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Vortices over Riemann surfaces and dominated splittings

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Abstract. We associate a flow ϕ with a solution of the vortex equations on a closed oriented Riemannian 2-manifold (M, g) of negative Euler characteristic and investigate its properties. We show that ϕ always admits a dominated splitting and identify special cases in which ϕ is Anosov. In particular, starting from holomorphic differentials of fractional degree, we produce novel examples of Anosov flows on suitable roots of the unit tangent bundle of (M, g).

Keywords: thermostats, vortex equations, Anosov flows, dominated splitting, fractional differentials
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1. *Introduction*

1.1. *Background.* This paper is concerned with the description and study of a class of dynamical systems determined by the solutions of a pair of partial differential equations (PDEs) naturally arising in Abelian gauge theories on a closed oriented Riemannian 2-manifold (M, g) of negative Euler characteristic.

Let SM denote the unit tangent bundle of (M, g). Given a smooth function $\lambda\lambda \in C^\infty(SM)$, we may consider the ordinary differential equation (ODE) for $\gamma : \mathbb{R} \to M$

$$\ddot{\gamma} = \lambda(\gamma, \dot{\gamma}) J\dot{\gamma}, \tag{1.1}$$

where $J : TM \to TM$ denotes rotation by $\pi/2$ according to the orientation of the surface, and the acceleration of γ is computed using the Levi-Civita connection of g. Equation (1.1) describes the motion of a particle on M driven by a force orthogonal to its velocity with magnitude determined by λ. As such, it is easy to see that the speed of γ remains constant and thus $(\gamma, \dot{\gamma})$ defines a flow in SM. If $\lambda = 0$, we obtain the geodesic flow of g, the prototype example of a conservative dynamical system. If λ only depends on position (i.e. it is the pullback of a function on M), we still obtain a volume-preserving flow (a magnetic flow), but the situation changes if λ is allowed to depend on velocities. For instance, we may take λ as the restriction to SM of a 1-form on M and, in that case, we obtain a *Gaussian thermostat* as studied in [33, 34]. In general, these flows are not volume preserving and here we are concerned with thermostat flows as defined by (1.1) when λ arises from a higher-order differential on M.

The dynamical properties that we shall investigate are *hyperbolicity* and *domination*. Hyperbolicity has played a prominent role in dynamics [31], but weaker forms of hyperbolicity, such as domination, have, in recent decades, come under intense focus [6]. The notion of dominated splitting was introduced by Mañé in the context of the proof of the stability conjecture (cf. [30]), but it has appeared in several other contexts and under different names. It can be regarded as a projective form of hyperbolicity and it can also be characterized in terms of the singular value decomposition of the linear Poincaré flow [4]. The notion is particularly relevant in our setting: for volume-preserving flows on 3-manifolds domination is equivalent to hyperbolicity, but for dissipative thermostats this is no longer the case. Thus, in the results below, some effort will be spent in studying when we can upgrade our flows from having a dominated splitting to being Anosov.

Let us give the benchmark example that motivates our construction. Let A be a *holomorphic* cubic differential on M so that $\overline{\partial} A = 0$, and suppose that the pair (g, A) is linked by the additional equation

$$K_g = -1 + 2|A|_g^2,$$

where K_g is the Gauss curvature. By the work of Labourie [22] and Loftin [23], such a pair gives rise to a properly convex projective structure on M and hence to an associated divisible strictly convex set $\tilde{M} \subset \mathbb{RP}^2$. The set $\tilde{M}$ comes equipped with a distance function, the so-called *Hilbert metric* (see, for instance, [20] for details) while g is known as the *Blaschke metric*. The Hilbert metric is the distance function of a Finsler metric whose geodesic flow is known to be Anosov [3]. If we choose λ to be the imaginary part of A—regarded as a function on SM—then the thermostat flow determined by (1.1) is a suitable reparametrization of the geodesic flow of the Hilbert metric. While the work of Labourie interprets the pair of equations $\overline{\partial} A = 0$ and $K_g = -1 + 2|A|_g^2$ as an instance of Hitchin's Higgs bundle equations [17], they may also be interpreted as an example of the so-called Abelian vortex equations [11]. One can, in fact, consider similar equations for differentials of any order, not just three, and investigate the dynamical properties of the associated thermostat. This was done in [25], but here we uncover a larger landscape that allows, for example, the consideration of holomorphic differential of *fractional order*, that is, holomorphic sections of $K^{m/n}$, where K is the canonical line bundle of (M, g). The natural habitat of our thermostats is not the unit sphere bundle any more, but rather *root bundles* covering SM to accommodate for the fractional degrees.

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1.2. *Vortices.* We now proceed to describe in detail the geometric setting for our pair of PDEs.

Let $L \to M$ be a complex line bundle of positive degree $\deg(L)$. For a triple consisting of a Hermitian bundle metric $\hbar$ on L, a del-bar operator $\overline{\partial}_L$ on L and a (1,0)-form φ on M with values in L, we consider the pair of equations

$$R(\mathrm{D}) + \tfrac{1}{2}\varphi \wedge \varphi^* + i\ell\Omega_g = 0 \quad \text{and} \quad \overline{\partial}_L \varphi = 0. \tag{1.2}$$

Here we write $\ell := \deg(L)/|\chi(M)|$, D denotes the Chern connection on L with respect to $(\hbar, \overline{\partial}_L)$, $R(\mathrm{D})$ is its curvature, Ω_g is the area form of g and $\varphi^* := \hbar(\cdot, \varphi)$. We assume $\hbar$ to be conjugate linear in the second variable, so that φ^* is a (0,1)-form on M with values in the dual L^{-1} of L.

The pair of equations (1.2) are a minor variation of the Abelian vortex equations on a Riemann surface, and hence we refer to them as *vortex equations* as well. The usual Abelian vortex equations concern a triple $(\hbar, \overline{\partial}_{L'}, \Phi)$, where Φ is a section of a complex line bundle L' over an oriented Riemannian 2-manifold (M, g). Besides Φ being holomorphic, one requests that the Chern connection D determined by $(\hbar, \overline{\partial}_{L'})$ satisfies

$$i\Lambda R(\mathrm{D}) + \frac{1}{2}\Phi \otimes \Phi^* - \frac{c}{2} = 0, \tag{1.3}$$

where c is some real constant and Λ denotes the L^2-adjoint of wedging with the area form Ω_g. The Abelian vortex equations are a modification of the Ginzburg–Landau model for superconductors and were first studied by Noguchi [28] and Bradlow [7] (for background, see also [19]). A general framework for the so-called symplectic vortices over closed Riemann surfaces was described in [9].

1.3. *Vortex thermostats.* Since L has positive degree and $\chi(M) < 0$, there exist unique positive coprime integers (m, n) so that we have an isomorphism $L^n \simeq K^m$ of complex line bundles. We fix an nth root $SM^{1/n}$ of the unit tangent bundle $\pi : SM \to M$ of (M, g). By this we mean a principal SO(2)-bundle $\pi_n : SM^{1/n} \to M$, which is an equivariant n-fold cover of $\pi : SM \to M$ (see §2.2 below for details)

Following [8], we call three linearly independent vector fields (X, H, V) on a smooth 3-manifold N a *generalized Riemannian structure* if they satisfy the commutator relations

$$[V, X] = H, \quad [V, H] = -X, \quad [X, H] = K_g V,$$

for some smooth function K_g on N. A *(generalized) thermostat* is a flow ϕ on N that is generated by a vector field of the form $X + \lambda V$, where λ is a smooth function on N. The root $SM^{1/n}$ is equipped with a generalized Riemannian structure by pulling back the natural Riemannian structure on SM determined by g and the orientation (where X is the geodesic vector field and V the vertical vector field). In §4, we show how to associate a thermostat to a solution $(\hbar, \overline{\partial}_L, \varphi)$ of the vortex equations on $L \to (M, g)$. We call such flows *vortex thermostats*.

In the special case where L is the canonical bundle equipped with its standard complex structure and Hermitian metric induced by g and where φ vanishes identically, the vortex equations (1.2) are equivalent to g being hyperbolic. The case where g has non-constant

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negative Gauss curvature can be dealt with by modifying the complex structure on K. In particular, suitably reparametrized, our family of flows include the geodesic flow of metrics of negative Gauss curvature and, more generally, the so-called W-flows of Wojtkowski [33, 34] (in the case of negative curvature, c.f. [25, Remark 4.10]).

1.4. *Results.* Our goal is to establish hyperbolicity properties for the general class of vortex thermostats. We first show the following theorem.

THEOREM A. *Every vortex thermostat admits a dominated splitting. Moreover, if all closed orbits of ϕ are hyperbolic saddles, then ϕ is Anosov.*

The choice of an nth root $SM^{1/n}$ of SM gives a corresponding nth root $K^{1/n}$ of K and hence an isomorphism $\mathscr{Z} : L \to K^{m/n}$ of complex line bundles. While $\mathscr{Z}$ is, in general, not an isomorphism of holomorphic line bundles, we can upgrade Theorem A as follows.

THEOREM B. *Suppose $\mathscr{Z} : L \to K^{m/n}$ is an isomorphism of holomorphic line bundles. Then the associated vortex thermostat is Anosov.*

We do not know if there is a vortex thermostat which is not Anosov.

As in the case of the usual vortex equations, the equations (1.2) are invariant under a suitable action of the complex gauge group of L, that is, the group $\mathrm{G}_{\mathbb{C}}$ of automorphisms of L. We show that, by possibly applying a complex gauge transformation, we can assume without losing generality that $\hbar = \hbar_0$, where $\hbar_0$ denotes the natural Hermitian bundle metric on $L \simeq K^{m/n}$ determined by g. The 1-form φ is a section of $K \otimes L \simeq K^{1+\ell}$ and hence we may think of φ/ℓ as a differential A of fractional degree $1 + \ell > 1$. Furthermore, since $K^{m/n} \simeq L$ as complex line bundles, there exists a unique 1-form θ on M so that $\overline{\partial}_{K^{m/n}} - \overline{\partial}_L = \ell\theta^{0,1}$, where $\theta^{0,1}$ denotes the (0,1)-part of θ. By construction, the above isomorphism $\mathscr{Z}$ of complex line bundles is an isomorphism of holomorphic line bundles if and only if θ vanishes identically. In terms of the triple (g, A, θ) the vortex equations (1.2) are equivalent to

$$K_g - \delta_g\theta = -1 + \ell|A|_g^2 \quad \text{and} \quad \overline{\partial}A = \ell\, \theta^{0,1} \otimes A,$$

where $|\cdot|_g$ denotes the pointwise norm induced on $K^{1+\ell}$ by g and δ_g is the co-differential. Thus, we recover the main equations from [25] (see also [24]), but now in the more general setting of fractional differentials. In particular, Theorem A and Theorem B above generalize the results from [25] to the case of differentials of fractional degree. Proving the Anosov property for fractional differential presents new obstacles, particularly those in the range $0 < \ell < 1$.

As in [25], our flows do not preserve a volume form, unless φ vanishes. More precisely, the proof of [25, Theorem 5.5] shows that under the hypotheses of Theorem B the associated vortex thermostat ϕ preserves an absolutely continuous measure if and only if φ vanishes identically. This property implies that vortex thermostats as in Theorem B with $\varphi \neq 0$ have positive entropy production and thus they provide interesting models in non-equilibrium statistical mechanics [13, 14, 29]. The moduli space of gauge equivalence

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classes of solutions of the usual vortex equations was described in [7, Theorem 4.6], and we expect a similar statement to hold in the case considered here; this may be taken up elsewhere.

In Appendix A, we briefly discuss the dominated splitting property for a thermostat that one can associate with the usual vortex equations.

2. *Preliminaries*

2.1. *The unit tangent bundle.* Let (M, g) be an oriented Riemannian 2-manifold and let $\pi : SM \to M$ denote its unit tangent bundle. Recall that SM is equipped with a coframing consisting of three linearly independent 1-forms $(\underline{\omega_1}, \underline{\omega_2}, \underline{\psi})$. The 1-forms $(\underline{\omega_1}, \underline{\omega_2})$ span the 1-forms on SM that are semi-basic for the basepoint projection π, that is, the forms that vanish when evaluated on vertical vector fields. Explicitly, for all $(x, v) \in SM$ and $\xi \in T_{(x,v)}SM$,

$$\underline{\omega_1}(\xi) = g(d\pi(\xi), v) \quad \text{and} \quad \underline{\omega_2}(\xi) = g(d\pi(\xi), Jv),$$

where $J : TM \to TM$ denotes rotation by $\pi/2$ in a counter-clockwise direction with respect to the fixed orientation. The third 1-form $\underline{\psi}$ is the Levi-Civita connection form of g so that we have the structure equations

$$d\underline{\omega_1} = -\underline{\omega_2} \wedge \underline{\psi}, \quad d\underline{\omega_2} = -\underline{\psi} \wedge \underline{\omega_1}, \quad d\underline{\psi} = -K_g \underline{\omega_1} \wedge \underline{\omega_2},$$

where K_g denotes the (pullback to SM of the) Gauss curvature of g. Denoting by $(\underline{X}, \underline{H}, \underline{V})$ the vector fields dual to $(\underline{\omega_1}, \underline{\omega_2}, \underline{\psi})$, the structure equations imply the commutator relations

$$[\underline{V}, \underline{X}] = \underline{H}, \quad [\underline{V}, \underline{H}] = -\underline{X}, \quad [\underline{X}, \underline{H}] = K_g \underline{V}. \tag{2.1}$$

The vector field $\underline{X}$ is the geodesic vector field of (M, g) and $\underline{V}$ is the generator of the SO(2) right action on SM, which we denote by $R_{e^{i\vartheta}}$ for $e^{i\vartheta} \in \mathrm{SO}(2)$.

Note that a complex-valued 1-form on M that is a (1,0)-form with respect to the Riemann surface structure defined by J pulls back to SM to become a complex multiple of the form $\underline{\omega} := \underline{\omega_1} + i\underline{\omega_2}$. The form $\underline{\omega}$ satisfies the equivariance property $(R_{e^{i\vartheta}})^*\underline{\omega} = e^{-i\vartheta}\underline{\omega}$ for all $e^{i\vartheta} \in \mathrm{SO}(2)$ and hence a section β of the canonical bundle K of M is represented by a complex-valued function $\boldsymbol{\beta}$ on SM satisfying the equivariance property $(R_{e^{i\vartheta}})^*\boldsymbol{\beta} = e^{i\vartheta}\boldsymbol{\beta}$. To recover the associated (1,0)-form on M, we observe that $\boldsymbol{\beta}\underline{\omega}$ is semi-basic and invariant under the SO(2)-right action, and hence the pullback of a unique (1,0)-form on M, which is β.

Remark 2.1. (Notation) We write $Y(f)$ for the (Lie-)derivative of a smooth real or complex-valued function f in the direction of a vector field Y. Whenever no confusion is possible about the argument of the linear differential operator Y, we will simply write Yf instead of $Y(f)$.

2.2. *Roots of the unit tangent bundle.* Let $n \in \mathbb{N}$ and $\pi_n : SM^{1/n} \to M$ be a principal right SO(2)-bundle whose right action we denote by $R_{e^{i\vartheta}}$ as well. Let $\pi : SM \to M$ denote the unit tangent bundle of the oriented Riemannian 2-manifold (M, g) and let $(\underline{\omega_1}, \underline{\omega_2}, \underline{\psi})$

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be its coframing. We call $\pi_n : SM^{1/n} \to M$ an *nth root of SM* if there exists an n-fold covering map $\rho : SM^{1/n} \to SM$ so that $\pi_n = \pi \circ \rho$ and so that

$$\rho \circ R_{e^{i\vartheta}} = R_{e^{in\vartheta}} \circ \rho$$

for all $e^{i\vartheta} \in \mathrm{SO}(2)$. We refer the reader to [15] for background about nth roots of SM. We write $\omega_i = \rho^*\underline{\omega_i}$ and $\psi = \rho^*\underline{\psi}$ and let $(X, H, \mathbb{V})$ denote the framing dual to $(\omega_1, \omega_2, \psi)$ on $SM^{1/n}$. The structure equations imply the usual commutator relations

$$[\mathbb{V}, X] = H, \quad [\mathbb{V}, H] = -X, \quad [X, H] = K_g\mathbb{V}. \tag{2.2}$$

Recall that a section β of the canonical bundle K of (M, g) is represented by a complex-valued function $\boldsymbol{\beta}$ on SM satisfying the equivariance property $(R_{e^{i\vartheta}})^*\boldsymbol{\beta} = e^{i\vartheta}\boldsymbol{\beta}$. Writing $\tilde{\boldsymbol{\beta}} := \boldsymbol{\beta} \circ \rho$, the function $\tilde{\boldsymbol{\beta}}$ satisfies $(R_{e^{i\vartheta}})^*\tilde{\boldsymbol{\beta}} = e^{in\vartheta}\tilde{\boldsymbol{\beta}}$ and hence we obtain an nth root $K^{1/n}$ of K whose sections are represented by complex-valued functions $\boldsymbol{B}$ on $SM^{1/n}$ satisfying $(R_{e^{i\vartheta}})^*\boldsymbol{B} = e^{i\vartheta}\boldsymbol{B}$ for all $e^{i\vartheta} \in \mathrm{SO}(2)$. Likewise, for each $m \in \mathbb{Z}$, the smooth sections of $K^{m/n}$ are represented by smooth complex-valued functions $\boldsymbol{B}$ on $SM^{1/n}$ satisfying

$$(R_{e^{i\vartheta}})^*\boldsymbol{B} = e^{im\vartheta}\boldsymbol{B} \tag{2.3}$$

for all $e^{i\vartheta} \in \mathrm{SO}(2)$. In particular, for each $m \in \mathbb{Z}$, we obtain a Hermitian bundle metric $\hbar_0$ on $K^{m/n}$ defined by

$$(\boldsymbol{B}_1, \boldsymbol{B}_2) \mapsto \boldsymbol{B}_1\overline{\boldsymbol{B}_2},$$

where $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ represent sections of $K^{m/n}$.

Furthermore, observe that, by definition, $\mathbb{V}$ is only $(1/n)$th of the generator V of the SO(2)-action on $SM^{1/n}$. As a consequence, the infinitesimal version of (2.3) becomes

$$\mathbb{V}\boldsymbol{B} = \frac{1}{n}V\boldsymbol{B} = i\left(\frac{m}{n}\right)\boldsymbol{B} \tag{2.4}$$

and hence the map

$$\boldsymbol{B} \mapsto d\boldsymbol{B} - i\left(\frac{m}{n}\right)\psi\boldsymbol{B}$$

equips $K^{m/n}$ with a connection ∇ whose connection form is $-i(m/n)\psi$. The (0,1)-part ∇'' of ∇ equips $K^{m/n}$ with a holomorphic line bundle structure $\overline{\partial}_{K^{m/n}}$, so that ∇ is the Chern connection of the Hermitian holomorphic line bundle $(K^{m/n}, \overline{\partial}_{K^{m/n}}, \hbar_0)$.

Finally, note that applying $\mathbb{V}$ again to (2.4) shows that we may write $\boldsymbol{B} = (n\mathbb{V}b/m) + ib$ for a unique real-valued function b on $SM^{1/n}$ satisfying $\mathbb{V}\mathbb{V}b = -(m/n)^2b$. Conversely, if a smooth real-valued function b on $SM^{1/n}$ satisfies $\mathbb{V}\mathbb{V}b = -(m/n)^2b$, then $\boldsymbol{B} := (n\mathbb{V}b/m) + ib$ represents a smooth section B of $K^{m/n}$.

2.3. *Thermostats.* Let N be a smooth 3-manifold equipped with three smooth vector fields (X, H, V) that are linearly independent at each point of N. The next definition follows [8].

Definition 2.2. We say that N carries a *generalized Riemannian structure* if (X, H, V) satisfy the commutator relations

$$[V, X] = H, \quad [V, H] = -X, \quad [X, H] = K_g V, \tag{2.5}$$

for some smooth function K_g on N.

Example 2.3. Let (M, g) be an oriented Riemannian 2-manifold and let $\pi_n : SM^{1/n} \to M$ be an nth root of its unit tangent bundle $\pi : SM \to M$. Then $(X, H, \mathbb{V})$, defined as in §2.2, equip $N = SM^{1/n}$ with a generalized Riemannian structure.

Suppose N carries a generalized Riemannian structure (X, H, V) with dual 1-forms $(\omega_1, \omega_2, \psi)$.

Definition 2.4. A *(generalized) thermostat* on N is a flow ϕ generated by a vector field of the form $F := X + \lambda V$, where $\lambda \in C^\infty(N)$.

3. *Dominated splittings and hyperbolicity*

In this section, we summarize the main dynamical set-up that we shall use; in the first three subsections, we follow closely the presentation in [**25**]. For background on the notion of dominated splittings, we refer to [**10**].

3.1. *Definitions.*

Let N be a smooth closed 3-manifold and let $\phi : N \times \mathbb{R} \to N$ be a continuous flow. A *cocycle over ϕ with values in GL$(2, \mathbb{R})$* is a continuous map $\Psi : N \times \mathbb{R} \to \mathrm{GL}(2, \mathbb{R})$ such that

$$\Psi_{t_1+t_2}(x) = \Psi_{t_1}(\phi_{t_2}(x))\Psi_{t_2}(x)$$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in N$. Note that the cocycle condition ensures that, on the trivial vector bundle $E = N \times \mathbb{R}^2$, we obtain a continuous linear flow $\rho : E \times \mathbb{R} \to E$ by defining

$$\rho_t((x, a)) = (\phi_t(x), \Psi_t(x)a)$$

for all $(x, a) \in E = N \times \mathbb{R}^2$ and $t \in \mathbb{R}$.

We say that E admits a continuous ρ-invariant splitting if there exist continuous ρ-invariant line bundles $E^{s,u}$ so that $E = E^u \oplus E^s$. We fix a norm $|\cdot|$ on $\mathbb{R}^2$.

Definition 3.1. The cocycle Ψ is said to be *hyperbolic* is there exists a continuous ρ-invariant splitting (E^s, E^u) and positive constants $C, \mu > 0$ so that

$$\|\Psi_t(x)|_{E^s(x)}\| \leqslant Ce^{-\mu t} \quad \text{and} \quad \|\Psi_{-t}(x)|_{E^u(x)}\| \leqslant Ce^{-\mu t}$$

for all $x \in N$ and $t > 0$.

Here $\|\cdot\|$ denotes the operator norm induced on $\mathrm{Hom}(E^{s,u}(x), E^{s,u}(\phi_t(x)))$ by the norm $|\cdot|$, respectively. A weaker notion than that of hyperbolicity is to ask that, for all $x \in N$, any direction not contained in the subspace $E^s(x)$ converges exponentially fast to $E^u(\phi_t(x))$ when applying $\rho_t(x)$. This condition is equivalent to the following notion.

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Definition 3.2. The cocycle Ψ is said to admit a *dominated splitting* if there exists a continuous ρ-invariant splitting (E^u, E^s) and positive constants $C, \mu > 0$ so that

$$\|\Psi_t(x)|_{E^s(x)}\| \|\Psi_{-t}(\phi_t(x))|_{E^u(\phi_t(x))}\| \leqslant Ce^{-\mu t} \tag{3.1}$$

for all $x \in N$ and $t > 0$.

3.2. *The derivative cocycle of a thermostat.* Suppose the closed 3-manifold N is equipped with a generalized Riemannian structure and a thermostat ϕ generated by the vector field $F = X + \lambda V$, as above. Using the bracket relations (2.5), it is straightforward to derive the ODEs dictating the behaviour of $d\phi_t$. Given an initial condition $\xi \in T_x N$ and if we write

$$d\phi_t(\xi) = w(t)F(\phi_t(x)) + y(t)H(\phi_t(x)) + u(t)V(\phi_t(x))$$

for real-valued functions w, y, u on $\mathbb{R}$, then

$$\dot{w} = \lambda y, \tag{3.2}$$

$$\dot{y} = u, \tag{3.3}$$

$$\dot{u} = V(\lambda)\dot{y} - \kappa y, \tag{3.4}$$

where

$$\kappa := K_g - H\lambda + \lambda^2. \tag{3.5}$$

In order to associate a cocycle with a thermostat, we consider the rank two quotient vector bundle $E = TN/\mathbb{R}F \simeq \mathbb{R}H \oplus \mathbb{R}V$. Elements in E will be denoted by $[\xi]$, where $\xi \in TN$. The mapping $d\phi_t$ descends to define a mapping

$$\rho : \mathbb{R} \times E \to E, \quad (t, [\xi]) \mapsto \rho(t, [\xi]) = [d\phi_t(\xi)],$$

which satisfies $\rho_{t_1} \circ \rho_{t_2} = \rho_{t_1+t_2}$ for all $t_1, t_2 \in \mathbb{R}$. This is sometimes called the *linear Poincaré flow*. The basis of vector fields (F, H, V) on N defines a vector bundle isomorphism $TN \simeq N \times \mathbb{R}^3$ and, consequently, an identification $E \simeq N \times \mathbb{R}^2$. Therefore, we obtain a cocycle $\Psi : N \times \mathbb{R} \to \mathrm{GL}(2, \mathbb{R})$ over ϕ by requiring that, for each $t \in \mathbb{R}$ and all $(x, a) \in E$,

$$\rho_t((x, a)) = (\phi_t(x), \Psi_t(x)a).$$

Explicitly, Ψ_t is the linear map whose action on $\mathbb{R}^2$ is

$$\Psi_t(x) : \begin{pmatrix} y(0) \\ \dot{y}(0) \end{pmatrix} \mapsto \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}$$

with

$$\ddot{y}(t) - (V\lambda)(\phi_t(x))\dot{y}(t) + \kappa(\phi_t(x))y(t) = 0.$$

Observe that, for thermostats, the 2-plane bundle spanned by H and V is, in general, *not* invariant under $d\phi_t$.

The cocycle Ψ_t is hyperbolic if and only if the thermostat flow ϕ_t is Anosov (cf., for instance, [33, Proposition 5.1]). We will say that ϕ_t admits a dominated splitting if Ψ_t admits a dominated splitting. This is the natural notion for flows (see [1, Definition 1]).

For the case of flows on 3-manifolds, as is our case, the existence of a dominated splitting can produce hyperbolicity if additional information on the closed orbits is available. Indeed [1, Theorem B] implies that if all closed orbits of ϕ are hyperbolic saddles, then $N = \Lambda \cup \mathcal{T}$, where Λ is a hyperbolic invariant set and $\mathcal{T}$ consists of finitely many normally hyperbolic irrational tori.

Flows with dominated splitting are also called *projectively Anosov flows*. We note that when the flow ϕ admits a dominated splitting we may write $TN = \tilde{E}^s + \tilde{E}^u$, where $\tilde{E}^{s,u}$ are continuous plane bundles invariant under $d\phi_t$ and whose intersection is $\mathbb{R}F$. In general, they are integrable but, unlike the Anosov case, they may not be uniquely integrable. Also note that the irrational tori in $\mathcal{T}$ must be tangential to $\tilde{E}^s$ or $\tilde{E}^u$ due to the domination condition. We refer to [2] and the references therein for a classification of these flows when the bundles $\tilde{E}^{s,u}$ are of class C^2 (in which case, they do determine codimension one foliations of class C^2).

3.3. *Infinitesimal generators and conjugate cocycles.* For a smooth cocycle $\Psi : N \times \mathbb{R} \to \mathrm{GL}(2, \mathbb{R})$, we define its infinitesimal generator $\mathbb{B} : N \to \mathfrak{gl}(2, \mathbb{R})$ as

$$\mathbb{B}(x) := -\frac{d}{dt}\bigg|_{t=0} \Psi_t(x).$$

The cocycle Ψ can be obtained from $\mathbb{B}$ as the unique solution to

$$\frac{d}{dt}\Psi_t(x) + \mathbb{B}(\phi_t(x))\Psi_t(x) = 0, \quad \Psi_0(x) = \mathrm{Id}.$$

In the case of thermostats, it is easy to check that

$$\mathbb{B} = \begin{pmatrix} 0 & -1 \\ \kappa & -V\lambda \end{pmatrix},$$

where $\kappa = K_g - H\lambda + \lambda^2$. Given a gauge, that is, a smooth map $\mathcal{P} : N \to \mathrm{GL}(2, \mathbb{R})$, we obtain a new cocycle by conjugation

$$\tilde{\Psi}_t(x) = \mathcal{P}^{-1}(\phi_t(x))\Psi_t(x)\mathcal{P}(x).$$

It is straightforward to check that the infinitesimal generator $\tilde{\mathbb{B}}$ of $\tilde{\Psi}_t$ is related to $\mathbb{B}$ by

$$\tilde{\mathbb{B}} = \mathcal{P}^{-1}\mathbb{B}\mathcal{P} + \mathcal{P}^{-1}F\mathcal{P}. \tag{3.6}$$

Below, we shall use gauges of a particular type. Consider a gauge transformation $\mathcal{P} : N \to \mathrm{GL}(2, \mathbb{R})$ given by

$$\mathcal{P} = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix},$$

where p is a smooth real-valued function on N. A computation using (3.6) shows that the conjugate cocyle $\tilde{\Psi}_t$ via P has infinitesimal generator given by

$$\tilde{\mathbb{B}} = \begin{pmatrix} -p & -1 \\ \kappa_p & -V\lambda + p \end{pmatrix},$$

where $\kappa_p := \kappa + Fp + p(p - V\lambda)$. Since the cocycles Ψ_t and $\tilde{\Psi}_t$ are conjugate, they have the same dominated splitting/hyperbolicity properties, but the form of $\tilde{\mathbb{B}}$ will expose the

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origins of these properties when $\kappa_p < 0$ (cf. [35, Introduction]). In both cases, the trace of the matrix is $-V\lambda$ (minus divergence of F), giving an indication that F may not preserve volume.

3.4. *Conditions ensuring domination and hyperbolicity.* We have [25, Theorem 3.7].

THEOREM 3.3. *Let N be a closed 3-manifold that is equipped with a generalized Riemannian structure (X, H, V) and a thermostat flow ϕ generated by $F = X + \lambda V$. Suppose there exists a smooth function $p : N \to \mathbb{R}$ such that*

$$\kappa_p = \kappa + Fp + p(p - V\lambda) < 0.$$

Then ϕ admits a dominated splitting with $V \notin E^{s,u}$.

Remark 3.4. More precisely, in [25, Theorem 3.7], only the case of a thermostat on the unit tangent bundle of an oriented Riemannian 2-manifold (M, g) is considered. However, it is easy to check that the arguments in [25, Theorem 3.7] also prove Theorem 3.3. In [25], we employ quadratic forms to establish this result; instead, we could have used a cone-field criterion as described, for instance, in [10, Theorem 2.6].

The fact that $V \notin E^{s,u}$ implies that there are uniquely defined continuous (in fact, Hölder) functions $r^{s,u} : N \to \mathbb{R}$ such that $H + r^{s,u}V \in E^{s,u}$. The invariance of the bundles $E^{s,u}$ translates into Riccati equations for $r^{s,u}$ of the form

$$Fr + r^2 - rV\lambda + \kappa = 0.$$

Observe that $h := r - p$ satisfies the Riccati equation

$$Fh + h^2 + h(2p - V\lambda) + \kappa_p = 0. \tag{3.7}$$

Moreover, the functions $r^{u,s}$ can be constructed using a limiting procedure, as follows. Fix $x \in N$ and consider, for each $R > 0$, the unique solution u_R to the Riccati equation along $\phi_t(x)$

$$\dot{u} + u^2 - uV\lambda + \kappa = 0$$

satisfying $u_R(-R) = \infty$. Then

$$r^u(x) = \lim_{R\to\infty} u_R(0). \tag{3.8}$$

Note that $r^u(\phi_t(x)) = \lim_{R\to\infty} u_R(t)$.

Finally, under the assumption in Theorem 3.3 that $\kappa_p < 0$, we get the important additional information that $h^u := r^u - p > 0$ and $h^s := r^s - p < 0$. We call these the positive and negative Hopf solutions given that they play a similar role to the solutions introduced by E. Hopf in [18] for the geodesic flow.

The property $V \notin E^{s,u}$ allows a convenient visualization of the domination condition in terms of the behaviour of solutions to the Riccati equation, as depicted in Figure 1. The reader might find this figure useful when following some of the arguments below, particularly the proof of Lemma 5.1. To prove that our flows are Anosov, we shall use

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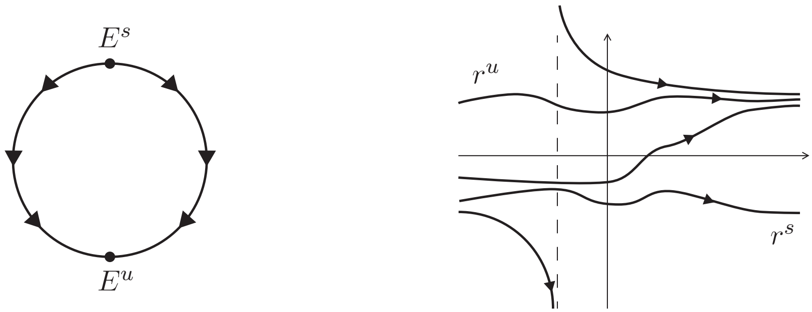

FIGURE 1. Dominated splitting property.

the following lemma that 'upgrades' the domination condition to hyperbolicity under additional information on the solutions $r^{s,u}$.

LEMMA 3.5. *Under the same assumptions as in Theorem 3.3, suppose, in addition, that either:*

(1) $r^u > 0$ *and* $r^s < 0$*; or*

(2) $V\lambda - p - (\kappa_p/(r^u - p)) > 0$ *and* $V\lambda - p - (\kappa_p/(r^s - p)) < 0$.

Then ϕ_t *is Anosov.*

Proof. We first consider (1). For a given initial condition $(y(0), \dot{y}(0)) \in E^u$, we know, that under the cocycle Φ_t, we have $\dot{y} = r^u y$. If $r_u > 0$, we can find a uniform constant $\mu > 0$ such that $|y(-t)| \leqslant e^{-\mu t}|y(0)|$ for $t > 0$. This gives uniform exponential growth for Ψ_t on E^u. Arguing with $r^s < 0$, we get uniform exponential contraction for Ψ_t on E^s and thus show that Ψ_t is hyperbolic.

Now assume condition (2) and consider a solution with initial conditions $(y(0), \dot{y}(0)) \in E^u$. Then $\dot{y} = r^u y$ and let $z := (r^u - p)y$ (recall that $r^u - p > 0$). Then a calculation shows that $\dot{z} = (V\lambda - p)z - \kappa_p y = (V\lambda - p - (\kappa_p/(r^u - p)))z$. This gives exponential growth for z and hence the desired exponential growth for Ψ_t on E^u. Arguing in a similar way with E^s, we deduce that Ψ_t is hyperbolic. □

Remark 3.6. In [25], we used condition (1) to prove that thermostat flows with $\theta = 0$ are Anosov when ℓ is an integer ≥ 1. Remarkably, for the case of fractional differentials in the range $0 < \ell < 1$, we will crucially need alternative (2).

Although we do not use the next proposition, it complements Theorem 3.3 quite nicely and it gives an indication of the importance of the property $V \notin E^{s,u}$.

PROPOSITION 3.7. *Suppose that the thermostat determined by* λ *is such that* Ψ_t *admits a continuous invariant splitting* $E = E^u \oplus E^s$ *with* $V \notin E^{u,s}$*. Then the splitting is dominated and there exists a hyperbolic* $SL(2, \mathbb{R})$*-cocycle* Ψ_t^{hyp} *such that*

$$\Psi_t = e^{\frac{1}{2}\int_0^t V\lambda}\, \Psi_t^{hyp}.$$

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Proof. We know that the existence of a splitting with $V \notin E^{u,s}$ gives rise to two continuous functions $r^{u,s} : N \to \mathbb{R}$ satisfying the Riccati equation

$$Fr + r^2 - rV\lambda + \kappa = 0.$$

Moreover, $r^u - r^s \neq 0$.

Recall that the infinitesimal generator for the cocycle Ψ_t is

$$\mathbb{B} = \begin{pmatrix} 0 & -1 \\ \kappa & -V\lambda \end{pmatrix}.$$

Consider a gauge transformation $\mathscr{P} : N \to \mathrm{GL}(2, \mathbb{R})$ given by

$$\mathscr{P} = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$$

with $p = V\lambda/2$. Then the conjugate cocycle $\tilde{\Psi}_t$ via $\mathscr{P}$ has infinitesimal generator given by

$$\tilde{\mathbb{B}} = -\frac{1}{2}V\lambda\ \mathrm{Id} + \begin{pmatrix} 0 & -1 \\ \kappa_p & 0 \end{pmatrix}.$$

To complete the proof, we need to prove that the cocycle generated by

$$\begin{pmatrix} 0 & -1 \\ \kappa_p & 0 \end{pmatrix}$$

is hyperbolic. Note that $h^{u,s} := r^{u,s} - p$ satisfies the Riccati equation

$$Fh + h^2 + \kappa_p = 0.$$

The quadratic form

$$Q(a, b) = 2ab - ([h^u]^2 + [h^s]^2)a^2$$

has the property that

$$\dot{Q} = (b - h^u a)^2 + (b - h^s a)^2 > 0$$

unless $a = b = 0$. (Note that $\dot{b} + \kappa_p a = 0$ and $\dot{a} = b$.) Now the hyperbolicity follows, for instance, from [**35**, Proposition 4.1 & Theorem 4.4]. □

Remark 3.8. We do not know of any example of a thermostat as in Proposition 3.7 that is not Anosov.

3.5. *Bi-contact structures.* It is possible to recast the discussion of §3.4 in terms of the notion of a bi-contact structure, which was introduced by Eliashberg and Thurston [**12**] and further studied by Mitsumatsu [**27**] in the context of projective Anosov flows.

If N is a closed 3-manifold, we shall say that a *bi-contact pair* is a pair of contact forms (τ_+, τ_-) such that $\tau_+ \wedge d\tau_+$ and $\tau_- \wedge d\tau_-$ give rise to opposite orientations and $\ker \tau_+ \cap \ker \tau_-$ is 1-dimensional at every point. It turns out (cf. [**12**, **27**]) that the flow of a non-zero vector field F has a dominated splitting (or is a projective Anosov flow) if and only if there is a bi-contact pair (τ_+, τ_-) such that $F \in \ker \tau_+ \cap \ker \tau_-$.

Now suppose that N is endowed with a generalized Riemannian structure (X, H, V) and that $\lambda, p \in C^\infty(N)$ are given functions. We consider a new frame (F, H_p, V),

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where $F := X + \lambda V$ and $H_p = H + pV$. If we denote by (α, β, ψ) the co-frame dual to (X, H, V), then a simple computation shows that $(\alpha, \beta, \tilde{\psi})$ is the co-frame dual to (F, H_p, V), where

$$\tilde{\psi} = -\lambda\alpha - p\beta + \psi.$$

Then we have the following lemma.

LEMMA 3.9. *The pair $(\beta, \tilde{\psi})$ is a bi-contact pair if and only if $\kappa_p < 0$.*

We omit the proof of the lemma (which is a fairly straightforward computation), since we will not use it in subsequent sections. Since $F \in \ker \beta \cap \ker \tilde{\psi}$, we see that with this lemma we essentially recover Theorem 3.3. The conditions appearing in Lemma 3.5 can now be rephrased in a more pleasing way in terms of the bi-contact pair $(\beta, \tilde{\psi})$. Indeed, condition (1) is equivalent to

$$d\beta(F, H + r^u V) > 0 \quad \text{and} \quad d\beta(F, H + r^s V) < 0,$$

while condition (2) is equivalent to

$$d\tilde{\psi}(F, H + r^u V) > 0 \quad \text{and} \quad d\tilde{\psi}(F, H + r^s V) < 0.$$

Again, we omit the verification of these equivalences as they will not be used in what follows.

4. *Thermostats from vortices*

4.1. *The vortex equations.* Let (M, g) be a closed oriented Riemannian 2-manifold of negative Euler characteristic and let $\nu : L \to M$ be a complex line bundle of positive degree. For a triple consisting of a Hermitian bundle metric $\hbar$ on L, a del-bar operator $\overline{\partial}_L$ on L and a (1,0)-form φ on M with values in L, we consider the pair of equations

$$R(\mathrm{D}) + \tfrac{1}{2}\varphi \wedge \varphi^* + i\ell\Omega_g = 0 \quad \text{and} \quad \overline{\partial}_L \varphi = 0.$$

Here we write $\ell := \deg(L)/|\chi(M)|$, D denotes the Chern connection on L with respect to $(\hbar, \overline{\partial}_L)$, $R(\mathrm{D})$ is its curvature, Ω_g is the area form of g and the 1-form φ^* with values in the dual L^{-1} of L is defined by

$$\varphi^*(v)(\xi) := \hbar(\xi, \varphi(v))$$

for all $x \in M$, $v \in TM$ and $\xi \in \nu^{-1}(\{x\})$. We assume $\hbar$ to be conjugate linear in the second variable, so that φ^* is an L^{-1}-valued (0,1)-form. We extend the wedge-product to bundle-valued forms in the standard way, so that, for $\varphi \in \Omega^1(L)$ and $\varrho \in \Omega^1(L^{-1})$,

$$(\varphi \wedge \varrho)(v, w) = \varrho(w)\varphi(v) - \varrho(v)\varphi(w)$$

for all $x \in M$ and $v, w \in T_x M$. In particular, we obtain

$$\begin{aligned}(\varphi \wedge \varphi^*)(v, w) &= \hbar(\varphi(v), \varphi(w)) - \hbar(\varphi(w), \varphi(v)) \\ &= \hbar(\varphi(v), \varphi(w)) - \overline{\hbar(\varphi(v), \varphi(w))} = 2i \operatorname{Im} \hbar(\varphi(v), \varphi(w))\end{aligned}$$

so that $\varphi \wedge \varphi^*$ is a purely imaginary (1,1)-form on M.

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The *complex gauge group* $G_{\mathbb{C}}$ of L is the group of automorphisms of L (covering the identity on M) and the *gauge group* G of $(L, \mathfrak{h})$ consists of the automorphisms of L that are unitary with respect to $\mathfrak{h}$. Since an automorphism of a one-dimensional complex vector space is just a non-vanishing complex number, we have $\mathrm{G}_{\mathbb{C}} \simeq C^{\infty}(M, \mathbb{C}^*)$ and $\mathrm{G} \simeq C^{\infty}(M, \mathrm{U}(1))$, the smooth functions on M with values in the one-dimensional unitary group $\mathrm{U}(1)$. An element $\tau \in \mathrm{G}_{\mathbb{C}}$ acts on a Hermitian bundle metric $\mathfrak{h}$ on L by the rule

$$\tau \cdot \mathfrak{h} = |\tau|^2 \mathfrak{h} \tag{4.1}$$

and on $\varphi \in \Omega^{p,q}(L)$ by the rule

$$\tau \cdot \varphi = \tau^{-1}\varphi. \tag{4.2}$$

We define an action on the space of del-bar operators on L by

$$\tau \cdot \overline{\partial}_L = \overline{\partial}_L + \tau^{-1}\overline{\partial}\tau. \tag{4.3}$$

Writing $\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}$ for the Chern connection on L determined by the Hermitian metric $\mathfrak{h}$ and del-bar operator $\overline{\partial}_L$, we obtain the following lemma.

LEMMA 4.1. *For a Hermitian holomorphic line bundle $(L, \mathfrak{h}, \overline{\partial}_L)$ and $\tau \in G_{\mathbb{C}}$, we have the following identities.*

(i) $R(\mathrm{D}_{\tau\cdot\mathfrak{h},\overline{\partial}_L}) = R(\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}) - 2\partial\overline{\partial}\log|\tau|$.

(ii) $R(\mathrm{D}_{\mathfrak{h},\tau\cdot\overline{\partial}_L}) = R(\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}) + 2\partial\overline{\partial}\log|\tau|$.

Proof. (i): Let $s : U \to L$ be a local non-vanishing holomorphic section of L. We write $u := \mathfrak{h}(s, s)$ and let $\theta \in \Omega^1_U$ denote the connection form of the Chern connection $\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}$ with respect to s. Recall that $\theta = u^{-1}\partial u$. Therefore, the connection form θ' of the Chern connection $\mathrm{D}_{\tau\cdot\mathfrak{h},\overline{\partial}_L}$ with respect to s satisfies

$$\theta' = (|\tau|^2 u)^{-1}\partial(|\tau|^2 u) = \theta + 2\partial\log|\tau|.$$

The curvature thus becomes

$$d\theta' = d\theta - 2\overline{\partial}\partial\log|\tau|,$$

which proves (i). In order to prove (ii), we first remark that the connection

$$\mathrm{D} = \mathrm{D}_{\mathfrak{h},\overline{\partial}_L} + \tau^{-1}\overline{\partial}\tau$$

satisfies $\mathrm{D}'' = \mathrm{D}''_{\mathfrak{h},\tau\cdot\overline{\partial}_L}$ and thus so does

$$\nabla = \mathrm{D}_{\mathfrak{h},\overline{\partial}_L} + \tau^{-1}\overline{\partial}\tau - \overline{\tau}^{-1}\partial\overline{\tau}$$

as we have added a (1,0)-form. By definition, the Chern connection $\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}$ is compatible with $\mathfrak{h}$ and hence so is ∇ as we have added a purely imaginary 1-form. Therefore ∇ is compatible with $\mathfrak{h}$ and satisfies $\nabla'' = \mathrm{D}''_{\mathfrak{h},\tau\cdot\overline{\partial}_L}$, so it must be the Chern connection $\mathrm{D}_{\mathfrak{h},\tau\cdot\overline{\partial}_L}$. For the curvature we obtain

$$R(\mathrm{D}_{\mathfrak{h},\tau\cdot\overline{\partial}_L}) = R(\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}) + d(\tau^{-1}\overline{\partial}\tau - \overline{\tau}^{-1}\partial\overline{\tau}) = R(\mathrm{D}_{\mathfrak{h},\overline{\partial}_L}) + 2\partial\overline{\partial}\log|\tau|. \qquad \square$$

We now have the following proposition.

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PROPOSITION 4.2. *Let $L \to M$ be a complex line bundle on the oriented Riemannian 2-manifold (M, g) and let $\ell := \deg(L)/|\chi(M)|$. Then the triple $(\mathcal{h}, \overline{\partial}_L, \varphi)$ satisfies*

$$R(\mathrm{D}) + \tfrac{1}{2}\varphi \wedge \varphi^* + i\ell\Omega_g = 0 \quad \textit{and} \quad \overline{\partial}_L \varphi = 0$$

if and only if $(\tau \cdot \mathcal{h}, \tau \cdot \overline{\partial}_L, \tau \cdot \varphi)$ does.

Proof. We observe that, for all $v, w \in TM$,

$$((\tau \cdot \varphi) \wedge (\tau \cdot \varphi)^{*_{\tau \cdot \mathcal{h}}})(v, w) = |\tau|^2 \mathcal{h}(\tau^{-1}\varphi(w), \tau^{-1}\varphi(v))$$
$$- |\tau|^2 \mathcal{h}(\tau^{-1}\varphi(v), \tau^{-1}\varphi(w)) = |\tau|^2 \tau^{-1}\overline{\tau^{-1}}(\varphi \wedge \varphi^{*_{\mathcal{h}}})(v, w) = (\varphi \wedge \varphi^{*_{\mathcal{h}}})(v, w)$$

so that $\varphi \wedge \varphi^* \in \Omega^{1,1}$ is invariant under complex gauge transformations. Now Lemma 4.1 immediately implies that $R(\mathrm{D}_{\mathcal{h},\overline{\partial}_L}) = R(\mathrm{D}_{\tau \cdot \mathcal{h}, \tau \cdot \overline{\partial}_L})$ and thus shows the invariance of the first equation. Likewise, we immediately obtain

$$(\tau \cdot \overline{\partial}_L)(\tau \cdot \varphi) = \tau \cdot \overline{\partial}_L \varphi,$$

so that the equation

$$\overline{\partial}_L \varphi = 0$$

is preserved under the action of the complex gauge group. □

4.2. *The vortex equations on a root of SM.* Since L has positive degree and $\chi(M) < 0$, there exist unique positive coprime integers (m, n) so that we have an isomorphism $L^n \simeq K^m$ of complex line bundles. We fix an nth root $SM^{1/n}$ of the unit tangent bundle SM of (M, g) and let $K^{1/n}$ denote the corresponding nth root of K, so that we have an isomorphism $\mathcal{Z} : L \to K^{m/n}$ of complex line bundles. Note that such a root exists since n divides $\chi(M)$. We equip $SM^{1/n}$ with the generalized Riemannian structure $(X, H, \mathbb{V})$ as in Example 2.3. We may write $\mathcal{h} = e^{2f}\mathcal{h}_0$ for a unique smooth real-valued function f on M. Abusing notation, we also use the letter f to denote the pullback of f to $SM^{1/n}$. Recall that the space of del-bar operators on a line bundle $L \to M$ is an affine space modelled on $\Omega^{0,1}$. Therefore, without losing generality, we can assume that there exists a 1-form θ on M so that

$$\overline{\partial}_L = \overline{\partial}_{K^{m/n}} - \ell\,\theta^{0,1}, \tag{4.4}$$

where $\theta^{0,1} = \frac{1}{2}(\theta - i \star_g \theta) \in \Omega^{0,1}$ denotes the (0,1)-part of θ and $\star_g$ is the Hodge-star with respect to g. We may also think of θ as a real-valued function on SM and, abusing notation, we also write θ to denote its pullback to $SM^{1/n}$. Note that the function θ on $SM^{1/n}$ satisfies $\mathbb{V}\mathbb{V}\theta = -\theta$. The pullback of $\theta^{0,1}$ to $SM^{1/n}$ can be expressed as $\frac{1}{2}(\theta + i\mathbb{V}\theta)\overline{\omega}$, where we write $\omega = \omega_1 + i\omega_2$ and $\overline{\omega} = \omega_1 - i\omega_2$. Therefore, the connection form ζ on $SM^{1/n}$ of the Chern connection D of $(L, \overline{\partial}_L, \mathcal{h})$ can be written as

$$\zeta = -i\ell\psi + w\omega - \frac{\ell}{2}(\theta + i\mathbb{V}\theta)\overline{\omega}$$

for some unique complex-valued function w on $SM^{1/n}$. On $SM^{1/n}$, the condition that D preserves $\mathcal{h} = e^{2f}\mathcal{h}_0$ translates to

$$d(e^{2f}\boldsymbol{B}_1\overline{\boldsymbol{B}_2}) = e^{2f}((d\boldsymbol{B}_1 + \zeta\boldsymbol{B}_1)\overline{\boldsymbol{B}_2} + \boldsymbol{B}_1(d\overline{\boldsymbol{B}_2} + \overline{\zeta}\overline{\boldsymbol{B}_2})),$$

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where $\boldsymbol{B}_1, \boldsymbol{B}_2$ represent arbitrary smooth sections of L. A straightforward calculation yields

$$\zeta = -i\ell\psi + \left(\frac{\ell}{2}(\theta - i\mathbb{V}\theta) + Xf - iHf\right)\omega - \frac{\ell}{2}(\theta + i\mathbb{V}\theta)\overline{\omega}.$$

The (1,0)-form φ with values in L is a section of $K \otimes L \simeq K^{(n+m)/n}$, so that, on $SM^{1/n}$, the form φ is represented by a complex-valued 1-form $\boldsymbol{\varphi}$, which we may write as

$$\boldsymbol{\varphi} = \ell\left(\frac{\mathbb{V}a}{1+\ell} + ia\right)\omega,$$

where the real-valued function a satisfies $\mathbb{V}\mathbb{V}a = -(1+\ell)^2 a$, since $\ell = m/n$.

LEMMA 4.3. *We have* $\overline{\partial}_L\varphi = 0$ *if and only if*

$$0 = X\mathbb{V}a - (1+\ell)Ha - \ell\theta\mathbb{V}a + \ell(1+\ell)a\mathbb{V}\theta. \tag{4.5}$$

Proof. Since M is complex one-dimensional, the condition $\overline{\partial}_L\varphi = 0$ is equivalent to φ being covariant constant with respect to the Chern connection D of $(L, \mathfrak{h}, \overline{\partial}_L)$. On $SM^{1/n}$, this translates to

$$0 = d\boldsymbol{\varphi} + \zeta \wedge \boldsymbol{\varphi}.$$

Since ζ defines a connection on L, terms involving ψ will cancel each other out and hence we can compute modulo ψ. We obtain

$$\zeta \wedge \boldsymbol{\varphi} = \frac{\ell^2}{2}\left(\frac{\mathbb{V}a}{(1+\ell)} + ia\right)(\theta + i\mathbb{V}\theta)\,\omega \wedge \overline{\omega} \mod \psi.$$

We define

$$W_\pm = \tfrac{1}{2}(X \mp iH).$$

Note that $(W_+, W_-, \mathbb{V})$ is the dual basis to $(\omega, \overline{\omega}, \psi)$. Hence, we obtain

$$d\boldsymbol{\varphi} = \ell W_-\left(\frac{\mathbb{V}a}{1+\ell} + ia\right)\overline{\omega} \wedge \omega = -\frac{\ell}{2}(X + iH)\left(\frac{\mathbb{V}a}{1+\ell} + ia\right)\omega \wedge \overline{\omega} \mod \psi.$$

The vanishing of the imaginary part of $d\boldsymbol{\varphi} + \zeta \wedge \boldsymbol{\varphi}$ is thus equivalent to

$$\begin{aligned} 0 &= \frac{\ell}{1+\ell}X\mathbb{V}a - \ell Ha - \frac{\ell^2}{1+\ell}\theta\mathbb{V}a + \ell^2 a\mathbb{V}\theta \\ &= \frac{\ell}{1+\ell}(X\mathbb{V}a - (1+\ell)Ha - \ell\theta\mathbb{V}a + \ell(1+\ell)a\mathbb{V}\theta), \end{aligned}$$

as claimed.

Conversely, if a, θ satisfy (4.5), then by applying $\mathbb{V}$ and using the commutator relations (2.2), as well as $\mathbb{V}\mathbb{V}a = -(1+\ell)^2 a$ and $\mathbb{V}\mathbb{V}\theta = -\theta$, we easily recover that the real part of $d\boldsymbol{\varphi} + \zeta \wedge \boldsymbol{\varphi}$ must vanish as well. □

Writing

$$\boldsymbol{A} := \frac{\mathbb{V}a}{1+\ell} + ia,$$

we obtain the following lemma.

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LEMMA 4.4. *We have* $R(\mathrm{D}) + \frac{1}{2}\varphi \wedge \varphi^* + i\ell\Omega_g = 0$ *if and only if*

$$K_g + X\theta + H\mathbb{V}\theta = -1 + \ell e^{2f}|\boldsymbol{A}|^2 - \frac{1}{\ell}(XXf + HHf). \tag{4.6}$$

Proof. Observe that $\varphi^* \in \Omega^{0,1}(L^{-1})$ is represented by

$$\boldsymbol{\varphi}^* = e^{2f}\overline{\boldsymbol{\varphi}} = e^{2f}\ell\overline{\boldsymbol{A}}\overline{\omega}$$

so that $\varphi \wedge \varphi^*$ is represented by

$$\boldsymbol{\varphi} \wedge \boldsymbol{\varphi}^* = \ell^2 e^{2f}|\boldsymbol{A}|^2 \omega \wedge \overline{\omega}.$$

Note that the pullback to $SM^{1/n}$ of the area form Ω_g of g becomes $(i/2)\omega \wedge \overline{\omega}$. Again, since ζ is the connection form of a connection, the ψ-terms will cancel each other out in the curvature expression $d\zeta$. We obtain

$$\begin{aligned} d\zeta &= -\frac{\ell}{2}\bigg(K_g + W_-(\theta - i\mathbb{V}\theta) + W_+(\theta + i\mathbb{V}\theta) + \frac{4}{\ell}W_-W_+f\bigg)\omega \wedge \overline{\omega} \\ &= -\frac{\ell}{2}\bigg(K_g + X\theta + H\mathbb{V}\theta + \frac{1}{\ell}(XXf + HHf)\bigg)\omega \wedge \overline{\omega}, \end{aligned}$$

where we use that $Xf - iHf = 2W_+f$ and the structure equation

$$d\psi = -\tfrac{i}{2}K_g\omega \wedge \overline{\omega}.$$

In total, we get

$$\begin{aligned} d\zeta + \frac{1}{2}\boldsymbol{\varphi} \wedge \boldsymbol{\varphi}^* + i\ell\frac{i}{2}\omega \wedge \overline{\omega} = -\frac{\ell}{2}\bigg(&K_g + X\theta + H\mathbb{V}\theta + \frac{1}{\ell}(XXf + HHf) \\ &- \ell e^{2f}|\boldsymbol{A}|^2 + 1\bigg)\omega \wedge \overline{\omega} = 0, \end{aligned}$$

which proves the claim. □

4.3. *Fractional differentials.* Note that we may think of φ/ℓ as a section of $K \otimes L \simeq K^{(n+m)/n}$, which we denote by A. Thus, we may interpret A as a differential of fractional degree $(n+m)/n = 1 + \ell$. Recall that the choice of an nth root $SM^{1/n}$ of SM equips $K^{(n+m)/n}$ with a Hermitian bundle metric which we denote by $\hbar_0$. Defining $|A|_g^2 := \hbar_0(A, A)$, the pullback of the function $|A|_g^2$ to $SM^{1/n}$ is $|\boldsymbol{A}|^2$. Moreover, the co-differential $\delta_g\theta$ of θ with respect to g pulls back to $SM^{1/n}$ to become $-X\theta - H\mathbb{V}\theta$ and the Laplacian $\Delta_g f$ of f with respect to g pulls back to $SM^{1/n}$ to become $XXf + HHf$. Using this notation, equation (4.6) can be written as

$$K_g - \delta_g\theta = -1 + \ell e^{2f}|A|_g^2 - \frac{1}{\ell}\Delta_g f.$$

Observe also that, since $\overline{\partial}_L\varphi = 0$, the equation (4.4) implies

$$\overline{\partial}_{K^{1+\ell}}A = \ell\,\theta^{0,1} \otimes A.$$

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4.4. *The thermostat.* In order to associate a thermostat on $SM^{1/n}$ with a solution of the vortex equation, we first consider, as a motivating example, the case $L = K^2$. In this case, $n = 1$ and $m = 2$ so that no choice of a root of SM is necessary. We may take $\overline{\partial}_L$ to be the del-bar operator on K^2 induced by the metric g: that is, we choose θ to vanish identically. Furthermore, we choose $\hbar$ to be $\hbar_0$ so that f vanishes identically as well. Thinking of φ as a section of $K \otimes L \simeq K^3$, we obtain a cubic differential A, and the vortex equations become

$$K_g = -1 + 2|A|_g^2 \quad \text{and} \quad \overline{\partial}_{K^3} A = 0.$$

In particular, the cubic differential A is holomorphic with respect to the standard holomorphic line bundle structure on K^3. Now observe that L admits a square root $L^{1/2} \simeq K$ and hence we may interpret $\varphi/2$ as a section of $K \otimes \mathrm{Hom}(L^{-1/2}, L^{1/2})$. Using the Hermitian metric induced by $\hbar_0$ on $L^{1/2} \simeq K$, we may identify $L^{1/2} \simeq \overline{L^{-1/2}}$. As a real vector bundle, $\overline{L^{-1/2}}$ is isomorphic to $L^{-1/2}$. Therefore, we may interpret $\varphi/2$ as a 1-form on M with values in the endomorphisms of $L^{-1/2}$, thought of as a real vector bundle. Identifying $\mathbb{C} \simeq \mathbb{R}^2$ in the usual way, multiplication with the complex number z, thought of as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, has matrix representation

$$\begin{pmatrix} \mathrm{Re}\, z & -\mathrm{Im} z \\ \mathrm{Im} z & \mathrm{Re}\, z \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^2$. Taking into account the identification $L^{1/2} \simeq \overline{L^{-1/2}}$, which just amounts to complex conjugation, the 1-form $\varphi/2$ is thus represented by

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{Re}\, \boldsymbol{\varphi} & -\mathrm{Im} \boldsymbol{\varphi} \\ \mathrm{Im} \boldsymbol{\varphi} & \mathrm{Re}\, \boldsymbol{\varphi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathrm{Re}\, \boldsymbol{\varphi} & -\mathrm{Im} \boldsymbol{\varphi} \\ -\mathrm{Im} \boldsymbol{\varphi} & -\mathrm{Re} \boldsymbol{\varphi} \end{pmatrix}.$$

The Chern connection on L induces a connection on $L^{-1/2}$ whose connection form is $-(1/2)\zeta$. Adding $\varphi/2$ to this connection, thought of as a connection on the real vector bundle $L^{-1/2}$, we obtain a connection ∇ with connection form

$$\Upsilon = (\Upsilon^i_j) = -\frac{1}{2}\begin{pmatrix} \mathrm{Re}(\zeta - \boldsymbol{\varphi}) & -\mathrm{Im}(\zeta - \boldsymbol{\varphi}) \\ \mathrm{Im}(\zeta + \boldsymbol{\varphi}) & \mathrm{Re}(\zeta + \boldsymbol{\varphi}) \end{pmatrix}.$$

Since $L^{-1/2} \simeq K^{-1}$, the vector bundle $L^{-1/2}$, as a real vector bundle, is isomorphic to the tangent bundle of M. Thus Υ defines a connection ∇ on TM, and in [26, Lemma 3.1] it is shown that the orbits of the thermostat ϕ on SM defined by the condition $F \lrcorner \Upsilon^2_1 = 0$ project to M to become the geodesics of ∇, when ignoring the parametrization.

Remark 4.5. The connection ∇ defines a properly convex projective structure on M whose associated Hilbert geodesic flow is a C^1 reparametrization of ϕ. We refer the reader to [25] and references therein for details.

In general, L will not admit a square root, but we may nonetheless formally carry out the same construction, except that now the identification $L^{1/2} \simeq \overline{L^{-1/2}}$ needs to amount to the metric $e^f h_0$ induced by h on the formal root $L^{1/2}$. We may thus define

$$\Upsilon = (\Upsilon^i_j) = -\frac{1}{2}\begin{pmatrix}\mathrm{Re}\,\zeta & -\mathrm{Im}\zeta\\ \mathrm{Im}\,\zeta & \mathrm{Re}\,\zeta\end{pmatrix} + \frac{1}{2}\begin{pmatrix}e^f & 0\\ 0 & -e^f\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,\boldsymbol{\varphi} & -\mathrm{Im}\boldsymbol{\varphi}\\ \mathrm{Im}\,\boldsymbol{\varphi} & \mathrm{Re}\,\boldsymbol{\varphi}\end{pmatrix}$$
$$= -\frac{1}{2}\begin{pmatrix}\mathrm{Re}(\zeta - e^f\boldsymbol{\varphi}) & -\mathrm{Im}(\zeta - e^f\boldsymbol{\varphi})\\ \mathrm{Im}(\zeta + e^f\boldsymbol{\varphi}) & \mathrm{Re}(\zeta + e^f\boldsymbol{\varphi})\end{pmatrix}. \tag{4.7}$$

Note that the vortex equations can be written as

$$d\zeta = \frac{\ell}{2}\omega\wedge\overline{\omega} - \frac{1}{2}e^{2f}\boldsymbol{\varphi}\wedge\overline{\boldsymbol{\varphi}} \quad \text{and} \quad d\boldsymbol{\varphi} = -\zeta\wedge\boldsymbol{\varphi}. \tag{4.8}$$

We also obtain

$$d\omega = \big(\zeta/\ell + \tfrac{1}{2}(\theta + i\mathbb{V}\theta)\overline{\omega}\big)\wedge\omega.$$

From (4.8), we easily conclude that

$$d\Upsilon + \Upsilon\wedge\Upsilon = \frac{i}{4}\begin{pmatrix}0 & -\ell\\ \ell & 0\end{pmatrix}\omega\wedge\overline{\omega}.$$

Again, in formal analogy to the case $L = K^2$, we obtain a thermostat ϕ on $SM^{1/n}$ by requiring that $F \lrcorner \Upsilon^2_1 = 0$. Using the notation above,

$$\lambda = e^f a - \mathbb{V}\theta - \frac{1}{\ell}Hf.$$

Remark 4.6. (Gauge invariance) Recall that the vortex equations are invariant under the action of the complex gauge group $\mathrm{G}_{\mathbb{C}}$. It is thus natural to ask how the gauge group affects the associated thermostat. Choosing $\tau = e^w$ for some smooth real-valued function w on M, the equations (4.1), (4.2) and (4.3) imply that the triple (A, θ, f) is replaced by

$$(A, \theta, f) \mapsto (\hat{A}, \hat{\theta}, \hat{f}) = \Big(e^{-w}A, \theta - \frac{1}{\ell}dw, f + w\Big).$$

Let $\hat{\lambda}$ be defined with respect to $(\hat{A}, \hat{\theta}, \hat{f})$. Then we obtain

$$\hat{\lambda} = e^{\hat{f}}\hat{a} - \mathbb{V}\hat{\theta} - \frac{1}{\ell}H\hat{f} = e^{f+w}e^{-w}a - \mathbb{V}\Big(\theta - \frac{1}{\ell}dw\Big) - \frac{1}{\ell}H(f + w) = \lambda,$$

where we use that $\mathbb{V}dw = Hw$ when we think of dw as a function on $SM^{1/n}$. It follows that the thermostat associated with a solution of the vortex equations is invariant under the action of the real part of the gauge group $\mathrm{G}_{\mathbb{C}}$. Therefore, without losing generality, we can assume that f vanishes identically: that is, $h = h_0$. Note, however, that the unitary part G does affect the associated thermostat.

5. *Proof of Theorems A and B*

Summarizing §4, given a solution $(h, \overline{\partial}_L, \varphi)$ to the vortex equations for a complex line bundle $L \to (M, g)$ and upon fixing an nth root $SM^{1/n}$ of SM, we obtain a *vortex thermostat* on $SM^{1/n}$. After possibly applying a (non-unitary) gauge transformation to $(h, \overline{\partial}_L, \varphi)$, we can assume that the thermostat ϕ arises from $\lambda = a - \mathbb{V}\theta$, where a encodes a fractional differential on M, that is, a section A of $K^{(m+n)/n}$ and θ a 1-form on M so

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that we have the equations

$$K_g - \delta_g \theta = -1 + \ell |A|_g^2 \quad \text{and} \quad \overline{\partial} A = \ell\, \theta^{0,1} \otimes A, \tag{5.1}$$

where, for simplicity of notation, we write $\overline{\partial}$ for $\overline{\partial}_{K^{(m+n)/n}}$ and where $\ell = m/n$. Thus, by Lemma 4.3 and Lemma 4.4, our set-up consists of $(X, H, \mathbb{V})$ on $SM^{1/n}$ as well as real-valued functions a, θ satisfying $\mathbb{V}\mathbb{V}a = -(1+\ell)^2 a$ and $\mathbb{V}\mathbb{V}\theta = -\theta$ so that

$$K_g = -1 - X\theta - H\mathbb{V}\theta + \ell |\boldsymbol{A}|^2, \tag{5.2}$$

$$\frac{X\mathbb{V}a}{1+\ell} = Ha + \frac{\ell\theta\mathbb{V}a}{1+\ell} - \ell a \mathbb{V}\theta, \tag{5.3}$$

where ℓ is a positive rational number and $\boldsymbol{A} = (\mathbb{V}a/(1+\ell)) + ia$.

5.1. *Dominated splitting.* Applying Theorem 3.3 we obtain the following theorem.

THEOREM A. *Every vortex thermostat admits a dominated splitting. Moreover, if all closed orbits of ϕ are hyperbolic saddles, then ϕ is Anosov.*

Proof. Using Theorem 3.3, we need to show that there exists a smooth function $p : SM^{1/n} \to \mathbb{R}$ so that

$$\kappa_p = \kappa + Fp + p(p - \mathbb{V}\lambda) < 0.$$

Recall that $\lambda = a - \mathbb{V}\theta$. Taking $p = \theta + \mathbb{V}a/(1+\ell)$, we compute

$$\begin{aligned}
\kappa_p - \kappa &= F\left(\theta + \frac{\mathbb{V}a}{1+\ell}\right) - \left(\theta + \frac{\mathbb{V}a}{1+\ell}\right)\left(\theta + \frac{\mathbb{V}a}{1+\ell} - \mathbb{V}a + \mathbb{V}\mathbb{V}\theta\right) \\
&= X\theta + Ha + \frac{\ell\theta\mathbb{V}a}{1+\ell} - \ell a\mathbb{V}\theta + \lambda\mathbb{V}p - \ell\left(\theta + \frac{\mathbb{V}a}{1+\ell}\right)\frac{\mathbb{V}a}{1+\ell} \\
&= X\theta + Ha - (1+\ell)a^2 - (\mathbb{V}\theta)^2 + 2a\mathbb{V}\theta - \ell\left(\frac{\mathbb{V}a}{1+\ell}\right)^2 \\
&= X\theta + Ha - \ell|\boldsymbol{A}|^2 - a^2 - (\mathbb{V}\theta)^2 + 2a\mathbb{V}\theta \\
&= -1 - K_g - H\mathbb{V}\theta + Ha - a^2 - (\mathbb{V}\theta)^2 + 2a\mathbb{V}\theta \\
&= -1 - (K_g - H\lambda + \lambda^2) = -1 - \kappa,
\end{aligned}$$

where we have used that $\mathbb{V}\mathbb{V}\theta = -\theta$ and $\mathbb{V}\mathbb{V}a = -(1+\ell)^2 a$ as well as (3.5), (5.2) and (5.3). We conclude that $\kappa_p = -1$ and the existence of a dominated splitting follows.

Finally, the addendum regarding the Anosov property when the closed orbits of ϕ are hyperbolic saddles is a consequence of [1, Theorem B]. Indeed, in our situation, the invariant normally hyperbolic irrational tori cannot arise since V must be transversal to them. If we had one such torus T, then the projection map $\pi_n : SM^{1/n} \to M$ restricted to T would be a local diffeomorphism, which is absurd since $\chi(M) < 0$. □

5.2. *The Anosov property.* While we have an isomorphism $\mathcal{Z} : L \to K^{m/n}$ of complex line bundles, the two line bundles need not be isomorphic as holomorphic line bundles. We do, however, obtain the following theorem.

THEOREM B. *Suppose that $\mathcal{Z} : L \to K^{m/n}$ is an isomorphism of holomorphic line bundles. Then the associated vortex thermostat is Anosov.*

Recall from (4.4) that we write $\overline{\partial}_L = \overline{\partial}_{K^{m/n}} - \ell\,\theta^{0,1}$ for some 1-form θ on M. The isomorphism $\mathcal{Z}$ being an isomorphism of holomorphic line bundles translates to θ vanishing identically. Henceforth, we restrict to the case where $\theta \equiv 0$, so that the equations (5.1) become

$$K_g = -1 + \ell|A|_g^2 \quad \text{and} \quad \overline{\partial} A = 0. \tag{5.4}$$

We start with the following comparison lemma.

LEMMA 5.1. *Let h be the positive Hopf solution of $Fh + h^2 + Bh - 1 = 0$. Then*

$$\frac{-c + \sqrt{c^2+4}}{2} \leqslant h \leqslant \frac{c + \sqrt{c^2+4}}{2},$$

where $c = \max|B|$ and $B = ((1-\ell)/(1+\ell))\mathbb{V}a$.

Proof of Lemma 5.1. We fix $(x, v) \in SM^{1/n}$. Recall from §3 that the existence of a dominated splitting implies that the positive Hopf solution h may be constructed using the limiting procedure

$$h(x, v) = \lim_{R\to\infty} \eta_R(0),$$

where, for $R > 0$, the function η_R denotes the solution to the ODE

$$\dot{\eta}(t) + \eta^2(t) + B(\phi_t(x, v))\eta(t) - 1 = 0$$

with $\eta_R(-R) = 0$. Since $B \geqslant -c$ and h is positive,

$$\dot{\eta} = -\eta^2 - B\eta + 1 \leqslant -\eta^2 + c\eta + 1.$$

Hence, if γ solves the constant coefficients Riccati equation

$$\dot{\gamma} + \gamma^2 - c\gamma - 1 = 0,$$

then $\eta(t) \leqslant \gamma(t)$ for $t \geqslant t_0$ provided $\eta(t_0) = \gamma(t_0)$ by ODE comparison. The solution γ_R to $\dot{\gamma} + \gamma^2 - c\gamma - 1 = 0$ with $\gamma_R(-R) = 0$ is given by

$$\gamma_R(t) = \frac{1 - e^{(-R-t)/E}}{-C_- + C_+ e^{(-R-t)/E}},$$

where

$$C_\pm = \frac{c \pm \sqrt{c^2+4}}{2}$$

and $E = 1/(C_+ - C_-)$. Therefore

$$\eta_R(0) \leqslant \gamma_R(0) \to -1/C_- = C_+$$

as $R \to \infty$ and thus $h(x, v) \leqslant (c + \sqrt{c^2+4})/2$.

The lower bound can also be proved in the same way. Since $B \leqslant c$,

$$\dot{\eta} = -\eta^2 - B\eta + 1 \geqslant -\eta^2 - c\eta + 1.$$

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And now we compare with solutions of

$$\dot{\gamma} + \gamma^2 + c\gamma - 1 = 0,$$

in particular, those γ_R with $\gamma_R(-R) = \infty$. One gets

$$\eta_R(0) \geqslant \gamma_R(0) \to \frac{-c + \sqrt{c^2+4}}{2}$$

as $R \to \infty$ and thus $h(x, v) \geqslant (-c + \sqrt{c^2+4})/2$. □

For what follows, we need a bound on $|A|_g^2$.

LEMMA 5.2. *Suppose* (g, A) *satisfies* $K_g = -1 + \ell|A|_g^2$ *and* $\overline{\partial} A = 0$. *Then* $K_g < 0$.

In the case where A is a differential of integral degree $d \geqslant 2$, the lemma was proved in [25]. It is easy to check that the proof also holds in the case of a differential of fractional degree $d > 1$. We refer the reader to [25, Lemma 5.2] for details.

We are now ready to prove Theorem B.

Proof of Theorem B. We already know that the flow admits a dominated splitting. To prove the Anosov property, we shall use Lemma 3.5. We will prove that, in the range $\ell \geqslant 1$, our flows fit alternative (1) and that, for $0 < \ell \leqslant 1$, they fit alternative (2). We shall prove the claims for the unstable bundle. The proofs for the stable bundle are analogous.

We note that Lemma 5.2 gives

$$-1 < \frac{\sqrt{\ell}}{1+\ell}\mathbb{V}a < 1. \tag{5.5}$$

Also note that, for our thermostat $p = \mathbb{V}a/(1+\ell)$, $\kappa_p = -1$ and $h = r^u - p$.

First, assume that $\ell \geqslant 1$. We shall prove that $r^u > 0$. This is equivalent to

$$h + \frac{\mathbb{V}a}{1+\ell} > 0. \tag{5.6}$$

In view of (5.5) and (5.6), it is enough to prove that

$$h \geqslant 1/\sqrt{\ell}.$$

From the definition of c in Lemma 5.1 and the bound $(\sqrt{\ell}/(1+\ell))\mathbb{V}a < 1$, we derive that $c \leqslant (1-\ell)/\sqrt{\ell}$. Hence

$$\frac{-c + \sqrt{c^2+4}}{2} \geqslant 1/\sqrt{\ell}$$

and the desired bound follows from Lemma 5.1.

Now assume that $0 < \ell \leqslant 1$. Condition (2) in Lemma 3.5 for r^u becomes

$$\left(\frac{\ell}{1+\ell}\right)\mathbb{V}a + 1/h > 0. \tag{5.7}$$

In view of (5.5) and (5.7), it is enough to prove that

$$h \leqslant 1/\sqrt{\ell}.$$

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From the definition of c in Lemma 5.1 and the bound $(\sqrt{\ell}/(1+\ell))\mathbb{V}a < 1$, we derive that $c \leqslant (1-\ell)/\sqrt{\ell}$. Hence

$$\frac{c+\sqrt{c^2+4}}{2} \leqslant 1/\sqrt{\ell}$$

and the desired bound follows from Lemma 5.1. □

Remark 5.3. As we have mentioned in the introduction, in the special case where A is a cubic holomorphic differential, a solution (g, A) to (5.4) gives rise to a properly convex projective structure on M. The monodromy representation of such a properly convex projective structure is an example of an *Anosov representation* as introduced by Labourie [21]. In recent work [5], Bochi, Potrie and Sambarino show how Anosov representations can be used to construct certain cocycles admitting a dominated splitting. However, at the time of writing, it is unclear whether there is any relationship between [5] and our construction that goes beyond the special case of cubic holomorphic differentials.

6. *Examples*

Let M be a closed oriented surface equipped with a hyperbolic metric g_0. Assume, furthermore, that the unit tangent bundle SM of (M, g_0) admits an nth root $SM^{1/n}$ so that, correspondingly, we have an nth root $K^{1/n}$ of the canonical bundle K of (M, g_0). Let m be a positive integer and write $\ell = m/n$. We equip $K^{1+\ell}$ with the holomorphic structure determined by g_0: that is, in our previous notation, we choose $\theta \equiv 0$. Suppose A is a holomorphic differential of fractional degree $1+\ell$. Note that such differentials exist by the Riemann–Roch theorem. In order to obtain one of our Anosov flows, we must thus find a metric g in the conformal equivalence class of g_0 so that

$$K_g = -1 + \ell |A|_g^2.$$

Under a conformal change $g_0 \mapsto e^{2u} g_0$ with $u \in C^\infty(M)$, the norm $|A|^2_{g_0}$ changes as

$$|A|^2_{e^{2u}g_0} = e^{-2(1+\ell)u}|A|^2_{g_0}.$$

We also have the identity

$$K_{e^{2u}g_0} = e^{-2u}(-1-\Delta u)$$

for the change of the Gauss curvature under conformal change. Here Δ denotes the Laplace operator with respect to the hyperbolic metric. Writing $g = e^{2u}g_0$, we thus obtain the PDE

$$\Delta u = -1 + e^{2u} - \ell e^{-2\ell u}\alpha$$

with $\alpha := |A|^2_{g_0}$. Since $\alpha \geqslant 0$, this quasi-linear elliptic PDE admits a unique smooth solution which can be obtained by standard methods (see, for instance, [32, Proposition 1.9]). Therefore, we obtain a solution to the vortex equations and an associated Anosov flow.

Remark 6.1. Recall that every closed oriented hyperbolic Riemann surface (M, g_0) admits a *Fuchsian model*, which realizes its unit tangent bundle SM as a quotient $\Gamma \setminus \mathrm{PSL}(2,\mathbb{R})$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a *Fuchsian group*, that is, a discrete torsion-free subgroup

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of PSL(2, $\mathbb{R}$). Therefore, we obtain a square root $SM^{1/2} \simeq \tilde{\Gamma} \setminus \mathrm{SL}(2, \mathbb{R})$, where $\tilde{\Gamma} \subset \mathrm{SL}(2, \mathbb{R})$ denotes the preimage of Γ under the 2-fold cover $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$. Since the unit tangent bundles with respect to conformally equivalent metrics are isomorphic as principal SO(2)-bundles, we also obtain a square root of the unit tangent bundle for every metric in the conformal equivalence class of g_0. In particular, on every closed hyperbolic Riemann surface, we obtain an Anosov flow on $SM^{1/2}$ from a holomorphic differential A of fractional degree $1 + 1/2 = 3/2$. These flows are topologically orbit equivalent to the lift of a constant curvature geodesic flow [16], but they do not arise from the lift of a flow on SM.

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A. *Appendix. Variants of the vortex equations*

Instead of our variant of the vortex equations, we may also consider the following pair of equations on an oriented Riemannian 2-manifold (M, g) of negative Euler characteristic

$$K_g - \delta_g \theta = -1 + \ell e^{2f} |A|_g^2 - \frac{1}{k} \Delta_g f \quad \text{and} \quad \overline{\partial} A = k\, \theta^{0,1} \otimes A. \tag{A.1}$$

Here A is a differential of fractional degree $1 + \ell > 1$, $\theta \in \Omega^1$, $f \in C^\infty$ and k is a real constant. Notice that we recover our vortex equations by choosing $k = \ell$. We leave it as an exercise for the interested reader to check that, for the choice $c = 2(\ell + 1)$, the usual vortex equations (1.3) are equivalent to (A.1) when $k = \ell + 1$. Again, it is straightforward to verify that (A.1) are invariant under suitable gauge transformations. Namely, writing a gauge transformation as $\tau = e^{w + i\vartheta}$ for $w, \vartheta \in C^\infty$, we obtain a solution

$$\tau \cdot (A, \theta, f) = \left(e^{-(w + i\vartheta)} A, \theta - \frac{1}{k}(dw + \star_g d\vartheta), f + w \right)$$

to the above vortex equations from a solution (A, θ, f). As before, we obtain a thermostat on a suitable root $SM^{1/n}$ of SM, by defining

$$\lambda = e^f a - \mathbb{V}\theta - \frac{1}{\ell} Hf,$$

where we use notation as in §4. The thermostat is again invariant under real gauge transformations of the form $\tau = e^w$, so that we can assume that f vanishes identically. Thus

$$K_g - \delta_g \theta = -1 + \ell |A|_g^2 \quad \text{and} \quad \overline{\partial} A = k\, \theta^{0,1} \otimes A.$$

Taking $p = \theta + \mathbb{V}a/(1 + \ell)$, exactly as in the proof of Theorem A, we compute that

$$\kappa_p = -1 + (k - \ell) \operatorname{Re}((\theta + i\mathbb{V}\theta)\boldsymbol{A}),$$

where $\boldsymbol{A} = (\mathbb{V}a/(1 + \ell)) + ia$. For the usual vortex equations with $k = \ell + 1$, we thus obtain $\kappa_p = -1 + \operatorname{Re}((\theta + i\mathbb{V}\theta)\boldsymbol{A})$. Moreover, for the usual vortex equations, we have the bound $|A|^2 \leqslant 1/\ell$ (see [7, Proposition 5.2]). Thus, we still obtain a dominated splitting provided $|\theta + i\mathbb{V}\theta| < \sqrt{\ell}$.

Remark A.1. We do not know whether we still obtain a dominated splitting if the bound $|\theta + i\mathbb{V}\theta| < \sqrt{\ell}$ does not hold.

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