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Consistency of Natural Relations on Sets

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The natural relations for sets are those definable in terms of the emptiness of the subsets corresponding to Boolean combinations of the sets. For pairs of sets, there are just five natural relations of interest, namely, strict inclusion in each direction, disjointness, intersection with the universe being covered, or not. Let N denote $\{1, 2, \ldots, n\}$ and $\binom{N}{2}$ denote $\{(i, j) \mid i, j \in N \text{ and } i < j\}$. A function μ on $\binom{N}{2}$ specifies one of these relations for each pair of indices. Then μ is said to be *consistent on* $M \subseteq N$ if and only if there exists a collection of sets corresponding to indices in M such that the relations specified by μ hold between each associated pair of the sets. Firstly, it is proved that if μ is consistent on all subsets of N of size three then μ is consistent on N. Secondly, explicit conditions that make μ consistent on a subset of size three are given as generalized transitivity laws. Finally, it is shown that the result concerning binary natural relations can be generalized to r-ary natural relations for arbitrary $r \geqslant 2$.

1. Introduction

Let n be a natural number and N denote the set $\{1, \ldots, n\}$. Suppose that we are given some combinatorial object μ defined on N and that, for any subset M of N, the object, denoted by μ_M, is obtained by restricting μ to the subset M. We consider predicates P that are inheritable, in the sense that, for all subsets M of N, if P holds on μ_M then P holds on $\mu_{M'}$

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for any $M' \subset M$. Such a predicate P often turns out to be 'locally computable', that is, if P holds on all objects μ_M with M satisfying some conditions then P holds on the whole object μ_N. One typical example of such a predicate is given by Helly's theorem [4]. For example, in two dimensions, this states that a family of compact convex planar sets has a nonempty intersection if and only if every triple of the sets has a nonempty intersection.

In this paper we give another locally computable predicate for which we are only required to check the predicate on all objects μ_M for 'small' M. First we describe five *natural relations* between sets, denoted by $\subset$, $\supset$, $\|$, $\perp$, $\bowtie$, representing strict inclusion in each direction, disjointness, covering the universe, and the general case, respectively. Each can be defined in terms of the emptiness or otherwise of Boolean combinations of the sets, and the set of these five relations is denoted by $\mathcal{R}$. Let $\binom{N}{2}$ denote $\{(i, j) \in N^2 \mid i < j\}$. Each object μ we deal with is an assignment of a natural relation to each pair (i, j) in $\binom{N}{2}$, that is, a function from $\binom{N}{2}$ to $\mathcal{R}$.

The function μ is *compatible* with a collection of sets $S_1, \ldots, S_n$ if the relation $\mu(i, j)$ holds between S_i and S_j for all (i, j) in $\binom{N}{2}$. If μ is compatible with some such collection then μ is said to be *consistent*. For any subset $M \subseteq N$, the object μ_M is simply the restriction of μ to $\binom{M}{2}$. If μ_M is consistent then μ is said to be *consistent on* M. Our main result is that if μ is consistent on every subset of N of size three, then μ is consistent. Conditions that make μ consistent are given explicitly in terms of the natural relations that may hold for any three subsets. The main result for binary natural relations can be generalized to that for r-ary natural relations for arbitrary $r \geqslant 2$.

The problem of characterizing a predicate on graphs, or equivalently a family of graphs satisfying the predicate, is also discussed in Fellows and Langston [2] in a different context. It was pointed out in that paper that local computability which only requires locally available information is essential in architectural requirements of parallel computing. Some combinatorial aspects of inclusion and exclusion and their relation to Boolean complexity are also discussed by Linial and Nisan in [5].

In Section 2, we introduce natural relations on sets, and give generalized transitivity constraints on natural relations of μ which guarantee that μ is consistent. In order to prove the statement, a set of vectors is used as a model for μ. It is also shown that there exists a feasible algorithm which, given a partial function μ from $\binom{N}{2}$ to $\mathcal{R}$, decides whether or not μ can be extended to obtain a consistent total function. In Section 3, we give an alternative graph model for the inclusion relations of μ and verify the same result as in Section 2. In Section 4, we generalize our results to r-ary natural relations for arbitrary $r \geqslant 2$, and introduce a local condition on μ, called the *inheritance* property. It is shown that, if transitivity constraints are replaced by the inheritance condition, then the results in the previous sections can be generalized to the case of r-ary natural relations for arbitrary $r \geqslant 2$. In Section 5, some considerations on the time complexity of the consistency problem for binary natural relations are given, together with concluding remarks.

2. Consistency conditions for natural relations

Let the universe U be nonempty. A *natural relation* for any set or sets is one that is defined in terms of the emptiness or otherwise of the subsets defined by Boolean combinations of

Table 1 Five natural relations.

subsets	relations $\subset$	$\supset$	$\parallel$	$\perp$	$\bowtie$
(1,1)	1	1	0	1	1
(1,0)	0	1	1	1	1
(0,1)	1	0	1	1	1
(0,0)	1	1	1	0	1

the sets. For one set, there are four cases, depending on the emptiness of the set and its complement. If both are empty then $U = \phi$, a case we have excluded. The remaining three cases correspond to the set being empty, equal to the universe and proper, respectively. A subset S of U is called *proper* if neither $S = \phi$ nor $S = U$. In Sections 2 and 3 we shall allow only proper subsets of U.

For two sets A and B, there are formally 16 possible relations. Under our assumptions that the universe is nonempty and both sets are proper, there remain just seven cases. One is $A = B$, another is $A = \overline{B} = U \neg B$. Both of these cases are special in that if they hold then one of the sets can be eliminated by substitution from the remaining relations. The remaining five natural relations constitute $\mathscr{R}$. Table 1 defines these five relations in terms of the emptiness, denoted by 0, or nonemptiness, denoted by 1, of four subsets. In the table, (a, b) indicates the subset $A^a \cap B^b$, where a and b are in $\{0, 1\}$, $S^1 = S$ and $S^0 = \overline{S}$.

Let Σ denote $\{0, 1\}$. For v in Σ^n, let $v^{(i)}$ denote the ith component of v. Given n subsets $\mathscr{S} = S_1, \ldots, S_n$ of U, we can determine whether each subset of the form

$$S_1^{v^{(1)}} \cap S_2^{v^{(2)}} \cdots \cap S_n^{v^{(n)}}$$

is empty or not, where $v = (v^{(1)}, \ldots, v^{(n)})$ is in Σ^n. Let $V(\mathscr{S})$ denote the set of vectors v in Σ^n such that $S_1^{v^{(1)}} \cap \cdots \cap S_n^{v^{(n)}}$ is nonempty. Furthermore, let T_i, for $1 \leqslant i \leqslant n$, denote the set of vectors v in $V(\mathscr{S})$ such that $v^{(i)} = 1$. Then it is easy to see that consistency of $S_1, \ldots, S_n$ for $\mathscr{R}$ is the same as that of $T_1, \ldots, T_n$. In other words, for any natural relation α, $S_i \alpha S_j$ holds if and only if $T_i \alpha T_j$ holds. So, without loss of generality, we can consider a set of vectors V rather than a collection of subsets as far as the consistency problem is concerned.

Relation α in Table 1 is considered to be a function from Σ^2 to Σ in the obvious way: for (a, b) in Σ^2, $\alpha(a, b) = 1$ if $A^a \cap B^b$ is nonempty in the relation, and $\alpha(a, b) = 0$ otherwise. Relation α can also be considered to be the set of vectors in Σ^2 for which the function takes value 1: for instance, $\subset\ = \{(1, 1), (0, 1), (0, 0)\}$. Let $v^{(i,j)}$ denote $(v^{(i)}, v^{(j)})$, and $V^{(i,j)}$ denote $\{v^{(i,j)} \mid v \in V\}$. This notation can be generalized in an obvious way to the case of more indices. Let μ be a function from $\binom{N}{2}$ to $\mathscr{R}$, and let $M \subseteq N$. We say that μ is *compatible with* V on M if and only if $V^{(i,j)} = \mu(i, j)$ for all (i, j) in $\binom{M}{2}$, and that μ is *consistent on* M if and only if there exists a subset V of Σ^n that is compatible with μ on M. In particular, when $M = N$, the phrase 'on M' in the definition may be dropped. For $M \subseteq N$, let V_M denote

$$\Sigma^n - \left\{ v \in \Sigma^n \mid \exists (i, j) \in \binom{M}{2}\ v^{(i,j)} \notin \mu(i, j) \right\}.$$

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In fact, the set V_M depends on μ, but we refrain from using μ in the notation because it will be clear from the context.

The next proposition says that V_M defined above is the largest set among the sets that are compatible with μ on M.

Proposition 1. *If μ is compatible with $V \subseteq \Sigma^n$ on $M \subseteq N$, then $V \subseteq V_M$.*

Proof. Assume to the contrary that $V \nsubseteq V_M$ holds. Then there exists v in Σ^n such that $v \in V$ and $v \notin V_M$. By $v \notin V_M$,

$$\exists (i,j) \in \binom{M}{2}\ v^{(i,j)} \notin \mu(i,j).$$

Thus we have $v^{(i,j)} \in V^{(i,j)}$ and $v^{(i,j)} \notin \mu(i,j)$, contradicting the assumption that $\mu(i,j) = V^{(i,j)}$. □

By definition the next proposition is clear.

Proposition 2. *For any $M \subseteq N$ with $|M| \geqslant 2$ and any (i,j) in $\binom{M}{2}$,*

$$\mu(i,j) \supseteq V_M^{(i,j)}.$$ □

Proposition 3. *The function μ is consistent on $M \subseteq N$ with $|M| \geqslant 2$ if and only if,*

$$\text{for all } (i,j) \in \binom{M}{2}, \quad \mu(i,j) = V_M^{(i,j)}.$$

Proof. For the proof of the 'only if' part, suppose that μ is compatible with $V \subseteq \Sigma^n$ on $M \subseteq N$. Then, by Propositions 1 and 2 and the definition of compatibility, we have

$$\mu(i,j) = V^{(i,j)} \subseteq V_M^{(i,j)} \subseteq \mu(i,j),$$

and hence $\mu(i,j) = V_M^{(i,j)}$ for any (i,j) in $\binom{M}{2}$. On the other hand, the condition of the proposition implies that μ is compatible with V_M on $M \subseteq N$. □

For u in Σ^n and $A \subseteq \Sigma^n$, let $u \oplus A = \{u \oplus v \mid v \in A\}$, where $u \oplus v$ denotes the vector obtained by taking the bit-wise 'exclusive or' of u and v. For u in Σ^n, the transformation φ_u on the set of functions from $\binom{N}{2}$ to $\mathscr{R}$ is defined as $\varphi_u(\mu)(i,j) = u^{(i,j)} \oplus \mu(i,j)$. Note that, in the definition, $\mu(i,j)$ is thought of as a subset of Σ^2.

Clearly we have the next proposition.

Proposition 4. *Let u be in Σ^n and $V \subseteq \Sigma^n$. Then μ is compatible with V if and only if $\varphi_u(\mu)$ is compatible with $u \oplus V$.* □

In view of Proposition 4, we have the next proposition, which says that the transformation φ_u preserves the consistency of μ for any u in Σ^n.

Proposition 5. *Let u be in Σ^n. Then μ is consistent if and only if $\varphi_u(\mu)$ is consistent.* □

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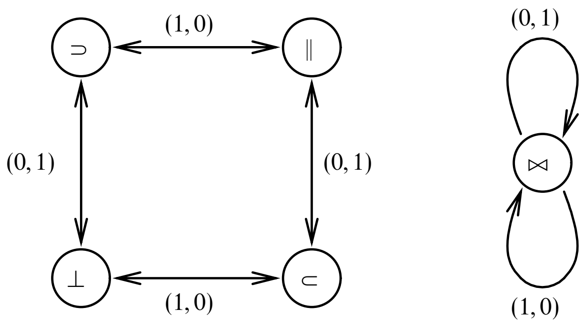

Figure 1 Transformations on natural relations.

Table 2 The eight transitivity constraints.

$u^{(i,j,k)}$	$\mu(i,j)$	$\mu(j,k)$	$\mu(i,k)$
(0,0,0)	$\subset$	$\subset$	$\subset$
(0,0,1)	$\subset$	$\parallel$	$\parallel$
(0,1,0)	$\parallel$	$\perp$	$\subset$
(0,1,1)	$\parallel$	$\supset$	$\parallel$
(1,0,0)	$\perp$	$\subset$	$\perp$
(1,0,1)	$\perp$	$\parallel$	$\supset$
(1,1,0)	$\supset$	$\perp$	$\perp$
(1,1,1)	$\supset$	$\supset$	$\supset$

Before proceeding to the main theorem, we show in Figure 1 how the five natural relations are transformed by φ_u for $u = (1,0)$ and $(0,1)$.

It is convenient to extend the definition of any object μ to $\{(j,i) \mid i < j\}$ in the obvious way, so that $\mu(i,j) = \subset$ if and only if $\mu(j,i) = \supset$, and $\mu(i,j) = \mu(j,i)$ if $\mu(i,j) \in \{\parallel, \perp, \bowtie\}$.

If μ is consistent, then the transitivity of inclusion implies that the following constraint holds for any distinct indices i, j and k.

$$\text{If} \quad \mu(i,j) = \subset \quad \text{and} \quad \mu(j,k) = \subset, \quad \text{then} \quad \mu(i,k) = \subset . \qquad (*)$$

By applying Proposition 5 for various choices of u, we can transform the constraint $(*)$ in various ways. For example, let $u^{(i,j,k)} = (1,0,1)$ and $\mu' = \phi_u(\mu)$. Now, if $\mu(i,j) = \perp$ and $\mu(j,k) = \parallel$ then $\mu'(i,j) = \subset$ and $\mu'(j,k) = \subset$, and so $\mu'(i,k) = \subset$, which implies that $\mu(i,k) = \supset$. In Table 2 we show the eight *transitivity constraints* which are obtained by taking various vectors as $u^{(i,j,k)}$. If μ satisfies these eight constraints it is said to be *transitive*.

The next theorem says that these conditions that are necessary to make μ consistent on any set of three indices turn out to be sufficient conditions to make μ consistent on the set of *all* indices.

Theorem 6. *If μ is transitive then μ is consistent.*

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Proof. We shall prove the statement of the theorem by induction on n. The statement holds trivially when $n = 2$. Assume that the statement holds for $n - 1$, where $n \geqslant 3$.

In view of Figure 1, it is easy to see that there exists some u in Σ^n such that $\varphi_u(\mu)(j,n) \in \{\subset, \supset, \bowtie\}$ for all $1 \leqslant j \leqslant n-1$. By Proposition 5 it suffices to show that $\varphi_u(\mu)$ is consistent. Furthermore, it is easy to see that if μ satisfies the conditions of the theorem then $\varphi_u(\mu)$ also satisfies the conditions. So, denoting $\varphi_u(\mu)$ again by μ, we may assume that μ satisfies the conditions of the theorem and that $\mu(j,n) \in \{\supset, \subset, \bowtie\}$ for all $1 \leqslant j \leqslant n - 1$. Let

$$N_i = N - \{i\}$$

for $1 \leqslant i \leqslant n$. Then, by the induction hypothesis and Proposition 3, μ is compatible with V_{N_i} on N_i for $1 \leqslant i \leqslant n$.

Lemma 1. *For any v in V_{N_i} at least one of $(v^{(1)}, \ldots, v^{(i-1)}, 0, v^{(i+1)}, \ldots, v^{(n)})$ and $(v^{(1)}, \ldots, v^{(i-1)}, 1, v^{(i+1)}, \ldots, v^{(n)})$ is in V_N.*

Proof. Without loss of generality, let $i = n$. Assume to the contrary that there exists v in V_{N_n} such that both $(v^{(1)}, \ldots, v^{(n-1)}, 0)$ and $(v^{(1)}, \ldots, v^{(n-1)}, 1)$ are not in V_N. Then, since $\mu(\ell, n) \in \{\supset, \subset, \bowtie\}$ for any $1 \leqslant \ell \leqslant n - 1$, there exist $1 \leqslant j,\, k \leqslant n - 1$ such that

$$\begin{aligned} \mu(j,n) &= \subset, \\ \mu(k,n) &= \supset, \\ v^{(j)} &= 1, \\ v^{(k)} &= 0. \end{aligned}$$

Hence, by transitivity we have $\mu(j,k) = \subset$, which, together with $v^{(j)} = 1$ and $v^{(k)} = 0$, implies that v does not belong to V_{N_n}, contradicting the assumption. □

By the induction hypothesis and Lemma 1 we have,

$$\text{for all } 1 \leqslant i \leqslant n \text{ and } (j,k) \in \binom{N_i}{2}, \quad \mu(j,k) = V_{N_i}^{(j,k)} = V_N^{(j,k)}.$$

On the other hand, since $n \geqslant 3$, for any (j,k) in $\binom{N}{2}$ there exists $1 \leqslant i \leqslant n$ such that $(j,k) \in \binom{N_i}{2}$, and hence $\mu(j,k) = V_N^{(j,k)}$, completing the proof. □

Before closing the section, we note that by using Theorem 6 we can construct a feasible algorithm which, given a partial function μ from $\binom{N}{2}$ to $\mathscr{R}$, decides whether or not μ can be extended to obtain a consistent total function. The algorithm works as follows. Given a partial function μ, check if it satisfies the transitivity constraints. If not, give the answer that μ is not consistent. Otherwise, extend μ using the transitivity constraints repeatedly until none of these constraints can be applied. In doing this, if there exists a pair to which different relations are assigned then give the answer that μ is not extensible consistently. Otherwise, conclude that μ is extensible consistently. In fact, if we assign $\bowtie$ to any pairs that remain unspecified at the end of the extension process, we obtain a total function. Clearly the total function obtained in this way is an extension of μ and is consistent in view of Theorem 6.

3. An intuitive model for μ

In this section we shall introduce another, intuitive, model for μ based directly on the natural relations, so that we can give another proof of Theorem 6.

Proposition 7. *If μ is transitive then there exists u in Σ^n such that $\varphi_u(\mu)(i,j) \in \mathscr{R}_0 = \{\subset, \supset, \|, \bowtie\}$ holds for any (i,j) in $\binom{N}{2}$.*

Proof. Let μ be transitive. Then it is easy to see that $\varphi_u(\mu)$ is also transitive for any u in Σ^n. We shall prove the conclusion of the proposition by induction on n. When $n = 2$, Figure 1 shows the result at once. Assume as inductive hypothesis that the result holds for $N_n = \{1, 2, \ldots, n-1\}$. Then there exists u' in Σ^n such that $\varphi_{u'}(\mu)(i,j) \in \mathscr{R}_0$ holds for any (i,j) in $\binom{N_n}{2}$. Now at most one of $\supset$ and $\perp$ appears in $\varphi_{u'}(\mu)(1,n)$, $\varphi_{u'}(2,n), \ldots, \varphi_{u'}(n-1,n)$. This is because, if there exist i,j in N_n such that $\varphi_{u'}(\mu)(i,n) = \perp$ and $\varphi_{u'}(\mu)(j,n) = \supset$, then by transitivity we have $\varphi_{u'}(i,j) = \perp$, contradicting the assumption. Thus, by taking $u = u' \oplus (0, \ldots, 0, 1)$ if there exists i in N_n such that $\varphi_{u'}(\mu)(i,n) = \perp$, and taking $u = u'$ otherwise, we see from Figure 1 that $\varphi_u(\mu)(i,j) \in \mathscr{R}_0$ holds for any (i,j) in $\binom{N}{2}$, completing the induction step. □

Let $\varphi_u(\mu)$ be as in Proposition 7. We note that, in order to obtain a model for μ, it is sufficient by Proposition 4 to obtain a model for $\varphi_u(\mu)$. We again denote $\varphi_u(\mu)$ by μ so that $\mu(i,j) \in \mathscr{R}_0$ for any (i,j) in $\binom{N}{2}$. We shall define a collection of subsets that is compatible with μ. To do this, consider the directed graph G' with vertex set $V' = \{x_1, \ldots, x_n\}$ and edge set $E' = \{(x_i, x_j) \mid \mu(i,j) = \supset\}$. Since μ is transitive, G' is an acyclic graph, and we define S'_i to be the set of x_i and its descendants. It is easy to see that $S'_i \supset S'_j$ if and only if $(x_i, x_j) \in E'$, and if and only if $\mu(i,j) = \supset$, so that G' already gives a model for the set containment relations of μ.

For a complete model for μ we need to extend G' with extra vertices. Define $G'' = (V'', E'')$, where $V'' = V' \cup \{x_{i,j} \mid \mu(i,j) = \bowtie \text{ and } i < j\}$ and $E'' = E' \cup \{(x_i, x_{i,j}), (x_j, x_{i,j}) \mid x_{i,j} \in V''\}$. As in the case of graph G', let S''_i be the set of x_i and its descendants in the graph G''. If there exist i and j in N such that $S''_i \cup S''_j = V''$ holds, then let $V = V'' \cup \{x_\infty\}$ and $E = E''$. Otherwise, let $V = V''$ and $E = E''$. The final graph G is defined to be (V, E). Now we define S_i to be the set consisting of x_i and its descendants in the graph G. The containment relation on the new sets is the same as in the graph G' and agrees with $\mu^{-1}(\supset)$. Therefore, if $\mu(i,j) = \bowtie$ then neither containment can hold between S_i and S_j but $x_{i,j} \in S_i \cap S_j$. Hence, since $S_i \cup S_j$ cannot be the whole set V, we have $S_i \bowtie S_j$. For the proof of the converse, suppose that $S_i \bowtie S_j$ and so $\mu(i,j) \in \{\|, \bowtie\}$. If there is some x_k in $S_i \cap S_j$ then $k \neq i, j$, so $S_k \subset S_i$ and $S_k \subset S_j$. Hence $\mu(k,i) = \mu(k,j)$ and the transitivity constraints imply that $\mu(i,j) \neq \|$. Otherwise there is some $x_{k,l}$ in $S_i \cap S_j$ where $S_k \subseteq S_i$ and $S_l \subseteq S_j$, and so

(I) $k = i$ or $\mu(k,i) = \subset$, and
(II) $l = j$ or $\mu(l,j) = \subset$.

The existence of $x_{k,l}$ implies that $\mu(k,l) = \bowtie$. Then (I) and (II) together with the transitivity constraints again imply that $\mu(i,j) \neq \|$. This completes the proof that the graph G is a model for μ.

4. Generalization

So far we have discussed the consistency of binary natural relations. In this section we generalize the situation to the case of r-ary natural relations for arbitrary $r \geqslant 2$. In doing so we use, instead of the transitivity constraints, some more restrictive local constraints, called the inheritance property. In this section we assume that the sets we deal with are any subsets of the universe, dropping the assumption of their being proper.

Most of the notation and the results in Section 3 are generalized as follows. An *r-ary natural relation* is a function from Σ^r to Σ. The set of r-ary natural relations is denoted by $\mathcal{G}$, that is, $\mathcal{G} = \{g : \Sigma^r \to \Sigma\}$. For $M \subseteq N$, let $\binom{M}{r}$ denote $\{(i_1, \ldots, i_r) \in M^r \mid i_1 < \cdots < i_r\}$. For v in Σ^n and $A = (i_1, \ldots, i_j)$ in $\binom{N}{j}$, let $v^{(A)}$ denote the vector $(v^{(i_1)}, \ldots, v^{(i_j)})$. Moreover, for $V \subseteq \Sigma^n$ let $V^{(A)} = \{v^{(A)} \mid v \in V\}$. The generalized object μ is a function from $\binom{N}{r}$ to $\mathcal{G}$. If $\mu(A) = V^{(A)}$ holds for any A in $\binom{M}{r}$, then μ is said to be *compatible with* $V \subseteq \Sigma^n$ *on* $M \subseteq N$. Recall that, when $\mu(A)$ is a function g from Σ^r to Σ, $\mu(A)$ denotes the set $\{u \in \Sigma^r \mid g(u) = 1\}$ as well as the function g. If there exists $V \subseteq \Sigma^n$ such that μ is compatible with V on $M \subseteq N$, then μ is said to be *consistent on* M. Recall that in the definitions above the phrase 'on M' may be dropped when M is equal to N. For $V \subseteq \Sigma^n$ and $M \subseteq N$, let V_M denote

$$\Sigma^n - \left\{v \in \Sigma^n \mid \exists A \in \binom{M}{r} \mu(A)(v^{(A)}) = 0\right\}.$$

Propositions 1, 2 and 3 are easily generalized to obtain Propositions 8, 9 and 10, respectively.

Proposition 8. *If μ is compatible with $V \subseteq \Sigma^n$ on $M \subseteq N$, then $V \subseteq V_M$.* □

Proposition 9. *For any $M \subseteq N$ with $|M| \geqslant r$ and any A in $\binom{M}{r}$,*

$$\mu(A) \supseteq V_M^{(A)}.$$ □

Proposition 10. *The function μ is consistent on $M \subseteq N$ with $|M| \geqslant r$ if and only if,*

$$\text{for all } A \in \binom{M}{r}, \quad \mu(A) = V_M^{(A)}.$$ □

To generalize Theorem 6 to the case of r-ary natural relations for arbitrary $r \geqslant 2$, we introduce a notion of inheritance which replaces the transitivity law for the case of $r = 2$. The function μ is said to have the *inheritance property* on $M \subseteq N$ if $V_{M'}^{(M')} = V_M^{(M')}$ holds for any $M' \subseteq M$ such that $|M'| \geqslant r$.

The next proposition says that some inheritance conditions can be restated in a seemingly weaker fashion.

Proposition 11. *The function μ has the inheritance property on every $M \subseteq N$ with $r \leqslant |M| \leqslant 2r-1$ if and only if μ has the inheritance property on every $M \subseteq N$ with $|M| = 2r-1$.*

Proof. The 'only if' part of the proposition is trivially true. For the proof of the 'if' part, let M', M'' and M be arbitrary subsets of N such that $M' \subseteq M'' \subseteq M$, $|M'| \geqslant r$ and $|M| = 2r-1$. By the condition of the proposition, $V_{M'}^{(M')} = V_M^{(M')}$ holds. On the other hand, by definition we have $V_{M'} \supseteq V_{M''} \supseteq V_M$, and hence $V_{M'}^{(M')} \supseteq V_{M''}^{(M')} \supseteq V_M^{(M')}$. Thus we have $V_{M'}^{(M')} = V_{M''}^{(M')}$, completing the proof of the 'if' part. □

Theorem 6 in Section 2 can be generalized to obtain the next theorem.

Theorem 12. *If μ has the inheritance property on every $M \subseteq N$ with $r \leqslant |M| \leqslant 2r-1$, then μ is consistent on N.*

Proof. The theorem will be proved by induction on n in a similar way to Theorem 6. The statement holds trivially when $r \leqslant |N| \leqslant 2r-1$. Assume that the statement holds for $n-1$, where $n \geqslant 2r$.

By the induction hypothesis and Proposition 10, μ is compatible with V_{N_i} on $N_i = N \neg \{i\}$ for $1 \leqslant i \leqslant n$.

Lemma 2. *Suppose that μ has the inheritance property on every $M \subseteq N$ with $r \leqslant |M| \leqslant 2r-1$ and that $\mu(A) = V_{N_i}^{(A)}$ for any A in $\binom{N_i}{r}$, where $1 \leqslant i \leqslant n$. Then, for any $(v^{(1)}, \dots, v^{(n)})$ in V_{N_i}, at least one of $(v^{(1)}, \dots, v^{(i-1)}, 0, v^{(i+1)}, \dots, v^{(n)})$ and $(v^{(1)}, \dots, v^{(i-1)}, 1, v^{(i+1)}, \dots, v^{(n)})$ belongs to V_N.*

Proof. Without loss of generality, let $i = n$. Let $v = (v^{(1)}, \dots, v^{(n)})$ be an arbitrary vector in V_{N_n}, and put

$$v_0 = (v^{(1)}, \dots, v^{(n-1)}, 0), \quad \text{and} \quad v_1 = (v^{(1)}, \dots, v^{(n-1)}, 1).$$

Assume in contradiction that $v_0 \notin V_N$, and $v_1 \notin V_N$. Then there exist B_0 and B_1 in $\binom{N}{r}$ such that

$$n \in B_0, n \in B_1, \ v_0^{(B_0)} \notin \mu(B_0), \quad \text{and} \quad v_1^{(B_1)} \notin \mu(B_1). \tag{4.1}$$

Put $B_0' = B_0 - \{n\}$, and $B_1' = B_1 - \{n\}$.

Case 1. $B_0 = B_1$.
Let j be an arbitrary integer in $N - B_0$. Put $M = B_0 \cup \{j\}$ and $M' = B_0' \cup \{j\}$. Then we have

$$\begin{aligned} V_M^{(M')} &= V_{M'}^{(M')} && \text{by the inheritance assumption,} \\ &= \mu(M') && \text{by definition,} \\ &= V_{N_n}^{(M')} && \text{by the assumption of the fact.} \end{aligned}$$

Therefore, since $v \in V_{N_n}$, we have

$$\exists v' \in V_M \ v^{(M')} = v'^{(M')},$$

which implies that either

$$v'^{(B_0)} = v_0^{(B_0)} \quad \text{or} \quad v'^{(B_0)} = v_1^{(B_1)}. \tag{4.2}$$

By the inheritance assumption, we have

$$v'^{(B_0)} \in V_M^{(B_0)} = V_{B_0}^{(B_0)} = \mu(B_0)(= \mu(B_1)),$$

contradicting the fact that both (4.1) and (4.2) hold.

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Case 2. $B_0 \neq B_1$.
Put

$$M = B_0 \cup B_1, \quad \text{and} \quad M' = B'_0 \cup B'_1 = M \neg \{n\}.$$

Since $r \leqslant |M| \leqslant 2r - 1$, we have

$$V_M^{(M')} = V_M^{(M)}$$

by the inheritance assumption. Because

$$v \in V_{N_n} \subseteq V_{M'},$$

we have

$$v^{(M')} \in V_{N_n}^{(M')} \subseteq V_{M'}^{(M')} = V_M^{(M')}. \tag{4.3}$$

Hence either $v_0^{(M)} \in V_M^{(M)}$ or $v_1^{(M)} \in V_M^{(M)}$. Without loss of generality, suppose $v_0^{(M)} \in V_M^{(M)}$. Then

$$v_0^{(B_0)} \in V_M^{(B_0)} \subseteq V_{B_0}^{(B_0)} = \mu(B_0).$$

This contradiction completes the proof of the fact. □

By the inductive hypothesis and Lemma 2 we have,

$$\text{for all } 1 \leqslant i \leqslant n \text{ and } A \in \binom{N_i}{r}, \quad \mu(A) = V_{N_i}^{(A)} = V_N^{(A)}.$$

On the other hand, since $|N| > r$, for any A in $\binom{N}{r}$ there exists $1 \leqslant i \leqslant n$ such that $A \in \binom{N_i}{A}$, and hence $\mu(A) = V_N^{(A)}$, completing the proof. □

In the remainder of this section, we shall discuss the relation between the transitivity and inheritance conditions.

Proposition 13. *Let $r = 2$ and let the range of μ be $\mathcal{R}$. Then μ is transitive if and only if μ satisfies the inheritance constraints: that is, for any M in $\binom{N}{3}$ and any M' in $\binom{M}{2}$,*

$$V_{M'}^{(M')} = V_M^{(M')}.$$

Proof. For the proof of the 'only if' part, assume that μ is transitive, and therefore consistent. Let M be an arbitrary element in $\binom{N}{3}$, and let $M = (i, j, k)$ and $M' = (i, j)$. Since $\mu(M') = V_{M'}^{(M')} \supseteq V_M^{(M')}$, all we have to verify is that $\mu(M') \subseteq V_M^{(M')}$. Let v be an arbitrary vector in $\mu(M')$. Without loss of generality we may take $v = (1, 1)$. Then, in any family of subsets consistent with μ, we have $S_i \cap S_j \neq \emptyset$. In terms of V_N, this means that $v^{(i,j)} = (1, 1)$ for some $v \in V_N \subseteq V_M$, that is, $(1, 1) \in V_M^{(M')}$.

For the proof of the 'if' part, assume that μ satisfies the inheritance constraints. We show that every transitivity is satisfied. Let $\mu(i, j)$ and $\mu(j, k)$ be one of the eight pairs given in Table 2. Then there exists a vector u in Σ^3 such that $\varphi_u(\mu)(i, j) = \subset$ and $\varphi_u(\mu)(j, k) = \subset$. Since the inheritance property is preserved under the transformation φ_u, $\varphi_u(\mu)$ also satisfies the inheritance constraints. It suffices to show that $\varphi_u(\mu)(i, k) = \subset$,

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which guarantees that the triple of relations $\mu(i,j), \mu(j,k), \mu(i,k)$ satisfies the transitivity constraint given in Table 2. Let us denote $\varphi_u(\mu)$ again by μ. Deleting all the vectors $u = (u^{(1)}, u^{(2)}, u^{(3)})$ such that $\mu(i,j)(u^{(1)}, u^{(2)}) = 0$ or $\mu(j,k)(u^{(2)}, u^{(3)}) = 0$ from Σ^3, we obtain

$$V_{\{i,j,k\}} \subseteq \{111, 011, 001, 000\}.$$

By the inheritance constraint we therefore have

$$V_{\{i,k\}}^{(i,k)} = V_{\{i,j,k\}}^{(i,k)} \subseteq \{11, 01, 00\}.$$

But, since $V_{\{i,k\}}^{(i,k)} = \mu(i,k)$ and $|\mu(i,k)| \geqslant 3$, we also have $|V_{\{i,k\}}^{(i,k)}| \geqslant 3$. Thus we have

$$V_{\{i,k\}}^{(i,k)} = \{11, 01, 00\},$$

establishing that $\mu(i,k) = V_{\{i,k\}}^{(i,k)} = \subset$. □

So far we have established that when $r = 2$ transitivity implies consistency, and that when $r \geqslant 2$ the inheritance condition implies consistency. Since consistency trivially implies transitivity, transitivity is equivalent to consistency when $r = 2$. One might ask if this is also the case for arbitrary r-ary natural relations. To discuss the problem, we need to generalize the notion of transitivity to the case of r-ary natural relations.

As the argument in the proof of Proposition 13 suggests, the transitivity condition may be generalized to the condition described as:

$$\text{for all } M \in \binom{N}{2r-1} \text{ and } M' \in \binom{M}{r}, \quad V_{M'}^{(M')} = V_{M}^{(M')}. \tag{C1}$$

In fact, Proposition 13 says that when $r = 2$ transitivity is equivalent to condition (C1). So the problem is stated as follows. Does condition (C1) imply consistency for r-ary natural relations? As in the case of $r = 2$, consistency trivially implies (C1) for r-ary natural relations. Hence, if we are able to answer the question affirmatively, then transitivity, namely condition (C1), is in general equivalent to consistency. So far we are not able to prove the implication.

On the other hand, as Proposition 11 shows, the inheritance condition is stated as:

$$\text{for all } M \in \binom{N}{2r-1}, \ r \leqslant i \leqslant 2r-1, \ \text{ and } M' \in \binom{M}{i}, \quad V_{M'}^{(M')} = V_{M}^{(M')}. \tag{C2}$$

Before closing this section we shall give an example of μ that satisfies (C1), but not (C2). So (C2) is a stronger condition than (C1).

Let $N = \{1, \ldots, 5\}$ and $r = 3$. Boolean functions g_0 and g_1 of three variables are defined as follows:

$$\begin{aligned} g_0(x_1, x_2, x_3) &= x_1 \vee x_2 \vee x_3, \\ g_1(x_1, x_2, x_3) &= x_1 \oplus x_2 \oplus x_3 \oplus 1. \end{aligned}$$

The function μ from $\binom{N}{3}$ to $\mathscr{G}$ is defined as

$$\mu(i,j,k) = \begin{cases} g_1, & \text{if } (i,j,k) = (1,2,5) \text{ or } (3,4,5), \\ g_0, & \text{otherwise.} \end{cases}$$

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Table 3 Vectors in $V_{\{1,2,3,4\}}$. Values 1 and 0 will be substituted for every occurrence of symbols $*_1, *_0$ and $\sharp$.

$v^{(1)}$	$v^{(2)}$	$v^{(3)}$	$v^{(4)}$	$v^{(5)}$
1	0	1	0	$*_1$
1	0	0	1	$*_1$
0	1	1	0	$*_1$
0	1	0	1	$*_1$
1	1	0	0	$*_0$
0	0	1	1	$*_0$
1	1	1	1	$*_0$
1	1	1	0	$\sharp$
1	1	0	1	$\sharp$
1	0	1	1	$\sharp$
0	1	1	1	$\sharp$

The vectors in $V_{\{1,2,3,4\}}$ are shown in Table 3, where both 1 and 0 are substituted for every occurrence of $*_1, *_0$, and $\sharp$ in the table. For vectors v in $V_{\{1,2,3,4\}}$ to belong to $V_{\{1,2,3,4,5\}}$, the parity of $v^{(1)} + v^{(2)} + v^{(5)}$ and $v^{(3)} + v^{(4)} + v^{(5)}$ must be even so that both $g_1(v^{(1)}, v^{(2)}, v^{(5)}) = 1$ and $g_1(v^{(3)}, v^{(4)}, v^{(5)}) = 1$ hold. To satisfy these conditions, symbols $*_1$ and $*_0$ in Table 3 have to be replaced with 1 and 0, respectively. On the other hand, none of 1 and 0 for the occurrence $\sharp$ satisfies both of these conditions. This is because the parity of $v^{(1)} + v^{(2)}$ is different from that of $v^{(3)} + v^{(4)}$ in the last four rows in the table. So, by checking further the conditions corresponding to the remaining triples of the indices, we can see that the set V_N consists of the first seven vectors in Table 3 with $*_1$ and $*_0$ being replaced by 1 and 0, respectively. Furthermore, we can check that $V_N^{(M')} = V_{M'}^{(M')} = \mu(M')$ holds for any M' in $\binom{N}{3}$. That is, taking N as M in condition (C1), we see that (C1) is satisfied for μ given above. So $V_{\{1,2,3,4\}}^{(1,2,3,4)}$ consists of vectors composed of the first four components of all of the vectors in Table 3, whereas $V_{\{1,2,3,4,5\}}^{(1,2,3,4)}$ consists of those corresponding to the first seven vectors. Namely, these sets are written as

$$V_{\{1,2,3,4\}}^{(1,2,3,4)} = \left\{(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}) \in \Sigma^4 \mid \sum_{i=1}^{4} v^{(i)} = 2, 3, \text{or } 4\right\},$$

$$V_{\{1,2,3,4,5\}}^{(1,2,3,4)} = \left\{(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}) \in \Sigma^4 \mid \sum_{i=1}^{4} v^{(i)} = 2 \text{ or } 4\right\}.$$

Since $V_{\{1,2,3,4\}}^{(1,2,3,4)} \neq V_{\{1,2,3,4,5\}}^{(1,2,3,4)}$, μ turns out to be an example of the function satisfying (C1), but not (C2).

5. Concluding remarks

We have investigated the problem of deciding whether or not a function μ that specifies the type of the natural r-ary relation for each collection of r sets is consistent, and proved that if μ satisfies a local consistency condition on each collection of $2r - 1$ sets

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then μ is consistent on the whole collection of sets. The local consistency condition is given as the transitivity constraints when $r = 2$, and as the inheritance property when $r \geqslant 2$. Consistency trivially implies transitivity, so, when $r = 2$, transitivity turns out to be equivalent to consistency. So far we are unable to verify that, when $r \geqslant 3$, the generalized transitivity constraints, which are weaker than the inheritance property, imply consistency. When $r = 2$, based on the transitivity constraints explicitly given, we gave a feasible algorithm which, given a partial function μ, decides whether or not μ can be extended to obtain a consistent total function.

When $r = 2$, the consistency of μ for n sets can be decided in time $O(n^3)$ by checking the transitivity constraints for all triples of sets. Furthermore, as pointed out by Jimbo [3], the problem can be solved in time $O(n^{2.37})$. To show this fact, let $M_\subset$ be the matrix whose (i, j) component is 1 if $\mu(i, j) = \subset$, and 0 otherwise. Then the constraint $(*)$ in Section 2 can be written as $M_\subset M_\subset \leqslant M_\subset$, where the matrix product is done using Boolean sum and product, and '$\leqslant$' holds between matrices if and only if '$\leqslant$' holds between all the corresponding components in the matrices. Likewise, we can rewrite the remaining transitivity constraints in Table 2 in matrix terms. Since, by the well-known result of Coppersmith and Winograd [1], the product of two $n \times n$ matrices can be computed in time $O(n^{2.37})$, the consistency problem for n sets can be computed in time $O(n^{2.37})$.

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