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LIMIT DISTRIBUTION FOR A CONSECUTIVE-k-OUT-OF-n: F SYSTEM

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Abstract

A consecutive-k-out-of-n: F system consists of n components ordered on a line. Each component, and the system as a whole, has two states: it is either functional or failed. The system will fail if and only if at least k consecutive components fail. The components are not necessarily equal and we assume that components' failures are stochastically independent. Using a result of Barbour and Eagleson (1984) we find a bound for the distance of the distribution of system's lifetime from the Weibull distribution. Subsequently, using this bound limit theorems are derived under quite general conditions.

POISSON LIMIT THEOREM; WEIBULL DISTRIBUTION

1. Introduction

A consecutive-k-out-of-n: F system consists of n linearly ordered components. The system will fail if and only if at least k consecutive components fail.

Recently, consecutive-k-out-of-n: F systems have been proposed to model telecommunication systems and oil pipelines [3], [4], vacuum systems in accelerators [5], computer ring networks [6] and spacecraft relay stations [2].

Throughout, we assume that failures of components are stochastically independent. In [7] Papastavridis proved that, for the case of equal components, the failure distribution of such a system approaches the Weibull distribution as $n \to \infty$.

In this paper we prove the same result under quite general and natural assumptions for the case of unequal components.

In Section 2 we prove Theorem 1 which provides a bound for the distance of the distribution of system's lifetime from the Weibull distribution. Using this result we get Theorem 2 which gives us a limiting result under quite general assumptions. Finally, Examples 1 and 2 indicate that practically all reasonable systems satisfy the assumptions of Theorem 2, so their lifetime approaches the Weibull distribution.

2. The main result

Let F_i be the failure distribution of the ith component (i.e. the probability that the component i will fail in time less than or equal to $t \geqq 0$), $i = 1, \dots, n$. We define $P_j(t) = F_j(t) \cdots F_{j+k-1}(t)$, $j = 1, \dots, n-k+1$, $p(t) = \max_{1 \leqq i \leqq n} F_i(t)$ and $\lambda(t) = \sum_{j=1}^{n-k+1} P_j(t)$. Finally we denote by T the life length of a consecutive-k-out-of-n: F system. We notice that the parameter n is suppressed from $P_j(t)$, $p(t)$, $\lambda(t)$ and T. Our main result is the following.

Theorem 1.

$$|P(T \leqq t) - (1 - e^{-\lambda(t)})| \leqq (2k-1)p^k(t) + (2k-2)p(t).$$

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Proof. Let t be fixed and let us consider the random variable X_j, $j = 1, \dots, n-k+1$, which takes the value 1 if and only if all the components $j, j+1, \dots, j+k-1$ fail and 0 in all other cases and the random variable $X = \sum_{j=1}^{n-k+1} X_j$. It is clear that the system fails if and only if $X > 0$. We observe that $E(X_j) = P_j(t)$, $j = 1, \dots, n-k+1$ and $E(X) = \sum_{j=1}^{n-k+1} P_j(t) = \lambda(t)$. Applying Theorem 2 of Barbour and Eagleson [1] we have that

$$\begin{aligned}|P(T \leqq t) - (1 - e^{-\lambda(t)})| &\leqq \min(1, 1/E(x)) \sum_{j=1}^{n-k+1} \Big[P_j^2(t) + \sum_{\substack{i=j-k+1 \\ i \neq j}}^{j+k-1} [P_j(t)P_i(t) + E(X_jX_i)] \Big] \\ &\leqq (1/E(X)) \Big[\Big(\sum_{j=1}^{n-k+1} P_j(t)p^k(t) \Big) + \Big(\sum_{j=1}^{n-k+1} P_j(t) \Big)(2k-2)p^k(t) \\ &\quad + \Big(\sum_{j=1}^{n-k+1} E(X_j) \Big)(2k-2)p(t) \Big] \\ &= (2k-1)p^k(t) + (2k-2)p(t).\end{aligned}$$

We now present a limit theorem which can be easily proved by using Theorem 1, but before stating it we need some preparation.

For the system discussed above let us consider the following assumptions:

(1) There are positive numbers λ_i, α_i and functions α_i so that $F_i(t) = (\lambda_i t)^{\alpha_i} + t^{\alpha_i}\phi_i(t)$, $i = 1, 2, \dots$ for $0 \leqq t < \delta$, where δ is a positive number.

(2) We assume that $\lim \phi_i(t) = 0$ as $t \to 0$, uniformly on i,

(3) $\lim \lambda_i = \lambda$ as $i \to \infty$.

We define $\alpha = \inf \alpha_i$ and we denote by T_n the life length of a consecutive-k-out-of-n: F system. (*Remark*: Beware that T_n and T are different notations for the same random variable.)

Theorem 2. Under the Assumptions 1–3

(a) If $\alpha = \alpha_i$ for $i = 1, 2, \dots$, then

$$\lim P(n^{1/k\alpha} T_n \leqq t) = 1 - \exp[-(\lambda t)^{\alpha k}] \quad \text{as} \quad n \to \infty.$$

(b) If $\alpha > 0$ and for every $i = 1, 2, \dots$ there is a j with $i \leqq j \leqq i + k - 1$ so that $\alpha_j > \alpha$, then

$$\lim P(n^{1/k\alpha} T_n \leqq t) = 0 \quad \text{as} \quad n \to \infty.$$

Proof. It is not difficult to prove that the bound provided by Theorem 1 goes to zero as $n \to \infty$. Furthermore, it is easy to derive that $\lambda(t) \to (\lambda t)^{\alpha k}$ as $n \to \infty$ in case (a), while $\lambda(t) \to 0$, as $n \to \infty$ in case (b).

To get an idea how the previous results work we present the following two simple examples:

(1) A specific example where Theorem 2 is applicable is the case of ith component $i = 1, \dots, n$ having the Weibull distribution as life length distribution i.e.

$$F_i(t) = 1 - \exp[-(\lambda_i t)^\alpha] = (\lambda_i t)^\alpha + t^\alpha O(1)$$

with $\lambda_i \to \lambda$ as $i \to \infty$.

(2) The following example mixes two kinds of well-known parametric families i.e. the Weibull and the gamma distribution. Let (λ_i) and (μ_i) be sequences of positive numbers converging respectively to λ and $\lambda(\Gamma(\alpha+1))^{1/\alpha}$. Let $x_i = (\lambda_i, \mu_i)$ and

$$G_{2i-1}(t) = 1 - \exp[-(\lambda_i t)^\alpha] = (\lambda_i t)^\alpha + t^\alpha o(1)$$

and

$$G_{2i}(t) = \int_0^t \frac{\mu_i^\alpha}{\Gamma(\alpha)} s^{\alpha-1} e^{-\mu_i s}\, ds = \frac{\mu_i^\alpha}{\Gamma(\alpha+1)} t^\alpha + t^\alpha o(1).$$

It is easy to check that Theorem 2 is still applicable and so $n^{(1/k\alpha)} T_n$ approaches a Weibull distributed random variable of parameters αk and λ.

Remark. Let us denote by T_n^c the life length of a circular consecutive-k-out-of-n: F system (see Derman et al. [4]). We then have the more or less obvious inequality

$$P(T_n \leqq t) \leqq P(T_n^c \leqq t) \leqq P(T_n \leqq t) + \sum_{j=n-k+2}^{n} \prod_{i=j}^{j+k-1} F_i(t),$$

which means that the analogous limit theorems for the circular case are valid too.

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