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Ergodic averages with prime divisor weights in L^1

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Abstract. We show that $\omega(n)$ and $\Omega(n)$, the number of distinct prime factors of n and the number of distinct prime factors of n counted according to multiplicity, are good weighting functions for the pointwise ergodic theorem in L^1. That is, if g denotes one of these functions and $S_{g,K} = \sum_{n\le K} g(n)$, then for every ergodic dynamical system $(X, \mathcal{A}, \mu, \tau)$ and every $f \in L^1(X)$,

$$\lim_{K\to\infty} \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n) f(\tau^n x) = \int_X f \, d\mu \quad \text{for } \mu \text{ almost every } x \in X.$$

This answers a question raised by Cuny and Weber, who showed this result for L^p, $p > 1$.

1. *Introduction*

In [**1**] Cuny and Weber investigated whether some arithmetic weights are good weights for the pointwise ergodic theorem in L^p. In this paper we show that the prime divisor functions ω and Ω are both good weights for the L^1 pointwise ergodic theorem. The same fact for the spaces L^p, $p > 1$ was proved in [**1**] and our paper answers a question raised in that paper. Recall that if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $\omega(n) = k$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_k$. We denote by g one of these functions. Given K, we put

$$S_{g,K} = \sum_{n\le K} g(n).$$

We suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $\tau : X \to X$ is a measure-preserving ergodic transformation. Given $f \in L^1(X)$, we consider the g-weighted ergodic averages

$$\mathcal{M}_{g,K} f(x) = \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n) f(\tau^n x). \tag{1}$$

We show that for $g = \omega$ or Ω these averages μ almost everywhere converge to $\int_X f\, d\mu$, that is, g is a good universal weight for the pointwise ergodic theorem in L^1. See Theorem 6. In proving this theorem we use maximal inequalities. Readers of our paper pointed out that the novel idea in our proof is that in our estimates instead of using a fixed moment of the appropriate averages (which is usually the first or the second moment) we use increasing averages. See Lemma 5 and the proof of Claim 9, especially (24) and (25). This approach of using different moment estimates might turn out to be useful in other situations as well.

For some similar ergodic theorems with other weights like the Möbius function, its absolute value or the Liouville function, we refer to the papers of El Abdalaoui *et al* [**3**] and of Rosenblatt and Wierdl [**8**].

2. *Preliminary results*

We recall [**5**, Theorem 430 on p. 72]

$$\sum_{n\le K} \omega(n) = K \log\log K + B_1 K + o(K) \quad \text{and} \tag{2}$$

$$\sum_{n\le K} \Omega(n) = K \log\log K + B_2 K + o(K). \tag{3}$$

Hence, for both cases we can assume that there exists a constant B (which depends on whether $g = \omega$ or $g = \Omega$) such that

$$\sum_{n\le K} g(n) = K \log\log K\bigg(1 + \frac{B}{\log\log K} + \frac{o(K)}{K \log\log K}\bigg). \tag{4}$$

From this it follows that there exists $C_S > 0$ such that, for all $K \in \mathbb{N}$,

$$\bigg(\sum_{n\le K} g(n)\bigg)^{\lfloor \log\ \log K\rfloor} = (S_{g,K})^{\lfloor \log\ \log K\rfloor} > C_S(K\lfloor \log\ \log K\rfloor)^{\lfloor \log\ \log K\rfloor}. \tag{5}$$

We need some information about the distribution of the functions ω and Ω. We use [**6**, (3.9) on p. 689] by Norton, which is based on a result of Halász [**4**], which is cited as [**6**, (3.8) Lemma]. Next we state [**6**, (3.9)] with $\delta = 0.1$ and $z = 2 - \delta = 1.9$.

PROPOSITION 1. *There exists a constant $\widetilde{C}$ such that, for every $K \ge 1$,*

$$\sum_{n\le K} 1.9^{\omega(n)} \le \sum_{n\le K} 1.9^{\Omega(n)} \le \widetilde{C} K \exp(0.9 \cdot E(K)) \tag{6}$$

where $E(K) = \sum_{p\le K} (1/p)$.

Recall that by [**5**, Theorem 427],

$$E(K) = \sum_{p\le K} \frac{1}{p} = \log\log K + B_1 + o(1). \tag{7}$$

The constant B_1 is the same one which appears in (2). The way we will use this is the following: there exists a constant C_P such that for $K > 3$,

$$E(K) = \sum_{p\le K} \frac{1}{p} < C_P \log\log K. \tag{8}$$

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Combining this with (6), we obtain that for $g = \omega$ or Ω we have for $K > 3$,

$$\sum_{n \le K} 1.9^{g(n)} < \widetilde{C} \cdot K \cdot \exp(0.9 \cdot C_P \log\log K) \le C_H \cdot K \exp(0.9 \cdot C_P \lfloor \log\ \log K \rfloor), \tag{9}$$

with a suitable constant C_H not depending on K.

In [**1**] a result of Delange [**2**] was used to deduce [**1**, Theorem 2.7]. The result of Delange is the following theorem.

THEOREM 2. *For every $m \ge 1$, we have*

$$\sum_{n \le K} g(n)^m = K(\log\log K)^m + O(K(\log\log K)^{m-1}).$$

We were unable to use this result since the constant in $O(K(\log\log K)^{m-1})$ cannot be chosen not depending on $m \ge 1$.

Hence, we use (9) in the proof of the following lemma.

LEMMA 3. *There exists a constant C_Ω such that, for all $K \ge 16$,*

$$\sum_{n \le K} \omega(n)^{\lfloor \log\ \log K \rfloor} \le \sum_{n \le K} \Omega(n)^{\lfloor \log\ \log K \rfloor} < K(C_\Omega \lfloor \log\ \log K \rfloor)^{\lfloor \log\ \log K \rfloor}. \tag{10}$$

We remark that the assumption $K \ge 16$ implies that $\log\log K > 1.01 > 1$.

Proof. Since $\omega(n) \le \Omega(n)$, the first inequality is obvious in (10).

We assume that $K \ge 16$ is fixed and for ease of notation we put $\nu = \lfloor \log\ \log K \rfloor$. Set

$$N_{l,K} = \{n \le K : 2^l \nu \le \Omega(n) < 2^{l+1}\nu\}. \tag{11}$$

By (9), $N_{l,K} \cdot 1.9^{2^l \nu} < C_H K \exp(0.9 \cdot C_P \nu)$. This implies that

$$N_{l,K} < C_H K \cdot \exp((0.9 \cdot C_P - 2^l \log 1.9)\nu). \tag{12}$$

Since $\log 1.9 > 0.6$, we can choose l_0 such that, for $l \ge l_0$,

$$0.9 \cdot C_P - 2^l \log 1.9 + (l+1)\log 2 < -0.5 \cdot 2^l = -2^{l-1}. \tag{13}$$

From (12) and (13), we infer that

$$\sum_{n \le K} \Omega(n)^\nu < K \cdot (2\nu)^\nu + \sum_{l=1}^{\infty} N_{l,K}(2^{l+1}\nu)^\nu$$

$$\le\ K \cdot (2\nu)^\nu \sum_{l=1}^{l_0-1} K(2^{l+1}\nu)^\nu + \sum_{l=l_0}^{\infty} C_H K \nu^\nu \exp(((\log 2^{l+1}) + 0.9C_P - 2^l \log 1.9)\nu)$$

(using (13) with a suitable constant $C_1 > 2$)

$$< C_1^\nu K \nu^\nu + \sum_{l=l_0}^{\infty} C_H K \nu^\nu \exp(-2^{l-1}\nu) \tag{14}$$

(recalling that $\nu = \lfloor \log\ \log K \rfloor \ge \lfloor \log\ \log 16 \rfloor = 1$, with a suitable constant C_Ω)

$$< K\nu^\nu \left(C_1^\nu + C_H \sum_{l=l_0}^{\infty} \exp(-2^{l-1}) \right) < C_\Omega^\nu K \nu^\nu = K(C_\Omega \lfloor \log\ \log K \rfloor)^{\lfloor \log\ \log K \rfloor}. \qquad \square$$

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We need an elementary inequality stated in Lemma 4, which, as Pavel Zorin-Kranich pointed out to me, is a consequence of the generalized Hölder's inequality: if $v_j > 0$, $j = 1, \ldots, \nu$ and $\sum_{j=1}^{\nu} (1/v_j) = 1$, then

$$\sum_{i=1}^{K} \prod_{j=1}^{\nu} |a_{i,j}| \leq \prod_{j=1}^{\nu} \sqrt[v_j]{\sum_{i=1}^{K} |a_{i,j}|^{v_j}}. \tag{15}$$

LEMMA 4. *Suppose that $K, \nu \in \mathbb{N}$, $b_1, \ldots, b_K$ are non-negative numbers and we have permutations $\pi_j : \{1, \ldots, K\} \to \{1, \ldots, K\}$, $j = 1, \ldots, \nu$. Then*

$$b_{\pi_1(1)} \cdots b_{\pi_\nu(1)} + \cdots + b_{\pi_1(K)} \cdots b_{\pi_\nu(K)} \leq b_1^\nu + \cdots + b_K^\nu. \tag{16}$$

Proof. Indeed, setting $v_j = \nu$ for all j, by using (15), we obtain

$$\sum_{i=1}^{K} \prod_{j=1}^{\nu} |b_{\pi_j(i)}| \leq \prod_{j=1}^{\nu} \sqrt[\nu]{\sum_{i=1}^{K} |b_{\pi_j(i)}|^\nu} = b_1^\nu + \cdots + b_K^\nu. \qquad \square$$

According to some of my colleagues, it is more natural to give an elementary proof of Lemma 4 based on mathematical induction and on the simple fact that if $A > B \geq 0$ and $C > D \geq 0$ then from $(A - B)(C - D) \geq 0$ it follows that $AC + BD \geq AD + BC$. An earlier version of our paper contained such an argument, but for this final version we preferred the above shorter proof.

We will use the transference principle and hence we need to consider functions on the integers. Suppose that $\varphi : \mathbb{Z} \to [0, +\infty)$ is a function on the integers with compact/bounded support. Again g will denote ω or Ω. Put

$$M_{g,K}\varphi(j) = \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n)\varphi(j + n) \quad \text{for } j \in \mathbb{Z}.$$

First we prove a 'localized' maximal inequality.

LEMMA 5. *There exists a constant $C_g > 0$ such that, for any $\varphi : \mathbb{Z} \to [0, +\infty)$, $K \geq 16$ and $k \in \mathbb{Z}$,*

$$\sum_{j=1}^{K} (M_{g,K}\varphi(k + j))^{\lfloor \log \log K \rfloor} \leq \left(\sum_{j=2}^{2K} \varphi(k + j)\right)\left(\frac{C_g}{K} \sum_{j=2}^{2K} \varphi(k + j)\right)^{\lfloor \log \log K \rfloor - 1}. \tag{17}$$

Proof. Without limiting generality, we can suppose that $k = 0$ and $K \geq 16$ is fixed. We use again the notation $\nu = \nu_K = \lfloor \log \log K \rfloor$. We put

$$\widetilde{g}(n) = \widetilde{g}_K(n) = \begin{cases} g(n) & \text{if } 1 \leq n \leq K, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

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We need to estimate

$$\begin{aligned}
&\sum_{j=1}^{K}\bigg(\frac{1}{S_{g,K}}\sum_{n=1}^{K} g(n)\varphi(j+n)\bigg)^{\nu}\\
&\quad=\frac{1}{S_{g,K}^{\nu}}\sum_{j=1}^{K}\sum_{n_1=1}^{K}\cdots\sum_{n_\nu=1}^{K} g(n_1)\cdots g(n_\nu)\cdot\varphi(j+n_1)\cdots\varphi(j+n_\nu)\\
&\quad=\frac{1}{S_{g,K}^{\nu}}\sum_{n'=1}^{K}\sum_{j_1=2}^{2K}\cdots\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\widetilde{g}(n')\widetilde{g}(n'+j_2-j_1)\cdots\widetilde{g}(n'+j_\nu-j_1)\\
&\quad=\frac{1}{S_{g,K}^{\nu}}\sum_{j_1=2}^{2K}\cdots\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=1}^{K}\widetilde{g}(n')\widetilde{g}(n'+j_2-j_1)\cdots\widetilde{g}(n'+j_\nu-j_1)\\
&\quad\le\frac{1}{S_{g,K}^{\nu}}\sum_{j_1=2}^{2K}\cdots\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=-2K+3}^{3K-2}(\widetilde{g}(n'))^{\nu}\quad\text{(using Lemma 4 and (18))}\\
&\quad=\frac{1}{S_{g,K}^{\nu}}\sum_{j_1=2}^{2K}\cdots\sum_{j_\nu=2}^{2K}\varphi(j_1)\cdots\varphi(j_\nu)\cdot\sum_{n'=1}^{K}(g(n'))^{\nu}\\
&\quad\le K\cdot C_{\Omega}^{\nu}\nu^{\nu}\frac{1}{S_{g,K}^{\nu}}\bigg(\sum_{j=2}^{2K}\varphi(j)\bigg)^{\nu}\quad\text{(by using Lemma 3)}\\
&\quad< K\cdot C_{\Omega}^{\nu}\nu^{\nu}\frac{1}{C_S(K\nu)^{\nu}}\bigg(\sum_{j=2}^{2K}\varphi(j)\bigg)^{\nu}\quad\text{(by (5))}\\
&\quad<\bigg(\sum_{j=2}^{2K}\varphi(j)\bigg)\cdot\bigg(C_g\frac{1}{K}\sum_{j=2}^{2K}\varphi(j)\bigg)^{\nu-1}\quad\text{(with a suitable constant } C_g>0).\qquad\square
\end{aligned}$$

3. *Main result*

THEOREM 6. *For every ergodic dynamical system $(X, \mathcal{A}, \mu, \tau)$ and every $f \in L^1(X)$,*

$$\lim_{K\to\infty}\mathcal{M}_{g,K}f(x)=\int_X f\,d\mu\quad\textit{for } \mu \textit{ almost every } x\in X. \tag{19}$$

Proof. By [**1**, Theorem 2.5 and Remark 2.6] we know that ω and Ω are good weights for the pointwise ergodic theorem in L^p for $p > 1$. This means that we have a dense set of functions in L^1 for which the pointwise ergodic theorem holds. In [**1**, Theorem 2.5] it is not stated explicitly that the limit function of the averages $\mathcal{M}_{g,K}f$ is $\int_X f\,d\mu$, but from the proof of this theorem it is clear that $\mathcal{M}_{g,K}f$ not only converges almost everywhere, but its limit is indeed $\int_X f\,d\mu$ (at least for $f \in L^\infty(\mu)$). Indeed, from [**1**, (2.2)] it follows that $\mathcal{M}_{g,K}f$ can be written as the sum of an ordinary Birkhoff average of f and an error term which tends to zero as $K \to \infty$.

Hence, by a standard application of Banach's principle (see, for example, [**7**, p. 91]), the following weak L^1-maximal inequality proves Theorem 6. □

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PROPOSITION 7. *There exists a constant C_m such that, for every ergodic dynamical system $(X, \mathcal{A}, \mu, \tau)$ and for every $f \in L^1(\mu)$ and $\lambda \geq 0$,*

$$\mu\Big\{x : \sup_{K\geq 1} \mathcal{M}_{g,K} f(x) > \lambda\Big\} \leq C_m \frac{\|f\|_1}{\lambda}. \tag{20}$$

Proof of Proposition 7. By standard transference arguments, see, for example, [**8**, Ch. III], it is sufficient to establish a corresponding weak maximal inequality on the integers with $\lambda = 1$ for non-negative functions with compact support. Hence, this proof will be completed by Proposition 8 below. □

Thus, we need to state and prove the following maximal inequality.

PROPOSITION 8. *There exists a constant C_m such that, for every $\varphi : \mathbb{Z} \to [0, \infty)$ with compact support,*

$$\#\Big\{j : \sup_{K\in\mathbb{N}} M_{g,K}\varphi(j) > 1\Big\} \leq C_m \|\varphi\|_{\ell_1}.$$

It is enough to prove Proposition 8 along the subsequence $K = 2^l$, $l = 1, 2, \ldots$. This will be done in the following Claim 9. Set $M_l = M_{g,2^l}$.

CLAIM 9. *There exists a constant C'_m such that, for every $\varphi : \mathbb{Z} \to [0, +\infty)$ with compact support,*

$$\#\Big\{j : \sup_{l\in\mathbb{N}} M_l\varphi(j) > 1\Big\} \leq C'_m \|\varphi\|_{\ell_1}. \tag{21}$$

Proof of Proposition 8 based on Claim 9. Given $K \in \mathbb{N}$, choose $l_K \in \mathbb{N}$ such that $2^{l_K-1} < K \leq 2^{l_K}$. By (2) or (3), there exists a constant $C_R > 0$ not depending on K such that $S_{g,2^{l_K}} \leq C_R S_{g,K}$. We have

$$\begin{aligned} 1 < M_{g,K}\varphi(j) &= \frac{1}{S_{g,K}} \sum_{n=1}^{K} g(n)\varphi(j+n) \\ &\leq \frac{S_{g,2^{l_K}}}{S_{g,K}} \cdot \frac{1}{S_{g,2^{l_K}}} \sum_{n=1}^{2^{l_K}} g(n)\varphi(j+n) \leq C_R M_{g,2^{l_K}}\varphi(j). \end{aligned}$$

Hence, $1 < M_{g,K}\varphi(j)$ implies that $1/C_R < M_{g,2^{l_K}}\varphi(j) = M_{l_K}\varphi(j)$.

For any $\widetilde{\varphi} : \mathbb{Z} \to [0, +\infty)$ with compact support taking $\varphi = C_R\widetilde{\varphi}$, by Claim 9, we obtain

$$\begin{aligned} \#\Big\{j : \sup_{K\in\mathbb{N}} M_{g,K}\widetilde{\varphi}(j) > 1\Big\} &\leq \#\Big\{j : \sup_{l\in\mathbb{N}} M_l\varphi(j) > 1\Big\} \\ &\leq C'_m \|\varphi\|_{\ell_1} = C'_m C_R \|\widetilde{\varphi}\|_{\ell_1}. \end{aligned}$$ □

Proof of Claim 9. For ease of notation in this proof, when we speak about intervals we will speak about subintervals of the integers; for example, when we speak about a dyadic interval $(r2^l, (r+1)2^l]$ then in fact we mean the interval $(r2^l, (r+1)2^l] \cap \mathbb{Z}$.

For small l, say for $1 \le l \le 4$, it is easy to obtain an estimate needed for (20). Since $\varphi \ge 0$ is of compact support,

$$\|M_l\varphi\|_{\ell_1} = \sum_{j=-\infty}^{\infty} \frac{1}{S_{g,2^l}} \sum_{n=1}^{2^l} g(n)\varphi(j+n)$$
$$= \frac{1}{S_{g,2^l}} \sum_{n=1}^{2^l} g(n) \sum_{j=-\infty}^{\infty} \varphi(j+n) = \|\varphi\|_{\ell_1}$$

and hence $\#\{j : M_l\varphi(j) > 1\} \le \|M_l\varphi\|_{\ell_1} = \|\varphi\|_{\ell_1}$, which implies that

$$\#\Big\{j : \sup_{1\le l\le 4} M_l\varphi(j) > 1\Big\} \le 4\|\varphi\|_{\ell_1}. \tag{22}$$

Next suppose that $l > 4$. We consider the dyadic intervals $(r2^l, (r+1)2^l]$, $r \in \mathbb{Z}$. Recall that the constant C_g appeared in Lemma 5. We say that $r \in R_{l,+}$ if

$$\frac{1}{2^l} \sum_{j=r2^l+1}^{r2^l+2\cdot 2^l} \varphi(j) > \frac{1}{100 \cdot C_g}. \tag{23}$$

Otherwise, if $r \notin R_{l,+}$, we say that $r \in R_{l,-}$.

For $r \in R_{l,-}$, we use Lemma 5 and the negation of (23) to deduce that for $l > 4$,

$$\sum_{j=1}^{2^l} (M_l\varphi(r2^l + j))^{\lfloor \log\ \log 2^l \rfloor} < \bigg(\sum_{j=2}^{2\cdot 2^l} \varphi(r2^l + j)\bigg) \cdot \bigg(\frac{1}{100}\bigg)^{\lfloor \log\ \log 2^l \rfloor - 1}$$
$$\le 100^2 \bigg(\sum_{j=2}^{2\cdot 2^l} \varphi(r2^l + j)\bigg) \cdot \bigg(\frac{1}{100}\bigg)^{\log\log 2^l}$$
$$\le 100^2 \bigg(\sum_{j=2}^{2\cdot 2^l} \varphi(r2^l + j)\bigg) \cdot \exp(-(\log 100) \cdot \log\log 2^l)$$
$$\le 100^2 \bigg(\sum_{j=2}^{2\cdot 2^l} \varphi(r2^l + j)\bigg) \cdot \frac{6}{l^2}, \tag{24}$$

where we used that $4.61 \ge \log 100 \ge 4.60517$ and $\log\log 2 > -0.37$ implies that

$$\exp(-(\log 100) \cdot \log\log 2^l) = \exp(-(\log 100)((\log l) + \log\log 2))$$
$$= \exp(-(\log 100) \log\log 2) \cdot \exp(-(\log 100) \log l) < \frac{6}{l^2}.$$

Set $\mathcal{M}_l^* = \{j : M_l\varphi(j) > 1\}$ and $\mathcal{M}^* = \bigcup_l \mathcal{M}_l^*$.

If $r \in R_{l,-}$, then by (24),

$$\#(\mathcal{M}_l^* \cap (r2^l, (r+1)2^l]) \le \sum_{j=1}^{2^l} (M_l\varphi(r2^l + j))^{\lfloor \log\ \log 2^l \rfloor}$$
$$\le 6 \cdot 100^2 \cdot \frac{1}{l^2} \bigg(\sum_{j=2}^{2\cdot 2^l} \varphi(r2^l + j)\bigg).$$

Hence,

$$\#\left(\mathcal{M}_l^* \cap \bigcup_{r\in R_{l,-}} (r2^l, (r+1)2^l]\right) \le 12 \cdot 100^2 \frac{1}{l^2} \|\varphi\|_{\ell_1}$$

and

$$\#\left(\bigcup_l \left(\mathcal{M}_l^* \cap \bigcup_{r\in R_{l,-}} (r2^l, (r+1)2^l]\right)\right) \le 12 \cdot 100^2 \frac{\pi^2}{6} \|\varphi\|_{\ell_1}. \tag{25}$$

On the other hand,

$$\bigcup_{l>4} \bigcup_{r\in R_{l,+}} (r2^l, (r+1)2^l] \subset \bigcup_{l>4} \bigcup_{r\in R_{l,+}} [r2^l, (r+2)2^l]. \tag{26}$$

By the well-known property of coverings by subintervals, we can select a subsystem $\mathcal{I}_+^*$ of the intervals $\mathcal{I}_+ = \{[r2^l, (r+2)2^l] : l > 4, r \in R_{l,+}\}$ such that the subsystem $\mathcal{I}_+^*$ covers the same set as $\mathcal{I}_+$ and a point is covered by no more than two elements of $\mathcal{I}_+^*$, that is,

$$\sum_{I\in\mathcal{I}_+^*} \chi_I(j) \le 2 \quad \text{for all } j \in \mathbb{Z} \quad \text{and} \quad \bigcup_{I\in\mathcal{I}_+} I = \bigcup_{I\in\mathcal{I}_+^*} I. \tag{27}$$

From (23), it follows that if $[r2^l, (r+2)2^l] = I \in \mathcal{I}_+^*$, then, recalling that in this proof intervals denote subintervals of integers,

$$C_g \cdot 400 \sum_{j\in I} \varphi(j) > 4 \cdot 2^l > \#I.$$

Thus, by (27),

$$\#\left(\bigcup_{I\in\mathcal{I}_+} I\right) = \#\left(\bigcup_{I\in\mathcal{I}_+^*} I\right) < C_g \cdot 800 \|\varphi\|_{\ell_1}.$$

Hence, by (26),

$$\#\left(\bigcup_{l>4} \bigcup_{r\in R_{l,+}} (r2^l, (r+1)2^l]\right) \le C_g \cdot 800 \|\varphi\|_{\ell_1}.$$

From this, (22) and (25), it follows that

$$\#\mathcal{M}^* \le \left(4 + 12 \cdot 100^2 \frac{\pi^2}{6} + 800 C_g\right) \|\varphi\|_{\ell_1} = C'_m \|\varphi\|_{\ell_1}. \qquad \square$$

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REFERENCES

[**1**] C. Cuny and M. Weber. Ergodic theorems with arithmetical weights. *Israel J. Math.* **217** (2017), 139–180.
[**2**] H. Delange. Sur des formules de Atle Selberg. *Acta Arith.* **19** (1971), 105–146.
[**3**] E. H. El Abdalaoui, J. Kułaga-Przymus, M. Lemańczyk and T. De La Rue. The Chowla and the Sarnak conjectures from ergodic theory point of view. *Discrete Contin. Dyn. Syst.* **37**(6) (2017), 2899–2944.
[**4**] G. Halász. On the distribution of additive and the mean values of multiplicative arithmetic functions. *Studia Sci. Math. Hungar.* **6** (1971), 211–233.
[**5**] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 6th edn. Oxford University Press, Oxford, 2008.
[**6**] K. K. Norton. On the number of restricted prime factors of an integer. I. *Illinois J. Math.* **20**(4) (1976), 681–705.
[**7**] K. Petersen. *Ergodic Theory (Cambridge Studies in Advanced Mathematics, 2)*. Cambridge University Press, Cambridge, 1983.
[**8**] J. M. Rosenblatt and M. Wierdl. Pointwise ergodic theorems via harmonic analysis. *Ergodic Theory and its Connections with Harmonic Analysis (Alexandria, 1993) (London Mathematical Society Lecture Note Series, 205)*. Cambridge University Press, Cambridge, 1995, pp. 3–151.