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THE ROGERS–RAMANUJAN CONTINUED FRACTION AND RELATED ETA-QUOTIENT REPRESENTATIONS

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Abstract

We construct eta-quotient representations of two families of q-series involving the Rogers–Ramanujan continued fraction by establishing related recurrence relations. We also display how these eta-quotient representations can be utilised to dissect certain q-series identities.

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1. Introduction

Throughout, we adopt the customary q-series notation:

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k), \qquad (A;q)_\infty := \prod_{k=0}^{\infty}(1-Aq^k)$$

and

$$\left(\begin{matrix} A_1, A_2, \ldots, A_n \\ B_1, B_2, \ldots, B_m \end{matrix}; q\right)_\infty := \frac{(A_1;q)_\infty (A_2;q)_\infty \cdots (A_n;q)_\infty}{(B_1;q)_\infty (B_2;q)_\infty \cdots (B_m;q)_\infty}.$$

The Rogers–Ramanujan continued fraction was discovered by Rogers [16], independently by Ramanujan [14], and also independently by Schur [18]. In the literature (see, for example, [1, 6, 10]), it often refers to the generalised continued fraction

$$\frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots},$$

but in this paper we will drop the factor of $q^{1/5}$. That is, we define

$$R(q) := \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots}.$$

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It is known (see [12, page 145]) that $R(q)$ can be represented as an infinite product,

$$R(q) = \begin{pmatrix} q, q^4 \\ q^2, q^3 \end{pmatrix}; q^5 \Big)_\infty .$$

Modular equations for the Rogers–Ramanujan continued fraction have been studied extensively by many mathematicians, including Rogers and Ramanujan themselves [2, 14, 15, 17, 21]. For example, [12, Equation (40.1.10)] states that

$$(R(q^2) - R(q)^2)(1 + qR(q)R(q^2)^2) = 2qR(q)R(q^2)^3$$

and [12, Equation (40.1.12)] states that

$$(R(q^3) - R(q)^3)(1 + q^2R(q)R(q^3)^3) = 3qR(q)^2R(q^3)^2. \tag{1.1}$$

Recall that the Dedekind eta function is defined by

$$\eta(q) := q^{1/24}(q; q)_\infty.$$

In this paper, we investigate eta-quotient representations of two families of q-series involving the Rogers–Ramanujan continued fraction by establishing the following recurrence relations. First, for $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, we define

$$P(\alpha, \beta) = \frac{1}{q^\alpha R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta} q^\alpha R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta}. \tag{1.2}$$

THEOREM 1.1. *Let*

$$K = \frac{\eta(q^2)\eta(q^5)^5}{\eta(q)\eta(q^{10})^5} = q^{-1}\frac{(q^2; q^2)_\infty (q^5; q^5)_\infty^5}{(q; q)_\infty (q^{10}; q^{10})_\infty^5}.$$

Then the following recurrence relations hold:

$$P(\alpha, \beta + 1) = 4K^{-1}P(\alpha, \beta) + P(\alpha, \beta - 1), \tag{1.3}$$

$$P(\alpha + 2, \beta) = KP(\alpha + 1, \beta) + P(\alpha, \beta). \tag{1.4}$$

We also have initial values:

$$P(0, 0) = 2, \tag{1.5}$$

$$P(0, 1) = 4K^{-1}, \tag{1.6}$$

$$P(1, 0) = K, \tag{1.7}$$

$$P(1, -1) = 4K^{-1} - 2 + K. \tag{1.8}$$

Next, for $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, we define

$$Q(\alpha, \beta) = \frac{1}{q^\alpha R(q)^{2\alpha+3\beta} R(q^3)^{\alpha-\beta}} + (-1)^\alpha q^\alpha R(q)^{2\alpha+3\beta} R(q^3)^{\alpha-\beta}. \tag{1.9}$$

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THEOREM 1.2. *Let*

$$S = \frac{\eta(q)^3\eta(q^3)^3}{\eta(q^5)^3\eta(q^{15})^3} = q^{-2}\frac{(q;q)_\infty^3(q^3;q^3)_\infty^3}{(q^5;q^5)_\infty^3(q^{15};q^{15})_\infty^3}$$

and

$$T = \frac{\eta(q^3)\eta(q^5)^5}{\eta(q)\eta(q^{15})^5} = q^{-2}\frac{(q^3;q^3)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{15};q^{15})_\infty^5}.$$

Then the following recurrence relations hold:

$$Q(\alpha,\beta+1) = (2+9T^{-1})Q(\alpha,\beta) - Q(\alpha,\beta-1), \tag{1.10}$$

$$Q(\alpha+2,\beta) = (-\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T + \tfrac{3}{2})Q(\alpha+1,\beta) + Q(\alpha,\beta). \tag{1.11}$$

We also have initial values:

$$Q(0,0) = 2, \tag{1.12}$$

$$Q(0,1) = 2 + 9T^{-1}, \tag{1.13}$$

$$Q(1,0) = -\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T + \tfrac{3}{2}, \tag{1.14}$$

$$Q(1,-1) = -\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T - \tfrac{3}{2}. \tag{1.15}$$

REMARK 1.3. Some of the initial values in Theorems 1.1 and 1.2 are already known. For example, to derive dissection identities of $(-q;q)_\infty$, Baruah and Begum [4, Equations (1.19)–(1.21)] proved (1.6)–(1.8). Also, (1.13) is due to Gugg [11, Theorem 5.1(iv)]. However, the two complicated identities (1.14) and (1.15) appear to be new.

As a by-product of Theorem 1.2, we obtain a modular equation involving S and T.

THEOREM 1.4. *We have*

$$81 + 144T + 46T^2 - 16T^3 + T^4 - 18ST - 2ST^3 + S^2T^2 = 0. \tag{1.16}$$

Finally, we remark that by (1.5) and (1.6) together with the recurrence relation (1.3), it is possible to represent $P(0,\beta)$ in terms of K for each $\beta \in \mathbb{Z}$. There are similar representations of $P(1,\beta)$ for each $\beta \in \mathbb{Z}$. Further, the recurrence relation (1.4) reveals that $P(\alpha,\beta) \in \mathbb{Z}[K,K^{-1}]$ for each $\alpha \geq 2$ and $\beta \in \mathbb{Z}$. In Table 1 we list the representations of $P(\alpha,\beta)$ in terms of K with $0 \leq \alpha \leq 2$ and $-3 \leq \beta \leq 3$.

Similar arguments can be applied to $Q(\alpha,\beta)$ to show that $Q(\alpha,\beta) \in \mathbb{Q}[S,T,T^{-1}]$ for each $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$. Since such eta-quotient representations of $Q(\alpha,\beta)$ are much more convoluted, we will not list them concretely in the manner of Table 1.

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TABLE 1. Representations of $P(\alpha,\beta)$ in $\mathbb{Z}[K,K^{-1}]$.

$\beta \backslash \alpha$	0	1
-3	$-64K^{-3}-12K^{-1}$	$64K^{-3}-32K^{-2}+20K^{-1}-6+K$
-2	$16K^{-2}+2$	$-16K^{-2}+8K^{-1}-4+K$
-1	$-4K^{-1}$	$4K^{-1}-2+K$
0	2	K
1	$4K^{-1}$	$4K^{-1}+2+K$
2	$16K^{-2}+2$	$16K^{-2}+8K^{-1}+4+K$
3	$64K^{-3}+12K^{-1}$	$64K^{-3}+32K^{-2}+20K^{-1}+6+K$

$\beta \backslash \alpha$	2
-3	$-64K^{-3}+64K^{-2}-44K^{-1}+20-6K+K^2$
-2	$16K^{-2}-16K^{-1}+10-4K+K^2$
-1	$-4K^{-1}+4-2K+K^2$
0	$2+K^2$
1	$4K^{-1}+4+2K+K^2$
2	$16K^{-2}+16K^{-1}+10+4K+K^2$
3	$64K^{-3}+64K^{-2}+44K^{-1}+20+6K+K^2$

Let $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, where $\mathbb{H}$ is the upper half complex plane. For any positive integer N, let $\Gamma_0(N)$ be the Hecke congruence subgroup of level N defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ c \equiv 0 \pmod{N} \right\}.$$

Let $K_0(N)$ denote the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. A result of Newman [13] indicates that K is in $K_0(10)$ and S and T are both in $K_0(15)$. Thus, we have the following results.

COROLLARY 1.5. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, we have $P(\alpha,\beta) \in \mathbb{Z}[K,K^{-1}]$ and therefore $P(\alpha,\beta) \in K_0(10)$.*

COROLLARY 1.6. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, we have $Q(\alpha,\beta) \in \mathbb{Q}[S,T,T^{-1}]$ and therefore $Q(\alpha,\beta) \in K_0(15)$.*

2. Proofs

2.1. Proofs of the recurrences. We shall prove the following identities, from which the recurrences (1.3), (1.4), (1.10) and (1.11) follow as immediate consequences.

$$P(\alpha,\beta)P(0,1) = P(\alpha,\beta+1) - P(\alpha,\beta-1), \tag{2.1}$$

$$P(\alpha+1,\beta)P(1,0) = P(\alpha+2,\beta) - P(\alpha,\beta), \tag{2.2}$$

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$$Q(\alpha,\beta)Q(0,1) = Q(\alpha,\beta+1) + Q(\alpha,\beta-1), \tag{2.3}$$

$$Q(\alpha+1,\beta)Q(1,0) = Q(\alpha+2,\beta) - Q(\alpha,\beta). \tag{2.4}$$

PROOF OF (2.1) AND (2.2). It follows from (1.2) that

$$\begin{aligned}
&P(\alpha,\beta)P(0,1)\\
&= \Big(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\Big)\Big(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\Big)\\
&= \Big(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q^2)}{R(q)^2} - (-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q)^2}{R(q^2)}\Big)\\
&\quad -\Big(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q)^2}{R(q^2)} - (-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q^2)}{R(q)^2}\Big)\\
&= \Big(\frac{1}{q^\alpha R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}} + (-1)^{\alpha+(\beta+1)}q^\alpha R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}\Big)\\
&\quad -\Big(\frac{1}{q^\alpha R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}} + (-1)^{\alpha+(\beta-1)}q^\alpha R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}\Big)\\
&= P(\alpha,\beta+1) - P(\alpha,\beta-1),
\end{aligned}$$

from which we arrive at (2.1). Also, (2.2) follows by a similar argument. □

PROOF OF (2.3) AND (2.4). It follows from (1.9) that

$$\begin{aligned}
&Q(\alpha,\beta)Q(0,1)\\
&= \Big(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}} + (-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\Big)\Big(\frac{R(q^3)}{R(q)^3} + \frac{R(q)^3}{R(q^3)}\Big)\\
&= \Big(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q^3)}{R(q)^3} + (-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q)^3}{R(q^3)}\Big)\\
&\quad +\Big(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q)^3}{R(q^3)} + (-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q^3)}{R(q)^3}\Big)\\
&= \Big(\frac{1}{q^\alpha R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}} + (-1)^{\alpha}q^\alpha R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}\Big)\\
&\quad +\Big(\frac{1}{q^\alpha R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}} + (-1)^{\alpha}q^\alpha R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}\Big)\\
&= Q(\alpha,\beta+1) + Q(\alpha,\beta-1).
\end{aligned}$$

This therefore proves (2.3). One may derive (2.4) by the same procedure. □

2.2. Proofs of (1.14) and (1.15). As we have commented in Remark 1.3, the only (and true) difficulty is proving (1.14) and (1.15). Let us begin with an interesting relation between $Q(1,0)$ and $Q(1,-1)$.

LEMMA 2.1. *We have*

$$Q(1,0) - Q(1,-1) = 3. \tag{2.5}$$

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PROOF. Notice that

$$Q(1,0)-Q(1,-1)=\left(\frac{1}{qR(q)^2R(q^3)}-qR(q)^2R(q^3)\right)-\left(\frac{R(q)}{qR(q^3)^2}-\frac{qR(q^3)^2}{R(q)}\right)$$
$$=\frac{(R(q^3)-R(q)^3)(1+q^2R(q)R(q^3)^3)}{qR(q)^2R(q^3)^2}.$$

Thanks to the modular equation (1.1), we arrive at (2.5). □

Lemma 2.1 implies that if one of (1.14) and (1.15) is proved, then the other follows automatically.

Now recall that $K_0(N)$ is the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. For $f(\tau)\in K_0(N)$ with Fourier expansion

$$f(\tau)=\sum_{n=n_0}^{\infty}a_nq^n,$$

we define the U-operator by

$$U(f)=\sum_{n=\lceil n_0/5\rceil}^{\infty}a_{5n}q^n. \tag{2.6}$$

A standard result [3, pages 80–82] states that for any positive integer N, if $f\in K_0(5N)$, then $U(f)\in K_0(N)$.

For notational convenience, let us write

$$E(q)=(q;q)_\infty.$$

Our proof of (1.14) relies on a surprisingly neat 5-dissection identity as follows.

LEMMA 2.2. *We have*

$$U\left(\frac{E(q^3)^2}{E(q)}\right)=\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})}. \tag{2.7}$$

PROOF. It follows from Newman [13] that

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3}\in K_0(15)\quad\text{and}\quad q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\in K_0(75).$$

If we compare the Fourier expansions of

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3}\quad\text{and}\quad U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right),$$

which are both in $K_0(15)$, it can be observed that

$$U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right)=q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3},$$

from which (2.7) follows. □

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We are now in a position to prove (1.14). Recall that the 5-dissection formulas for $E(q)$ and $1/E(q)$ (see [12, Equations (8.1.1) and (8.4.4)]) are respectively given by

$$E(q) = E(q^{25})\Big(\frac{1}{R(q^5)} - q - q^2R(q^5)\Big) \tag{2.8}$$

and

$$\frac{1}{E(q)} = \frac{E(q^{25})^5}{E(q^5)^6}\Big(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Big). \tag{2.9}$$

Therefore,

$$\frac{E(q^3)^2}{E(q)} = \frac{E(q^{25})^5E(q^{75})^2}{E(q^5)^6} \times\Big(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Big)\Big(\frac{1}{R(q^{15})} - q^3 - q^6R(q^{15})\Big)^2,$$

from which we extract

$$U\Big(\frac{E(q^3)^2}{E(q)}\Big) = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}\bigg(\Big(\frac{1}{q^2R(q)^4R(q^3)^2} + q^2R(q)^4R(q^3)^2\Big) - 4\Big(\frac{1}{qR(q)^2R(q^3)} - qR(q)^2R(q^3)\Big) - 3\Big(\frac{R(q)}{qR(q^3)^2} - \frac{qR(q^3)^2}{R(q)}\Big) + 2\Big(\frac{R(q^3)}{R(q)^3} + \frac{R(q)^3}{R(q^3)}\Big) - 5\bigg).$$

Thus,

$$\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})} = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}(Q(2,0) - 4Q(1,0) - 3Q(1,-1) + 2Q(0,1) - 5),$$

that is,

$$S = Q(2,0) - 4Q(1,0) - 3Q(1,-1) + 2Q(0,1) - 5.$$

It follows from (2.4) and (1.12) that

$$Q(2,0) = Q(1,0)^2 + Q(0,0) = Q(1,0)^2 + 2$$

and from (2.4) and (2.12) that

$$Q(1,-1)Q(1,0) = Q(2,-1) - Q(0,-1) = -9T^{-1} - 4 + T.$$

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Also, (2.5) states that

$$Q(1,0) - Q(1,-1) = 3.$$

Therefore,

$$\begin{aligned} S &= (Q(1,0)^2 + 2) - 4Q(1,0) - 3Q(1,-1) + 2Q(0,1) - 5 \\ &= Q(1,0)(Q(1,-1) + 3) - 4Q(1,0) - 3(Q(1,0) - 3) + 2Q(0,1) - 3 \\ &= -4Q(1,0) + Q(1,0)Q(1,-1) + 2Q(0,1) + 6 \\ &= -4Q(1,0) + (-9T^{-1} - 4 + T) + 2(2 + 9T^{-1}) + 6 \\ &= -4Q(1,0) + 9T^{-1} + 6 + T, \end{aligned}$$

from which (1.14) follows. Finally, (1.15) follows from (1.14) and (2.5).

2.3. Proof of Theorem 1.4. It follows from (1.12), (1.13) and the recurrence relation (1.10) that

$$Q(0,-1) = (2 + 9T^{-1})Q(0,0) - Q(0,1) = 2 + 9T^{-1}. \tag{2.10}$$

Therefore, by (1.11), (1.15) and (2.10),

$$\begin{aligned} Q(2,-1) &= (-\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T + \tfrac{3}{2})Q(1,-1) + Q(0,-1) \\ &= (-\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T + \tfrac{3}{2})(-\tfrac{1}{4}S + \tfrac{9}{4}T^{-1} + \tfrac{1}{4}T - \tfrac{3}{2}) + (2 + 9T^{-1}). \end{aligned} \tag{2.11}$$

On the other hand, Gugg [11, Theorem 5.1(v)] proved that

$$Q(2,-1) = -2 + T. \tag{2.12}$$

We therefore arrive at Theorem 1.4 by equating (2.11) and (2.12).

3. Applications

In this section we will explain how to take advantage of the eta-quotient representations of $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ to prove q-series identities and congruences.

Applications of Theorem 1.1 have been used extensively in several recent papers. For example, in [9], the authors used Theorem 1.1 to give an elementary proof of congruences modulo 25 for broken k-diamond partitions that were first discovered in [19, Theorem 2]. Also, by Theorem 1.1, the second author [20] derived several congruences modulo 25 for the 5-dots bracelet partition function. The first author and Hirschhorn [8] utilised the eta-quotient representations of $P(\alpha,\beta)$ to give an elementary proof of an infinite family of congruences modulo powers of 5 for partitions into distinct parts. A similar treatment was used for 1-shell totally symmetric plane partitions [7]. (See [7, Section 2.1] for a detailed account of such applications.)

For applications of the eta-quotient representations of $Q(\alpha,\beta)$, we prove the following q-series identity as an illustration.

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THEOREM 3.1. *Define the U-operator as in (2.6). Then*

$$U\Big(q^{-2}\frac{E(q^3)^3}{E(q)^2}\Big) = \frac{5}{4}q^{-1}\frac{E(q^3)^4E(q^5)^{12}}{E(q)^{10}E(q^{15})^5} - \frac{5}{4}q^{-1}\frac{E(q^3)^6E(q^5)^4}{E(q)^6E(q^{15})^3} + \frac{5}{2}q\frac{E(q^3)^3E(q^5)^7}{E(q)^9} - \frac{495}{4}q^3\frac{E(q^3)^2E(q^5)^2E(q^{15})^5}{E(q)^8}. \tag{3.1}$$

In [22], Zhang and Shi showed that if we expand the sixth-order mock theta function

$$\beta(q) = \sum_{n=0}^{\infty}\frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}} =: \sum_{n=0}^{\infty}p_\beta(n)q^n,$$

then

$$\sum_{n=0}^{\infty}p_\beta(3n+1)q^n = \frac{E(q^3)^3}{E(q)^2},$$

from which Zhang and Shi deduced that

$$p_\beta(15n+7) \equiv 0 \pmod 5. \tag{3.2}$$

We see that (3.1) is a strengthening of (3.2).

PROOF. Substituting the 5-dissection identities of $E(q)$ and $1/E(q)$, that is, (2.8) and (2.9), into $E(q^3)^3/E(q)^2$ and applying the U-operator,

$$U\Big(q^{-2}\frac{E(q^3)^3}{E(q)^2}\Big) = \frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}U(\Pi),$$

where

$$\begin{aligned}\Pi = q^{-2}\Big(&\frac{1}{R(q^{15})} - q^3 - q^6R(q^{15})\Big)^3\\ \times\Big(&\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5)\\ &+ 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Big)^2.\end{aligned}$$

We expand the products in Π and find that $U(\Pi)$ has terms

$$\begin{aligned}U(\Pi) = &\frac{5}{R(q)^6R(q^3)^3} - \frac{60q}{R(q)^4R(q^3)^2} + \frac{20q}{R(q)R(q^3)^3} + \frac{50q^2}{R(q)^5}\\ &+ \frac{60q^2R(q)}{R(q^3)^2} + \frac{20q^2R(q)^4}{R(q^3)^3} - \frac{15q^3R(q^3)^2}{R(q)^6} + 75q^3\\ &- \frac{15q^3R(q)^6}{R(q^3)^2} + \frac{20q^4R(q)^3}{R(q^3)^4} - \frac{60q^4R(q^3)^2}{R(q)} - 50q^4R(q)^5\\ &+ 20q^5R(q)R(q^3)^3 - 60q^5R(q)^4R(q^3)^2 - 5q^6R(q)^6R(q^3)^3.\end{aligned}$$

In light of the definition (1.9), grouping the first and last terms gives $5q^3Q(3,0)$, and grouping the second and second last terms gives $-60q^3Q(2,0)$. Thus,

$$U(\Pi) = 5q^3(Q(3,0) - 12Q(2,0) + 4Q(2,-1) + 10Q(1,1) + 12Q(1,-1) + 4Q(1,-2) - 3Q(0,2) + 15),$$

from which we conclude that

$$U\Big(q^{-2}\frac{E(q^3)^3}{E(q)^2}\Big) = 5q^3\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}(Q(3,0) - 12Q(2,0) + 4Q(2,-1) + 10Q(1,1) + 12Q(1,-1) + 4Q(1,-2) - 3Q(0,2) + 15).$$

If we apply Theorem 1.2 to write each summand $Q(\cdot,\cdot)$ in terms of S and T, then

$$U\Big(q^{-2}\frac{E(q^3)^3}{E(q)^2}\Big) = \frac{5q^3}{64T^3}\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}} \times A(q),$$

where

$$\begin{aligned}A(q) = 729 &+ 1458T + 783T^2 + 92T^3 + 23T^4 - 14T^5 + T^6\\ &- 243ST - 1764ST^2 - 50ST^3 + 28ST^4 - 3ST^5\\ &+ 27S^2T^2 - 14S^2T^3 + 3S^2T^4 - S^3T^3.\end{aligned}$$

Recalling (1.16),

$$81 + 144T + 46T^2 - 16T^3 + T^4 - 18ST - 2ST^3 + S^2T^2 = 0,$$

we have

$$\begin{aligned}A(q) &= A(q) - (81 + 144T + 46T^2 - 16T^3 + T^4 - 18ST - 2ST^3 + S^2T^2)\\ &\qquad \times (T^2 + 2T + 9 - ST)\\ &= -1584ST^2 + 32ST^3 + 16ST^4 - 16S^2T^3.\end{aligned}$$

It follows that

$$U\Big(q^{-2}\frac{E(q^3)^3}{E(q)^2}\Big) = 5q^3\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}\Big(-\frac{99}{4}ST^{-1} + \frac{1}{2}S + \frac{1}{4}ST - \frac{1}{4}S^2\Big)$$

which is exactly (3.1). □

REMARK 3.2. In a private communication with Nayandeep Deka Baruah, we were informed that Baruah, Begum and Das [5] recently derived a number of dissection identities for several partition functions. For instance, they showed that

$$U\Big(q^{-1}\frac{1}{E(q)E(q^3)}\Big) = \frac{E(q^5)^5}{E(q)^6E(q^{15})} + 10q\frac{E(q^5)^{10}}{E(q)^7E(q^3)^5} + q^2\frac{E(q^{15})^5}{E(q^3)^6E(q^5)} + 45q^3\frac{E(q^5)^5E(q^{15})^5}{E(q)^6E(q^3)^6} - 90q^5\frac{E(q^{15})^{10}}{E(q)^5E(q^3)^7}.$$

These identities could also be shown with the assistance of Theorem 1.2 and (1.16).

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