1
Introduction

We introduce here quarks and gluons. The analogy with electrodynamics at short distances disappears at larger distances with the emergence of the string tension, the force that confines the quarks and gluons permanently into bound states called hadrons.

Subsequently we introduce the simplest relativistic field theory, the classical scalar field.

1.1 QED, QCD, and confinement

Quantum electrodynamics (QED) is the quantum theory of photons (γ) and charged particles such as electrons ($e^{\pm}$), muons ($\mu^{\pm}$), protons (p), pions ($\pi^{\pm}$), etc. Typical phenomena that can be described by perturbation theory are Compton scattering ($\gamma + e^- \to \gamma + e^-$), and pair annihilation/production such as $e^+ + e^- \to \mu^+ + \mu^-$. Examples of non-perturbative phenomena are the formation of atoms and molecules. The expansion parameter of perturbation theory is the fine-structure constant[1] $\alpha = e^2/4\pi$.

Quantum chromodynamics (QCD) is the quantum theory of quarks (q) and gluons (g). The quarks u, d, c, s, t and b ('up', 'down', 'charm', 'strange', 'top' and 'bottom') are analogous to the charged leptons ν_e, e, ν_μ, μ, ν_τ, and τ. In addition to electric charge they also carry 'color charges', which are the sources of the gluon fields. The gluons are analogous to photons, except that they are self-interacting because they also carry color charges. The strength of these interactions is measured by $\alpha_{\mathrm{s}} = g^2/4\pi$ (alpha strong), with g analogous to the electromagnetic charge e. The 'atoms' of QCD are $q\bar{q}$ ($\bar{q}$ denotes the antiparticle of q)

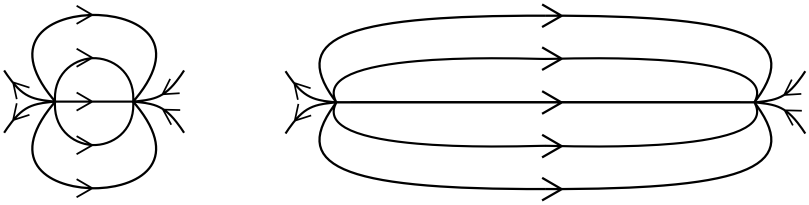

Fig. 1.1. Intuitive representation of chromoelectric field lines between a static quark–antiquark source pair in QCD: (a) Coulomb-like at short distances; (b) string-like at large distances, at which the energy content per unit length becomes constant.

bound states called mesons† (π, K, η, η', ρ, K^*, ω, ϕ, $\ldots$) and $3q$ bound states called baryons (the nucleon N, and furthermore Σ, Λ, Ξ, Δ, Σ^*, $\Lambda^*, \ldots$). The mesons are bosons and the baryons are fermions. There may be also multi-quark states analogous to molecules. Furthermore, there are expected to be glueballs consisting mainly of gluons. These bound states are called 'hadrons' and their properties as determined by experiment are recorded in the tables of the Particle Data Group [2].

The way that the gluons interact among themselves has dramatic effects. At distances of the order of the hadron size, the interactions are strong and $\alpha_{\rm s}$ effectively becomes arbitrarily large as the distance scale increases. Because of the increasing potential energy between quarks at large distances, it is not possible to have single quarks in the theory: they are permanently confined in bound states.

For a precise characterization of confinement one considers the theory with gluons only (no dynamical quarks) in which static external sources are inserted with quark quantum numbers, a distance r apart. The energy of this configuration is the quark–antiquark potential $V(r)$. In QCD confinement is realized such that $V(r)$ increases linearly with r as $r \to \infty$,

$$V(r) \approx \sigma r, \quad r \to \infty. \tag{1.1}$$

The coefficient σ is called the string tension, because there are effective string models for $V(r)$. Such models are very useful for grasping some of the physics involved (figure 1.1).

Because of confinement, quarks and gluons cannot exist as free parti-

† The quark content of these particles is given in table 7.1 in section 7.5.

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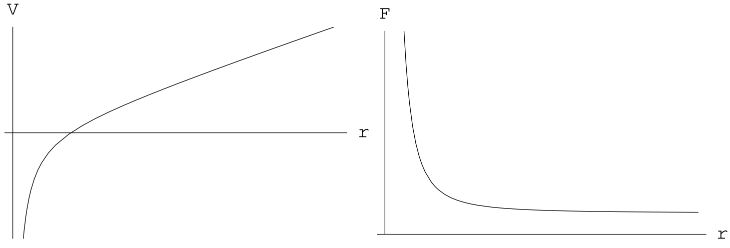

Fig. 1.2. Shape of the static $q\bar{q}$ potential and the force $F = \partial V/\partial r$.

cles. No such free particles have been found. However, scattering experiments at high momentum transfers (corresponding to short distances) have led to the conclusion that there are quarks and gluons inside the hadrons. The effective interaction strength α_s is *small* at short distances. Because of this, perturbation theory is applicable at short distances or large momentum transfers. This can also be seen from the force derived from the $q\bar{q}$ potential, $F = \partial V/\partial r$. See figure 1.2. Writing conventionally

$$F(r) = \frac{4}{3}\frac{\alpha_s(r)}{r^2}, \tag{1.2}$$

we know that $\alpha_s \to 0$ very slowly as the distance decreases,

$$\alpha_s(r) \approx \frac{4\pi}{11\ln(1/\Lambda^2 r^2)}. \tag{1.3}$$

This is called *asymptotic freedom*. The parameter Λ has the dimension of a mass and may be taken to set the dimension scale in quark-less 'QCD'. For the glueball mass m or string tension σ we can then write

$$m = C_m\Lambda, \qquad \sqrt{\sigma} = C_\sigma\Lambda. \tag{1.4}$$

Constants like C_m and C_σ, which relate short-distance to long-distance properties, are non-perturbative quantities. They are pure numbers whose computation is a challenge to be met by the theory developed in the following chapters.

The value of the string tension σ is known to be approximately $(400\ \text{MeV})^2$. This information comes from a remarkable property of the hadronic mass spectrum, the fact that, for the leading spin states, the spin J is approximately linear in the squared mass m^2,

$$J = \alpha_0 + \alpha' m^2. \tag{1.5}$$

See figure 1.3. Such approximately straight 'Regge trajectories' can be

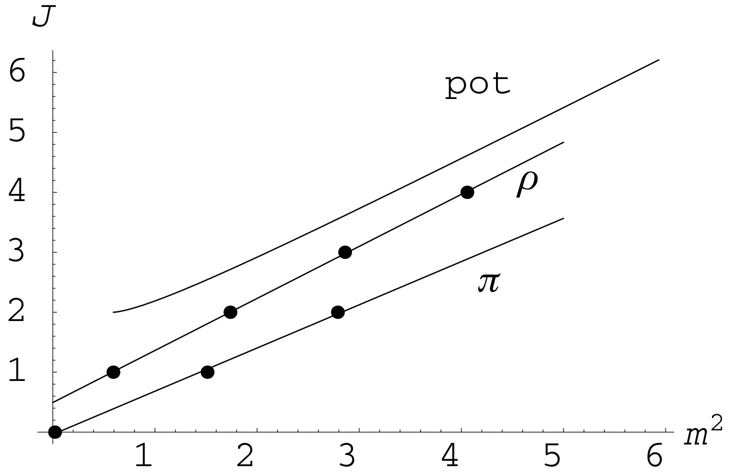

Fig. 1.3. Plot of spin J versus m^2 (GeV^2) for ρ- and π-like particles. The dots give the positions of particles, the straight lines are fits to the data, labeled by their particles with lowest spin. The line labeled 'pot' is L versus H^2 for the solution (1.10), for clarity shifted upward by two units, for $m_q = m_\rho/2$, $\sigma = 1/8\alpha'_\rho$.

understood from the following simple effective Hamiltonian for binding of a $q\bar{q}$ pair,

$$H = 2\sqrt{m_q^2 + p^2} + \sigma r. \tag{1.6}$$

Here m_q is the mass of the constituent quarks, taken to be equal for simplicity, $p = |\mathbf{p}|$ is the relative momentum, $r = |\mathbf{r}|$ is the relative separation, and the spin of the quarks is ignored. The potential is taken to be purely linear, because we are interested in the large-mass bound states with large relative angular momentum L, for which one expects that only the long-distance part of $V(r)$ is important.

For such states with large quantum number L the classical approximation should be reasonable. Hence, consider the classical Hamilton equations,

$$\frac{dr_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial r_k}. \tag{1.7}$$

and the following Ansatz for a circular solution:

$$\begin{aligned} r_1 &= a\cos(\omega t), \quad r_2 = a\sin(\omega t), \quad r_3 = 0, \\ p_1 &= -b\sin(\omega t), \quad p_2 = b\cos(\omega t), \quad p_3 = 0. \end{aligned} \tag{1.8}$$

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Substituting (1.8) into (1.7) we get relations among ω, a, and b, and expressions for p and r, which can be written in the form

$$p = b = \sigma\omega^{-1}, \quad r = a = 2s^{-1}\sigma^{-1}p, \quad s \equiv \sqrt{1 + m_q^2/p^2}, \tag{1.9}$$

such that L and H can be written as

$$L = rp = 2s^{-1}\sigma^{-1}p^2, \quad H = 2(s + s^{-1})p. \tag{1.10}$$

For $p^2 \gg m_q^2$, $s \approx 1$, $L \propto p^2$ and $H \propto p$. Then $L \propto H^2$ and, because $H = m$ is the mass (rest energy) of the bound state, we see that

$$\alpha' \equiv \left[LH^{-2}\right]_{p/m_q \to \infty} = (8\sigma)^{-1}. \tag{1.11}$$

It turns out that L is approximately linear in H^2 even for quite small p^2, such that $L < 1$, as shown in figure 1.3. Of course, the classical approximation is suspect for L *not* much larger than unity, but the same phenomenon appears to take place quantum mechanically in nature, where the lower spin states are also near the straight line fitting the higher spin states.[2]

With $\alpha' = 1/8\sigma$, the experimental value $\alpha' \approx 0.90\ \mathrm{GeV}^{-2}$ gives $\sqrt{\sigma} \approx$ 370 MeV. The effective string model (see e.g. [3] section 10.5) leads approximately to the same answer: $\alpha' = 1/2\pi\sigma$, giving $\sqrt{\sigma} \approx 420$ MeV. The string model is perhaps closer to reality if most of the bound-state energy is in the string-like chromoelectric field, but it should be kept in mind that both the string model and the effective Hamiltonian give only an approximate representation of QCD.

1.2 Scalar field

We start our exploration of field theory with the scalar field. Scalar fields $\varphi(x)$ ($x = (\mathbf{x}, t)$, $t \equiv x^0$) are used to describe spinless particles. Particles appearing elementary on one distance scale may turn out to be be composite bound states on a smaller distance scale. For example, protons, pions, etc. appear elementary on the scale of centimeters, but composed of quarks and gluons on much shorter distance scales. Similarly, fields may also be elementary or composite. For example, for the description of pions we may use elementary scalar fields $\varphi(x)$, or composite scalar fields of the schematic form $\bar{\psi}(x)\gamma_5\psi(x)$, where $\psi(x)$ and $\bar{\psi}(x)$ are quark fields and γ_5 is a Dirac matrix. Such composite fields can still be approximately represented by elementary $\varphi(x)$, which are then called effective fields. This is useful for the description of effective

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interactions, which are the result of more fundamental interactions on a shorter distance scale.

A basic tool in the description is the action $S = \int dt\, L$, with L the Lagrangian. For a nonrelativistic particle described by coordinates q_k, $k = 1, 2, 3$, the Lagrangian has the form kinetic energy minus potential energy, $L = \dot{q}_k\dot{q}_k/2m - V(q)$.† For the anharmonic oscillator in three dimensions the potential has the form $V(q) = \omega^2 q^2/2 + \lambda(q^2)^2/4$, $q^2 \equiv q_k q_k$. In field theory a simple example is the action for the φ^4 theory,

$$S = \int_M d^4x\, \mathcal{L}(x), \quad d^4x = dx^0\, dx^1\, dx^2\, dx^3, \tag{1.12}$$

$$\mathcal{L}(x) = \tfrac{1}{2}\partial_t\varphi(x)\partial_t\varphi(x) - \tfrac{1}{2}\nabla\varphi(x)\cdot\nabla\varphi(x) - \tfrac{1}{2}\mu^2\varphi(x)^2 - \tfrac{1}{4}\lambda\varphi(x)^4, \tag{1.13}$$

Here M is a domain in space–time, $\varphi(x)$ is a scalar field, $\mathcal{L}(x)$ is the action density or Lagrange function, and λ and μ^2 are constants (λ is dimensionless and μ^2 has dimension $(\text{mass})^2 = (\text{length})^{-2}$). Note that the index $\mathbf{x}$ is a continuous analog of the discrete index k: $\varphi(\mathbf{x}, t) \leftrightarrow q_k(t)$.

Requiring the action to be stationary under variations $\delta\varphi(x)$ of $\varphi(x)$, such that $\delta\varphi(x) = 0$ for x on the boundary of M, leads to the equation of motion:

$$\begin{aligned}\delta S &= \int d^4x \left[-\partial_t^2\varphi(x) + \nabla^2\varphi(x) - \mu^2\varphi(x) - \lambda\varphi(x)^3\right] \delta\varphi(x) \\ &= 0 \quad \Rightarrow \quad (\partial_t^2 - \nabla^2 + \mu^2)\varphi + \lambda\varphi^3 = 0.\end{aligned} \tag{1.14}$$

In the first step we made a partial integration. In classical field theory the equations of motion are very important (e.g. Maxwell theory). In quantum field theory their importance depends very much on the problem and method of solution. The action itself comes more to the foreground, especially in the path-integral description of quantum theory.

Various states of the system can be characterized by the energy $H = \int d^3x\, \mathcal{H}$. The energy density has the form kinetic energy plus potential energy, and is given by

$$\mathcal{H} = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\nabla\varphi)^2 + U, \tag{1.15}$$

$$U = \tfrac{1}{2}\mu^2\varphi^2 + \tfrac{1}{4}\lambda\varphi^4. \tag{1.16}$$

The field configuration with lowest energy is called the ground state. It has $\dot{\varphi} = \nabla\varphi = 0$ and minimal U. We shall assume $\lambda > 0$, such that $\mathcal{H}$ is

† Unless indicated otherwise, summation over repeated indices is implied, $\dot{q}_k\dot{q}_k \equiv \sum_k \dot{q}_k\dot{q}_k$.

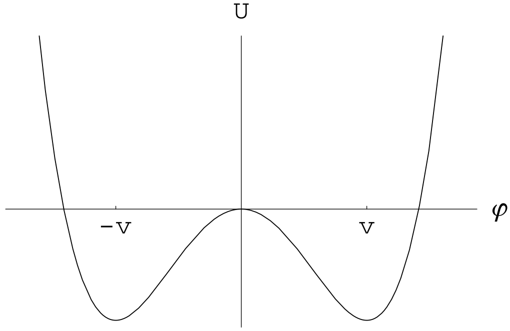

Fig. 1.4. The energy density for constant fields for $\mu^2 < 0$.

bounded from below for all φ. From a graph of $U(\varphi)$ (figure 1.4) we see that the cases $\mu^2 > 0$ and $\mu^2 < 0$ are qualitatively different:

$$\mu^2 > 0: \quad \varphi_{\rm g} = 0, \quad U_{\rm g} = 0;$$
$$\mu^2 < 0: \quad \varphi_{\rm g} = \pm v, \quad v^2 = -\frac{\mu^2}{\lambda}, \quad U_{\rm g} = -\frac{1}{4}\frac{\mu^2}{\lambda}. \tag{1.17}$$

So the case $\mu^2 < 0$ leads to a doubly degenerate ground state. In this case the symmetry of S or $\mathcal{H}$ under $\varphi(x) \to -\varphi(x)$ is broken, because a non-zero $\varphi_{\rm g}$ is not invariant, and one speaks of spontaneous (or dynamical) symmetry-breaking.

Small disturbances away from the ground state propagate and disperse in space and time in a characteristic way, which can be found by linearizing the equation of motion (1.14) around $\varphi = \varphi_{\rm g}$. Writing $\varphi = \varphi_{\rm g} + \varphi'$ and neglecting $O(\varphi'^2)$ gives

$$(\partial_t^2 - \nabla^2 + m^2)\varphi' = 0, \tag{1.18}$$

$$m^2 = U''(\varphi_{\rm g}) = \begin{cases} \mu^2, & \mu^2 > 0; \\ \mu^2 + 3\lambda v^2 = -2\mu^2, & \mu^2 < 0. \end{cases} \tag{1.19}$$

Wavepacket solutions of (1.18) propagate with a group velocity $\mathbf{v} = \partial\omega/\partial\mathbf{k}$, where $\mathbf{k}$ is the average wave vector and $\omega = \sqrt{m^2 + \mathbf{k}^2}$. In the quantum theory these wavepackets are interpreted as particles with energy–momentum $(\omega, \mathbf{k})$ and mass m. The particles can scatter with an interaction strength characterized by the coupling constant λ. For $\lambda = 0$ there is no scattering and the field is called 'free'.