can be *embedded* in a zero-dimensional system. We will take care of surjectivity later.

THEOREM 3.2. *Assume that Φ is an aperiodic assignment on a zero-dimensional Bauer simplex K and that Φ can be embedded in a topological dynamical system. Then Φ can be embedded in a zero-dimensional system.*

Proof. Let (X, T) be a topological dynamical system. Assume that K is a zero-dimensional Bauer simplex which is a face in $\mathcal{M}_T(X)$ and that K contains no periodic measures. We need to construct a zero-dimensional system (X'', T'') and a face K'' of $\mathcal{M}_{T''}(X'')$ such that the assignments on K and K'' obtained as the restrictions of the natural assignments arising from (X, T) and (X'', T''), respectively, are equivalent.

We start by inductively constructing a one-parameter family $(\mathcal{P}_t^{(k)})_{k \in \mathbb{N}}$ with $t \in [0, 1]$ of sequences of partitions of X such that

$$\lim_k \mathsf{diam}(\mathcal{P}_t^{(k)}) = 0$$

uniformly in t.

Fix a decreasing to zero sequence $(r_1^{(k)})_{k \in \mathbb{N}}$. For each k, choose a finite open cover $\mathcal{U}_1^{(k)}$ of X which consists of open balls $B^{(k)}(x_1^{(k)}, r_1^{(k)}), \ldots, B^{(k)}(x_{m_k}^{(k)}, r_1^{(k)})$ of radius $r_1^{(k)}$. There exists a positive number $r_0^{(k)} < r_1^{(k)}$ such that the balls $B^{(k)}(x_1^{(k)}, r_0^{(k)}), \ldots,$ $B^{(k)}(x_{m_k}^{(k)}, r_0^{(k)})$ still form a cover $\mathcal{U}_0^{(k)}$ of X. The covers $\mathcal{U}_t^{(k)}$ for $t \in [0, 1]$ are then constituted by the balls $B^{(k)}(x_1^{(k)}, r_t^{(k)}), \ldots, B^{(k)}(x_{m_k}^{(k)}, r_t^{(k)})$ with radii $r_t^{(k)} \in [r_0^{(k)}, r_1^{(k)}]$ depending continuously and increasingly on t.

Next, for each $k \in \mathbb{N}$ and $t \in [0, 1]$, we let $\mathcal{P}_t^{(k)}$ be the finite partition of X consisting of all intersections of the form

$$P_t^\eta = \bigcap_{i=1}^{m_k} (B_i)^{\eta(i)},$$

where $\eta : \{1, \ldots, m_k\} \to \{0, 1\}$, $\mathcal{U}_t^{(k)} = \{B_1, \ldots, B_{m_k}\}$, $(B_i)^0 = B_i$ and $(B_i)^1 = X \setminus B_i$ (some of the sets P_t^η may be empty). Since for $\eta \equiv 1$ the set P_t^η is empty (and we agree that the empty set has diameter zero), all members P_t^η of $\mathcal{P}_t^{(k)}$ have diameters at most $2r_1^{(k)}$. Thus, $\lim_k \mathsf{diam}(\mathcal{P}_t^{(k)}) = 0$, uniformly in t, as required.

For a family $\mathcal{P}$ of subsets of X, we denote by $\partial \mathcal{P}$ the union of all boundaries of the members of $\mathcal{P}$. Notice that $\partial \mathcal{P}_t^{(k)} = \partial \mathcal{U}_t^{(k)}$, which equals the union of boundaries of finitely many balls of radius $r_t^{(k)}$ and with centers not depending on t. Since the boundary of a ball of radius r is contained in the sphere of radius r, the boundaries of balls with a common center and different radii are pairwise disjoint (in a zero-dimensional space, two balls with a common center and of different radii may be equal, but then their boundary is necessarily empty). Thus, at most countably many of the boundaries of balls with a common center

may have positive measure for a fixed probability measure μ. It follows that the set

$$I_\mu^+ = \left\{ t \in [0, 1] : \mu\left(\bigcup_{k \in \mathbb{N}} \partial \mathcal{P}_t^{(k)}\right) > 0 \right\}$$

is at most countable. Thus, for any $\mu \in \mathcal{M}_T^e(X)$, the set

$$I_\mu^0 = \mathfrak{C} \setminus I_\mu^+$$

is a dense subset of the classical Cantor set $\mathfrak{C} \subset [0, 1]$.

We suspend the main proof for a while and prove two auxiliary lemmas. The first one is about the following upper semicontinuity.

LEMMA 3.3. *For every $k \in \mathbb{N}$, the function*

$$\psi^{(k)} : \mathcal{M}_T^e(X) \times [0, 1] \to [0, 1], \quad (\mu, t) \mapsto \mu(\partial \mathcal{P}_t^{(k)})$$

is upper semicontinuous (of two variables).

Proof. Fix $k \in \mathbb{N}$. We have

$$\partial \mathcal{P}_t^{(k)} = \bigcup_{i=1}^{m_k} \partial B^{(k)}(x_i^{(k)}, r_t^{(k)}).$$

Hence,

$$\mathbb{1}_{\partial \mathcal{P}_t^{(k)}} = \max_{1 \le i \le m_k} \mathbb{1}_{\partial B^{(k)}(x_i^{(k)}, r_t^{(k)})}.$$

Given $t \in [0, 1]$ and $d > 0$, we denote by $h_{t,d}^{(k)} : \mathbb{R} \to [0, 1]$ the continuous tent function assuming the value 1 at $r_t^{(k)}$ and 0 outside the interval $[r_t^{(k)} - d, r_t^{(k)} + d]$. Now, for $1 \le i \le m_k$, we define, for all $x \in X$,

$$g_{t,d,i}^{(k)}(x) = h_{t,d}^{(k)}(\text{dist}(x, x_i^{(k)})) \quad \text{and} \quad g_{t,d}^{(k)} = \max_{1 \le i \le m_k} g_{t,d,i}^{(k)}.$$

Clearly, for each t and d, $g_{t,d}^{(k)}$ is a continuous function; moreover, for fixed $d > 0$, the family of functions $\{g_{t,d}^{(k)} : t \in [0, 1]\}$ is equicontinuous, which easily implies the double continuity of

$$(\mu, t) \mapsto \int g_{t,d}^{(k)} \, d\mu$$

on $\mathcal{M}_T^e(X) \times [0, 1]$.

Further, as $d \to 0$, $g_{t,d}^{(k)}$ tends non-increasingly to $\mathbb{1}_{\partial \mathcal{P}_t^{(k)}}$; hence, by the dominated Lebesgue theorem, $\int g_{t,d}^{(k)} \, d\mu$ tends non-increasingly to $\mu(\partial \mathcal{P}_t^{(k)}) = \psi^{(k)}(\mu, t)$. Since a non-increasing limit of continuous functions is upper semicontinuous, we have completed the proof of the lemma. $\qquad\square$

Our next goal is to prove the following 'continuous selector lemma'.

LEMMA 3.4. *Let K be as in the formulation of Theorem 3.2. Then there exists a continuous function $s : \mathsf{ex}K \to \mathfrak{C}$ satisfying $s(\mu) \in I_\mu^0$ for every $\mu \in \mathsf{ex}K$.*

Proof. We can assume that $\mathsf{diam}(\mathsf{ex}K) = 1$. For every natural number n, we will inductively define a finite clopen partition $\{K_1^{(n)}, \ldots, K_{l_n}^{(n)}\}$ of $\mathsf{ex}K$ and a finite family $\{W_1^{(n)}, \ldots, W_{l_n}^{(n)}\}$ of clopen subsets of $\mathfrak{C}$ having, for each $n \in \mathbb{N}$, the following properties:

(1) $\max\{\mathsf{diam}(K_i^{(n)}), \mathsf{diam}(W_i^{(n)})\} \le 2^{1-n}$ for every $1 \le i \le l_n$;

(2) if $n > 1$, then for each $1 \le i \le l_n$ there is some $1 \le j \le l_{n-1}$ such that $K_i^{(n)} \times W_i^{(n)}$ is a subset of $K_j^{(n-1)} \times W_j^{(n-1)}$;

(3) $\psi^{(k)}(K_i^{(n)} \times W_i^{(n)}) \subset [0, 2^{1-n}]$ for every $0 \le k \le n$ and $1 \le i \le l_n$.

To begin with, we set $l_1 = 1$, $K_1^{(1)} = \mathsf{ex}K$ and $W_1^{(1)} = \mathfrak{C}$. Clearly, the conditions (1) and (3) are fulfilled. Next, we fix an $n \ge 2$ and suppose that $\{K_1^{(n-1)}, \ldots, K_{l_{n-1}}^{(n-1)}\}$ and $\{W_1^{(n-1)}, \ldots, W_{l_{n-1}}^{(n-1)}\}$ have been defined. For each $\mu \in \mathsf{ex}K$, there is a unique $1 \le j_\mu \le l_{n-1}$ with $\mu \in K_{j_\mu}^{(n-1)}$.

By Lemma 3.3 (noting that $\mathsf{ex}K \subset \mathcal{M}_T^e(X)$) and since I_μ^0 is dense in $\mathfrak{C}$, there exist clopen subsets $K_\mu^{(n)} \subset K_{j_\mu}^{(n-1)}$ and $W_\mu^{(n)} \subset W_{j_\mu}^{(n-1)}$ with $\mu \in K_\mu^{(n)}$ and

$$\max\{\mathsf{diam}(K_\mu^{(n)}), \mathsf{diam}(W_\mu^{(n)})\} < 2^{1-n}$$

satisfying

$$\psi^{(k)}(K_\mu^{(n)} \times W_\mu^{(n)}) \subset [0, 2^{1-n}]$$

for all $0 \le k \le n$.

By compactness, $\mathsf{ex}K$ can be covered by finitely many sets $K_{\mu_{n,1}}^{(n)}, \ldots, K_{\mu_{n,l_n}}^{(n)}$. Since these sets are clopen, by subsequent subtracting we can make them disjoint (and still covering $\mathsf{ex}K$) and we denote them as $K_1^{(n)}, \ldots, K_{l_n}^{(n)}$. Correspondingly, we also enumerate $W_{\mu_{n,1}}^{(n)}, \ldots, W_{\mu_{n,l_n}}^{(n)}$ as $W_1^{(n)}, \ldots, W_{l_n}^{(n)}$. Clearly, the properties (1)–(3) are now satisfied for n.

Once the induction is completed, we continue as follows. Given $n \in \mathbb{N}$, let $s_n : \mathsf{ex}K \to \mathfrak{C}$ be a simple function assuming on each set $K_i^{(n)}$ a constant value $t_i^{(n)} \in W_i^{(n)}$ ($1 \le i \le l_n$). Since the sets $K_i^{(n)}$ are clopen, s_n is continuous. Finally, let $s : \mathsf{ex}K \to \mathfrak{C}$ be the limit function of $(s_n)_{n \in \mathbb{N}}$. By (1) and (2), s is a uniform limit of a sequence of continuous functions and hence s is continuous as well. Also, the conditions (2) and (3) imply that $\psi^{(k)}(\mu, s(\mu)) \le 2^{-n}$ for each $\mu \in \mathsf{ex}K$, $n \in \mathbb{N}$ and every $k \le n$. This yields $\psi^{(k)}(\mu, s(\mu)) = 0$ for all $\mu \in \mathsf{ex}K$ and every $k \in \mathbb{N}$. Thus, for any $\mu \in \mathsf{ex}K$, we have $s(\mu) \in I_\mu^0$, which ends the proof of the lemma. $\square$

We return to the proof of Theorem 3.2. Recall that we aim to construct a zero-dimensional system (X'', T'') and a continuous affine injection $\pi'' : K \to \mathcal{M}_{T''}(X'')$ such that for any $\mu \in K$ and $\mu'' = \pi''(\mu)$, the systems (X, μ, T) and (X'', μ'', T'') are measure-theoretically isomorphic. The mapping π'' is not going to be surjective and its image is going to be a face K'' of $\mathcal{M}_{T''}(X'')$.

For a fixed $k \in \mathbb{N}$ and any $\eta \in \{0, 1\}^{\{1, \ldots, m_k\}}$, we let

$$P^\eta = \bigcup_{t \in \mathfrak{C}} (P_t^\eta \times \{t\}).$$

Now, the collection

$$\mathcal{P}^{(k)} = \{P^\eta : \eta \in \{0, 1\}^{\{1, \ldots, m_k\}}\}$$

is a finite measurable partition of $X \times \mathfrak{C}$ labeled by the elements η. Let $N(x, t)$ be the array-name of the point $(x, t) \in X \times \mathfrak{C}$ under the action $T \times \mathsf{id}$ with respect to the sequence of partitions $(\mathcal{P}^{(k)})_{k \in \mathbb{N}}$ of $X \times \mathfrak{C}$, which uses the labels $\eta \in \{0, 1\}^{\{1, \ldots, m_k\}}$ as the alphabet in the kth row of an array (notice that if P_t^η is empty, the symbol η will not appear in $N(x, t)$ for any $x \in X$). Set

$$X'' = \overline{\{(N(x, t), t) : (x, t) \in X \times \mathfrak{C}\}}.$$

Let $T'' : X'' \to X''$ be given by $T'' = \sigma \times \mathsf{id}$, where σ denotes the shift on arrays. By a standard argument, the dynamical system (X'', T'') is a zero-dimensional extension of $X \times \mathfrak{C}$ via a factor mapping $\pi_0 : X'' \to X \times \mathfrak{C}$ which is one-to-one (has singleton fibers) except for points (x, t) whose orbits visit the boundary of some member of $\mathcal{P}^{(k)}$. The mapping π_0 yields a continuous mapping π_0^* from $\mathcal{M}_{T''}(X'')$ onto $\mathcal{M}_{T \times \mathsf{id}}(X \times \mathfrak{C})$. So, by a simple argument (see [**Dow05**, Lemma 4.1]), the inverse mapping defined on the set of measures which have a unique preimage is a homeomorphism in the relative topologies.

We note here that the section at a level t of $\partial \mathcal{P}^{(k)}$ is contained in $\partial \mathcal{U}_t^{(k)} = \partial \mathcal{P}_t^{(k)}$. In fact, if $x \in X \setminus \partial \mathcal{P}_t^{(k)}$, then x belongs to some P_t^η together with some δ-ball around x, which implies that for all t' sufficiently close to t, the $(\delta/2)$-ball around x is contained in $P_{t'}^\eta$. This implies that (x, t) is not in ∂P^η, i.e., x is not in the t-section of this boundary.

Let s be the continuous selector function defined in the statement of Lemma 3.4. For $\mu \in \mathsf{ex} K$, we have $\mu \times \delta_{s(\mu)} \in \mathcal{M}_{T \times \mathsf{id}}^e(X \times \mathfrak{C})$, where δ_t denotes the Dirac measure at t. Note that

$$\mu \times \delta_{s(\mu)} \left(\bigcup_{k \in \mathbb{N}} \partial \mathcal{P}^{(k)} \right) \le \mu \left(\bigcup_{k \in \mathbb{N}} \partial \mathcal{P}_{s(\mu)}^{(k)} \right) = 0.$$

Therefore, the mapping π_0 is one-to-one on a set of full measure $\mu \times \delta_{s(\mu)}$, which implies that $\mu \times \delta_{s(\mu)}$ has a unique preimage by π_0^* (which we denote by μ'') and, moreover, the system (X'', μ'', T'') is measure-theoretically isomorphic to $(X \times \mathfrak{C}, \mu \times \delta_{s(\mu)}, T \times \mathsf{id})$ (and hence, trivially, via the projection onto the first coordinate, to (X, μ, T)). Clearly, the mapping $(\mu, t) \mapsto \mu \times \delta_t$ is continuous from $K \times \mathfrak{C}$ to $\mathcal{M}_{T \times \mathsf{id}}(X \times \mathfrak{C})$. Further, the graph of s is a compact subset of the domain of this mapping and maps bijectively onto the set $\{\mu \times \delta_{s(\mu)} : \mu \in \mathsf{ex} K\}$; hence, the latter set is also compact. Since, as we have observed earlier, the points in the latter set have singleton preimages by π_0, the inverse of π_0^* is a homeomorphism between the set $\{\mu \times \delta_{s(\mu)} : \mu \in \mathsf{ex} K\}$ and its preimage by π_0^* (i.e., the set $\{\mu'' : \mu \in \mathsf{ex} K\}$), which is hence compact as well. The composition

$$\mu'' \mapsto \mu \times \delta_{s(\mu)} \mapsto (\mu, s(\mu)) \mapsto \mu$$

serves as a homeomorphism between $\{\mu'' : \mu \in \mathsf{ex} K\}$ and $\mathsf{ex} K$; moreover, the measures corresponding to each other by this mapping are isomorphic.

Using the ergodic decomposition, the above mapping can be prolonged† to an affine and continuous mapping from the compact convex hull spanned by the set $\{\mu'' : \mu \in \mathsf{ex} K\}$ onto K. Since the measures μ'' are ergodic (being isomorphic to the corresponding ergodic measures μ), this compact convex hull is a face of $\mathcal{M}_{T''}(X'')$. Since both simplices are Bauer, the prolongation is injective (see e.g. [**Dow11**, Appendix A.2.5]) and hence a

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† We use the term 'prolongation' (of a function) for what is customarily described as 'extension', because the word 'extension' has in this note a defined meaning (opposite to 'factor').

homeomorphism (which is obviously affine) and maintains the property that the measures corresponding to each other by this mapping are isomorphic. Now, we can define π'' as the inverse of the above prolonged mapping. $\qquad \square$

Proof of Theorem 3.1. Theorem 3.1 now becomes a direct consequence of Theorem 3.2 and [**Dow08**, Theorem 4.1] applied to the system (X'', T'') and the compact convex hull spanned by the set $\{\mu'' : \mu \in K\}$, which is a face K'' in $\mathcal{M}_{T''}(X'')$. Each measure μ'' is isomorphic to some invariant measure $\mu \in K$, where K is assumed to not contain periodic measures. So, μ'' is aperiodic. Thus, the face K'' contains no periodic measures, as required in the cited theorem. $\qquad \square$

4. *Proof of the main result*
The proof of Theorem 1.2 relies on the following three lemmas.

LEMMA 4.1. *The simplex of invariant measures $\mathcal{M}_T(X)$ of any aperiodic zero-dimensional system (X, T) 'appears densely' in the universal simplex, namely, inside any open subset of the universal simplex, we can find a face $\widetilde{K}$ such that the natural assignment of the universal system restricted to $\widetilde{K}$ is a copy of $\mathcal{M}_T(X)$ (we will briefly say that $\widetilde{K}$ is a copy of $\mathcal{M}_T(X)$).*

Proof. Take any aperiodic zero-dimensional system (X, σ). Since (X, σ) is conjugate to a subsystem of the universal system, we can assume that X is an invariant closed subset of $\mathfrak{C}^{\mathbb{Z}}$ and σ denotes the shift. Let (k_i, n_i) be a sequence of integer-valued vectors such that both coordinates increase to infinity as i grows and let $\mathcal{R}_i$ denote the collection of all $(k_i \times n_i)$-rectangles. Note that all open sets in the universal simplex of the form

$$U(\mu, i_0, \epsilon) = \{\nu : |\mu(R) - \nu(R)| < \epsilon \text{ for all } R \in \mathcal{R}_i \text{ and } i \leq i_0\}$$

with μ ranging over ergodic measures, $i_0 \in \mathbb{N}$ and $\epsilon > 0$ form a base of the weak-star topology. Fix a set $U(\mu, i_0, \epsilon)$ in the universal simplex. There exist a point $x_0 \in \mathfrak{C}^{\mathbb{Z}}$ generic for μ and a constant $N_0 > \max\{2n_{i_0}/\epsilon, k_{i_0}\}$ such that for each $N \geq N_0$, the frequency of occurrences of any $R \in \mathcal{R}_i$ with $i \leq i_0$ in the rectangle $[1, N_0] \times [0, N]$ of x_0 equals $\mu(R)$ up to an error of $\epsilon/2$. In X there exists a clopen N_0-marker F visited by the orbit of each $x \in X$ with gaps ranging between N_0 and $2N_0 - 1$. For each $x \in X$, we define a new array $\widetilde{x}$ with rows enumerated from $-N_0 + 1$ to $+\infty$ as follows: for $i > 0$ and any $n \in \mathbb{Z}$, $\widetilde{x}(i, n) = x(i, n)$. The contents of the rows with indices in $[-N_0 + 1, 0]$ are described below: let n and $n + N$ be two consecutive times of visits of the orbit of x in F. We define $\widetilde{x}$ on the rectangle $[-N_0 + 1, 0] \times [n, n + N - 1]$ by

$$\widetilde{x}(i - N_0, n + k) = x_0(i, k), \quad i \in [1, N_0], k \in [0, N - 1].$$

Let $\phi(x)$ denote the array $\widetilde{x}$ with the enumeration of rows shifted so that the rows of $\phi(x)$ are indexed from 1 to $+\infty$. From the construction of ϕ, we see that $\phi : X \to \phi(X) \subset \mathfrak{C}^{\mathbb{Z}}$ is continuous and one-to-one and thus $\phi(X)$ is topologically conjugate to X, so $\mathcal{M}_\sigma(\phi(X))$ is a copy of $\mathcal{M}_\sigma(X)$. By the choice of N_0, it is not hard to estimate that for any $x \in X$, any rectangle $R \in \mathcal{R}_i$ with $i \leq i_0$ occurs in $\phi(x)$ with frequency equal to $\mu(R)$ up to an error of ϵ. This proves that $\mathcal{M}_\sigma(\phi(X))$ is a subset of $U(\mu, n_0, \epsilon)$. $\qquad \square$

LEMMA 4.2. *Suppose that E is a σ-compact zero-dimensional set represented as a countable union $\bigcup_{n \geq 1} E_n$, where every set E_n is compact. Then $E = \bigcup_{n \geq 1} \widetilde{E}_n$, where the sets $\widetilde{E}_n$ are compact, pairwise disjoint and each $\widetilde{E}_n$ is contained in some E_m.*

Proof. We let $\widetilde{E}_{0,0} = E_1$. Note that $E_2 \setminus E_1$ is relatively open in E_2 and thus can be represented as a countable disjoint union $E_2 \setminus E_1 = \bigcup_{i \geq 1} \widetilde{E}_{2,i}$, where every set $\widetilde{E}_{2,i}$ is clopen in E_2 and thus is closed in E. We proceed analogously countably many times, namely, for each $n \geq 2$, the difference $E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$ is relatively open in E_n and thus can be represented as a countable disjoint union $\bigcup_{i \geq 1} \widetilde{E}_{n,i}$, where every $\widetilde{E}_{n,i}$ is clopen in E_n and thus is closed in E. It now suffices to rearrange the double sequence $(\widetilde{E}_{n,i})_{n,i}$ into a single sequence $(\widetilde{E}_n)_n$. $\qquad\square$

LEMMA 4.3. *Let K be a simplex and let F be a face of K which is ϵ-dense in K (i.e., for each $x \in K$, there is some $y \in F$ with $\mathsf{dist}(x, y) < \epsilon$). Then there exists an affine continuous retraction (i.e., a map which is the identity on its range), $\theta : K \to F$, such that $\mathsf{dist}(\theta(x), x) \leq \epsilon$ for all $x \in K$.*

Proof. For each $x \in K \setminus F$, let $\Theta(x) = \{y \in F : \mathsf{dist}(x, y) < \epsilon\}$ and $\bar{\Theta}(x) = \overline{\Theta(x)}$. For $x \in F$, define $\Theta(x) = \bar{\Theta}(x) = \{x\}$. By ϵ-density and convexity of F, and convexity of the metric, the multifunction Θ has non-empty convex images and $\bar{\Theta}$ has non-empty compact and convex images. Moreover, $\bar{\Theta}$ is lower hemicontinuous, i.e., for every relatively open set $U \subset F$, the 'preimage' $\{x \in K : \bar{\Theta}(x) \cap U \neq \emptyset\}$ is open. Indeed, we have

$$\{x \in K : \bar{\Theta}(x) \cap U \neq \emptyset\} = \{x : \Theta(x) \cap U \neq \emptyset\} = \{x : \mathsf{dist}(x, U) < \epsilon\},$$

which is open (regardless of the topological properties of U, which in this case is neither open nor closed in K). Also note that convexity of the metric and the fact that F is a face (hence no point in F is a convex combination involving points from outside F) imply that Θ is *convex*, i.e., satisfies, for any $x, y \in K$ and $\alpha \in (0, 1)$, the inclusion

$$\alpha\Theta(x) + (1 - \alpha)\Theta(y) \subset \Theta(\alpha x + (1 - \alpha)y).$$

Clearly, $\bar{\Theta}$ is convex as well.

At this point we can apply the well-known Lazar–Michael selection theorem (see [**La60**, **FLP01**]), which asserts that $\bar{\Theta}$ admits an affine and continuous selector (i.e., a function $\theta : K \to F$ such that $\theta(x) \in \bar{\Theta}(x)$ for all $x \in K$). Clearly, θ satisfies the assertion of the lemma. $\qquad\square$

Proof of Theorem 1.2. As in the proof of Theorem 3.1, most of the effort will be devoted to *embedding* Φ on K in a zero-dimensional system (which is the same as embedding it in the universal simplex). Surjectivity will be taken care of in the last paragraph of the proof.

Using the fact that the universal simplex is Poulsen, and that the Poulsen simplex is universal in the sense that every simplex is affinely homeomorphic to a face of the Poulsen simplex (see [**LOS78**]), we can assume that K is a face of the universal simplex. From now on, for any face F of the universal simplex, we will abbreviate 'the restriction to F of the natural assignment coming from the universal simplex' shortly as the 'restricted natural assignment on F'. So, on K we have two assignments: Φ and the restricted natural assignment (which, at this stage, is beyond our control).

Recall that $\mathsf{ex}K = \bigcup_{n \geq 1} E_n$, where each E_n is zero-dimensional and compact and the restriction of Φ to the simplex K_n spanned by E_n is embeddable in a dynamical system. Lemma 4.2 provides a partition of $\mathsf{ex}K$ into disjoint compact sets $\widetilde{E}_n$, each contained in some E_m. Then Φ restricted to each $\widetilde{E}_n$ remains embeddable in a dynamical system. In other words, we can assume from the start that the sets E_n are disjoint. We choose a summable sequence $(\epsilon_k)_{k \geq 1}$ of positive numbers and continue by induction, as follows.

Step 1. For a sufficiently large n_1, the (disjoint) union $E_1 \cup E_2 \cup \cdots \cup E_{n_1}$ is ϵ_1-dense in $\mathsf{ex}K$. Then the simplex L_1 spanned by this union is ϵ_1-dense in K. Each of the sets E_n $(1 \leq n \leq n_1)$ can be partitioned into a finite union of disjoint clopen sets of diameters smaller than ϵ_1. We denote the finitely many sets obtained in this manner from all the sets $E_1, \ldots, E_{n_1}$ by $E_{1,1}, \ldots, E_{1,m_1}$ and we let $K_{1,1}, \ldots, K_{1,m_1}$ be the Bauer simplices spanned by $E_{1,1}, \ldots, E_{1,m_1}$, respectively. In a simplex, faces with disjoint sets of extreme points are disjoint. This implies that the faces $K_{1,i}$ are disjoint. All of these simplices have diameters smaller than ϵ_1 and their union spans L_1. The restriction of Φ to each $K_{1,i}$ $(i = 1, \ldots, m_1)$ is embeddable in a dynamical systemand, by Theorem 3.1, it is realizable in a zero-dimensional system.

Now, we apply the affine continuous retraction of Lemma 4.3, which we denote by $\mathsf{Ret}_1 : K \to L_1$, and which moves points by less than ϵ_1.

Next, using Lemma 4.1, for each $i = 1, \ldots, m_1$, in the ϵ_1-neighborhood of the set $K_{1,i}$ (within the universal simplex) we find a face $K'_{1,i}$ affinely homeomorphic to $K_{1,i}$ and such that the restricted natural assignment on $K'_{1,i}$ is equivalent to $\Phi|_{K_{1,i}}$. Moreover, since every proper face of any simplex is nowhere dense in that simplex†, we can easily arrange that the faces $K'_{1,i}$ are pairwise disjoint and disjoint from K. For each i, we choose an affine homeomorphism from $K_{1,i}$ onto $K'_{1,i}$ which establishes the equivalence between $\Phi|_{K_{1,i}}$ and the restricted natural assignment on $K'_{1,i}$. The union of these maps is then prolonged harmonically to an affine homeomorphism π_1 from L_1 onto the simplex L'_1 spanned by the union $K'_{1,1} \cup \cdots \cup K'_{1,m_1}$. Notice that π_1 does not move points more than $2\epsilon_1$ (this is obvious for points in each $K_{1,i}$; then the property passes to the harmonic prolongation by convexity of the metric). The composition $\phi_1 = \pi_1 \circ \mathsf{Ret}_1$ maps K onto L'_1 moving points by less than $3\epsilon_1$ and on L_1 it coincides with π_1 and establishes an equivalence between $\Phi|_{L_1}$ and the restricted natural assignment on L'_1.

Step $k + 1$. Suppose that for some $k \geq 1$ we have constructed an affine continuous map $\phi_k : K \to L'_k$ onto some face of the universal simplex disjoint from K, so that on the face L_k of K spanned by the union $E_1 \cup \cdots \cup E_{n_k}$ it coincides with an affine homeomorphism $\pi_k : L_k \to L'_k$ which establishes an equivalence between $\Phi|_{L_k}$ and the restricted natural assignment on L'_k. We start by applying the convex combination

$$\phi'_k = (1 - \alpha_k)\phi_k + \alpha_k \mathsf{id},$$

where α_k is positive, but small enough so that at each point of K, ϕ'_k differs from ϕ_k (in the distance) by less than ϵ_{k+1}. Contrary to the non-injective map ϕ_k, ϕ'_k is easily seen (using disjointness of K and L'_k) to be an affine homeomorphism between K and its image (however, the good assignment on the image of L_k is now lost; moreover, the image,

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† Indeed, let $x \in F$, where F is a proper face in a simplex K and let $e \in \mathsf{ex}K \setminus \mathsf{ex}F$. Then the sequence $(1/n)e + (n - 1/n)x$ tends to x from outside F.

although remaining a simplex, is no longer a face of the universal simplex). Let L_{k+1} be a face of K spanned by a disjoint union of L_k and sufficiently many sets $E_{n_k+1}, \ldots, E_{n_{k+1}}$, so that $\phi'_k(L_{k+1})$ is ϵ_{k+1}-dense in $\phi'_k(K)$. As before, we partition the sets E_n ($n_k < n \leq n_{k+1}$) into smaller clopen sets denoted by $E_{k+1,1}, \ldots, E_{k+1,m_{k+1}}$ such that, for each $i = 1, \ldots, m_{k+1}$, $\phi'_k(E_{k+1,i})$ has diameter smaller than ϵ_{k+1}. Clearly, Φ restricted to each $E_{k+1,i}$ is embeddable in a topological system and, thus, by Theorem 3.1, realizable in a zero-dimensional system.

We can now apply the retraction

$$\mathsf{Ret}_{k+1} : \phi'_k(K) \to \phi'_k(L_{k+1}),$$

which moves points by less than ϵ_{k+1} and leaves the points of $\phi'_k(L_{k+1})$ invariant.

In the ϵ_{k+1}-neighborhood of each set $\phi'_k(K_{k+1,i})$ we find a face (of the universal simplex) $K'_{k+1,i}$ affinely homeomorphic to $K_{k+1,i}$, on which the restricted natural assignment is equivalent to $\Phi|_{K_{k+1,i}}$. Moreover, we can arrange that these faces are disjoint from each other, from L'_k and from K. We let L'_{k+1} be the simplex spanned by the union $L'_k \cup K'_{k+1,1} \cup \cdots \cup K'_{k+1,m_{k+1}}$. This simplex is affinely homeomorphic to L_{k+1} (and thus also to $\phi'_k(L_{k+1})$) and we can select an affine homeomorphism $\pi_{k+1} : \phi'_k(L_{k+1}) \to L'_{k+1}$ so that:

- on $\phi'_k(L_k)$, it acts by the formula $\pi_{k+1}(\phi'_k(x)) = \phi_k(x)$. Then it sends $\phi'_k(L_k)$ onto L'_k 'forgetting' the effect of the convex combination with the identity and it establishes an equivalence between $(\phi'_k \circ \Phi)|_{\phi'_k(L_k)}$ (the transported by ϕ'_k assignment $\Phi|_{L_k}$) and the restricted natural assignment on L'_k;
- on each set $\phi'_k(K_{k+1,i})$, it establishes an equivalence between $(\phi'_k \circ \Phi)|_{\phi'_k(K_{k+1,i})}$ (the transported by ϕ'_k assignment $\Phi|_{K_{k+1,i}}$) and the restricted natural assignment on $K'_{k+1,i}$; notice that on each set $\phi'_k(K_{k+1,i})$, π_{k+1} moves points by less than $2\epsilon_{k+1}$.

Then we define ϕ_{k+1} as the composition $\pi_{k+1} \circ \mathsf{Ret}_{k+1} \circ \phi'_k$ (see Figure 1). This map ϕ_{k+1} has the following properties:

(1) it is an affine continuous map from K onto L'_{k+1} which on L_k coincides with ϕ_k;

(2) on L_{k+1}, it establishes an equivalence between $\Phi|_{L_{k+1}}$ and the restricted natural assignment on L'_{k+1};

(3) it differs from ϕ'_k by less than $3\epsilon_{k+1}$ and thus from ϕ_k by less than $4\epsilon_{k+1}$.

When the induction is complete, we define ϕ as the pointwise limit of the maps ϕ_k. By (3) and summability of the sequence $(\epsilon_k)_{k \geq 1}$, the limit exists and is uniform, so ϕ is continuous (and it is clearly affine). On each set L_k, the limit is achieved in step k (further mappings coincide with ϕ_k), hence $\phi(L_k) = L'_k$, and ϕ establishes an equivalence between $\Phi|_{L_k}$ and the restricted natural assignment on L'_k. In particular, ϕ is injective on L_k. Thus, ϕ is injective on $\mathsf{ex}K = \bigcup_{k \geq 1} \mathsf{ex}L_k$ and it sends extreme points of K to points extreme in the universal simplex and thus extreme in the image K'. This implies that ϕ is injective on the entire simplex K and hence is an affine homeomorphism between K and K'. Also, it establishes an equivalence between $\Phi|_{\mathsf{ex}K}$ and the restricted natural assignment on $\mathsf{ex}K'$, which, by harmonicity of both assignments, implies that ϕ establishes an equivalence between Φ and the restricted natural assignment on K'. This ends the proof of embeddability of Φ in the universal simplex, i.e., in a zero-dimensional system.
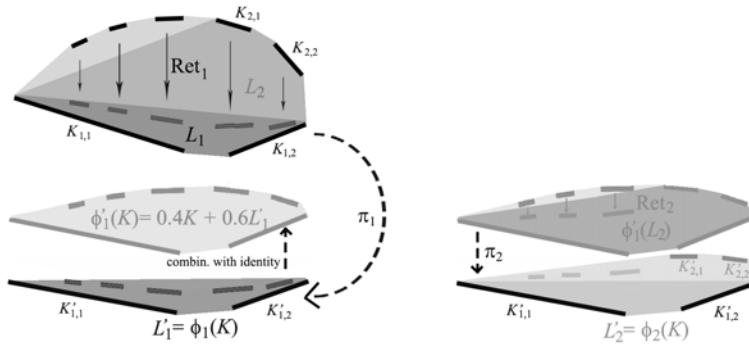
FIGURE 1. Construction of the map ϕ_2.

To complete the proof, we apply once again [**Dow08**, Theorem 4.1], this time to the universal simplex and its face K'. Each ergodic measure $\mu' \in \mathsf{ex}K'$ is isomorphic to some ergodic measure in the assignment Φ and hence is aperiodic. Thus, according to the above-cited theorem, the restricted natural assignment on K' can be realized in a zero-dimensional system. Since we have just constructed an equivalence between Φ and the restricted natural assignment on K', the assignment Φ can be realized in a zero-dimensional system as well.

5. *Final remarks*

(1) Both main Theorems 3.1 and 1.2 hold for non-invertible topological dynamical systems (X, T) (where $T : X \to X$ is a continuous map, not necessarily a homeomorphism). One has to extend the notion of an assignment admitting as values measure-preserving endomorphisms (rather than automorphisms). The proof of Theorem 3.1 does not depend on invertibility (also [**Dow08**, Theorem 4.1], used in the last stage, holds for endomorphisms). Krieger's marker lemma (Theorem 2.1) also remains valid for non-invertible systems (see e.g. [**Dow06**, Lemma 1]). The only delicate place is in the proof of Lemma 4.1, where we define the additional rows of $\widetilde{x}$ by placing there certain rectangles *to the right* of the N_0-markers. In the non-invertible case, all arrays have only non-negative column numbers and the first marker usually occurs at some positive position. Then we have no indication as to what should be placed in the additional rows *to the left* of the first marker. To cope with this problem, we first fill the new rows only to the right of the first N_0-marker and then we shift the contents of these rows to the left by $2N_0 - 1$ units. In this manner the unfilled left section will certainly disappear.

(2) It seems that the proof of Theorem 1.2 could be used to show the following.

(A) Suppose that a simplex K equals the closed convex hull of a countable family of its faces: $K = \overline{\mathsf{conv}}\big(\bigcup_{n \geq 1} K_n\big)$, where the faces K_n are disjoint and their diameters tend to zero. Let Φ be an aperiodic assignment on K such that $\Phi|_{K_n}$ is embeddable in a zero-dimensional system. Then Φ can be realized in a zero-dimensional system.

Alas, there is one place where the proof does not pass: in step $k + 1$ we need to be able to partition the sets $E_n = \mathsf{ex}K_n$ with $n_k < n \leq n_{k+1}$ into finitely many separated pieces,

each spanning a face of K, and whose images by ϕ'_k have small diameters. This may be impossible if the sets E_n are not compact zero-dimensional. We leave (A) as a conjecture.

(3) Initially the inductive part of the proof of Theorem 1.2 was supposed to be based on the proof of [**Dow08**, Corollary 5.2]. While working on the details we have discovered a serious gap in that proof. This paper fixes the gap: a correct proof of [**Dow08**, Corollary 5.2] is obtained by applying Theorem 1.2 in the case where all sets E_n (and thus K_n) are singletons.

(4) Anticipating obvious inquiries, we confess that we do not have any example of an aperiodic system with a zero-dimensional sigma-compact (or Bauer) simplex of invariant measures, for which the realizability in a zero-dimensional system would not follow by more direct reasons (for example from the SBP). So, one might criticize our results for lack of evident applicability. This is true. However, we treat this work as an opportunity to develop and practice new tools and methods in handling the difficult problem of realizing assignments in zero-dimensional systems. As mentioned in the introduction, the ultimate target is to prove (or disprove) such realizability for the natural assignments in all aperiodic topological dynamical systems.

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