TOROIDAL q-OPERS

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Abstract We define and study the space of q-opers associated with Bethe equations for integrable models of XXZ type with quantum toroidal algebra symmetry. Our construction is suggested by the study of the enumerative geometry of cyclic quiver varieties, in particular the ADHM moduli spaces. We define $(\overline{GL(\infty)},q)$-opers with regular singularities and then, by imposing various analytic conditions on singularities, arrive at the desired Bethe equations for toroidal q-opers.

1. Introduction: Geometric facets of Bethe equations

1.1. Integrable models and Bethe ansatz

The study of 1-dimensional quantum integrable models fueled modern mathematics with a variety of interesting ideas, in particular the discovery of the quantum groups and related structures. A particularly useful tool in the study of integrable models is the so-called algebraic Bethe ansatz method (see, e.g., [35, 54]), having its roots in the original papers of Hans Bethe from the 1930s.

Let us briefly describe here the modern mathematical perspective on how algebraic Bethe ansatz works for integrable models of specific type, namely the spin chains. Let $\mathfrak{g}$ be a simple Lie algebra and $\hat{\mathfrak{g}}_{k=0} = \mathfrak{g}\left[t^{\pm 1}\right]$ be the corresponding loop algebra (affine algebra with vanishing central charge $k = 0$). The finite-dimensional modules $\{V_i\}$ of $\mathfrak{g}$ give rise to the so-called evaluation modules $\{V_i(a_i)\}$, where $a_i \in \mathbb{C}^{\times}$ stand for the value of the loop parameter t. These modules generate a tensor category–namely, every finite-dimensional representation of $\hat{\mathfrak{g}}$ can be written as a tensor product of evaluation modules. Passing from $\mathfrak{g}\left[t^{\pm 1}\right]$ to the corresponding quantum affine algebra $U_{\hbar}(\hat{\mathfrak{g}})$ or the Yangian $Y_{\hbar}(\mathfrak{g})$, one obtains a deformation of such a tensor category, known as a braided tensor category [9]. This object features a new intertwining operator (not invertible in general)

$$R_{V_i(a_i),V_j(a_j)} : V_i(a_i) \otimes V_j(a_j) \longrightarrow V_j(a_j) \otimes V_i(a_i),$$

satisfying the famous Yang–Baxter equation. We note that in the deformed case, the analogue of the evaluation map exists only in type A, and the $\{V_i(a_i)\}$ stand here for modules appropriately 'twisted' by a_i finite-dimensional representations of $U_{\hbar}(\hat{\mathfrak{g}})$.

To describe the integrable model, we choose a specific object in such a braided tensor category,

$$\mathcal{H} = V_{i_1}(a_{i_1}) \otimes \cdots \otimes V_{i_n}(a_{i_n}),$$

which we refer to as a *physical space*, and the vectors in this space are called *states*. For a given module $W(u)$ called an *auxiliary module*, with parameter u known as a *spectral parameter*, we define the *transfer matrix*

$$T_{W(u)} = \mathrm{Tr}_{W(u)} \left[(Z \otimes 1) P R_{W(u),\mathcal{H}} \right].$$

Here the *twist* Z is given by $Z = \prod_{i=1}^{r} z_i^{\check{\alpha}_i} \in e^{\mathfrak{h}}$, where $\mathfrak{h}$ is the Cartan subalgebra in $\mathfrak{g}$, $\{\check{\alpha}_i\}_{i=1,\ldots,r}$ are the simple coroots of $\mathfrak{g}$ and P is a permutation operator. The *monodromy matrix* $M_{W(u)}^Z = (Z \otimes 1) P R_{W(u),\mathcal{H}}$ is an operator in $W(u) \otimes \mathcal{H}$. Notice that the transfer matrix $T_{W(u)}$ is an operator acting on the physical space $\mathcal{H}$. The Yang–Baxter equation implies that transfer matrices corresponding to various choices of $W(u)$ form a commutative algebra, known as a *Bethe algebra*. The commutativity of the Bethe algebra implies *integrability*, and the expansion coefficients of the transfer matrix yield (nonlocal) *Hamiltonians* of the XXX or XXZ spin chain depending on whether we deal with the Yangian or the quantum affine algebra. From now on, we will primarily focus on the quantum affine algebra and the XXZ model, although most of the construction applies to the Yangian and the XXX models as well.

The classic example of the XXZ Heisenberg magnet corresponds to the quantum algebra $U_\hbar\left(\hat{\mathfrak{sl}}(2)\right)$ in which the physical space $\mathcal{H}$ is constructed from $V_i(a_i) = \mathbb{C}^2(a_i)$–the standard 2-dimensional evaluation modules of $U_\hbar\left(\hat{\mathfrak{sl}}(2)\right)$.

The solution of the integrable model implies finding the eigenvalues and eigenvectors of simultaneously diagonalised Hamiltonians–that is, elements of the Bethe algebra. One way to accomplish the task is to follow the old-fashioned procedure from the 1980s known as *algebraic Bethe ansatz*. It implies that the eigenvalues of the transfer matrices (upon rescaling) are symmetric functions of the roots of the system of algebraic equations, known as Bethe ansatz equations. Although this approach is straightforward and effective, we will explore other modern techniques which provide insights into representation-theoretic aspects of the problem.

1.2. Modern approach to Bethe ansatz

1.2.1. Quantum Knizhnik–Zamolodchikov equations.
The intertwining operators for the quantum affine algebra $U_\hbar(\hat{\mathfrak{g}})$ and thus the matrix elements of their products, known as *conformal blocks*, satisfy certain difference equations known as quantum Kniznik–Zamolodchikov (qKZ) equations (also known as Frenkel–Reshetikhin equations) [25].

Explicitly, qKZ equations can be written as follows: difference equations

$$\Psi(a_{i_1}, \ldots, q a_{i_k}, \ldots, a_{i_n}, \{z_i\}) = H_{i_k}^{(q)} \Psi(a_{i_1}, \ldots, a_{i_n}, \{z_i\}), \tag{1.1}$$

where the solutions Ψ take values in $\mathcal{H}$ and operators $H_i^{(q)}$ are expressed in terms of products of R-matrices. The analytic properties of the solutions of qKZ equations will be discussed later in Section 1.3.2. There is also a commuting system of equations in $\{z_i\}$-variables for Ψ, known as *dynamical equations* (see, e.g., [58, 59]).

The solution to the qKZ equation is given by an integral expression, so that the integrand has the following asymptotic behaviour in the limit $q \to 1$ (or $\eta = \log(q) \to 0$):

$$e^{\frac{Y(\{a_i\},\{z_i\},\{x_i\})}{\eta}} \left[\phi_0\left(\{a_i\},\{z_i\},\{x_i\}\right) + O(\eta)\right], \tag{1.2}$$

where $\{x_i\}$ are the variables of integration. In the limit $q \to 1$, the stationary phase approximation gives $\Psi = e^{\frac{S}{\eta}}(\Psi_0 + O(\eta))$, where $S = Y|_{\sigma_i}$, with σ_i the solutions of the equations $\partial_{x_i} Y = 0$ which need to be solved with respect to the variables $\{x_i\}$. These equations coincide with the Bethe equations, and Ψ_0 is the eigenvector for operators $H_i^{(1)}$, known as the nonlocal Hamiltonians of the corresponding XXZ model: they emerge as coefficients from the expansion of the transfer matrices with respect to the spectral parameter–that is, $H_i^{(1)}\Psi_0 = e^{p_i}\Psi_0$, where $p_i = a_i\partial_{a_i}S$.

1.2.2. *QQ*-systems and Baxter operators.

When we earlier discussed the transfer matrices $T_{W(u)}$, we considered $W(u)$ to be a finite-dimensional module of $U_\hbar(\hat{\mathfrak{g}})$. We notice that the *universal R-matrix*, which produces particular braiding operators $R_{V_i(a_i),V_j(a_j)}$, belongs to the completion of the tensor product $U_\hbar\left(\hat{\mathfrak{b}}_+\right) \otimes U_\hbar\left(\hat{\mathfrak{b}}_-\right)$, where $U_\hbar\left(\hat{\mathfrak{b}}_\pm\right)$ are the Borel subalgebras of $U_\hbar(\hat{\mathfrak{g}})$. Therefore, there is no obstruction in taking auxiliary representations $W(u)$ to be representations of $U_\hbar\left(\hat{\mathfrak{b}}_+\right)$.

The purpose of that is as follows. There exist prefundamental representations of $U_\hbar\left(\hat{\mathfrak{b}}_+\right)$ which are infinite-dimensional. If one extends the braided tensor category of finite-dimensional modules by such representations, the Grothendieck ring of the resulting category is generated by those modules.

The corresponding transfer matrices turn out to be well defined, and moreover, the eigenvalues of the transfer matrices are polynomials of the spectral parameter, generating elementary symmetric functions of the solutions of Bethe equations. Such transfer matrices were originally introduced by Baxter and thus are known as *Baxter operators* ad hoc via their eigenvalues. Their representation-theoretic meaning was realised much later, in papers by Frenkel and Hernandez [21], following earlier ideas of Bazhanov, Lukyanov and Zamolodchikov [4] and Hernandez and Jimbo [31].

There are two series of prefundamental representations $\left\{V_+^i(u)\right\}_{i=1,\dots,r}$, $\left\{V_-^i(u)\right\}_{i=1,\dots,r}$ and the associated Baxter operators $\left\{Q_\pm^i(u)\right\}_{i=1,\dots,r}$. They obey the following key relation [22]:

$$\widetilde{\xi}_i Q_-^i(u) Q_+^i(\hbar u) - \xi_i Q_-^i(\hbar u) Q_+^i(u) = \Lambda_i(u) \prod_{j \neq i} \left[\prod_{k=1}^{-a_{ij}} Q_+^j\left(\hbar^{b_{ij}^k} u\right)\right]$$

$$i = 1,\dots,r,\ b_{ij}^k \in \mathbb{Z}.$$

Here polynomials $\Lambda_i(u)$ are known as Drinfeld polynomials, characterising the representation $\mathcal{H}$ of $U_\hbar(\hat{\mathfrak{g}})$, and $\xi_i, \widetilde{\xi_i}$ are some monomials of $\{z_i\}$.

This system of equations–known as the *QQ-system*, considered as equations on $\left\{Q_\pm^i(u)\right\}_{i=1,\dots,r}$ and subject to some nondegeneracy conditions–is equivalent to the Bethe ansatz equations.

We note that a similar construction and an analogue of the QQ-system should also exist for Yangians, with some progress being made in [3].

We mention that the QQ-systems emerged recently in a seemingly different context, the so-called ODE/IM correspondence [11, 5]. The statement of the correspondence can be roughly formulated as follows. The vacuum eigenvalues of the Baxter-operator quantum Korteweg–De Vries model associated with the affine Lie algebra $\hat{\mathfrak{g}}$ appear as spectral determinants of certain singular differential operators associated with the so-called affine opers associated with $^L\hat{\mathfrak{g}}$. In a particular case of standard quantum Korteweg–De Vries, these operators are just singular Sturm–Liouville operators. As shown in [43, 44], they turn out to be the solution of the QQ-system with different analyticity conditions on entire Q-functions (which are generally nonpolynomial in this case).

1.3. Geometric interpretations

1.3.1. Quantum *K*-theory of Nakajima varieties. The relation between enumerative algebraic geometry and integrability has been known for some time. Starting from the pioneering works of Witten and Dubrovin, it flourished in the works of A. Givental and his school in the 1990s. Recently, progress in understanding supersymmetric gauge theory merged with developments in geometric representation theory. In particular, the study of so-called *symplectic resolutions* from the representation-theoretic point of view gave a new life to this fruitful relationship in work of A. Okounkov and his collaborators [8]. It has been observed that some integrable systems based on quantum groups, specifically XXX and XXZ models, naturally emerge from enumerative geometry for a large class of algebraic varieties, known as *Nakajima quiver varieties* [47].

Let us recall this connection in the simplest nontrivial examples of such varieties, namely the cotangent bundles over Grassmannians $T^*\mathbf{Gr}_{k,n}$. The standard objects in the enumerative geometry are the appropriate deformations of the cup product and the tensor product in the equivariant cohomology and K-theory, correspondingly, where the deformation is characterised by the series in *Kähler parameters*, with coefficients being produced by curve counting.

The physics results of Nekrasov and Shatashvili [50] lead to the following conjecture about the equivariant quantum K-theory $K_T(T^*\mathbf{Gr}_{k,n})$: quantum multiplication by the generating function for the exterior algebra of the tautological bundle coincides with the Baxter Q-operator for the Heisenberg XXZ spin chain. Also, since tautological bundles generate the entire quantum K-theory, one can describe the equivariant quantum K-theory ring as the ring of symmetric functions of Bethe roots.

The proof of that conjecture was given in [53]. It uses the theory of quasimaps to Nakajima varieties as the 'curve counting', which is different from the older approach to quantum K-theory given by Givental. To relate the quantum equivariant K-theory

with spin chains, it is not enough to consider the operators of quantum multiplication by classical K-theory classes: in fact, both multiplication in the equivariant K-theory and the tautological classes should be deformed simultaneously–in our case of $T^* \mathbf{Gr}_{k,n}$, by just one Kähler parameter z. One can define elements $\hat{\mathcal{V}}^\tau \in K_T(T^* \mathbf{Gr}_{k,n})[[z]]$, which we call quantum tautological bundles. In the classical limit $z \to 0$, these elements coincide with the corresponding classical bundles V^τ, which is a certain tensorial polynomial of standard tautological bundles, corresponding to the symmetric polynomial τ in k variables in the standard K-theory. The localised equivariant quantum K-theory $K_T^{loc}(T^* \mathbf{Gr}_{k,n})$ can be identified with appropriate weight subspace in the space $\mathcal{H}$ of the XXZ Heisenberg magnet, so that considering the union of such spaces for all k, one obtains the entire space of states $\mathcal{H}$.

To prove this conjecture, one needs to define and compute *vertex functions*, which are quasimap analogues of Givental's *I-functions*. These are certain Euler characteristics which count quasimaps and determine the quantum K-theory classes. Such vertex functions satisfy the quantum difference equations which coincide with qKZ and dynamical equations [52], which were discussed in Section 1.2.1. To understand the action of the operators of quantum multiplication by the quantum tautological bundles, one has to study the $q \to 1$ asymptotics of such solutions of qKZ. That allows us to identify quantum tautological classes with the elements of the Bethe algebra, thereby leading to the proof of the conjecture of Nekrasov and Shatashvili.

These results have been proven for larger classes of Nakajima varieties–for example, partial flag varieties (see, e.g., [37, 41]).

Notice that this approach gives a geometric interpretation to qKZ and dynamical equations associated to $\mathfrak{g}$ of simply laced type for the specific physical spaces $\mathcal{H}$ emerging from quiver varieties. Moreover, each of the Q-operators on its own has a geometric meaning. The Q_+^i-operators correspond to the exterior powers of tautological bundles. Their Q_-^i counterparts correspond to the exterior powers of tautological bundles of Nakajima varieties with a different choice of stability parameters. In the case of flag varieties, such a change in the stability parameters is provided by the action of Weyl reflection.

However, the QQ-system relations themselves do not arise naturally, since in particular $Q_\pm^i$-operators do not act in the same space. In the next section, we will discuss another geometric viewpoint on Bethe ansatz, specifically related to the geometric interpretation of the QQ-system.

1.3.2. Quantum q-Langlands correspondence.

To oversee the geometric interpretation of QQ-systems, we take several steps back in time and deformation-wise. Earlier we described the construction of the XXZ spin chain. There is a certain scaling limit of the XXZ model which is called the Gaudin model. This limit can be understood quasiclassically as the $\hbar \to 0$ of equation (1.1), so that $q = \hbar^{k+h^\vee}$, where $h^\vee$ is a dual Coxeter element. Then the qKZ equation turns into the differential equation, known as the Knizhnik–Zamolodchikov (KZ) equation:

$$(k + h^\vee) \partial_{a_i} \Psi = H_i \Psi, \tag{1.3}$$

where H_i emerge as coefficients of $\log \hbar$ in $H^{(q)}$. The solution Ψ belongs to the classical limit of $\mathcal{H}$–that is, the tensor product of some evaluation representations of the Lie algebra $\hat{\mathfrak{g}}$ with evaluation parameters a_i: $\mathcal{H}^{cl} = V_1(a_1) \otimes \cdots \otimes V_n(a_n)$. The mutually commuting Gaudin Hamiltonians H_i have easy-to-read expressions

$$H_i = \sum_{j=1, i \neq j}^{n} \frac{t_i^\alpha \otimes t_j^\alpha}{a_i - a_j} + \mathcal{Z}_i,$$

where t^α form an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form, $\mathcal{Z}$ belongs to Cartan subalgebra of $\mathfrak{g}$ and the indices i, j indicate on which of the representations V_i these elements act.

One can see that in the limit $k \to -\hbar^\vee$, known as the *critical level* limit, this equation turns into an eigenvalue problem for Gaudin Hamiltonians. It is possible to interpret the solutions of the KZ equations in a particular analyticity region in evaluation parameters $|a_1| > |a_2| > \cdots > |a_n|$ as the equations for the intertwiners of $\hat{\mathfrak{g}}$ with central charge k. Moreover, the Gaudin Hamiltonians were shown in [15] to be part of a bigger structure, namely the centre $Z(U(\hat{\mathfrak{g}}))$ of $U(\hat{\mathfrak{g}})$ at the critical level $k = -h^\vee$.

There is a natural Poisson structure on $Z(U(\hat{\mathfrak{g}}))$ arising from standard commutators away from the critical level. The famous theorem of Feigin and Frenkel [14] provides isomorphism of this Poisson algebra and the classical limit of the *W-algebra* $W(^L\mathfrak{g})$, associated to the Langlands dual Lie algebra $^L\mathfrak{g}$, also known as the *Gelfand–Dickey algebra* of pseudodifferential operators in the case $\mathfrak{g} = \mathfrak{sl}_n$.

Later this statement was reformulated [20] in terms of special connections for principal $^L\tilde{G}$-bundles, known as *oper connections*, on the punctured disk, where $^L\tilde{G}$ is an adjoint Lie group associated to $^L\mathfrak{g}$. The reformulated Feigin–Frenkel theorem implies that there is an isomorphism between $Z(U(\hat{\mathfrak{g}}))$ and the space of functions on $^L\tilde{G}$-oper connections on a punctured disk. The path from such connections to Gelfand–Dickey pseudodifferential operators is given by a well-known construction, known as *Drinfeld–Sokolov reduction* [12].

Let us return to the eigenvalue problem for Gaudin Hamiltonians, arising from the critical level limit of equation (1.3). Frenkel's theorem [19] gives a geometric description of the spectrum in terms of opers. Explicitly, it states that there is a one-to-one correspondence between the space of *Miura oper* connections with regular singularities with trivial monodromies around them on $\mathbb{P}^1$ in the case when $\mathcal{Z} = 0$. The word 'Miura' there means that there is an extra condition on such oper connections: they have to preserve the reduction of the $^L\tilde{G}$-bundle to a Borel subgroup. Later, this theorem was generalised for $\mathcal{Z} \neq 0$ by adding irregular singularity at $\infty \in \mathbb{P}^1$ [16].

The constraints on such connections could be expressed in terms of Wronskian-type relations, which are particularly manifest in the case of $SL(N)$. That suggests that the QQ-system, which is a deformation of the Wronskian relation, should arise from an appropriate $\hbar$-deformation of Miura opers. We will provide further motivation and hints in this direction.

The pseudodifferential operators, corresponding to such Miura oper connections with regular singularities through Drinfeld–Sokolov reduction, describe the constraints on the

conformal blocks and intertwiners of the $W\left({}^{L}\mathfrak{g}\right)$-algebra in the limit when the central charge $c \to \infty$. The most famous such constraint is known as the Belavin–Polyakov–Zamolodchikov (BPZ) equation (essentially the Sturm–Liouville problem with singular potential) for conformal blocks of Virasoro algebra, which is the case when $\mathfrak{g} = \mathfrak{sl}(2)$.

Naturally, that led to the quantum Langlands correspondence linking conformal blocks of $W\left({}^{L}\mathfrak{g}\right)$-algebras and $\hat{\mathfrak{g}}$-conformal blocks away from the critical level. Recently, a q-deformation of this correspondence was proposed in [1]. The proof provided in the case of simply laced $\mathfrak{g}$ is based on the enumerative geometry approach which we touched on briefly in Section 1.3.1. The key to that is to further deform this correspondence–namely, to identify conformal blocks for the quantum affine algebra $U_{\hbar}(\hat{\mathfrak{g}})$ and the deformed W-algebra $W_{q,t}\left({}^{L}\mathfrak{g}\right)$, which is the 2-parametric deformation of the Gelfand–Dickey algebra [25].

The conformal blocks for $U_{\hbar}(\hat{\mathfrak{g}})$, as we discussed in Section 1.2.1, satisfy the qKZ equation. We remark that as in the classical case, they correspond to the solution of qKZ, analytic in the region $|a_1| > |a_2| > \cdots > |a_n|$. However, the solutions of qKZ which are provided by enumerative geometry–that is, vertex functions–are analytic in $\{z_i\}$-variables. Also, it turns out that they are the ones producing the conformal blocks of the $W_{q,t}\left({}^{L}\mathfrak{g}\right)$-algebra. The transition between two families of solutions is crucial for establishing the exact correspondence between such conformal blocks. We refer to [1] for the details.

1.3.3. Miura $\hbar$-opers. A natural question is to understand the difference analogues of BPZ-type equations which serve as constraints for the conformal blocks of $W_{q,t}\left({}^{L}\mathfrak{g}\right)$. As we have discussed, the differential BPZ equations on the critical level correspond to the classical objects, namely ${}^{L}\tilde{G}$-oper connections with regular singularities on $\mathbb{P}^1$. Let ${}^{L}G$ be the simply conected group with Lie algebra ${}^{L}\mathfrak{g}$. There is a natural classical object, the $\hbar$-difference connection, locally a meromorphic ${}^{L}G$-valued function $A(z)$ on a Zariski open set of $\mathbb{P}^1$, which transforms upon trivialisation change $A(z) \longrightarrow g(\hbar z)A(z)g^{-1}(z)$.

In [24], following the constructions in [38] for $SL(N)$, we developed the $\hbar$-difference analogue of opers as such $\hbar$-difference connections for any simply connected semisimple Lie group ${}^{L}G$ with a fixed Borel subgroup ${}^{L}B_{-}$. Locally, these $\hbar$-connections have the form $A(z) = n'(z)\prod_{i=1}^{r} s_i \phi_i^{\check{\alpha}_i}(z)n(z)$. Here $n(z), n'(z) \in G(z)$, $\phi_i(z) \in \mathbb{C}(z)$ and s_i are the lifts of the fundamental Weyl reflections to ${}^{L}G$. In other words, $A(z) \in B_{-}(z)cB_{-}(z)$, where $c = \prod_{i=1}^{r} s_i$ is a Coxeter element.

Moreover, we defined such $\left({}^{L}G, \hbar\right)$-opers and their Miura versions with regular singularities, which amounts to the connections of this type which preserve the opposite Borel subgroup of B_{+} and taking $\phi_i(z) = \Lambda_i(z) \in \mathbb{C}[z]$. We proved several structural theorems about them.

One of the major statements we make in [24] is devoted to the explicit relation of these objects to the QQ-systems and Bethe ansatz.[1] To do that, we work with two versions of what we call *Z-twisted conditions* for Miura opers. The simplest Z-twisted condition is

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[1]We note here that the QQ-system we obtained in [24] differs from the ones we discussed in Section 1.2.2 in the case not simply laced (the difference is in the coefficients $\{b_{ij}\}$). We mention an upcoming paper [23], where the integrable model corresponding to such a QQ-system is

implies that the $\left({}^L G, \hbar\right)$-oper connection can be $\hbar$-gauge-equivalent to a semisimple element $Z \in H \subset^L G$, where H is the Cartan subgroup. That means $A(z) = g(\hbar z) Z g^{-1}(z)$. This condition is a difference version of the zero-monodromy condition and double-pole irregular singularity at the ∞ point of $\mathbb{P}^1$.

The relaxed version of this Z-twisted condition is as follows. Given the principal $^L G$-bundle, one can construct an associated bundle for any fundamental representation V_{ω_i} for the fundamental weight ω_i. It turns out one can associate a $(GL(2), \hbar)$-oper to any such pair of an $\left({}^L G, \hbar\right)$-oper and V_{ω_i}: this is done by restricting the Miura $\left({}^L G, \hbar\right)$-oper to the 2-dimensional subspace spanned by the two top weights in V_{ω_i}. This is possible, since a Miura $\left({}^L G, \hbar\right)$-oper preserves the reduction to a positive Borel subgroup $^L B_+ \subset^L G$.

We say that the resulting Miura oper is a *Z-twisted Miura–Plücker* $\left({}^L G, \hbar\right)$-oper if every such $(GL(2), \hbar)$-oper is $\hbar$-gauge-equivalent to the restriction of Z to the corresponding 2-dimensional space.

In [24] we showed that Z-twisted Miura–Plücker $\left({}^L G, \hbar\right)$-opers with mild nondegeneracy conditions are in one-to-one correspondence with certain solutions of QQ-systems, and that does not depend on the order in the Coxeter element. In the simply laced case, such QQ-systems are equivalent to standard Bethe ansatz equations. The case not simply laced is more involved (see the discussion in [24] and the upcoming [23]).

While it immediately follows that any Z-twisted Miura oper is indeed a Z-twisted Miura–Plücker one, the opposite statement, however, is highly nontrivial. In [24] we introduce a chain of $\hbar$-gauge transformations, which we refer to as $\hbar$-Bäcklund transformations, which on the level of QQ-systems amounts to $Q_+^i(z) \to Q_-^i(z)$, $Z \to s_i(Z)$, where s_i is the elementary Weyl reflection. However, at every step, in order to progress further, we have to impose the nondegeneracy condition on the QQ-system and the associated Miura oper. We have shown that if one can proceed with this transformation to a Z-twisted Miura–Plücker oper corresponding to the $w_0(Z)$, where w_0 is the longest Weyl group element, then such a Z-twisted Miura–Plücker Miura oper is Z-twisted. We call such Miura–Plücker opers and the associated QQ-system w_0-generic.

We also discuss the explicit version of the $\hbar$-version of Drinfeld–Sokolov reduction, following the ideas of [56]. The scalar difference equations emerging this way from Z-twisted Miura $\left({}^L G, \hbar\right)$-opers and the correspondence with the difference equations from the conformal blocks for $W_{q,t}\left({}^L \mathfrak{g}\right)$-algebras remains an interesting open problem.

1.4. Our goals in this paper

1.4.1. Two approaches to $(SL(r+1), \hbar)$-opers. In this paper, we are investigating several problems. The first one is devoted to the correspondence between the results of [38], where we work with the $SL(r+1)$ case only, and the more general approach of [24]. In [38] we used a definition of a (Miura) $\hbar$-oper which is very specific to $SL(r+1)$. It can be deduced from the 'universal' definition of an $(SL(r+1), \hbar)$-oper as an $\hbar$-connection for the principal $SL(r+1)$-bundle, which we discussed in the previous section, with the standard order of reflections in the corresponding Coxeter element (following the order

discussed. In the present paper we will be working mostly with the QQ-system associated with $\mathfrak{sl}(n)$, and we refer the reader to [24] for more details.

in the Dynkin diagram), so that in the defining representation it is represented as the matrix with 0s above its superdiagonal.

Considering the associated bundle corresponding to the defining representation, one can reinterpret the oper condition in the following way. Namely, it is the condition on the oper action in the complete flag of subbundles of this associated bundle, which reflects its matrix structure already described. In particular, that implies that on a Zariski-dense subset in $\mathbb{P}^1$, the total space of the flag can be re-created by the consecutive action of the $(SL(r+1),\hbar)$-oper connection on the section of the line bundle. The Miura condition can be reformulated as the constraint that the connection preserve a different complete flag of subbundles.

Such a definition leads to another approach to the derivation of the QQ-systems from Z-twisted Miura $(SL(r+1),\hbar)$-opers with regular singularities. This is done using $\hbar$-deformed Wronskian matrices. Their matrix elements are components of the nontrivial section of the line bundle in the trivialisation when the oper connection is represented by the regular semisimple twist element Z and describes the relative position of two flags of subbundles. It turns out that the points where these flags are in a nongeneric position correspond to Bethe roots and QQ-systems, as we demonstrated in [38]. Here we show that the extension of the QQ-system by $\hbar$-Bäcklund transformations is provided by various minors in this $\hbar$-Wronskian matrix.[2]

More importantly, we explicitly construct the element $g(z)$ such that the connection takes the form $A(z) = g(\hbar z)Zg^{-1}(z)$. This element can be represented both in abstract Lie-theoretic form and explicitly in matrix notation, which uses polynomials of the extended QQ-system. As a consequence, we obtain that the w_0-generic condition, which was needed in general for a Z-twisted Miura–Plücker $\left({}^L G,\hbar\right)$-oper to be just Z-twisted, is not needed for ${}^L G = SL(N)$.

1.4.2. Completion to $\left(\overline{GL}(\infty),\hbar\right)$-opers.

Following the calculations of $(SL(r+1),\hbar)$-opers, it is not hard to extend this construction to $SL(\infty)$–the group of infinite-dimensional matrices with unit determinant with a finite number of nonzero off-diagonal entries and a finite number of nonunit elements on the diagonal. However, for any Miura $(SL(\infty),\hbar)$-oper, the corresponding QQ-system will always be finite. Let us explain how to construct a Miura oper which corresponds to the 'complete' QQ-system associated with the Dynkin diagram of A_∞. We note that $SL(\infty)$ has a well-defined set of fundamental representations based on semi-infinite wedge spaces, which has an interpretation in terms of the Dirac sea, and the generators of the Lie algebra $\mathfrak{sl}(\infty)$ are represented via quadratic expressions of the fermionic operators of exterior and interior multiplication, thereby generating a Clifford algebra.

One can complete the corresponding Lie algebra $\mathfrak{sl}(\infty)$ by allowing infinite sums of generators. The resulting Lie algebra, endowed by central extension equal to 1, has fundamental representations realised in the same spaces as $\mathfrak{sl}(\infty)$. This is an important

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[2] We note here that such extensions of the QQ-systems have been introduced in various circumstances–for example, [3, 43, 44, 22]–and studied systematically in the case $Z = 1$ by Mukhin and Varchenko [46].

construction, which plays a central role in the celebrated *boson–fermion* correspondence [26, 33].

To address related Miura opers, we take a certain completion of $SL(\infty)$, which will be sufficient to put an infinite number of terms in the QQ-system. Namely, we construct the group corresponding to the completion of the upper Borel subgroup in the Bruhat decomposition of $SL(\infty)$. The resulting object, denoted by $\overline{GL}(\infty)$, is the group of the infinite matrices with an infinite number of elements above the diagonal and an infinite number of nonunital elements on the diagonal, while the number of elements below the diagonal remains finite. It has the same set of fundamental representations realised in the same set of Dirac sea spaces already described.

The resulting Miura $\big(\overline{GL}(\infty),\hbar\big)$-opers satisfy similar properties as the $SL(r+1)$ ones. One can define Z-twisted and Z-twisted Miura–Plücker opers and explicitly construct the operator from the completed upper Borel subalgebra, diagonalising the corresponding connection matrix. As before, it is constructed from the elements of the extended QQ-system.

As an application of this construction, we can build the main novel objects of this paper, namely *toroidal opers*.

1.4.3. Toroidal opers and the q-Langlands correspondence for toroidal algebras.
There is a natural family of automorphisms of an $sl(\infty)$-algebra corresponding to the Dynkin-diagram translations through n vertices. On the group-theoretic level, such transformations are realised via the nth power of the 'completed' Coxeter element c (infinite matrix with the only nonzero elements being units on the superdiagonal). Imposing the condition $c^n A(z) c^{-n} = A(pz)$ for Z-twisted $\big(\overline{GL}(\infty),\hbar\big)$, where p is a new parameter, we obtain that the resulting constrained infinite QQ-system generates Bethe equations for the toroidal algebras $\widehat{\widehat{\mathfrak{gl}}}(n)$.

While the corresponding QQ-system for toroidal algebras [22] has yet to emerge from the perspective of prefundamental representations and Grothendieck rings, the Bethe equations for the toroidal algebra $\widehat{\widehat{\mathfrak{gl}}}(n)$ in a representation-theoretic setting emerged through a shortcut, namely the Baxter TQ-relation [17, 32]–the relation between the Q-operator and the transfer matrix.

However, a more natural approach to generating Bethe equations for toroidal algebras emanates from enumerative geometry. In Section 1.3.1, we discussed elementary examples of quiver varieties, namely $T^*\mathbf{Gr}_{k,n}$ and, in general, cotangent bundles to partial flag varieties. The corresponding quantum K-theory ring reproduces the Bethe algebra for the XXZ model related to $\widehat{\mathfrak{sl}}(n)$. Another set of varieties which have been extensively studied are the framed cyclic quiver varieties, which are related to $\widehat{\widehat{\mathfrak{gl}}}(n)$ toroidal algebras, where n is the number of vertices. In the simplest situation of one vertex, such a variety is identified with the space of ADHM instantons [55]. More details on algebraic properties of quantum toroidal algebras and their geometric realisation can be found in recent reviews [49].

According to general construction, the $q \to 1$ asymptotic of z-analytic solutions of the resulting qKZ equations reproduces the Bethe equations, which serve as constraints for the quantum K-theory ring. These are exactly the equations we reproduce from toroidal

opers. Given that t_1, t_2 are the standard deformation parameters of $U_{t_1, t_2}\left(\widehat{\widehat{\mathfrak{gl}}}(n)\right)$, we obtain the following exchange of parameters:

$$(\hbar, p) \leftrightarrow \left((t_1 t_2)^{-1}, t_1\right), \tag{1.4}$$

which serves as the first example of the analogue q-Langlands correspondence for toroidal algebras.

1.4.4. String-theory motivation. In string-theory literature it is common to study limits when the number of objects, such as branes, becomes infinite. The most relevant example for this paper is the topological holography program initiated by Gopakumar and Vafa [29]. According to that paper, a topological phase transition can be regarded as an interpolation between two desingularisations of the conifold geometry–the deformed conifold $T^* S^3$ and the resolved conifold $\mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1}$.

The M-theory description of the former phase, in the presence of certain defects and flux through one of the complementary complex directions, after dimensional reduction, leads to a 3-dimensional quiver gauge theory on $S^1 \times \mathbb{C}_q$. The massive spectrum of such 3D theories is described by the equivariant quantum K-theory of the corresponding quiver varieties, which we discussed earlier. The parameter $\hbar$ from before plays the role of an $\mathcal{N} = 4$ R-symmetry-equivariant parameter.

The latter, resolved phase yields 5-dimensional gauge theory. The moduli space of instantons in this 5D theory is given by the ADHM quiver, which later in this paper will be discussed in connection with toroidal q-opers.

The topological phase transition from the deformed phase to the resolved phase occurs when the number of branes which wrap the S^3 cycle in the deformed geometry and determines the number of gauge groups in the 3D theory becomes infinite. In addition, a certain quantisation condition between the Ω-background parameters of the 3D gauge theory and other mass parameters of the problem must be satisfied. Namely, if s_i and s_{i+1} are complexified gauge-field vacuum expectation values of vector superfields of the ith and $(i+1)$th gauge groups, respectively, then the condition reads $\frac{s_{i+1}}{s_i} = p^n$, where n is an integer. On the resolved side of the transition, p becomes an equivariant parameter of the K-theory of the ADHM moduli space. The same parameters already appeared in formula (1.4).

Representation-theoretic aspects of the Gopakumar–Vafa transition in connection with quantum geometry of quiver varieties of A-type were discussed in [36]. The present paper provides an alternative description of the same physics in terms of bona fide classical objects: q-opers. By combining our results with those of the first author in [36], we can establish the *quantum/classical* duality between the quantum XXZ spin chain of $\widehat{A}_0$-type, whose Bethe equations coincide with relations in quantum equivariant K-theory of the ADHM quiver variety, and the so-called 1-toroidal q-opers. This correspondence can be regarded as the large-rank limit of the quantum/classical duality which has been discussed in both the physics [27, 39] and mathematics [38] literature.

We also note that oper-related structures in A-type, as well as their super analogues, have appeared in recent physics literature on integrability in the AdS/CFT correspondence [34, 10] as well as some earlier work [42].

1.4.5. Structure of the paper. In Section 2 we give two equivalent definitions of $(SL(r+1),q)$-opers and their Miura versions as q-connections, which were introduced in [24] and [38], correspondingly. The first definition uses the Lie-theoretic approach and the second one uses complete flags of subbundles.

In Sections 3–5 we elaborate on the Lie-theoretic definition and recall basic constructions of [24]. Section 3 is devoted to Z-twisted $(SL(r+1),q)$-opers, which are q-gauge-equivalent to a diagonal matrix. In Section 4 a milder version of the Z-twisted condition is introduced which is related to associated bundles leading to the notion of Z-twisted Miura–Plücker $(SL(r+1),q)$-opers. We also discuss nondegeneracy conditions for these objects. Section 5 addresses the one-to-one correspondence between Z-twisted Miura–Plücker opers and the nondegenerate solutions of the QQ-systems (and thus Bethe ansatz equations) as well as their extension. We also prove that Z-twisted Miura–Plücker $(SL(r+1),q)$-opers are Z-twisted, and relate the extended QQ-system to quantum Bäcklund transformations, introduced in [24].

In Section 6 we use the second definition of Miura $(SL(r+1),q)$-opers and show how the Z-twisted condition and quantum Bäcklund transformations can be reformulated in terms of q-Wronskian matrices, extending the results of [38].

In Section 7 we describe the fermionic realisation of Z-twisted Miura $(SL(r+1),q)$-opers using the realisation of the fundamental representations in the fermionic Fock space. We then use it as a motivation to write an infinite-rank formula. To do that we introduce the group $\overline{GL}(\infty)$ and its representations in the fermionic Dirac sea–that is, semi-infinite wedge space–and then in Sections 8 and 9 we extend the finite-dimensional notions of $(SL(r+1),q)$-oper theory from earlier sections to the case of $\overline{GL}(\infty)$. In particular, we show the relation between the corresponding infinite generalisation of the QQ-system and Z-twisted Miura $\left(\overline{GL}(\infty),q\right)$-opers.

Finally, Section 10 is devoted to the main target of the paper: the toroidal opers. These are nondegenerate Z-twisted Miura opers with certain periodicity conditions. The main goal of the section is to show that they are in one-to-one correspondence with the nondegenerate solutions of the QQ-system for toroidal algebras. We also discuss the relation to the enumerative geometry of ADHM spaces and generalisations to framed cyclic quiver varieties.

2. $(SL(r+1),q)$-opers

2.1. Group-theoretic data and notations

Consider $SL(r+1)$ to be the simple algebraic group of invertible $(r+1)\times(r+1)$ matrices over $\mathbb{C}$. We fix a Borel subgroup B_- with unipotent radical $N_- = [B_-,B_-]$ of lower triangular matrices and strictly lower triangular matrices correspondingly. The maximal torus is the corresponding set of diagonal matrices $H \subset B_-$. Let B_+ be the opposite Borel subgroup containing H. Let $\{\alpha_1,\dots,\alpha_r\}$ be the set of positive simple roots for the pair

$H \subset B_+$. Let $\{\check{\alpha}_1, \ldots, \check{\alpha}_r\}$ be the corresponding coroots. Then the elements of the Cartan matrix of the Lie algebra $\mathfrak{sl}(r+1)$ of G are given by $a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$. The Lie algebra $\mathfrak{sl}(r+1)$ has Chevalley generators $\{e_i, f_i, \check{\alpha}_i\}_{i=1,\ldots,r}$, so that $\mathfrak{b}_- = \mathrm{Lie}(B_-)$ is generated by the f_is and $\check{\alpha}_i$s, and $\mathfrak{b}_+ = \mathrm{Lie}(B_+)$ is generated by the e_is and $\check{\alpha}_i$s. In the defining representation $\check{\alpha}_i \equiv E_{ii} - E_{i+1,i+1}$, $e_i \equiv E_{i,i+1}$, $f_i \equiv E_{i-1,i}$, where E_{ij} stand for the matrix with the only nonzero element 1 in the ijth place. The fundamental weights $\omega_1, \ldots, \omega_r$ are defined by the condition $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$.

Let $W_{SL(r+1)} = N(H)/H \equiv S_{r+1}$ be the Weyl group of $SL(r+1)$. Let $w_i \in W_{SL(r+1)} (i = 1, \ldots, r)$ denote the simple reflection corresponding to α_i. We also denote by w_0 the longest element of W, so that $B_+ = w_0(B_-)$. Recall that a Coxeter element of W is a product of all simple reflections in a particular order. It is known that the set of all Coxeter elements forms a single conjugacy class in W_G. We will fix once and for all (unless specified otherwise) a particular ordering of the simple roots according to the natural ordering provided by the Dynkin diagram. Let $c = w_r w_{r-1} \cdots w_1$ be the Coxeter element associated to this ordering. In what follows (unless specified otherwise), all products over $i \in \{1, \ldots, r\}$ will be taken in this order; thus, for example, we write $c = \prod_i w_i$. We also fix representatives $s_i \in N(H)$ of w_i. In particular, $s = \prod_i s_i$ will be a representative of c in $N(H)$.

In the following we will denote the deformation parameter q instead of $\hbar$, for convenience.[3]

2.2. $(SL(r+1),q)$-opers: Two definitions

Consider the automorphism $M_q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ sending $z \mapsto qz$, where $q \in \mathbb{C}^\times$ is *not* a root of unity.

Given a principal $SL(r+1)$-bundle $\mathcal{F}_{SL(r+1)}$ over $\mathbb{P}^1$ (in Zariski topology), let $\mathcal{F}^q_{SL(r+1)}$ denote its pullback under the map $M_q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ sending $z \mapsto qz$. A meromorphic $(SL(r+1),q)$-connection on a principal $SL(r+1)$-bundle $\mathcal{F}_{SL(r+1)}$ on $\mathbb{P}^1$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{F}_{SL(r+1)}, \mathcal{F}^q_{SL(r+1)}\right)$, where U is a Zariski-open dense subset of $\mathbb{P}^1$. We can always choose U so that the restriction $\mathcal{F}_{SL(r+1)}|_U$ of $\mathcal{F}_{SL(r+1)}$ to U is isomorphic to the trivial $SL(r+1)$-bundle. Choosing such an isomorphism – that is, a trivialisation of $\mathcal{F}_{SL(r+1)}|_U$ – we also obtain a trivialisation of $\mathcal{F}_{SL(r+1)}|_{M_q^{-1}(U)}$. Using these trivialisations, the restriction of A to the Zariski-open dense subset $U \cap M_q^{-1}(U)$ can be written as a section of the trivial $SL(r+1)$-bundle on $U \cap M_q^{-1}(U)$, and hence as an element $A(z)$ of $SL(r+1)(z)$, where we set $K(z) = K(\mathbb{C}(z))$. Changing the trivialisation of $\mathcal{F}_{SL(r+1)}|_U$ via $g(z) \in SL(r+1)(z)$ changes $A(z)$ by the following *q-gauge transformation*:

$$A(z) \mapsto g(qz)A(z)g(z)^{-1}. \qquad (2.1)$$

---

[3] The notation mix-up is due to the fact that in the context of the q-deformed geometric Langlands correspondence, $\hbar$ is used for the deformation parameter of the quantum algebra, while q is the parameter in the qKZ equation (see the introduction). Since we do not use the qKZ equation anywhere in the paper, we have renamed $\hbar$ as q.

This shows that the set of equivalence classes of pairs $\big(\mathcal{F}_{SL(r+1)}, A\big)$ is in bijection with the quotient of $SL(r+1)(z)$ by these q-gauge transformations. Equivalently, one could consider the vector bundle E of rank $r+1$ over $\mathbb{P}^1$ associated to $\mathcal{F}_{SL(r+1)}$ and define the $(SL(r+1), q)$-connection as a section of $\mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$, which is invertible and has determinant 1.

Following [24], we define an $(SL(r+1), q)$-oper as follows:

Definition 2.1. A meromorphic $(SL(r+1), q)$-*oper* (or simply a q-*oper*) on $\mathbb{P}^1$ is a triple $\big(\mathcal{F}_{SL(r+1)}, A, \mathcal{F}_{B_-}\big)$, where A is a meromorphic $(SL(r+1), q)$-connection on an $SL(r+1)$-bundle $\mathcal{F}_{SL(r+1)}$ on $\mathbb{P}^1$ and $\mathcal{F}_{B_-}$ is the reduction of $\mathcal{F}_{SL(r+1)}$ to B_- satisfying the following condition: there exists a Zariski-open dense subset $U \subset \mathbb{P}^1$ together with a trivialisation $\imath_{B_-}$ of $\mathcal{F}_{B_-}$, such that the restriction of the connection $A : \mathcal{F}_{SL(r+1)} \longrightarrow \mathcal{F}^q_{SL(r+1)}$ to $U \cap M_q^{-1}(U)$, written as an element of $SL(r+1)(z)$ using the trivialisations of $\mathcal{F}_{SL(r+1)}$ and $\mathcal{F}^q_{SL(r+1)}$ on $U \cap M_q^{-1}(U)$ induced by $\imath_{B_-}$, takes values in the Bruhat cell

$$B_- \left(\mathbb{C}\left[U \cap M_q^{-1}(U) \right] \right) c B_- \left(\mathbb{C}\left[U \cap M_q^{-1}(U) \right] \right).$$

Thus, locally, any q-oper connection A can be written (using a particular trivialisation $\imath_{B_-}$) in the form

$$A(z) = n'(z) \prod_i \big(\phi_i(z)^{\check{\alpha}_i} s_i \big) n(z), \tag{2.2}$$

where $\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$ are such that their zeros and poles are outside the subset $U \cap M_q^{-1}(U)$ of $\mathbb{P}^1$.

However, we used another definition in [38]:

Definition 2.2. A meromorphic $(GL(r+1), q)$-*oper* on $\mathbb{P}^1$ is a triple $(\mathcal{A}, E, \mathcal{L}_\bullet)$, where E is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles

$$\mathcal{L}_{r+1} \subset \cdots \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset \cdots \subset \mathcal{L}_1 = E,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that the q-connection $\mathcal{A} \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$ satisfies the following conditions:

i) $\mathcal{A} \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$.

ii) There exists a Zariski-open dense subset $U \subset \mathbb{P}^1$ such that the restriction of $\mathcal{A} \in Hom(\mathcal{L}_\bullet, \mathcal{L}^q_\bullet)$ to $U \cap M_q^{-1}(U)$ is invertible and satisfies the condition that the induced maps $\bar{\mathcal{A}}_i : \mathcal{L}_i / \mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i-1} / \mathcal{L}_i$ are isomorphisms on $U \cap M_q^{-1}(U)$.

An $(SL(r+1), q)$-*oper* is a $(GL(r+1), q)$-oper with the condition that $det(\mathcal{A}) = 1$ on $U \cap M_q^{-1}(U)$.

The equivalence of Definitions 2.1 and 2.2 can be proven along the same lines as the equivalence of the analogous definitions in the case of classical opers. One can derive the second definition from the first by considering the associated bundle $E = \big(\mathcal{F}_{SL(r+1)} \times V_{\omega_1} \big) / SL(r+1)$, where V_{ω_1} in the defining representation of G. That immediately provides a flag of subbundles in E, preserved by B_-. From the chosen order

in the Coxeter element we obtain that the induced q-connection on E locally has the form of the matrix with coefficients in $\mathbb{C}(z)$ so that it has 0s above the superdiagonal. That immediately leads to Definition 2.2(i) and (ii). In this way, the q-connection A induces a q-connection $\mathcal{A}$ in Definition 2.2. Notice that the second definition implies the local formula (2.2) in the defining representation, and thus by faithfulness the first definition follows from the second. In the following we will use the same notation for A and $\mathcal{A}$: it will be clear from the context which q-connection is used.

2.3. Miura $(SL(r+1),q)$-opers

The Miura condition for the the q-opers corresponds to the introduction of an additional datum: reduction of the underlying $SL(r+1)$-bundle to the Borel subgroup B_+ (opposite to B_-) that is preserved by the oper q-connection.

Definition 2.3. A *Miura $(SL(r+1),q)$-oper* on $\mathbb{P}^1$ is a quadruple $\big(\mathcal{F}_{SL(r+1)},A,\mathcal{F}_{B_-},\mathcal{F}_{B_+}\big)$, where $\big(\mathcal{F}_{SL(r+1)},A,\mathcal{F}_{B_-}\big)$ is a meromorphic $(SL(r+1),q)$-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the $SL(r+1)$-bundle $\mathcal{F}_{SL(r+1)}$ to B_+ that is preserved by the q-connection A.

An equivalent definition using a flag of subbundles can be obtained by using the explicit identification of G/B_+ with the flag variety:

Definition 2.4. A *Miura $(SL(r+1),q)$-oper* on $\mathbb{P}^1$ is a quadruple $\big(E,A,\mathcal{L}_\bullet,\hat{\mathcal{L}}_\bullet\big)$, where $(E,A,\mathcal{L}_\bullet)$ is a meromorphic $(SL(r+1),q)$-oper on $\mathbb{P}^1$ and $\hat{\mathcal{L}}_\bullet = \{\mathcal{L}_i\}$ is another full flag of subbundles in E that is preserved by the q-connection A.

Forgetting $\mathcal{F}_{B_+}$, we associate an $(SL(r+1),q)$-oper to a given Miura $(SL(r+1),q)$-oper. We will refer to it as the $(SL(r+1),q)$-oper underlying this Miura $(SL(r+1),q)$-oper.

From the point of view of local consideration, let U be a Zariski-open dense subset on $\mathbb{P}^1$ as in Definition 2.1. Choosing a trivialisation ι_{B_-} of $\mathcal{F}_{SL(r+1)}$ on $U \cap M_q^{-1}(U)$, we can write the q-connection A in the form of equation (2.2). On the other hand, using the B_+-reduction $\mathcal{F}_{B_+}$ we can choose another trivialisation of $\mathcal{F}_{SL(r+1)}$ on $U \cap M_q^{-1}(U)$ such that the q-connection A acquires the form $\widetilde{A}(z) \in B_+(z)$. Hence there exists $g(z) \in SL(r+1)(z)$ such that

$$g(zq)n'(z)\prod_i \big(\phi_i(z)^{\check{\alpha}_i} s_i\big) n(z)g(z)^{-1} = \widetilde{A}(z) \in B_+(z). \tag{2.3}$$

Suppose we are given a principal $SL(r+1)$-bundle $\mathcal{F}_{SL(r+1)}$ on any smooth complex manifold X equipped with reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ to B_- and B_+, respectively. Then we assign to any point $x \in X$ an element of the Weyl group S_{r+1}. Namely, the fibre $\mathcal{F}_{SL(r+1),x}$ of $\mathcal{F}_{SL(r+1)}$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+, respectively. Choose any trivialisation of $\mathcal{F}_{SL(r+1),x}$–that is, an isomorphism of $SL(r+1)$-torsors $\mathcal{F}_{SL(r+1),x} \simeq SL(r+1)$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset SL(r+1)$ and $\mathcal{F}_{B_+,x}$ with bB_+. Then $a^{-1}b$ is a well-defined element of the double quotient $B_-\backslash SL(r+1)/B_+$, which is in bijection with $W_{SL(r+1)}$. Hence we obtain a well-defined element of $W_{SL(r+1)} = S_{r+1}$.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have a *generic relative position* at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset SL(r+1)$).

Using Bruhat decomposition $SL(r+1)(z) = \bigsqcup_{w \in W_{SL(r+1)}} B_+(z)wN_-(z)$, we claim that $g(z)$ from equation (2.3) lies in the $w = 1$ cell–namely, $g(z) \in B_+(z)N_-(z)$.

Using the notion of relative position, we can reformulate this local statement as the following theorem, which was proven in [24]:

Theorem 2.5. *For any Miura $(SL(r+1),q)$-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in generic relative position for all $x \in V$.*

Returning to the local expression (2.3), we now wish to characterise the explicit representatives for $\tilde{A}(z)$.

Theorem 2.6. *Every element of the set $N_-(z)\prod_i \phi_i(z)^{\check{\alpha}_i} s_i N_-(z) \cap B_+(z)$ can be written in the form*

$$\prod_i g_i^{\check{\alpha}_i} e^{\frac{t_i(z)\phi_i(z)}{g_i}e_i}, \quad g_i \in \mathbb{C}^\times(z), \tag{2.4}$$

where each $t_i(z) \in \mathbb{C}(z)$ is determined by the lifting s_i.

This fact was proven in higher generality in [24]. Note that in the case of $SL(r+1)$ for a given order of s_i, this follows directly from the matrix realisation.

From now on we consider the liftings s_i of simple reflections $w_i \in W$ in such a way that $t_i = 1$ for $(i = 1, \ldots, r)$.

3. Z-twisted Miura $(SL(r+1),q)$-opers

3.1. Z-twisted (Miura) opers

In this paper we consider a class of (Miura) q-opers that are gauge-equivalent to a constant element of $SL(r+1)$ (as $(SL(r+1),q)$-connections). Moreover, we assume that such an element Z is the regular element of the maximal torus H. One can express it as

$$Z = \prod_{i=1}^r \zeta_i^{\check{\alpha}_i}, \quad \zeta_i \in \mathbb{C}^\times. \tag{3.1}$$

Definition 3.1. A *Z-twisted $(SL(r+1),q)$-oper* on $\mathbb{P}^1$ is an $(SL(r+1),q)$-oper that is equivalent to the constant element $Z \in H \subset H(z)$ under the q-gauge action of $SL(r+1)(z)$–that is, if $A(z)$ is the meromorphic oper q-connection (with respect to a particular trivialisation of the underlying bundle), there exists $g(z) \in G(z)$ such that

$$A(z) = g(qz)Zg(z)^{-1}. \tag{3.2}$$

A *Z-twisted Miura $(SL(r+1),q)$-oper* is a Miura $(SL(r+1),q)$-oper on $\mathbb{P}^1$ that is equivalent to the constant element $Z \in H \subset H(z)$ under the q-gauge action of $B_+(z)$ –that is,

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z). \tag{3.3}$$

It follows from Definition 3.1 that any Z-twisted $(SL(r+1),q)$-oper is also Z'-twisted for any Z' in the S_{r+1}-orbit of Z. But if we endow it with the structure of a Z-twisted Miura $(SL(r+1),q)$-oper (by adding a B_+-reduction $\mathcal{F}_{B_+}$ preserved by the oper q-connection), then we fix a specific element in this S_{r+1}-orbit.

Thus we have the following proposition, which allows us to characterise Z-twisted Miura q-opers associated to Z-twisted q-opers:

Proposition 3.2. *Let $Z \in H$ be regular. For any Z-twisted $(SL(r+1),q)$-oper $\left(\mathcal{F}_{SL(r+1)}, A, \mathcal{F}_{B_-}\right)$ and any choice of B_+-reduction $\mathcal{F}_{B_+}$ of $\mathcal{F}_{SL(r+1)}$ preserved by the oper q-connection A, the resulting Miura $(SL(r+1),q)$-oper is Z'-twisted for a particular $Z' \in S_{r+1} \cdot Z$. The set of A-invariant B_+-reductions $\mathcal{F}_{B_+}$ on the $(SL(r+1),q)$-oper is in one-to-one correspondence with the elements of W.*

3.2. (Miura) *q*-opers with regular singularities

Let $\{\Lambda_i(z)\}_{i=1,\ldots,N-1}$ be a collection of nonconstant polynomials.

Definition 3.3. *An $(SL(r+1),q)$-oper with regular singularities determined by $\{\Lambda_i(z)\}_{i=1,\ldots,r}$ is a q-oper on $\mathbb{P}^1$ whose q-connection* (2.2) *may be written in the form*

$$A(z) = n'(z) \prod_i \left(\Lambda_i(z)^{\check{\alpha}_i} s_i\right) n(z), \quad n(z), n'(z) \in N_-(z). \tag{3.4}$$

A *Miura $(SL(r+1),q)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$* is a Miura $(SL(r+1),q)$-oper such that the underlying q-oper has regular singularities determined by $\{\Lambda_i(z)\}_{i=1,\ldots,r}$.

The following theorem follows from Theorem 2.6 and gives an explicit parameterisation of generic elements from the space of Miura opers:

Theorem 3.4. *For every Miura $(SL(r+1),q)$-oper with regular singularities determined by the polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$, the underlying q-connection can be written in the form*

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \quad g_i(z) \in \mathbb{C}(z)^{\times}. \tag{3.5}$$

3.3. Cartan connections

Consider a Miura $(SL(r+1),q)$-oper. By Theorem 3.4, the underlying $(SL(r+1),q)$-connection can be written in the form of equation (3.5). Since it preserves the B_+-bundle $\mathcal{F}_{B_+}$ underlying this Miura $(SL(r+1),q)$-oper (see Definition 2.3), it may be viewed as a meromorphic (B_+,q)-connection on $\mathbb{P}^1$. Taking the quotient of $\mathcal{F}_{B_+}$ by $N_+ = [B_+, B_+]$ and using the fact that $B/N_+ \simeq H$, we obtain an H-bundle $\mathcal{F}_{B_+}/N_+$ and the corresponding (H,q)-connection, which we denote by $A^H(z)$. According to equation (3.5), it is given by the formula

$$A^H(z) = \prod_i g_i(z)^{\check{\alpha}_i}. \tag{3.6}$$

We call $A^H(z)$ the *associated Cartan q-connection* of the Miura q-oper $A(z)$.

Now, if our Miura q-oper is Z-twisted (see Definition 3.1), then we also have $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_+(z)$. Since $v(z)$ can be written as

$$v(z) = \prod_i y_i(z)^{\check{\alpha}_i} n(z), \quad n(z) \in N_+(z), \ y_i(z) \in \mathbb{C}(z)^\times, \tag{3.7}$$

the Cartan q-connection $A^H(z)$ has the form

$$A^H(z) = \prod_i y_i(qz)^{\check{\alpha}_i} Z \prod_i y_i(z)^{-\check{\alpha}_i}, \tag{3.8}$$

and hence we will refer to $A^H(z)$ as the *Z-twisted Cartan q-connection*. This formula shows that $A^H(z)$ is completely determined by Z and the rational functions $y_i(z)$. Indeed, comparing this equation with equation (3.6) gives

$$g_i(z) = \zeta_i \frac{y_i(qz)}{y_i(z)}. \tag{3.9}$$

We note that $A^H(z)$ determines the $y_i(z)$s uniquely up to scalar. Indeed, if there is another choice $\tilde{y}_i(z)$, we obtain that $h_i(z) = y_i(z)\tilde{y}_i^{-1}(z)$ satisfies the equation $h_i(qz) = h_i(z)$. Given the condition that q is not a root of unity, $h_i(z)$ has to be constant.

4. Miura–Plücker $(SL(r+1), q)$-opers

In this section we will talk about the notion of nondegeneracy and will relax the Z-twisted condition slightly (we refer to [24] for details). We will associate to the given $(SL(r+1), q)$-Miura oper a collection of $(GL(2), q)$-opers and require that all of them be Z-twisted with some nondegeneracy conditions. This will lead to the notion of Z-twisted Miura–Plücker q-opers. It turns out, as we will find out in Section 5, that these objects, supplied by the nondegeneracy condition, are in one-to-one correspondence with solutions of certain equations called a QQ-system, which are in turn related to Bethe ansatz equations. Also, in the next section we will show that for $SL(r+1)$, this relaxed Z-twisted Miura–Plücker condition is equivalent to the original Z-twisted condition.

4.1. The associated Miura $(GL(2), q)$-opers and Miura–Plücker condition

Let V_i be the irreducible representation of $SL(r+1)$ with the highest weight ω_i. Notice that the 1- and 2-dimensional subspaces L_i and W_i of V_i spanned by the weight vectors ν_{ω_i} (the highest weight vector) and $\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}$ are B_+-invariant subspaces of V_i.

Now let $(\mathcal{F}_{SL(r+1)}, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura $(SL(r+1), q)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$ (see Definition 3.3). Recall that $\mathcal{F}_{B_+}$ is a B_+-reduction of a G-bundle $\mathcal{F}_{SL(r+1)}$ on $\mathbb{P}^1$ preserved by the $(SL(r+1), q)$-connection A. Therefore for each $i = 1, \ldots, r$, the vector bundle

$$\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_{SL(r+1)} \underset{SL(r+1)}{\times} V_i$$

associated to V_i contains a rank 2 subbundle

$$\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle

$$\mathscr{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$$

associated to $L_i \subset W_i$.

Denote by $\phi_i(A)$ the q-connection on the vector bundle $\mathcal{V}_i$ corresponding to this Miura q-oper connection A. Since by definition A preserves $\mathcal{F}_{B_+}$, we obtain that $\phi_i(A)$ preserves the subbundles $\mathscr{L}_i$ and $\mathcal{W}_i$ of $\mathcal{V}_i$ and thus produces a $(GL(2),q)$-oper on $\mathcal{W}_i$. Let us denote such a q-oper by A_i.

If we trivialise $\mathcal{F}_{B_+}$ on a Zariski-open subset of $\mathbb{P}^1$ so that $A(z)$ has the form of equation (3.5) with respect to this trivialisation (see Theorem 3.4). This trivialises the bundles $\mathcal{V}_i$, $\mathcal{W}_i$ and $\mathscr{L}_i$, so that the q-connection $A_i(z)$ becomes a 2×2 matrix whose entries are in $\mathbb{C}(z)$. Moreover, $\mathcal{W}_i$ decomposes into a direct sum of two subbundles–$\hat{\mathscr{L}}_i$, preserved by B_+, and $\mathscr{L}_i$–with respect to which it satisfies the $(GL(2),q)$-oper condition. We can unify all that in the following proposition:

Proposition 4.1 ([24]). *The quadruple $\big(A_i, \mathcal{W}_i, \mathscr{L}_i, \hat{\mathscr{L}}_i\big)$ forms a Miura $(GL(2),q)$-oper, so that explicitly*

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j \neq i} g_j(z)^{-a_{ji}} \end{pmatrix}, \tag{4.1}$$

where we use the ordering of the simple roots determined by the Coxeter element c.

Now we impose the condition of equation (3.8) on the corresponding A^H-connection, namely

$$g_i = \zeta_i \frac{y_i(qz)}{y_i(z)}.$$

Let $G_i \cong \mathrm{SL}(2)$ be the subgroup of $SL(r+1)$ corresponding to the $\mathfrak{sl}(2)$-triple spanned by $\{e_i, f_i, \check{\alpha}_i\}$ which preserves W_i. Performing the gauge transformation via diagonal matrix for equation (4.1), we can represent the resulting connection as follows:

$$\begin{aligned} \widetilde{A}_i(z) = u(qz) A_i(z) u^{-1}(z) &= \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j \neq i} \zeta_j^{-a_{ji}} \end{pmatrix} \mathcal{A}_i(z) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j \neq i} \zeta_j^{-a_{ji}} \end{pmatrix} g_i^{\check{\alpha}_i}(z) e^{\frac{\rho_i(z)}{g_i(z)} e_i}, \end{aligned} \tag{4.2}$$

where

$$\rho_i(z) = \Lambda_i(z) \prod_{j>i} (\zeta_j y_j(qz))^{-a_{ji}} \prod_{j<i} y_j(z)^{-a_{ji}}. \tag{4.3}$$

Thus, under the assumption of equation (3.8), our Miura $(SL(r+1),q)$-oper $A(z)$ gives rise to a collection of meromorphic Miura $(SL(2),q)$-opers $\mathcal{A}_i(z)$ for $i = 1,\ldots,N-1$. It should be noted that it has regular singularities in the sense of Definition 3.3 if and only if $\rho_i(z)$ is a polynomial. For example, this holds for all i if all $y_j(z), j = 1,\ldots,N-1$, are polynomials, an observation we will use later.

Now we are ready to relax the Z-twisted condition as follows:

Definition 4.2 ([24]). A *Z-twisted Miura–Plücker $(SL(r+1),q)$-oper* is a meromorphic Miura $(SL(r+1),q)$-oper on $\mathbb{P}^1$ with the underlying q-connection $A(z)$, such that there exists $v(z) \in B_+(z)$ such that for all $i = 1,\ldots,r$, the Miura $(GL(2),q)$-opers $A_i(z)$ associated to $A(z)$ by equation (4.1) can be written in the form

$$A_i(z) = v(zq)Zv(z)^{-1}\big|_{W_i} = v_i(zq)Z_iv_i(z)^{-1}, \tag{4.4}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$.

Note that it follows from this definition that the (H,q)-connection $A^H(z)$ associated to a Z-twisted Miura–Plücker $(SL(r+1),q)$-oper can be written in the same form–equation (3.8)–as the (H,q)-connection associated to a Z-twisted Miura $(SL(r+1),q)$-oper.

However, while it is true that every Z-twisted Miura $(SL(r+1),q)$-oper is automatically a Z-twisted Miura–Plücker $(SL(r+1),q)$-oper, the converse is not necessarily true if $r \neq 1$.

4.2. Nondegeneracy conditions

In what follows, we will say that $v,w \in \mathbb{C}^{\times}$ are *q-distinct* if $q^{\mathbb{Z}}v \cap q^{\mathbb{Z}}w = \varnothing$.

In this subsection we introduce two nondegeneracy conditions for Z-twisted Miura–Plücker q-opers. The first, called the H-nondegeneracy condition, is applicable to arbitrary Miura q-opers with regular singularities. Recall from Theorey 3.4 that the underlying q-connection can be represented in the form of equation (3.5).

Definition 4.3 ([24]). A Miura $(SL(r+1),q)$-oper $A(z)$ of the form in equation (3.5) is called *H-nondegenerate* if the corresponding (H,q)-connection $A^H(z)$ can be written in the form of equation (3.8), where for all i,j,k with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros and poles of $y_i(z)$ and $y_j(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$.

Next we define the second nondegeneracy condition. This condition applies to Z-twisted Miura–Plücker $(SL(r+1),q)$-opers. We start from $(SL(2),q)$-opers.

Consider a Miura $(SL(2),q)$-oper given by equation (3.5), which reads in this case

$$A(z) = g(z)^{\check{\alpha}}\exp\left(\frac{\Lambda(z)}{g(z)}e\right),$$

so that the corresponding Cartan q-connection $A^H(z)$ is $A^H(z) = g(z)^{\check{\alpha}}$, where $y(z)$ is a rational function. Let us assume that $A(z)$ is H-nondegenerate (see Definition 4.3). This means that the zeros of $\Lambda(z)$ are q-distinct from the zeros and poles of $y(z)$.

If we apply to $A(z)$ a q-gauge transformation by an element of $h(z)^{\check{\alpha}} \in H[z]$, we obtain a new q-oper connection

$$\widetilde{A}(z) = \widetilde{g}(z)^{\check{\alpha}} \exp\left(\frac{\widetilde{\Lambda}(z)}{\widetilde{g}(z)} e\right), \tag{4.5}$$

where $\widetilde{g}(z) = g(z)h(zq)h(z)^{-1}$, $\widetilde{\Lambda}(z) = \Lambda(z)h(zq)h(z)$. It also has regular singularities, but for a different polynomial $\widetilde{\Lambda}(z)$, and $\widetilde{A}(z)$ may no longer be H-nondegenerate. However, it turns out there is an essentially unique gauge transformation from $H[z]$ for which the resulting $\widetilde{A}(z)$ is H-nondegenerate $\widetilde{A}^H(z)$ and $\widetilde{y}(z)$ is a polynomial. This choice allows us to fix the polynomial $\Lambda(z)$ determining the regular singularities of our $(SL(2),q)$-oper.

Lemma 4.4 ([24]).

(1) *There is an $(SL(2),q)$-oper $\widetilde{A}(z)$ in the $H[z]$-gauge class of $A(z)$ for which $\widetilde{A}^H(z) = \widetilde{g}(z)^{\check{\alpha}}$ is nondegenerate and the rational function $\widetilde{y}(z)$ is a polynomial. This oper is unique up to a scalar $a \in \mathbb{C}^\times$ that leaves $\widetilde{g}(z)$ unchanged but multiplies $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ by a and a^2, respectively.*

(2) *This $(SL(2),q)$-oper $\widetilde{A}(z)$ may also be characterised by the property that $\widetilde{\Lambda}(z)$ has maximal degree subject to the constraint that it is H-nondegenerate.*

This motivates the following definition:

Definition 4.5 ([24]). A Z-twisted Miura $(SL(2),q)$-oper is called *nondegenerate* if it is H-nondegenerate and the rational function $y(z)$ appearing in equation (3.8) is a polynomial.

We now turn to the general case. Recall Definition 4.2 of Z-twisted Miura–Plücker $(SL(r+1),q)$-opers.

Definition 4.6 ([24]). Suppose that $r > 1$. A Z-twisted Miura–Plücker $(SL(r+1),q)$-oper $A(z)$ is called *nondegenerate* if its associated Cartan q-connection $A^H(z)$ is nondegenerate and each associated Z_i-twisted Miura $(SL(2),q)$-oper $\mathcal{A}_i(z)$ is nondegenerate.

It turns out that this simply means that in addition to $A^H(z)$ being nondegenerate, each $y_i(z)$ from equation (3.8) is a polynomial. Here we provide the complete proof, since we will need it for the infinite-dimensional case.

Proposition 4.7 ([24]). *Suppose that $r > 1$ and let $A(z)$ be a Z-twisted Miura–Plücker $(SL(r+1),q)$-oper. The following statements are equivalent:*

(1) *$A(z)$ is nondegenerate.*

(2) *The Cartan q-connection $A^H(z)$ is nondegenerate, and each $A_i(z)$ has regular singularities–that is, $\rho_i(z)$ given by equation (4.3) is in $\mathbb{C}[z]$.*

(3) *Each $y_i(z)$ from equation (3.8) is a polynomial, and for all i,j,k with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros of $y_i(z)$ and $y_j(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$.*

Proof. To prove that (2) implies (3), we need only show that if each $\rho_i(z)$ given by equation (4.3) is in $\mathbb{C}[z]$, then the $y_i(z)$s are polynomials. Suppose $y_i(z)$ is not a polynomial, and choose $j \neq i$ such that $a_{ij} \neq 0$. Then $-a_{ij} > 0$, and so the denominator of $y_i(z)$ or $y_i(qz)$ appears in the denominator of $\rho_j(z)$. Moreover, since the poles of $y_i(z)$ are q-distinct from the zeros of $\Lambda_j(z)$ and the other $y_k(z)$s, the poles of $y_i(z)$ or $y_i(qz)$ would give rise to poles of $\rho_j(z)$. But then $A_j(z)$ would not have regular singularities.

Next, assume (3). Then $A^H(z)$ is nondegenerate by Definition 4.3. Since all the $y_i(z)$s are polynomials, the same is true for the $\rho_i(z)$s. (Here we are using the fact that the off-diagonal elements of the Cartan matrix, a_{ij} with $i \neq j$, are less than or equal to 0.) Since $\rho_i(z)$ is a product of polynomials whose roots are q-distinct from the roots of $y_i(z)$, we see that the Cartan q-connection associated to $A_i(z)$ is nondegenerate.

Finally, (2) is a trivial consequence of (1). $\qquad\square$

If we apply a q-gauge transformation by an element $h(z) \in H[z]$ to $A(z)$, we get a new Z-twisted Miura–Plücker $(SL(r+1),q)$-oper. However, the following proposition shows that it is only nondegenerate if $h(z) \in H$ is constant with respect to z. As a consequence, the Λ_ks of a nondegenerate q-oper are determined up to scalar multiples.

Proposition 4.8 ([24]). *If $A(z)$ is a nondegenerate Z-twisted Miura–Plücker $(SL(r+1),q)$-oper and $h(z) \in H[z]$, then $h(qz)A(z)h(z)^{-1}$ is nondegenerate if and only if $h(z)$ is a constant element of H.*

5. Z-twisted Miura $(SL(r+1),q)$-opers and QQ-systems

5.1. QQ-systems and Miura–Plücker $(SL(r+1),q)$-opers

One of the main results of the previous section was the explicit structure of the nondegenerate Miura–Plücker $(SL(r+1),q)$-oper with regular singularities defined by $\{\Lambda_i(z)\}_{i=(1,\dots,r)}$ and associated with regular element $Z = \prod_i \zeta_i^{\check{\alpha}_i}$. Following Proposition 4.7, the local expression $A(z)$ can be expressed as follows:

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \quad g_i(z) = \zeta_i \frac{Q_i^+(qz)}{Q_i^+(z)}, \tag{5.1}$$

where $Q_i^+(z)$ are monic polynomials (here we changed the notation $y_i(z) \equiv Q_i^+(z)$). From now on, we will assume that Z satisfies the following property:

$$\prod_{i=1}^r \zeta_i^{a_{ij}} = \frac{\zeta_j^2}{\zeta_{j+1}\zeta_{j-1}} \notin q^{\mathbb{Z}}, \quad \forall\, j = 1,\dots,r, \tag{5.2}$$

where a_{ij} are matrix elements of the Cartan matrix for $\mathfrak{sl}_{r+1}$. Since $\prod_{i=1}^r \zeta_i^{a_{ij}} \neq 1$ is a special case of equation (5.2), this implies that Z is *regular semisimple*.

5.2. The $SL(r+1)$ QQ-system

In [24] the following statement was proven (we specialise the result to the case of $SL(r+1)$):

Theorem 5.1. *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura–Plücker $(SL(r+1),q)$-opers and the set of nondegenerate polynomial*

solutions of the QQ-system

$$\xi_i Q_i^+(qz)Q_i^-(z) - \xi_{i+1}Q_i^+(z)Q_i^-(qz) = \Lambda_i(z)Q_{i-1}^+(z)Q_{i+1}^+(qz), \quad i=1,\dots,r, \qquad (5.3)$$

subject to the boundary conditions $Q_0^\pm(z) = Q_{r+1}^\pm(z) = 1$ *and* $\xi_0 = \xi_{r+2} = 1$ *so that*

$$\xi_1 = \zeta_1, \qquad \xi_2 = \frac{\zeta_2}{\zeta_1}, \qquad \dots, \qquad \xi_r = \frac{\zeta_r}{\zeta_{r-1}}, \qquad \xi_{r+1} = \frac{1}{\zeta_r}.$$

Note that ξ_i is the ith element on the diagonal of Z from equation (3.1).

We will say that a polynomial solution $\{Q_i^+(z), Q_i^-(z)\}_{i=1,\dots,r}$ of equation (5.3) is *nondegenerate* if the following conditions are satisfied: equation (5.2) holds, and for $i \neq j$ the zeros of $Q_i^+(z)$ and $Q_j^-(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$ for $|i-k|=1$, $|j-k|=1$.

For convenience we rewrite equation (5.3) as

$$\xi_i \phi_i(z) - \xi_{i+1}\phi_i(qz) = \rho_i(z), \qquad (5.4)$$

where

$$\phi_i(z) = \frac{Q_i^-(z)}{Q_i^+(z)}, \quad \rho_i(z) = \Lambda_i(z)\frac{Q_{i-1}^+(qz)Q_{i+1}^+(z)}{Q_i^+(z)Q_i^+(qz)}. \qquad (5.5)$$

5.3. Extended QQ-system and Z-twisted $(SL(r+1),q)$-opers

As was demonstrated in [24], for a simply connected simple Lie group G the set of nondegenerate Z-twisted Miura–Plücker q-opers includes as a subset the set of Z-twisted Miura (G,q)-opers. The opposite inclusion was deemed possible provided that Z-twisted Miura–Plücker q-opers are in addition w_0-generic [24, Theorem 7.10]. We will discuss this notion in detail later, in Section 5.6.

In this section we shall demonstrate that when G is a special linear group, we do not need this extra condition and the corresponding Z-twisted Miura–Plücker $(SL(r+1),q)$-oper will be a Z-twisted Miura q-oper; namely, there exists $v(z) \in B_+(z)$ such that the q-connection $A(z)$ reduces to an element of the form of Equation (3.1)–or equivalently,

$$v(qz)^{-1}A(z) = Zv(z)^{-1}. \qquad (5.6)$$

Moreover, we will construct an explicit expression for $v(z)$.

The following statement is a generalisation of the result of [46] to Z-twisted q-opers:

Theorem 5.2. *Let* $A(z)$ *be as in equation (5.1) and* Z *as in equation (3.1). Suppose* $Q_{i,i+1,\dots,j}^-(z)$ *(* $i,j \in \mathbb{Z}$*,* $i < j$*) are polynomials satisfying equations:*

$$\xi_i \phi_i(z) - \xi_{i+1}\phi_i(qz) = \rho_i(z), \qquad\qquad i=1,\dots,r,$$
$$\xi_i \phi_{i,i+1}(z) - \xi_{i+2}\phi_{i,i+1}(qz) = \rho_{i+1}(z)\phi_i(z), \qquad\qquad i=1,\dots,r-1,$$
$$\vdots \qquad\qquad\qquad (5.7)$$
$$\xi_i \phi_{i,\dots,r-2+i}(z) - \xi_{r+i-1}\phi_{i,\dots,r-2+i}(qz) = \rho_{r-1}(z)\phi_{i,\dots,r-3+i}(qz), \qquad i=1,2,$$
$$\xi_1 \phi_{1,\dots,r}(z) - \xi_{r+1}\phi_{1,\dots,r}(qz) = \rho_r(z)\phi_{1,\dots,r-1}(qz),$$

where for all $j > i$,

$$\phi_{i,\ldots,j}(z) = \frac{Q_{i,\ldots,j}^-(z)}{Q_j^+(z)}. \tag{5.8}$$

Then there exist $v(z) \in B_+(z)$ such that equation (5.6) holds and is given by

$$v(z) = \prod_{i=1}^r Q_i^+(z)^{\check{\alpha}_i} \cdot \prod_{i=1}^r V_i(z), \tag{5.9}$$

where

$$V_i(z) = \prod_{j=i}^r \exp\left(-\phi_{i,\ldots,j}(z)e_{i,\ldots,j}\right), \quad e_{i,\ldots,j} = [\ldots[[e_i, e_{i+1}], e_{i+2}] \ldots e_j]. \tag{5.10}$$

We shall prove a more general statement in Section 9 about $(\overline{GL}(\infty), q)$-opers which will contain Theorem 5.2 as a corollary. Here, to illustrate how the theorem works, we will regard some low-rank examples.

Notice that although the expression for $v(z)$ in equation (5.9) is rather complicated, the inverse $v(z)^{-1}$ can be succinctly presented as

$$v(z)^{-1} = \begin{pmatrix} \frac{1}{Q_1^+(z)} & \frac{Q_1^-(z)}{Q_2^+(z)} & \frac{Q_{12}^-(z)}{Q_3^+(z)} & \cdots & \frac{Q_{1,\ldots,r-1}^-(z)}{Q_r^+(z)} & Q_{1,\ldots,r}^-(z) \\ 0 & \frac{Q_1^+(z)}{Q_2^+(z)} & \frac{Q_2^-(z)}{Q_3^+(z)} & \cdots & \frac{Q_{2,\ldots,r-1}^-(z)}{Q_r^+(z)} & Q_{2,\ldots,r}^-(z) \\ 0 & 0 & \frac{Q_2^+(z)}{Q_3^+(z)} & \cdots & \frac{Q_{3,\ldots,r-1}^-(z)}{Q_r^+(z)} & Q_{3,\ldots,r}^-(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \frac{Q_{r-1}^+(z)}{Q_r^+(z)} & Q_r^-(z) \\ 0 & \cdots & \cdots & \cdots & 0 & Q_r^+(z) \end{pmatrix}. \tag{5.11}$$

Before we continue, the following statement will be needed:

Lemma 5.3. *The following relations hold for any $u, v \in \mathbb{C}$ and $i, j = 1, \ldots, r$:*

$$u^{\check{\alpha}_i} e^{ve_j} = \exp\left(u^{a_{ji}} v e_i\right) u^{\check{\alpha}_i}. \tag{5.12}$$

In general, if $[X, Y] = sY$ we have

$$u^X e^{vY} = \exp(u^s v Y) u^X. \tag{5.13}$$

Using this lemma we can rewrite the q-connection of equation (5.1) such that the roots of $SL(r+1)$ are placed in decreasing order.

Lemma 5.4. *Let*

$$\rho_i(z) = \Lambda_i(z) \frac{Q_{i-1}(qz) Q_{i+1}(z)}{Q_i(qz) Q_i(z)}. \tag{5.14}$$

Then the $(SL(r+1), q)$-oper reads

$$A(z) = \prod_{i=r}^1 Q_i^+(qz)^{\check{\alpha}_i} \cdot \prod_{i=r}^1 e^{\frac{\zeta_i}{\zeta_{i+1}}\rho_i(z)e_i} \cdot \prod_{i=r}^1 \zeta_i^{\check{\alpha}} Q_i^+(z)^{-\check{\alpha}_i}, \tag{5.15}$$

or as a matrix,

$$A(z) = \begin{pmatrix} g_1(z) & \Lambda_1(z) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{g_2(z)}{g_1(z)} & \Lambda_2(z) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{g_3(z)}{g_2(z)} & \Lambda_3(z) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \Lambda_{r-1}(z) & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \frac{g_r(z)}{g_{r-1}(z)} & \Lambda_r(z) \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{1}{g_r(z)} \end{pmatrix}. \tag{5.16}$$

At this point our choice of the order of simple roots may seem unsubstantiated, but it will be justified in later sections, where we consider $\left(\overline{GL(\infty)}, q\right)$-opers.

5.4. Examples

5.4.1. Miura $(SL(2), q)$-oper. The twist element $Z = \zeta^{\check{\alpha}} = \mathrm{diag}\left(\zeta, \zeta^{-1}\right) = \mathrm{diag}(\xi_1, \xi_2)$. The q-connection of equation (5.15) reads

$$A(z) = Q^+(qz)^{\check{\alpha}} \cdot e^{\zeta \rho(z) e} \cdot \zeta^{\check{\alpha}} Q^+(z)^{-\check{\alpha}} = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}. \tag{5.17}$$

We look for the gauge transformation in the form

$$v(z) = Q^+(z)^{\check{\alpha}} e^{-\phi(z) e}, \tag{5.18}$$

where $\phi(z) = \frac{Q^-(z)}{Q^+(z)}$. The left-hand side of equation (5.6) reads

$$v(qz)^{-1} A(z) = e^{\phi(qz)e} e^{\zeta \rho(z) e} \cdot \zeta^{\check{\alpha}} Q^+(z)^{-\check{\alpha}}, \tag{5.19}$$

where $\rho(z) = \frac{\Lambda(z)}{Q^+(z) Q^+(qz)}$. Meanwhile, the right-hand side equals

$$Z v(z)^{-1} = \zeta^{\check{\alpha}} e^{-\phi(z)e} Q^+(z)^{-\check{\alpha}} = e^{-\zeta^2 \phi(z) e} \zeta^{\check{\alpha}} Q^+(z)^{-\check{\alpha}}, \tag{5.20}$$

where we use Lemma 5.3 in the last step. Comparing these two expressions yields the desired QQ-system equation

$$\zeta \phi(z) - \zeta^{-1} \phi(qz) = \rho(z), \tag{5.21}$$

or equivalently,

$$\xi_1 \phi(z) - \xi_2 \phi(qz) = \rho(z). \tag{5.22}$$

5.4.2. Miura $(SL(3),q)$-oper. Consider $Z = \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} = \mathrm{diag}\left(\zeta_1, \frac{\zeta_2}{\zeta_1}, \frac{1}{\zeta_2}\right) = \mathrm{diag}(\xi_1, \xi_2, \xi_3)$.
The q-connection is given by

$$
\begin{aligned}
A(z) &= Q_1^+(qz)^{\check{\alpha}_1} Q_2^+(qz)^{\check{\alpha}_2} \cdot e^{\zeta_2 \rho_2(z) e_2} e^{\frac{\zeta_1 \rho_1(z)}{\zeta_2} e_1} \cdot \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} Q_1^+(z)^{-\check{\alpha}_1} Q_2^+(z)^{-\check{\alpha}_2} \\
&= \begin{pmatrix} g_1(z) & \Lambda_1(z) & 0 \\ 0 & \frac{g_2(z)}{g_1(z)} & \Lambda_2(z) \\ 0 & 0 & \frac{1}{g_2(z)} \end{pmatrix},
\end{aligned} \tag{5.23}
$$

and the gauge transformation reads

$$
v(z) = Q_1^+(z)^{\check{\alpha}_1} Q_2^+(z)^{\check{\alpha}_2} e^{-\phi_1(z) e_1} e^{-\phi_{12}(z)[e_1,e_2]} e^{-\phi_2(z) e_2}. \tag{5.24}
$$

Thus the left-hand side of equation (5.6) becomes

$$
\begin{aligned}
v(qz)^{-1} A(z) &= e^{\phi_2(qz) e_2} e^{\phi_{12}(qz)[e_1,e_2]} e^{\phi_1(qz) e_1} \cdot e^{\zeta_2 \rho_2(z) e_2} e^{\frac{\zeta_1 \rho_1(z)}{\zeta_2} e_1} \cdot \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} Q_1^+(z)^{-\check{\alpha}_1} Q_2^+(z)^{-\check{\alpha}_2} \\
&= e^{(\phi_2(qz) + \zeta_2 \rho_2) e_2} e^{(\phi_{12}(qz) + \zeta_2 \rho_2(z) \phi_1(qz))[e_1,e_2]} e^{\left(\phi_1(qz) + \frac{\zeta_1}{\zeta_2} \rho_1(z)\right) e_1} \cdot \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} Q_1^+(z)^{-\check{\alpha}_1} Q_2^+(z)^{-\check{\alpha}_2}.
\end{aligned} \tag{5.25}
$$

Meanwhile, the right-hand side equals

$$
Zv(z)^{-1} = \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} \cdot e^{\phi_2(qz) e_2} e^{\phi_{12}(qz)[e_1,e_2]} e^{\phi_1(qz) e_1} Q_1^+(z)^{-\check{\alpha}_1} Q_2^+(z)^{-\check{\alpha}_2}. \tag{5.26}
$$

Now we need to move all Cartan elements from the front of this expression to its rear using Lemma 5.3:

$$
Zv(z)^{-1} = e^{\frac{\zeta_2^2 \phi_2(z)}{\zeta_1} e_2} e^{\zeta_1 \zeta_2 \phi_{12}(z)[e_1,e_2]} e^{\frac{\zeta_1^2 \phi_1(z)}{\zeta_2} e_1} \cdot \zeta_1^{\check{\alpha}_1} \zeta_2^{\check{\alpha}_2} Q_1^+(z)^{-\check{\alpha}_1} Q_2^+(z)^{-\check{\alpha}_2}. \tag{5.27}
$$

By comparing equations (5.25) and (5.27), we get

$$
\begin{aligned}
\zeta_1 \phi_1(z) - \frac{\zeta_2}{\zeta_1} \phi_1(qz) &= \rho_1(z), \\
\frac{\zeta_2}{\zeta_1} \phi_2(z) - \frac{1}{\zeta_2} \phi_2(qz) &= \rho_2(z), \\
\zeta_1 \phi_{12}(z) - \frac{1}{\zeta_2} \phi_{12}(qz) &= \rho_2(z) \phi_1(qz),
\end{aligned} \tag{5.28}
$$

or equivalently,

$$
\begin{aligned}
\xi_1 \phi_1(z) - \xi_2 \phi_1(qz) &= \rho_1(z), \\
\xi_2 \phi_2(z) - \xi_3 \phi_2(qz) &= \rho_2(z), \\
\xi_1 \phi_{12}(z) - \xi_3 \phi_{12}(qz) &= \rho_2(z) \phi_1(qz).
\end{aligned} \tag{5.29}
$$

5.5. The extended QQ-system and Bethe ansatz

The first line of equation (5.7) is the $SL(r+1)$ QQ-system of equation (5.4). In the rest of the equations we introduce new functions (5.8). Notice that

$$
\rho_{i+1}(z) \phi_i(qz) = \Lambda_{i+1}(z) \frac{Q_i^-(qz) Q_{i+2}^+(z)}{Q_{i+1}^+(z) Q_{i+1}^+(qz)} =: \rho_{i,i+1}(z),
$$

where $\rho_{i,i+1}(z)$ is $\rho_{i+1}(z)$ with $Q_i^+(z)$ replaced by $Q_i^-(z)$. In terms of this new notation we can rewrite equation (5.7) as follows:

$$
\begin{aligned}
\xi_i \phi_i(z) - \xi_{i+1} \phi_i(qz) &= \rho_i(z), & i &= 1, \ldots, r, \\
\xi_i \phi_{i,i+1}(z) - \xi_{i+2} \phi_{i,i+1}(qz) &= \rho_{i,i+1}(z), & i &= 1, \ldots, r-1,
\end{aligned}
$$
$$
\vdots
$$
$$
\begin{aligned}
\xi_i \phi_{i,\ldots,r-2+i}(z) - \xi_{r+i-1} \phi_{i,\ldots,r-2+i}(qz) &= \rho_{i,\ldots,r-1+i}(z), & i &= 1, 2, \\
\xi_1 \phi_{1,\ldots,r}(z) - \xi_{r+1} \phi_{1,\ldots,r}(qz) &= \rho_{1,\ldots,r}(z).
\end{aligned}
\tag{5.30}
$$

For future reference, let us rewrite these equations in terms of the Q-polynomials:

$$
\begin{aligned}
\xi_i Q_i^+(qz) Q_i^-(z) - \xi_{i+1} Q_i^+(z) Q_i^-(qz) &= \Lambda_i(z) Q_{i-1}^+(qz) Q_{i+1}^+(z), \\
\xi_i Q_{i+1}^+(qz) Q_{i,i+1}^-(z) - \xi_{i+2} Q_{i+1}^+(z) Q_{i,i+1}^-(qz) &= \Lambda_{i+1}(z) Q_i^-(qz) Q_{i+2}^+(z),
\end{aligned}
$$
$$
\vdots
$$
$$
\begin{aligned}
\xi_i\, Q_{r-2+i}^+(qz) Q_{i,\ldots,r-2+i}^-(z) - \xi_{r-1+i} Q_{r-2+i}^+(z) Q_{i,\ldots,r-2+i}^-(qz) &= \Lambda_{r-1+i}(z) Q_{i,\ldots,r-1+i}^-(qz) Q_{r+i}^+(z), \\
\xi_1 Q_r^+(qz) Q_{1,\ldots,r}^-(z) - \xi_{r+1} Q_r^+(z) Q_{1,\ldots,r}^-(qz) &= \Lambda_r(z) Q_{1,\ldots,r-1}^-(qz).
\end{aligned}
\tag{5.31}
$$

We shall refer to equation (5.31) as the *extended QQ-system* for $SL(r+1)$. We call its solution *nondegenerate* if the resulting solution of the original QQ-system is nondegenerate.

Let us now show that starting from the solution of the nondegenerate QQ-system, we obtain solutions for the extended QQ-system as well. To do that we need the following result (which is true for other simply laced groups) of [24]:

Theorem 5.5. *The solutions of the nondegenerate $SL(r+1)QQ$-system are in one-to-one correspondence with the solutions of the Bethe ansatz equations for the $\mathfrak{sl}(r+1)$ XXZ spin chain:*

$$
\frac{Q_i^+\left(q w_k^i\right)}{Q_i^+\left(q^{-1} w_i^k\right)} \frac{\xi_i}{\xi_{i+1}} = -\frac{\Lambda_i\left(w_k^i\right) Q_{i+1}^+\left(q w_k^i\right) Q_{i-1}^+\left(w_k^i\right)}{\Lambda_i\left(q^{-1} w_k^i\right) Q_{i+1}^+\left(w_k^i\right) Q_{i-1}^+\left(q^{-1} w_k^i\right)},
\tag{5.32}
$$

where $i = 1, \ldots, r$ and $k = 1, \ldots, m_i$.

We extend the statement of this theorem as follows:

Theorem 5.6. *There is a one-to-one correspondence between the set of nondegenerate solutions of the extended QQ-system (5.31), the set of nondegenerate solutions of the QQ-system (5.3) and the set of solutions of the Bethe ansatz equations (5.32).*

Proof. Consider the first line of equation (5.31) which is also presented in equation (5.3). If the QQ-system is nondegenerate, then according to Theorem 5.5, there is a bijection between its polynomial nondegenerate solutions and Bethe equations. We will now show recursively that given the nondegenerate solution of the QQ-system, one can construct elements $Q_{i,i+1,\ldots,j}$ satisfying the equations of the extended QQ-system. Let us immediately consider the degenerate case, when $Q_i^-(z)$ and $Q_{i+1}^+(z)$ have common roots: without loss of generality, let us now assume that $Q_i^-(z)$ and $Q_{i+1}^+(z)$ have just one

common root u. Now we show that we can construct the solution to the line of equation (5.31), namely $Q_{i,i+1}^+$. Introducing the notation

$$Q_{i+1}^+(z) = (z-u)\check{Q}_{i+1}^+(z), \tag{5.33}$$

we see from the nondegeneracy condition on the QQ-system–namely the fact that $Q_i^+(z)$ and $Q_{i+1}^+(z)$ have q-distinct roots–that we have

$$Q_i^-(z) = (z-u)\left(q^{-1}z - u\right)\widetilde{Q}_i^-(z). \tag{5.34}$$

Now consider the following equation from the second line of equation (5.30):

$$\xi_i \phi_{i,i+1}(z) - \xi_{i+1}\phi_{i,i+1}(qz) = \Lambda_{i+1}(z)\frac{Q_i^-(qz)Q_{i+2}^+(z)}{Q_{i+1}^+(z)Q_{i+1}^+(qz)}.$$

Substituting equations (5.33) and (5.34), we get

$$\xi_i \frac{Q_{i,i+1}^-(z)}{(z-u)\check{Q}_{i+1}^+(z)} - \xi_{i+2}\frac{Q_{i,i+1}^-(qz)}{(qz-u)\check{Q}_{i+1}^+(qz)} = \Lambda_{i+1}(z)\frac{\widetilde{Q}_i^-(qz)Q_{i+2}^+(z)}{\check{Q}_{i+1}^+(z)\check{Q}_{i+1}^+(qz)}. \tag{5.35}$$

From the residue decomposition of both sides of this equation, we conclude that u must be the root of the polynomial $Q_{i,i+1}^-(z)$:

$$Q_{i,i+1}^-(z) = (z-u)\check{Q}_{i,i+1}^-(z).$$

Thus, one can represent the resulting system as follows:

$$\rho_{i,i+1}(z) = h_i(z) + \sum_{k=1,k\neq s}^{m_{i+1}}\frac{b_k}{z-w_k^{i+1}} + \sum_{k=1,k\neq s}^{m_{i+1}}\frac{c_k}{qz-w_k^{i+1}}, \tag{5.36}$$

$$\phi_{i,i+1}(z) = \widetilde{\phi}_i(z) + \sum_{k=1,k\neq s}^{m_{i+1}}\frac{d_k}{z-w_k^{i+1}}, \tag{5.37}$$

where $w_s^{i+1} = u$, $h_i(z)$ and $\widetilde{\phi}_i(z)$ are polynomials. By matching the polar and polynomial parts of equation (5.35) we can readily find coefficients d_k and polynomials $\widetilde{\phi}_i(z)$ and hence $Q_{i,i+1}^+(z)$.

The only constraint we need to satisfy is the one on b_k, c_k–namely, $\frac{b_k}{\xi_{i+2}} + \frac{c_k}{\xi_i} = 0$, where $k \neq s$. These equations are explicitly given by

$$\frac{Q_{i+1}^+\left(qw_k^{i+1}\right)}{Q_{i+1}^+\left(q^{-1}w_k^{i+1}\right)}\frac{\xi_i}{\xi_{i+2}} = -\frac{\Lambda_{i+1}\left(w_k^{i+1}\right)Q_{i+2}^+\left(qw_k^{i+1}\right)Q_i^-\left(w_k^{i+1}\right)}{\Lambda_{i+1}\left(q^{-1}w_k^{i+1}\right)Q_{i+2}^+\left(w_k^{i+1}\right)Q_i^-\left(q^{-1}w_k^{i+1}\right)}, \quad k \neq s. \tag{5.38}$$

At the same time, the ith equation can be rewritten as

$$\xi_i \frac{Q_i^+(qz)}{Q_i^+(z)} - \xi_{i+1}\frac{Q_i^-(qz)}{Q_i^-(z)} = \frac{\Lambda_i(z)Q_{i-1}^+(qz)Q_{i+1}^+(z)}{Q_i^+(z)Q_i^-(z)},$$

which leads to

$$\xi_i \frac{Q_i^+ \left(q w_k^{i+1} \right)}{Q_i^+ \left(w_k^{i+1} \right)} = \xi_{i+1} \frac{Q_i^- \left(q w_k^{i+1} \right)}{Q_i^- \left(w_k^{i+1} \right)},$$

where w_k^{i+1} are the roots of $Q_{i+1}^+(z)$ for $k \neq s$.

Thus equation (5.38) is equivalent to the Bethe equations emerging from the QQ-system:

$$\frac{Q_{i+1}^+ \left(q w_k^{i+1} \right)}{Q_{i+1}^+ \left(q^{-1} w_k^{i+1} \right)} \frac{\xi_{i+1}}{\xi_{i+2}} = -\frac{\Lambda_{i+1} \left(w_k^{i+1} \right) Q_{i+2}^+ \left(q w_k^{i+1} \right) Q_i^+ \left(w_k^{i+1} \right)}{\Lambda_{i+1} \left(q^{-1} w_k^{i+1} \right) Q_{i+2}^+ \left(w_k^{i+1} \right) Q_i^+ \left(q^{-1} w_k^{i+1} \right)}. \tag{5.39}$$

Therefore we have found that equation (5.35) follows from the XXZ Bethe equations.

This step can be iterated if the $Q_{i,\dots,j}^-$ polynomials have coincident roots with Q_j^+. Therefore we have shown that any QQ-system in the form of equation (5.30) with such degeneracies is equivalent to a nondegenerate QQ-system.

In the next section we shall present a different proof of this theorem, exploring the definition of Miura $(SL(r+1,q)$-opers involving flags of subbundles.

5.6. The extended QQ-system, Bethe equations and Bäcklund transformations

We would like to understand the representation-theoretic meaning of the extended QQ-system a bit better. In fact, motivated by [24] we can demonstrate that starting from the original QQ-system (the first line of equation (5.31), or equation (5.3)), under certain assumptions one can recover all the remaining equations of the entire extended QQ-system by *Bäcklund transformations.*

Bäcklund transformations were introduced for Miura q-opers in [24] and were associated to the ith simple reflection from the Weyl group:

Proposition 5.7. *Consider the q-gauge transformation of the q-connection given by equation (5.1):*

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \mu_i(z) = \frac{Q_{i-1}^+(z) Q_{i+1}^+(z)}{Q_+^i(z) Q_-^i(z)}. \tag{5.40}$$

Then $A^{(i)}(z)$ can be obtained from $A(z)$ by substituting in equation (5.1)

$$Q_+^j(z) \mapsto Q_+^j(z), \quad j \neq i, \tag{5.41}$$

$$Q_+^i(z) \mapsto Q_-^i(z), \qquad Z \longrightarrow s_i(Z) \quad \left(\zeta_i \mapsto \frac{\zeta_{i-1} \zeta_{i+1}}{\zeta_i} \right). \tag{5.42}$$

It is possible that after the transformation, the resulting operator gives rise to the nondegenerate QQ-system. Denoting the QQ-system after the Bäcklund transformation

as $\left\{\widetilde{Q}_i^\pm\right\}_{i=1,\dots,r}$, we obtain

$$\left\{\widetilde{Q}_j^+\right\}_{j=1,\dots,r} = \left\{Q_1^+,\dots,Q_{i-1}^+,Q_i^-,Q_{i+1}^+,\dots,Q_+^r\right\}, \tag{5.43}$$

$$\left\{\widetilde{Q}_j^-\right\}_{j=1,\dots,r} = \left\{Q_1^-,\dots,Q^{*\,-}_{i-1},-Q_i^+,Q_{i,i+1}^-,\dots,Q_r^-\right\},$$

$$\left\{\widetilde{\zeta}_j\right\}_{j=1,\dots,r} = \left\{\zeta_1,\dots,\zeta_{i-1},\frac{\zeta_{i-1}\zeta_{i+1}}{\zeta_i},\dots,\zeta_r\right\}.$$

The last line can be also rewritten in terms of ξ variables as

$$\left\{\widetilde{\xi}_j\right\}_{j=1,\dots,r} = \left\{\xi_1,\dots,\xi_{i-1},\xi_{i+1},\xi_i,\xi_{i+2},\dots,\xi_{r+1}\right\}.$$

Here we note that the notation $Q_{i,i+1}^-$ was used for Q_{i+1}^-, since the equation this new polynomial satisfies is the second one from the extended QQ-system. At the same time, the new polynomial $Q^{*\,-}_{i-1}(z)$ does not belong to what we called the extended QQ-system.

As an example, if we apply the first Bäcklund transformation

$$Q_1^+ \mapsto Q_1^-, \qquad Q_1^- \mapsto -Q_1^+, \qquad \xi_1 \mapsto \xi_2, \qquad \xi_2 \mapsto \xi_1, \qquad Q_2^- \mapsto Q_{1,2}^-,$$

to the QQ system for $SL(3)$,

$$\xi_1 Q_1^+(qz)Q_1^-(z) - \xi_2 Q_1^+(z)Q_1^-(qz) = \Lambda_1(z)Q_2^+(z),$$
$$\xi_2 Q_2^+(qz)Q_2^-(z) - \xi_3 Q_2^+(z)Q_2^-(qz) = \Lambda_2(z)Q_1^+(qz), \tag{5.44}$$

the first equation will not change; however, the second will become

$$\xi_1 Q_2^+(qz)Q_{1,2}^-(z) - \xi_3 Q_2^+(z)Q_{1,2}^-(qz) = \Lambda_2(z)Q_1^-(qz), \tag{5.45}$$

which completes its extended QQ-system (5.29).

In general, one can talk about successive Bäcklund transformations associated with the Weyl group element w. If such transformations are possible–namely, if after each of the elementary Bäcklund transformation one arrives at the nondegenerate q-oper (i.e., the nondegenerate solution of the QQ-system)–such an oper is called w-generic in [24]. As one can see, the equations of the extended QQ-system emerge as a part of the QQ-system equations obtained after every Bäcklund transformation if the Weyl group element is constructed by successive reflections along the order in the Dynkin diagram: $w = s_i s_{i+1} \cdots s_{j-1} s_j$.

We will refer to the collection of QQ-system equations obtained via Bäcklund transformations for all Weyl group elements w as the *full QQ-system*.

One of the applications of Bäcklund transformations which was proven in [24] is that a Z-twisted Miura–Plücker (G,q)-oper is a Z-twisted Miura (G,q)-oper if it is w_0-generic, where w_0 is the longest root.

Here we show that a stronger result holds for $(SL(r+1),q)$-opers. Combining Theorems 5.2 and 5.6, we obtain the following theorem, which is the central result of this section:

Theorem 5.8. *The nondegenerate Z-twisted Miura–Plücker $(SL(r+1),q)$-opers are Z-twisted Miura $(SL(r+1),q)$-opers. They are in one-to-one correspondence with the nondegenerate solutions of the QQ-system and thus $\mathfrak{sl}(r+1)$ XXZ Bethe equations.*

6. *q*-Opers via quantum Wronskians

6.1. Sections of line bundles and *q*-Wronskians

In this section we will make use of an alternative definition of Miura $(SL(r+1),q)$-opers (see Definition 2.4) to describe Z-twisted Miura q-opers with regular singularities, following [38]. Namely, we have a complete flag of subbundles $\mathcal{L}_\bullet$ such that the q-connection A maps $\mathcal{L}_i$ into $\mathcal{L}_{i-1}^q$ and the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i-1}^q/\mathcal{L}_i^q$ are isomorphisms for $i = 1,\ldots,r$ on $U \cap M_q^{-1}(U)$, where U is the Zariski-open dense subset. Explicitly, considering the determinants

$$\left(\prod_{j=0}^{i-2} \left(A\left(q^{i-2-j}z\right)\right) s(z) \wedge \cdots \wedge A\left(q^{i-2}z\right) s\left(q^{i-2}z\right) \wedge s\left(q^{i-1}z\right) \right)\Bigg|_{\Lambda^i \mathcal{L}_{r-i+2}^{q^{i-1}}} \tag{6.1}$$

for $i = 1,\ldots,r+1$, where s is a local section of $\mathcal{L}_{r+1}$, we claim that $(E,A,\mathcal{L}_\bullet)$ is an $(SL(r+1),q)$-oper if and only if at every point of $U \cap M_q^{-1}(U)$ there exists a local section for which each such determinant is nonzero [38]. In the case of regular singularities (see Section 3.2), each $\bar{A}_i$ is an isomorphism except at zeros of Λ_i, and thus we require the determinants to vanish at zeros of the polynomial

$$W_k(s) = P_1(z) \cdot P_2(q^2 z) \cdots P_k(q^{k-1}z), \quad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z). \tag{6.2}$$

Now we discuss the Z-twisted Miura condition. Recall from Section 2.3 that the Miura condition implies that there exists a flag $\hat{\mathcal{L}}_\bullet$ which is preserved by the q-connection A. The Z-twisted condition implies that in the gauge where A is given by a fixed semisimple diagonal element $Z \in H$, such a flag is formed by the standard basis $e_1,\ldots,e_{r+1}$.

The relative position between two flags is generic on $U \cap M_q^{-1}(U)$. The regular singularity condition implies that *quantum Wronskians*, namely determinants

$$\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Z s\left(q^{k-2}\right) \wedge s\left(q^{k-1}z\right), \tag{6.3}$$

have a subset of zeros which coincide with those of $\mathcal{W}_k(s)$. To be more explicit, for $k = 1,\ldots,r+1$, we have nonzero constants α_k and polynomials

$$\mathcal{V}_k(z) = \prod_{a=1}^{r_k} (z - v_{k,a}) \tag{6.4}$$

for which

$$
\det\begin{pmatrix}
1 & \cdots & 0 & \xi_1^{k-1}s_1(z) & \cdots & \xi_1 s_1\left(q^{k-2}z\right) & s_1\left(q^{k-1}z\right) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & \xi_k^{k-1}s_{r+1-k}(z) & \cdots & \xi_k s_{r+1-k}\left(q^{k-2}z\right) & s_k\left(q^{k-1}z\right) \\
0 & \cdots & 0 & \xi_{k+1}^{k-1}s_{r+1-k+1}(z) & \cdots & \xi_{r+1-k+1}s_{k+1}\left(q^{k-2}z\right) & s_{k+1}\left(q^{k-1}z\right) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \xi_{r+1}^{k-1}s_{r+1}(z) & \cdots & \xi_{r+1}s_{r+1}\left(q^{k-2}z\right) & s_{r+1}\left(q^{k-1}z\right)
\end{pmatrix} = \alpha_k W_k \mathcal{V}_k.
$$

$$(6.5)$$

Since $\mathcal{D}_{r+1}(s) = \mathcal{W}_{r+1}(s)$, we have $\mathcal{V}_{r+1} = 1$. We also set $\mathcal{V}_0 = 1$; this is consistent with the fact that equation (6.3) also makes sense for $k = 0$, giving $\mathcal{D}_0 = e_1 \wedge \cdots \wedge e_{r+1}$.

We can also rewrite equation (6.5) as

$$
\det_{i,j}\left[\xi_{r+1-k+i}^{k-j}s_{r+1-k+i}\left(q^{j-1}z\right)\right] = \alpha_k W_k \mathcal{V}_k,
\tag{6.6}
$$

where $i,j = 1, \ldots, k$.

Note that these determinants have a slightly different form those of [38]–twist parameters ξ_i entered in different powers. This is due to a different order of the simple roots in the definition of the q-oper.

Theorem 6.1 ([38]). *The polynomials $\{\mathcal{V}_k(z)\}_{k=1,\ldots,r}$ give the solution to the QQ-system (5.3) so that $Q_j^+(z) = \mathcal{V}_j(z)$ under the nondegeneracy condition that for all i,j,k with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros of $\mathcal{V}_i(z)$ and $\mathcal{V}_j(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$.*

Remark 6.2. Technically, we used stronger conditions in [38], namely that zeros of $\{\Lambda_i\}_{i=1\ldots r}$ and $\{\mathcal{V}_j(z)\}_{j=1\ldots r}$ had to be q-disjoint to satisfy the QQ-system equations, but we can relax it easily and even more than we have in this theorem.

In the next subsection we will show that the extended QQ-system can be obtained from various minors in q-Wronskian matrices. Theorem 6.1 allows us to relate the section $s(z)$ generating the line bundle $\mathcal{L}_{r+1}$ with the elements of the extended QQ-system using the transformation of equation (5.6).

Proposition 6.3. *Let $v(z)$ be the gauge transformation from equation (5.6) and $s(z)$ be the section generating $\mathcal{L}_{r+1}$ in the definition of the $(SL(r+1),q)$-oper. Then the components of $s(z)$ in the gauge when the q-oper connection is equal to Z are given by*

$$
s_{r+1}(z) = Q_r^+(z), \qquad s_r(z) = Q_r^-(z), \qquad s_k(z) = Q_{k,\ldots,r}^-(z),
\tag{6.7}
$$

for $k = 1, \ldots, r-1$.

Proof. Starting from equation (5.11), the proposition follows directly after acting with $v(z)^{-1}$ on the basis vector $e_{r+1} = (0,0,\ldots,0,1)$.

In the next subsection we will show that the extended QQ-system can be obtained from various minors in q-Wronskian matrices.

6.2. Wronskians and extended QQ-systems

First we will rewrite the extended QQ-system in a more convenient way to relate it to the minors in the q-Wronskian matrix. Namely, we multiply Q-terms by certain polynomials to get rid of the Λ-polynomials in the right-hand side. This is done in the following lemma:

Lemma 6.4. *The system of equations (5.7) is equivalent to the following set of equations:*

$$\xi_i \mathscr{D}_i^+(qz)\mathscr{D}_i^-(z) - \xi_{i+1}\mathscr{D}_i^+(z)\mathscr{D}_i^-(qz) = (\xi_i - \xi_{i+1})\mathscr{D}_{i-1}^+(qz)\mathscr{D}_{i+1}^+(z),$$

$$\xi_i \mathscr{D}_{i+1}^+(qz)\mathscr{D}_{i,i+1}^-(z) - \xi_{i+2}\mathscr{D}_{i+1}^+(z)\mathscr{D}_{i,i+1}^-(qz) = (\xi_i - \xi_{i+2})\mathscr{D}_i^-(qz)\mathscr{D}_{i+2}^+(z),$$

$$\vdots$$

$$\xi_i \mathscr{D}_{r+i-2}^+(qz)\mathscr{D}_{i,\ldots,r-1+i}^-(z) - \xi_{r+i-1}\mathscr{D}_{r+i-2}^+(z)\mathscr{D}_{i,\ldots,r-1+i}^-(qz) = (\xi_i - \xi_{r+i-1})\mathscr{D}_{i,\ldots,r-2+i}^-(qz)\mathscr{D}_{r+i-1}^+(z),$$

$$\xi_1 \mathscr{D}_r^+(qz)\mathscr{D}_{1,\ldots,r}^-(z) - \xi_{r+1}\mathscr{D}_r^+(z)\mathscr{D}_{1,\ldots,r}^-(qz) = (\xi_1 - \xi_{r+1})\mathscr{D}_{1,\ldots,r-1}^-(qz), \tag{6.8}$$

where index i ranges between the same values as in the corresponding lines in equation (5.7) for the polynomials

$$\mathscr{D}_k^+ = Q_k^+ F_k, \qquad \mathscr{D}_k^- = Q_k^- F_k \eta_k, \qquad \mathscr{D}_{l,\ldots,k}^- = Q_{l,\ldots,k}^- F_k \eta_{l,\ldots,k}, \tag{6.9}$$

where

$$F_i(z) = W_{r-i}\left(q^{r-i}z\right), \qquad \eta_{l,\ldots,i} = \prod_{a=0}^{i-l}(\xi_l - \xi_{l+a+1}).$$

We shall refer to equation (6.8) as the *extended $\mathscr{D}\mathscr{D}$-system* for $SL(r+1)$ and to its first line specifically as merely the *$\mathscr{D}\mathscr{D}$-system*.

Proof. The proof is a direct extension of the proof of [38] to other lines in equation (5.31). Since all equations are treated analogously, let us consider the second line of equation (6.8), which we can write as

$$\xi_{i-1}\mathscr{D}_i^+(qz)\mathscr{D}_{i-1,i}^-(z) - \xi_{i+1}\mathscr{D}_i^+(z)\mathscr{D}_{i-1,i}^-(qz) = (\xi_{i-1} - \xi_{i+1})\mathscr{D}_{i-1}^-(qz)\mathscr{D}_{i+1}^+(z). \tag{6.10}$$

After replacing

$$\mathscr{D}_i^+ = Q_i^+ F_i, \qquad \mathscr{D}_i^- = Q_i^- F_i \eta_i, \qquad \mathscr{D}_{i-1,i}^- = Q_{i-1,i}^- F_i \eta_{i-1,i},$$

and assigning

$$\eta_{i-1} = \xi_{i-1} - \xi_i, \qquad \eta_{i-1,i} = (\xi_{i-1} - \xi_i)(\xi_{i-1} - \xi_{i+1}),$$

we can see that equation (6.10) is equivalent to the second line of equation (5.31), provided that the following difference equation is satisfied:

$$\frac{F_{i-1}(qz)F_{i+1}(z)}{F_i(qz)F_i(z)} \cdot \frac{\eta_{i-1}}{\eta_{i-1,i}}(\xi_{i-1} - \xi_{i+1}) = \Lambda_i(z).$$

The validity of this relation follows from the foregoine formulae and the definitions in equation (6.2):

$$\frac{F_{i-1}(qz)F_{i+1}(z)}{F_i(qz)F_i(z)} = \frac{W_{r-i+1}\left(q^{r-i+1}z\right)}{W_{r-i}\left(q^{r-i}z\right)}\frac{W_{r-i-1}\left(q^{r-i}z\right)}{W_{r-i}\left(q^{r-i+1}z\right)} = \frac{P_{r-i+1}\left(q^{r-i}z\right)}{P_{r-i}\left(q^{r-i}z\right)} = \Lambda_i(z).$$

As we shall see below, one can express the solutions of the QQ- and $\mathscr{D}\mathscr{D}$-systems in terms of the section $s(z)$ of subbundle $\mathcal{L}_{r+1}$. Following the discussion of [38], we consider the following matrices:

$$M_{i_1,\ldots,i_j} = \begin{pmatrix} \xi_{i_1}^{j-1}s_{i_1}(z) & \cdots & \xi_{i_1}s_{i_1}\left(q^{j-2}z\right) & s_{i_1}\left(q^{j-1}z\right) \\ \vdots & \ddots & \vdots & \vdots \\ \xi_{i_j}^{j-1}s_{i_j}(z) & \cdots & \xi_{i_j}s_{i_j}\left(q^{j-2}z\right) & s_{i_j}\left(q^{j-1}z\right) \end{pmatrix}, \quad V_{i_1,\ldots,i_j} = \begin{pmatrix} \xi_{i_1}^{j-1} & \cdots & \xi_{i_1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \xi_{i_j}^{j-1} & \cdots & \xi_{i_j} & 1 \end{pmatrix},$$

(6.11)

where s_i are polynomials and $V_{i_1,\ldots,i_j}$ is the Vandermonde-like matrix whose determinant is

$$\det V_{i_1,\ldots,i_j} = \prod_{i<j}\left(\xi_i - \xi_j\right). \tag{6.12}$$

In [38] the following proposition was proven (in slightly different notation), which allows us to express solutions of the QQ-system in terms of q-Wronskians of equation (6.11). Here we provide a sketch of the proof for completeness.

Proposition 6.5. *Given polynomials $\mathscr{D}_i^+, \mathscr{D}_i^-$ for $i = 1,\ldots,r$ satisfying the first line of equation (6.8), there exist unique polynomials $s_1,\ldots,s_{r+1}$ such that*

$$\mathscr{D}_i^+(z) = \frac{\det M_{r+2-i,\ldots,r+1}(z)}{\det V_{r+2-i,\ldots,r+1}} \quad and \quad \mathscr{D}_i^-(z) = \frac{\det M_{r+1-i,r+3-i,\ldots,r+1}(z)}{\det V_{r+1-i,r+3-i,\ldots,r+1}}, \quad (6.13)$$

where matrix M is given in equation (6.11).

Proof. The proof is based on the determinant Desnanot–Jacobi identity, which holds for any $l \times l$ matrix M. In this proof we shall use this identity in the form

$$M_1^1 M_l^2 - M_l^1 M_1^2 = M_{1,l}^{1,2} M, \tag{6.14}$$

where M_b^a (resp., $M_{b,d}^{a,c}$) is the determinant of matrix M with row a and column b removed (resp., matrix M with rows a and c and column b and d removed). Note that the Desnanot–Jacobi identity holds for any pairs of indices $\{a,c\}$ and $\{b,d\}$ as long as $a \neq c$ and $b \neq d$.

In [38] it was shown using periodic properties of matrix M that the first line of equation (6.8) can be identified with equation (6.14) if $M_i = M_{r+1-i,\ldots,r+1}(z)$ is the determinant of the bottom right $i \times i$ submatrix of the $(r+1) \times (r+1)$ matrix $M_{1,\ldots,r+1}(z)$. It is easy

to see, for instance, that

$$(M_i)_1^1(z) = M_{r+2-i,\ldots,r+1}(qz), \qquad (M_i)_1^2(z) = M_{r+1-i,r+3-i,\ldots,r+1}(qz),$$

$$(M_i)_i^1(z) = \xi_{r+2-i}\left(\prod_{a=r+3-i}^{r+1}\xi_i\right)\cdot M_{r+2-i,\ldots,r+1}(z),$$

$$(M_i)_i^2(z) = \xi_{r+1-i}\left(\prod_{a=r+3-i}^{r+1}\xi_i\right)\cdot M_{r+1-i,r+3-i,\ldots,r+1}(z),$$

$$(M_i)_{1,i}^{1,2}(z) = \left(\prod_{a=r+3-i}^{r+1}\xi_i\right)\cdot M_{r+3-i,\ldots,r+1}(qz).$$

We can substitute these five relations into equation (6.14) and then divide both sides by $V_{r+2-i,\ldots,r+1}V_{r+1-i,r+3-i,\ldots,r+1}$. The first $\mathscr{DD}$-relation will follow after observing that

$$V_{r+3-i,\ldots,r+1}V_{r+1-i,\ldots,r+1} = (\zeta_{r+1-i} - \zeta_{r+2-i})V_{r+2-i,\ldots,r+1}V_{r+1-i,r+3-i,\ldots,r+1}.$$

In this proof we have derived an alternative presentation of $\mathscr{D}_i^\pm$ polynomials and their q-shifted counterparts in terms of minors M_i:

$$\mathscr{D}_i^+(z) = \frac{(M_i)_i^1(z)}{(V_i)_i^1}, \quad \mathscr{D}_i^-(z) = \frac{(M_i)_i^2(z)}{(V_i)_i^2}, \quad \mathscr{D}_i^+(qz) = \frac{(M_i)_1^1(z)}{(V_i)_1^1}, \quad \mathscr{D}_i^-(qz) = \frac{(M_i)_1^2(z)}{(V_i)_1^2},$$

(6.15)

where $V_i = V_{r+1-i,\ldots,r+1}(z)$ is the determinant of the bottom right $i \times i$ submatrix of the $(r+1) \times (r+1)$ matrix $V_{1,\ldots,r+1}(z)$. This way all polynomials which appear in the $\mathscr{DD}$-system can be universally presented as ratios of (unshifted) minors of two sets of matrices $\{M_i\}$ and $\{V_i\}$ for $i = 1,\ldots,r$.

Thus the ith equation of the $\mathscr{DD}$-system represents a Desnanot–Jacobi determinant identity for matrix M_i of the form in equation (6.11). In the following subsection we shall demonstrate that all equations of the extended $\mathscr{DD}$-system can also be thought of as determinant identities for matrices which are obtained from M_is by permutation of rows and columns. The latter is provided by Bäcklund transformations.

6.3. Bäcklund transformations and the extended $\mathscr{DD}$-system

We showed that the extended QQ-system is equivalent to the extended $\mathscr{DD}$-system in Lemma 6.4. Let us focus on the $\mathscr{DD}$-system, namely the equations corresponding to the first line in equation (6.8). We already mentioned that all of equation (5.31) can be obtained from the QQ-system by applying Bäcklund transformations. The same works for the $\mathscr{DD}$-system. The ith Bäcklund transformation replaces the data

$$\left\{\mathscr{D}_j^+, \mathscr{D}_j^-\right\}_{j=1,\ldots,r}, \qquad \{\xi_j\}_{j=1,\ldots,r+1},$$

with

$$\left\{\widetilde{\mathscr{D}}_j^+\right\}_{j=1,\dots,r} = \left\{\mathscr{D}_1^+,\dots,\mathscr{D}_{i-1}^+,\mathscr{D}_i^-,\mathscr{D}_{i+1}^+,\dots,\mathscr{D}_r^+\right\}, \tag{6.16}$$

$$\left\{\widetilde{\mathscr{D}}_j^-\right\}_{j=1,\dots,r} = \left\{\mathscr{D}_1^-,\dots,\mathscr{D}_{i-1}^{*-},\mathscr{D}_i^+,\mathscr{D}_{i-1,i}^-,\mathscr{D}_{i+2}^-,\dots,\mathscr{D}_r^-\right\},$$

$$\left\{\widetilde{\xi}_j\right\}_{j=1,\dots,r} = \{\xi_1,\dots,\xi_{i-1},\xi_{i+1},\xi_i,\dots,\xi_{r+1}\}.$$

Notice that in the QQ-system this rule works as $Q_i^- \mapsto -Q_i^+$. In the $\mathscr{D}\mathscr{D}$-system the sign disappears due to the presence of the multiplicative factor η_i between Q_i^-- and $\mathscr{D}$-functions.

We also note that polynomials $\mathscr{D}_{i-1}^{*-}$ do not belong to the extended $\mathscr{D}\mathscr{D}$-system; rather, they will be a part of the *full* $\mathscr{D}\mathscr{D}$-system. By applying Bäcklund transformations further we can readily find all polynomials from the full $\mathscr{D}\mathscr{D}$-system, which is in one-to-one correspondence with the full QQ-system we discussed in Section 5.

We can now find a similar presentation for other polynomials $\mathscr{D}_{i,\dots,j}^-$ in terms of ratios of determinants by combining the foregoing ideas and Proposition 6.3. In particular, we need to understand how Bäcklund transformations act on matrices (6.11).

Let us start with the $(r+1) \times (r+1)$ matrices $M_1 = M_{1,\dots,r+1}$ and $V_1 = V_{1,\dots,r+1}$ from equation (6.11). During the ith Bäcklund transformation (5.43), the functions $\mathscr{D}_i^+(z) \leftrightarrow \mathscr{D}_i^-(z)$, $\mathscr{D}_{i+1}^-(z) \leftrightarrow \mathscr{D}_{i,i+1}^-(z)$ and $\xi_i \leftrightarrow \xi_{i+1}$ in the extended $\mathscr{D}\mathscr{D}$-system get interchanged– which, using the identification of equation (6.7), amounts to acting by permutations r_i on the set of indices as

$$\{1,\dots,i,i+1,\dots,r+1\} \mapsto \{1,\dots,i+1,i,\dots,r+1\}.$$

Therefore we can define a new tuple of matrices $r_i(M_1)(z) = M_{1,\dots,i+1,i,\dots,r+1}$ and $r_i(V_1)(z) = V_{1,\dots,i+1,i,\dots,r+1}$ as well as their submatrices $r_i(M_j)(z)$ and $r_i(V_j)(z)$, which are obtained by excising the corresponding $(r-j+1) \times (r-j+1)$ bottom right blocks. Then by Proposition 6.5 we must have

$$\mathscr{D}_i^+(z) = \frac{\det r_i(M)_{r+2-i,\dots,r+1}(z)}{\det r_i(V)_{r+2-i,\dots,r+1}}, \qquad \mathscr{D}_{i,i+1}^-(z) = \frac{\det r_i(M)_{r+2-i,r+4-i,\dots,r+1}(z)}{\det r_i(V)_{r+2-i,r+4-i,\dots,r+1}}, \tag{6.17}$$

or equivalently,

$$\mathscr{D}_i^-(z) = \frac{r_i(M_i)_i^1(z)}{r_i(M_i)_i^1}, \qquad \mathscr{D}_{i,i+1}^-(z) = \frac{r_i(M_{i+1})_{i-1}^1(z)}{r_i(V_{i+1})_{i-1}^1}. \tag{6.18}$$

Notice that the second equality of equation (6.17) can be written in terms of the original matrix M:

$$\mathscr{D}_{i,i+1}^- = \frac{\det M_{i,i+3\dots,r+1}}{\det V_{i,i+3\dots,r+1}}. \tag{6.19}$$

In order to determine similar expressions for other $\mathscr{D}_{i,\dots,i+k}^-$, one needs to act by other elements of the Weyl group $W = S_{r+1}$. Essentially, the Bäcklund transformation, being associated with the elementary Weyl reflection, interchanges two rows in the q-Wronskian

matrix. This brings us to the following statement, which can be verified by a direct calculation along the lines of Proposition 6.5 for each set of equations of the extended $\mathscr{D}\mathscr{D}$-system:

Proposition 6.6. *The polynomials $\mathscr{D}^-_{i,\ldots i+k}$ from equation (6.8) read*

$$\mathscr{D}^-_{i,\ldots,i+k} = \frac{\det M_{i,i+k+2,\ldots,r+1}}{\det V_{i,i+k+2,\ldots,r+1}}, \quad i=1,\ldots,r,\ k=0,\ldots,r-i, \tag{6.20}$$

or equivalently,

$$\mathscr{D}^-_{i,\ldots,i+k} = \frac{r_{i+k-1}(\ldots r_{i+1}(r_i(M_{i+k}))\ldots)^{i+1}_{r+1}}{r_{i+k-1}(\ldots r_{i+1}(r_i(V_{i+k}))\ldots)^{i+1}_{r+1}}. \tag{6.21}$$

Although we do not discuss polynomials $\mathscr{D}^{*-}_{i-1}$ which belong to the full $\mathscr{D}\mathscr{D}$-system rather than to the extended $\mathscr{D}\mathscr{D}$-system, we can nevertheless provide a formula for them:

Proposition 6.7. *The polynomials $\mathscr{D}^{*-}_{i-1}$ from equation (6.8) read*

$$\mathscr{D}^{*-}_{i-1} = \frac{r_{i-1}(M_{i-1})^1_{i+1}(z)}{r_{i-1}(V_{i-1})^1_{i+1}}. \tag{6.22}$$

Proof. The $(i-1)$th equation of the full $\mathscr{D}\mathscr{D}$-system after applying the ith Bäcklund transformation reads

$$\xi_{i-1}\mathscr{D}^+_{i-1}(qz)\mathscr{D}^{*-}_{i-1}(z) - \xi_{i+1}\mathscr{D}^+_{i-1}(z)\mathscr{D}^{*-}_{i-1}(qz) = (\xi_{i-1}-\xi_{i+1})\mathscr{D}^+_{i-2}(qz)\mathscr{D}^-_i(z).$$

Given the description in equation (6.15) of $\mathscr{D}$ polynomials in terms of minors, it can be shown that this equation is equivalent to the Jacobi determinant identity of the form in equation (6.14) for matrix M_{i-1}.

This statement implies that the solutions of the full $\mathscr{D}\mathscr{D}$- and thus the full QQ-system are well defined if the original QQ-system is nondegenerate. Also, notice that all the other equations in the QQ-system correspond to all possible Miura $(SL(r+1),q)$-opers for a given $(SL(r+1),q)$-oper. Thus the following theorem, which generalises Theorem 5.8, is true:

Theorem 6.8.

(1) *The solution of the nondegenerate $(SL(r+1))QQ$-system can be extended to the solution of the the full QQ-system.*

(2) *This full QQ-system comprises $(r+1)!QQ$-systems, with Bäcklund transformations acting transitively between them.*

(3) *Each such QQ-system determines one of the $(r+1)!Z$-twisted Miura $(SL(r+1),q)$-opers, corresponding to a unique Z-twisted $(SL(r+1),q)$-oper.*

We can combine Lemma 6.4 with Propositions 6.5 and 6.6 to get the following proposition, which will be used in later sections to study infinite-dimensional q-opers:

Proposition 6.9. *The polynomials which appear in the extended QQ-system* (5.30) *are given by*

$$Q_i^+(z) = \frac{1}{F_i(z)} \cdot \frac{(M_i)_i^1(z)}{(V_i)_i^1}, \qquad Q_i^-(z) = \frac{1}{F_i(z)\eta_i} \cdot \frac{(M_i)_i^2(z)}{(V_i)_i^2},$$

$$Q_{i,\dots,i+k}^-(z) = \frac{1}{F_i(z)\eta_{i,\dots,k}} \cdot \frac{s_{i+k-1}(\dots s_{i+1}(s_i(M_{i+k}))\dots)_i^2(z)}{s_{i+k-1}(\dots s_{i+1}(s_i(V_{i+k}))\dots)_i^2}, \tag{6.23}$$

or equivalently,

$$Q_i^+(z) = \frac{1}{F_i(z)} \cdot \frac{\det M_{r+2-i,\dots,r+1}(z)}{\det V_{r+2-i,\dots,r+1}},$$

$$Q_i^-(z) = \frac{1}{F_i(z)\eta_i} \cdot \frac{\det M_{r+1-i,r+3-i,\dots,r+1}(z)}{\det V_{r+1-i,r+3-i,\dots,r+1}}, \tag{6.24}$$

$$Q_{i,\dots,i+k}^-(z) = \frac{1}{F_i(z)\eta_{i,\dots,k}} \cdot \frac{\det M_{i,i+k+2,\dots,r+1}(z)}{\det V_{i,i+k+2,\dots,r+1}}.$$

Note also the following expressions for shifted Q-functions, which will be used later:

$$Q_i^+(qz) = \frac{1}{F_i(z)} \cdot \frac{(M_i)_1^1(z)}{(V_i)_1^1}, \qquad Q_i^-(qz) = \frac{1}{F_i(z)\eta_i} \cdot \frac{(M_i)_1^2(z)}{(V_i)_1^2}. \tag{6.25}$$

6.4. Example: Miura $(SL(3),q)$-oper

Define matrices

$$M_1(z) = \begin{pmatrix} \xi_1^2 s_1(z) & \xi_1 s_1(qz) & s_1(q^2 z) \\ \xi_2^2 s_2(z) & \xi_2 s_2(qz) & s_2(q^2 z) \\ \xi_3^2 s_3(z) & \xi_3 s_3(qz) & s_3(q^2 z) \end{pmatrix}, \qquad V_1 = \begin{pmatrix} \xi_1^2 & \xi_1 & 1 \\ \xi_2^2 & \xi_2 & 1 \\ \xi_3^2 & \xi_3 & 1 \end{pmatrix},$$

$$M_2(z) = (M_1)_{23}(z) = \begin{pmatrix} \xi_2 s_2(z) & s_2(qz) \\ \xi_3 s_3(z) & s_3(qz) \end{pmatrix}, \qquad V_2 = (V_1)_{23} = \begin{pmatrix} \xi_2 & 1 \\ \xi_3 & 1 \end{pmatrix},$$

and the matrices obtained from them by the Weyl action:

$$r_1(M_1)(z) = \begin{pmatrix} \xi_2^2 s_2(z) & \xi_2 s_2(qz) & s_2(q^2 z) \\ \xi_1^2 s_1(z) & \xi_1 s_1(qz) & s_1(q^2 z) \\ \xi_3^2 s_3(z) & \xi_3 s_3(qz) & s_3(q^2 z) \end{pmatrix}, \qquad r_1(V_1) = \begin{pmatrix} \xi_2^2 & \xi_2 & 1 \\ \xi_1^2 & \xi_1 & 1 \\ \xi_3^2 & \xi_3 & 1 \end{pmatrix},$$

$$M_{1,2}(z) = r_1(M_1)_{23}(z) = \begin{pmatrix} \xi_1 s_1(z) & s_1(qz) \\ \xi_3 s_3(z) & s_3(qz) \end{pmatrix}, \qquad V_{1,2} = r_1(V_1)_{23} = \begin{pmatrix} \xi_1 & 1 \\ \xi_3 & 1 \end{pmatrix}.$$

Then the $\mathscr{D}\mathscr{D}$-system reads

$$\xi_1 \mathscr{D}_1^+(qz)\mathscr{D}_1^-(z) - \xi_2 \mathscr{D}_1^+(z)\mathscr{D}_1^-(qz) = (\xi_1 - \xi_2)\mathscr{D}_2^+(qz)W(z),$$

$$\xi_2 \mathscr{D}_2^+(qz)\mathscr{D}_2^-(z) - \xi_3 \mathscr{D}_2^+(z)\mathscr{D}_2^-(qz) = (\xi_2 - \xi_3)\mathscr{D}_1^+(z), \tag{6.26}$$

$$\xi_1 \mathscr{D}_2^+(qz)\mathscr{D}_{1,2}^-(z) - \xi_3 \mathscr{D}_2^+(z)\mathscr{D}_{1,2}^-(qz) = (\xi_1 - \xi_3)\mathscr{D}_1^-(z),$$

with

$$\mathscr{D}_1^+(z) = \frac{(M_1)_2^1(z)}{V_2^1}, \qquad \mathscr{D}_1^-(z) = \frac{(M_1)_3^2(z)}{V_3^2}, \qquad W(z) = \frac{\det M_1(z)}{\det V_1},$$

as well as

$$\mathscr{D}_2^+(z) = \frac{(M_2)_2^1(z)}{(V_2)_2^1} = s_3(z), \quad \mathscr{D}_2^-(z) = \frac{(M_2)_2^2(z)}{(V_2)_2^2} = s_2(z), \quad \mathscr{D}_{1,2}^-(z) = \frac{(M_{1,2})_2^2(z)}{(V_{1,2})_2^2} = s_1(z),$$

where $(M_i)_b^a(z)$ is the determinant of matrix $M_i(z)$ with row a and column b removed. The shifted $\mathscr{D}$-functions read

$$\mathscr{D}_1^+(qz) = \frac{(M_1)_1^1(z)}{(V_1)_1^1}, \qquad \mathscr{D}_1^-(qz) = \frac{(M_1)_1^2(z)}{(V_1)_1^2},$$

$$\mathscr{D}_2^+(qz) = \frac{(M_2)_1^1(z)}{(V_2)_1^1}, \qquad \mathscr{D}_2^-(qz) = \frac{(M_2)_1^2(z)}{(V_2)_1^2}, \quad \mathscr{D}_{1,2}^-(qz) = \frac{(M_{1,2})_1^2(z)}{(V_{1,2})_1^2}.$$

Then the solutions of the $SL(3)QQ$-system read

$$Q_1^+(z) = \frac{1}{F_1(z)(\xi_1 - \xi_2)} \cdot \frac{(M_1)_2^1(z)}{V_2^1}, \qquad Q_1^-(z) = \frac{1}{F_1(z)(\xi_1 - \xi_2)} \cdot \frac{(M_1)_3^2(z)}{V_3^2},$$

as well as

$$
\begin{aligned}
Q_2^+(z) &= \frac{1}{F_2(z)(\xi_2 - \xi_3)} \cdot \frac{(M_2)_2^1(z)}{(V_2)_2^1} = \frac{s_3(z)}{F_1(z)(\xi_1 - \xi_2)}, \\
Q_2^-(z) &= \frac{1}{F_2(z)(\xi_2 - \xi_3)} \cdot \frac{(M_2)_2^2(z)}{(V_2)_2^2} = \frac{s_2(z)}{F_2(z)(\xi_2 - \xi_3)}, \\
Q_{1,2}^-(z) &= \frac{1}{F_2(z)(\xi_2 - \xi_3)} \cdot \frac{(M_{1,2})_2^2(z)}{(V_{1,2})_2^2} = \frac{s_1(z)}{F_2(z)(\xi_2 - \xi_3)}.
\end{aligned}
\tag{6.27}
$$

6.5. Explicit formula for $(SL(r+1),q)$-oper via minors

We can now collect all the results of this section in order to present the Miura $(SL(r+1),q)$-oper of equation (5.16) in terms of trivialisation of the subbundle $\mathcal{L}_{r+1}$. Consider functions $g_i(z)$ which appear on the diagonal:

$$
\begin{aligned}
g_i(z) &= \zeta_i \frac{Q_i^+(qz)}{Q_i^+(z)} = \zeta_i \frac{F_i(z)}{F_i(qz)} \cdot \frac{(M_i)_1^1}{(M_i)_i^1} \frac{(V_i)_i^1}{(V_i)_1^1} \\
&= \zeta_i \cdot \left(\prod_{a=1}^i \xi_{r+2-a} \right) \cdot \prod_{b=1}^i \frac{\Lambda_{r-i+b}\left(q^{1-b}z\right)}{\Lambda_{r-i+b}(z)} \cdot \frac{(M_i)_1^1}{(M_i)_i^1}.
\end{aligned}
\tag{6.28}
$$

Then the diagonal entry of equation (5.16) becomes the meromorphic function

$$\frac{g_{i+1}}{g_i}(z) = \xi_{i+1} H_i^{(r)}(z,q) \cdot G_i^{(r)}(z,q), \tag{6.29}$$

where

$$H_i^{(r)}(z,q) = \prod_{b=1}^i \frac{\Lambda_{r-i+b}\left(q^{1-b}z\right)}{\Lambda_{r-i+b}\left(q^{-b}z\right)}, \qquad G_i^{(r)}(z,q) = \frac{(M_{i+1})_1^1}{(M_i)_1^1} \cdot \frac{(M_i)_i^1}{(M_{i+1})_{i+1}^1}. \tag{6.30}$$

6.6. Relation to Berenstein, Fomin and Zelevinsky's work on generalised minors

We devoted this section to the description of Miura $(SL(r+1),q)$-opers via various minors of the q-Wronskian matrix. That matrix is produced by the components of the section of the line bundle and the components of the constant regular element Z, representing the q-connection in the given trivialisation. One may wonder whether such a construction exists in the general case for a simply connected simple group G–namely, whether there exists an analogue of the q-Wronskian. Of course, in that case we do not have a line bundle, since the definition of (G,q) in terms of the flag of bundles is SL-specific. Nevertheless, there is a notion of generalised minors [6, 7, 18]. These are the functions on G, defined on the dense set, corresponding to the dense Bruhat cell N_-HN_+. For any $g=n_+hn_-$, the so-called *principal minors* $[g]^{\omega_i}$ are defined as the values of the multiplicative characters $[\cdot]^{\omega_i}: H \longrightarrow \mathbb{C}^*$ on h, namely $[h]^{\omega_i}$, corresponding to the fundamental weight ω_i for $i = 1,\dots,r$. Other generalised minors $\Delta_{u\omega_i,v\omega_i}$ are obtained by the action of the Weyl group elements on the left and the right of g, application of the appropriate lifts of Weyl group elements u,v on the right and the left and then application of $[\cdot]^{\omega_i}$. In the case of $SL(r+1)$, the nondegeneracy conditions imply that the full q-Wronskian matrix belongs to the dense Bruhat cell (i.e., it has Gauss decomposition) and the action of the Weyl group elements corresponds to the permutations of rows and columns.

One of the fundamental relations between generalised minors is as follows [18]. Set $u,v \in W$ such that for $i \in \{1,\dots,r\}$, $\ell(us_i)=\ell(u)+1$, $\ell(vs_i)=\ell(v)+1$. Then

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} - \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} = \prod_{j\neq i}\Delta_{u\omega_j,v\omega_j}^{-a_{ji}}.$$

When applied to the q-Wronskian matrix in the $SL(r+1)$ case, these equations reproduce the $\mathscr{D}\mathscr{D}$-system. In the case of general G, the left- and right-hand sides of this relation are very similar to the analogue of the $\mathscr{D}\mathscr{D}$-system [24]. Thus it is reasonable to assume the existence of the analogue of the Wronskian matrix as an element in $n_-(z)h(z)n_+(z) \in N_-(z)H(z)N_+(z) \subset G(z)$. We will discuss this in future work.

Note that one important feature of generalised minors is that relations between them give a cluster algebra structure for double Bruhat cells, so that our Bäcklund transformations descend from mutations for the cluster algebra elements.

We believe that these cluster structures stand behind known cluster structures relevant for Grothendieck rings of quantum affine algebras.

7. $\overline{GL}(\infty)$ and the fermionic Fock space

This section contains material on infinite-dimensional generalisations of $GL(N)$ and their representations which will be needed later. The reader may consult [38] for more details.

7.1. Miura $(SL(r+1),q)$-opers and the fermionic Fock space

First we note that given a defining representation V_{ω_1} of $SL(r+1)$, one can construct all other fundamental representations V_{ω_i} by considering wedge powers $\simeq \Lambda^i(V_{\omega_1})$. If $\nu_1,\dots,\nu_{r+1}$ are the standard basis vectors in V_{ω_1}, so that ν_1 is the highest weight, then

the highest weight vectors in V_{ω_i} are

$$\nu_i \wedge \nu_{i-1} \wedge \cdots \wedge \nu_1.$$

Introducing operators ψ_i of exterior multiplication on ν_i and ψ_i^* of interior multiplication by ν_i, we find that they satisfy a Clifford algebra relation

$$\psi_i^* \psi_j + \psi_j \psi_i^* = \delta_{ij}. \tag{7.1}$$

Using those operators we can realise the Chevalley generators as follows:

$$\check{\alpha}_i = \psi_i \psi_i^* - \psi_{i+1} \psi_{i+1}^*, \qquad e_i = \psi_i \psi_{i+1}^*, \qquad f_i = \psi_{i+1} \psi_i^*, \tag{7.2}$$

such that $[e_i, f_i] = \check{\alpha}_i$. We arrive at the following proposition:

Proposition 7.1. *In any fundamental representation, the q-connection, corresponding to the Miura $(SL(r+1),q)$-oper (5.1) reads as follows:*

$$A(z) = \prod_{i=r}^{1} g_i^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i} = g_r^{-\psi_{r+1} \psi_{r+1}^*} \prod_{i=r}^{1} e^{\Lambda_i(z) \psi_i \psi_{i+1}^*} \cdot \left[\frac{g_i}{g_{i-1}} \right]^{\psi_i \psi_i^*}, \tag{7.3}$$

where $g_0 = 1$.

Our goal in the following will be to make sense of the completion of this formula in the infinite-dimensional Fock space.

7.2. $SL(\infty)$ and its completions

In the following we review some basic facts from [33] on representations of infinite-dimensional Lie algebras and groups. We can define the group $GL(\infty)$ as a set of infinite-dimensional matrices characterised as follows:

$$GL(\infty) = \Big\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0 \Big\}.$$

$SL(\infty)$ is the subgroup of $GL(\infty)$ of unimodular matrices. The Lie algebra $\mathfrak{gl}(\infty)$ of $GL(\infty)$ is given by

$$\mathfrak{gl}(\infty) = \Big\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid a_{ij} = 0 \text{ for all but a finite number} \Big\} \tag{7.4}$$

and $\mathfrak{sl}(\infty)$ is the subalgebra of traceless matrices. The Lie algebra $\mathfrak{sl}(\infty)$ is the explicit realisation of the simple Kac–Moody algebra $\mathfrak{a}_\infty$, which one associates to the infinite Dynkin diagram A_∞. However, there exists a bigger algebra, known as $\bar{\mathfrak{a}}_\infty$, which consists of elements of the form

$$x = \sum_{i \in \mathbb{Z}} c_i \check{\alpha}_i + \sum_\alpha \eta_\alpha e_\alpha, \tag{7.5}$$

where e_α is an element of Cartan–Weyl basis corresponding to the root α with the height $\mathrm{ht}(\alpha)$ of $\mathfrak{sl}(\infty)$, so that the set

$$S_x = \{ k \in \mathbb{Z} \mid \exists \alpha, \eta_\alpha \neq 0, \ \mathrm{ht}(\alpha) = k \}$$

is finite. This algebra has two nontrivial central elements $c_1 = \sum_i i \check{\alpha}_i$ and $c_2 = \sum_i \check{\alpha}_i$. The explicit realisation of this algebra is given by the central extension of the algebra $\mathfrak{gl}_\infty$:

$$\overline{\mathfrak{gl}}_\infty = \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid a_{ij} = 0 \text{ for } |i-j| \gg 0 \right\}. \tag{7.6}$$

Namely, there exists a homomorphism from $\bar{\mathfrak{a}}_\infty$ to $\overline{\mathfrak{gl}}_\infty \oplus \mathbb{C}\mathbf{c}$ where c_1 is mapped to the identity matrix and $c_2 = \mathbf{c}$ corresponds to the central extension $\mathbf{c}$. Indeed, one can modify relations on the fundamental generators of $\mathfrak{a}_\infty$, namely

$$[e_0, f_0]_c = \check{\alpha}_0 + \mathbf{c}, \tag{7.7}$$

leaving all other relations between Chevalley generators intact. This leads to a nontrivial central extension for $\overline{\mathfrak{gl}}_\infty$, although for any $\mathfrak{gl}(n)$ subalgebra it is trivial.

To describe these algebraic structures, it is convenient to use matrix notation. Let us denote by E_{ij} the matrix whose (i,j) entry is 1 and all others are 0. These matrices obey the following commutation relations:

$$[E_{ij}, E_{mn}] = \delta_{jm} E_{in} - \delta_{ni} E_{mj}.$$

One can then represent

$$\check{\alpha}_i = E_{ii} - E_{i+1,i+1}, \qquad e_i = E_{i,i+1}, \qquad f_i = E_{i,i-1}.$$

Let us define

$$a_i = \sum_{k \in \mathbb{Z}} E_{k,k+i}, \quad i \neq 0,$$

and

$$a_0 = \sum_{k>0} E_{k,k} - \sum_{k \leq 0} E_{k,k}.$$

Then we have the following Heisenberg subalgebra:

$$[a_n, a_m] = n\mathbf{c}\delta_{n,-m}.$$

However, we will be interested in a smaller subalgebra $\bar{\mathfrak{a}}'_\infty \subset \bar{\mathfrak{a}}_\infty$, so that for every $x \in \bar{a}'_\infty$ in the form of equation (7.5), there are only a finite number of coefficients $\lambda_\alpha \neq 0$ for negative α. The corresponding subalgebra $\overline{\mathfrak{gl}}'_\infty \subset \overline{\mathfrak{gl}}_\infty$ is formed by matrices (7.6) with only a finite number of elements below the main diagonal. The corresponding Lie group is denoted $\overline{GL}(\infty)$:

$$\overline{GL}(\infty) = \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid a_{ij} = 0; k \text{ for } i > j \text{ for all but a finite number}; a_{ii} \neq 0 \; \forall \; i \in \mathbb{Z} \right\}.$$

Given the upper Borel part $\mathfrak{b}_+$ of $\overline{\mathfrak{gl}}_\infty$, generated by $\check{\alpha}_i$, e_i, one can construct an upper Borel subgroup $\overline{B}_+$ by exponentiating elements of b_+, which we denote as B_+:

$$\overline{B}_+ = \left\{ \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid a_{ij} = 0 \text{ for } i > j, \; a_{ii} \neq 0 \; \forall \; i \in \mathbb{Z} \right\}. \tag{7.8}$$

Combining it with B_-, the Borel subgroup of $SL(\infty)$, we get

$$
\begin{aligned}
\overline{B}_- = \Big\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid a_{ij} = 0 \text{ for } i < j, \\
a_{ii} \neq 0 \text{ for all } i, \ a_{ii} \neq 1 \text{ for all but a finite number,} \\
a_{ij} \neq 0 \text{ for } i > j \text{ for all but a finite number, } \det(A) = 1 \Big\}.
\end{aligned}
\tag{7.9}
$$

Then one can write the Bruhat decomposition $\overline{GL}(\infty) = \sqcup_{\bar{w}} \overline{B}_- \bar{w} \overline{B}_+$, where $\bar{w}$ is a Weyl group element inherited from a Weyl group element of $SL(k+1)$ subgroup for some finite k.

Now we can construct the appropriate generalisation of the q-connection in equation (3.5):

$$
A(z) = \prod_{i=+\infty}^{-\infty} g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \quad g_i(z) \in \mathbb{C}(z)^\times, \quad \Lambda_i(z) \in \mathbb{C}[z],
\tag{7.10}
$$

which is a well-defined element of $\overline{B}_+(z) \subset \overline{GL}(\infty)(z)$. Indeed, while it is an infinite product, the multiplication is well defined, giving the element of $\overline{B}_+(z) \subset \overline{GL}(\infty)(z)$ with nonzero elements on the diagonal and superdiagonal only. In the next subsection we give a simpler expression for equation (7.10) in the fundamental representations of $\overline{GL}(\infty)$.

Remark 7.2. In principle, it could be possible to consider further completions of $\overline{GL}(\infty)$ and make full use of the central extension. However, since we are interested in Miura q-opers, we only need to complete one of the Borels to arrive at this formula.

7.3. Infinite wedge-space representations for $\overline{GL}(\infty)$

Here we will explain the construction of the fundamental representations of $\bar{\mathfrak{a}}'_\infty$ with central charge 1, which will serve as fundamental representations for $\overline{GL}(\infty)$ as well.

Let $V = \oplus_{j \in \mathbb{Z}} \mathbb{C} \nu_j$ be the infinite-dimensional space where ν_j are basis elements. There is a natural action of $\mathfrak{sl}(\infty)$ on V as infinite-dimensional matrices. Consider the following expression:

$$
\Psi_m = \nu_m \wedge \nu_{m-1} \wedge \nu_{m-2} \wedge \cdots.
\tag{7.11}
$$

We will call it the highest weight vector in the vector space F_m. The other basis vectors in F_m have the form

$$
\Psi = \nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots,
\tag{7.12}
$$

where $i_m > i_{m-1} > i_{m-2} > \cdots$ and $i_k = k$ for $k \ll 0$. The action of the $\mathfrak{sl}(\infty)$-algebra on F_m is defined in the following way. We identify e_i, f_i, $\check{\alpha}_i$ with the matrix generators $E_{i,i+1}$, $E_{i+1,i}$, $E_{ii} - E_{i+1,i+1}$, respectively. Then we define the action of any element X of $\mathfrak{sl}(\infty)$ on F_m by the following formula:

$$
X\Psi = X\nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots + \nu_{i_m} \wedge X\nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots + \nu_{i_m} \wedge \nu_{i_{m-1}} \wedge X\nu_{i_{m-2}} \wedge \cdots.
$$

Remark 7.3. A famous representation of $\bar{\mathfrak{a}}_\infty$ with central charge $\mathbf{c} = 1$ is achieved in the following way. One has to modify the action of $\check{\alpha}_0$ via a shift $\check{\alpha}_0 \longrightarrow \check{\alpha}_0 - 1$:

$$\check{\alpha}_0 \Psi =$$
$$\check{\alpha}_0 \nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots + \nu_{i_m} \wedge \check{\alpha}_0 \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots + \nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \check{\alpha}_0 \nu_{i_{m-2}} \wedge \cdots$$
$$- v_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \cdots .$$

Notice that $\bar{\mathfrak{n}}_+ \Psi_m = 0$, where $\bar{\mathfrak{n}}_+ = \left[\bar{\mathfrak{b}}_+, \bar{\mathfrak{b}}_+ \right]$ and $\check{\alpha}_k \Psi_m = \delta_{k,m} \Psi_m$. Thus $\{F_m\}$ can be interpreted as fundamental representations of $\bar{\mathfrak{a}}'_\infty$ and fundamental representations of $\overline{GL}(\infty)$ as well. The group action is given by the formula

$$g \cdot \Psi = g\nu_{i_m} \wedge g\nu_{i_{m-1}} \wedge g\nu_{i_{m-2}} \wedge \cdots . \tag{7.13}$$

Using the formalism of the Clifford algebra in equation (7.1), we have again formulas (7.2) for the generators $\check{\alpha}$, e_i, f_i, where now $i \in \mathbb{Z}$. This allows us to write the expression for the element of $\overline{B}_+(z)$ from equation (7.10) acting on F_m as

$$A(z) = \prod_{i=+\infty}^{1} e^{\Lambda_i(z)\psi_i \psi_{i+1}^*} \left[\frac{g_i}{g_{i-1}} \right]^{\psi_i \psi_i^*} \cdot \prod_{i=0}^{-\infty} e^{\Lambda_i(z)\psi_i \psi_{i+1}^*} \left[\frac{g_i}{g_{i-1}} \right]^{-\psi_i^* \psi_i} . \tag{7.14}$$

8. $\left(\overline{GL}(\infty), q \right)$-opers

In this section and the next we generalise the definitions and theorems from Sections 2–5 to the infinite-dimensional case, namely, the (Miura) $\left(\overline{GL}(\infty), q \right)$-opers, Z-twisted and Z-twisted Miura–Plücker versions, as well as nondegeneracy conditions. Then we relate them to QQ-systems and explicitly describe the trivialising operator for the related Z-twisted Miura q-oper. In the process we have to take into account the generally infinite number of zeros and poles in the local expression. The explicit formulas will also change slightly.

A particularly interesting part of the infinite-dimensional case is the infinite flag in the associated bundle version of the definition of the q-oper, which will not involve the 'starting' line subbundle. Thus in the study of Z-twisted Miura $\left(\overline{GL}(\infty), q \right)$-opers we have to rely on the QQ-system only, without addressing the q-Wronskian approach.

8.1. Definitions of $\left(\overline{GL}(\infty), q \right)$-opers and the canonical form of Miura $\left(\overline{GL}(\infty), q \right)$-opers

Given a principal $\overline{GL}(\infty)$-bundle $\mathcal{F}_{\overline{GL}(\infty)}$ over $\mathbb{P}^1$, let $\mathcal{F}_{\overline{GL}(\infty)}^q$ denote its pullback under the map $M_q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ sending $z \mapsto qz$. A meromorphic $\left(\overline{GL}(\infty), q \right)$-connection on a principal $\overline{GL}(\infty)$-bundle $\mathcal{F}_{\overline{GL}(\infty)}$ on $\mathbb{P}^1$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{F}_{\overline{GL}(\infty)}, \mathcal{F}_{\overline{GL}(\infty)}^q \right)$, where U is an open dense subset of $\mathbb{P}^1$ in the standard topology. Notice that now the number of zeros and poles which we have to exclude from $\mathbb{P}^1$ could be infinite. We assume that the only two accumulations points possible are 0 and ∞. We can always choose U so that the restriction $\mathcal{F}_{\overline{GL}(\infty)}\big|_U$ of $\mathcal{F}_{\overline{GL}(\infty)}$ to U is isomorphic to the trivial $\overline{GL}(\infty)$-bundle. The restriction of A to the Zariski-open dense subset $U \cap M_q^{-1}(U)$ can be written as a

section of the trivial $\overline{GL}(\infty)$-bundle on $U \cap M_q^{-1}(U)$, and hence as an element $A(z)$ of $\overline{GL}(\infty)(z)$.

Definition 8.1. A meromorphic $\left(\overline{GL}(\infty),q\right)$-*oper* on $\mathbb{P}^1$ is a triple $\left(\mathcal{F}_{\overline{GL}(\infty)},A,\mathcal{F}_{\overline{B}_-}\right)$, where A is a meromorphic $\left(\overline{GL}(\infty),q\right)$-connection on a $\overline{GL}(\infty)$-bundle $\mathcal{F}_{\overline{GL}(\infty)}$ on $\mathbb{P}^1$ and $\mathcal{F}_{\overline{B}_-}$ is the reduction of $\mathcal{F}_{\overline{GL}(\infty)}$ to $\overline{B}_-$ satisfying the condition that there exists an open dense subset $U \subset \mathbb{P}^1$ together with a trivialisation $\imath_{\overline{B}_-}$ of $\mathcal{F}_{\overline{B}_-}$, such that the restriction of the connection $A : \mathcal{F}_{\overline{GL}(\infty)} \longrightarrow \mathcal{F}_{\overline{GL}(\infty)}^q$ to $U \cap M_q^{-1}(U)$, written as an element of $\overline{GL}(\infty)(z)$ using the trivialisations of $\mathcal{F}_{\overline{GL}(\infty)}$ and $\mathcal{F}_{\overline{GL}(\infty)}^q$ on $U \cap M_q^{-1}(U)$ induced by $\imath_{B_-}$, takes values in the infinite product of Bruhat cells $\prod_{i=+\infty}^{-\infty} \overline{B}_- \left(\mathbb{C}\left[U \cap M_q^{-1}(U)\right]\right) s_i \overline{B}_- \left(\mathbb{C}\left[U \cap M_q^{-1}(U)\right]\right)$, where the ordering in the product follows the infinite version of the one in $SL(r+1)$.

Therefore, any q-oper connection A can be written in the form

$$A(z) = \prod_{i=+\infty}^{-\infty} \left[n_i'(z) \left(\phi_i(z)^{\check{\alpha}_i} s_i \right) n_i(z) \right], \tag{8.1}$$

where $\phi_i(z) \in \mathbb{C}(z)$ and $n_i(z),n_i'(z) \in \bar{N}_- = \left[\bar{B}_-,\bar{B}_-\right](z)$ are such that their zeros and poles are outside the subset $U \cap M_q^{-1}(U)$ of $\mathbb{P}^1$. As we stated before, we require that the only accumulation points of zeros and poles of $\phi_i(z)$, $n_i(z),n_i'(z)$ be 0 and ∞.

We can give an alternative definition of the $\left(\overline{GL}(\infty),q\right)$-oper connection using associated bundles as well:

Definition 8.2. A meromorphic $\left(\overline{GL}(\infty),q\right)$-*oper* on $\mathbb{P}^1$ is a triple $(A,E,\mathcal{L}_\bullet)$, where E is an ambient vector bundle with the fibre being an infinite-dimensional vector space with countable basis and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles

$$\cdots \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset \cdots \subset E \quad --$$

that is, with the fibres for $\mathcal{L}_i$ being semi-infinite spaces so that $A \in \mathrm{Hom}_{\mathcal{O}_U}(E,E^q)$ satisfies the following conditions:

i) $\mathcal{A} \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$.

ii) There exists an open dense subset $U \subset \mathbb{P}^1$ such that the restriction of $\mathcal{A} \in \mathrm{Hom}(\mathcal{L}_\bullet,\mathcal{L}_\bullet^q)$ to $U \cap M_q^{-1}(U)$ belongs to $\overline{GL}(\infty)(z)$ and satisfies the condition that the induced maps $\bar{\mathcal{A}}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i-1}/\mathcal{L}_i$ are isomorphisms on $U \cap M_q^{-1}(U)$.

The equivalence between the two definitions can be established as in the finite-dimensional case, using the associated bundle for the defining representation and its faithfulness. We also will use the notation A for the associated version of the q-connection $\mathcal{A}$; it will be clear from context which one is meant.

Let us give two equivalent definitions of the Miura $\left(\overline{GL}(\infty),q\right)$-oper, as in the finite-dimensional case:

Definition 8.3.

i) A *Miura* $\left(\overline{GL}(\infty),q\right)$-*oper* on $\mathbb{P}^1$ is a quadruple $\left(\mathcal{F}_{\overline{GL}(\infty)},A,\mathcal{F}_{\overline{B}_-},\right.$ $\left.\mathcal{F}_{\overline{B}_+}\right)$, where $\left(\mathcal{F}_{\overline{GL}(\infty)},A,\mathcal{F}_{\overline{B}_-}\right)$ is a meromorphic $\left(\overline{GL}(\infty),q\right)$-oper on $\mathbb{P}^1$ and $\mathcal{F}_{\overline{B}_+}$ is a reduction of the $\overline{GL}(\infty)$-bundle $\mathcal{F}_{\overline{GL}(\infty)}$ to $\overline{B}_+$ that is preserved by the q-connection A.

ii) A *Miura* $\left(\overline{GL}(\infty),q\right)$-*oper* on $\mathbb{P}^1$ is a quadruple $\left(E,A,\mathcal{L}_\bullet,\hat{\mathcal{L}}_\bullet\right)$, where $(E,A,\mathcal{L}_\bullet)$ is a meromorphic $\overline{GL}(\infty)$-oper on $\mathbb{P}^1$ and $\hat{\mathcal{L}}_\bullet = \{\mathcal{L}_i\}$ is another full flag of subbundles in E that is preserved by the q-connection A.

As in $SL(r+1)$ case, we can define the relative position (see Section 2.3) between $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{B}_-}$ because of the Bruhat decomposition of G. We will say that $\mathcal{F}_{\overline{B}_-}$ and $\mathcal{F}_{\overline{B}_+}$ have a *generic relative position* at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $\overline{B}_- \cdot \overline{B}_+ \subset \overline{GL}(\infty)$).

We immediately have the following result, which is a generalisation of the finite-dimensional case:

Theorem 8.4. *For any Miura $\left(\overline{GL}(\infty),q\right)$-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{\overline{B}_-}$ and $\mathcal{F}_{\overline{B}_+}$ are in generic relative position for all $x \in V$.*

Proof. Notice that according to the local expression for the q-oper connection in equation (2.2) and the condition that it belong to $\overline{GL}(\infty)(z)$, there is a finite number of elements below the diagonal. This means that for some k,l we have

$$A(z) = \left[\prod_{i=+\infty}^{k} g_i^{\check{\alpha}_i}(z)e^{\frac{\phi_i(z)e_i}{g_i(z)}}\right] n'(z) \prod_{j=k-1}^{l+1} \left(\phi_j(z)^{\check{\alpha}_j}s_j\right) n(z) \left[\prod_{i=l}^{-\infty} g_i^{\check{\alpha}_i}(z)e^{\frac{\phi_i(z)e_i}{g_i(z)}}\right], \quad (8.2)$$

where $n(z),n'(z) \in \overline{N}_-(z)$ belong to the $SL(k+l)$ subgroup with H generated by $\{\check{\alpha}_j\}_{j=k-1}^{j=l+1}$. The expression in the middle, $A'(z) = n'(z)\prod_j \left(\phi_j(z)^{\check{\alpha}_j}s_j\right)n(z) \in SL(k+l)(z)$, is the local expression for the Miura $(SL(k+l),q)$-oper for which the generic property follows from the finite-dimensional case (see Theorem 2.5), and thus we have generic relative position for the Miura $\left(\overline{GL}(\infty),q\right)$-oper.

That leads to the following corollary:

Corollary 8.5. *For any Miura $\left(\overline{GL}(\infty),q\right)$-oper on $\mathbb{P}^1$, there exists a trivialisation of the underlying $\overline{GL}(\infty)$-bundle $\mathcal{F}_{\overline{GL}(\infty)}$ on an open dense subset of $\mathbb{P}^1$ for which the oper q-connection has the form*

$$\prod_{i=+\infty}^{-\infty} g_i^{\check{\alpha}_i}e^{\frac{\lambda_i t_i}{g_i}e_i}, \quad g_i \in \mathbb{C}(z)^\times, \quad (8.3)$$

where each $t_i \in \mathbb{C}(z)$ is determined by the lifting s_i and the order in the product is canonical.

As in the finite-dimensional case, we fix $t_i \equiv 1$ from now on.

8.2. *Z*-twisted Miura *q*-opers

Now we are ready to define the notion of the $\left(\overline{GL}(\infty),q\right)$-oper and the Miura $\left(\overline{GL}(\infty),q\right)$-oper, which are straightforward definitions of their $SL(r+1)$ counterparts. As in the $SL(r+1)$ case, let Z be the regular element of the maximal torus $\overline{H} = \overline{B}_+/\left[\overline{B}_+,\overline{B}_+\right]$. One can express it as follows:

$$Z = \prod_{i=+\infty}^{-\infty} \zeta_i^{\check{\alpha}_i}, \quad \zeta_i \in \mathbb{C}^\times. \tag{8.4}$$

Definition 8.6. A *Z-twisted* $\left(\overline{GL}(\infty),q\right)$-oper on $\mathbb{P}^1$ is a $\left(\overline{GL}(\infty),q\right)$-oper that is equivalent to the constant element $Z \in \overline{H} \subset \overline{H}(z)$ under the *q*-gauge action of $\overline{GL}(\infty)(z)$– that is, if $A(z)$ is the meromorphic oper *q*-connection (with respect to a particular trivialisation of the underlying bundle), there exists $g(z) \in \overline{GL}(\infty)(z)$ such that

$$A(z) = g(qz)Zg(z)^{-1}. \tag{8.5}$$

A *Z-twisted Miura* $\left(\overline{GL}(\infty),q\right)$-oper is a Miura $\left(\overline{GL}(\infty),q\right)$-oper on $\mathbb{P}^1$ that is equivalent to the constant element $Z \in \overline{H} \subset \overline{H}(z)$ under the *q*-gauge action of $\overline{B}_+(z)$–that is,

$$A(z) = v(qz)Zv(z)^{-1}, \quad v(z) \in \overline{B}_+(z). \tag{8.6}$$

Naturally, we have a proposition addressing the characterisation of *Z*-twisted Miura *q*-opers associated to *Z*-twisted q-opers:

Proposition 8.7. *Let* $Z \in \overline{H}$ *be regular. For any Z-twisted* $\left(\overline{GL}(\infty),q\right)$-oper $\left(\mathcal{F}_{\overline{GL}(\infty)},A,\mathcal{F}_{\overline{B}_-}\right)$ *and any choice of* $\overline{B}_+$-reduction $\mathcal{F}_{\overline{B}_+}$ *of* $\mathcal{F}_{\overline{GL}(\infty)}$ *preserved by the oper q-connection A, the resulting Miura* $\left(\overline{GL}(\infty),q\right)$-oper *is* Z'-twisted *for a particular* $Z' \in S_\infty \cdot Z$. *The set of A-invariant* $\overline{B}_+$-reductions $\mathcal{F}_{\overline{B}_+}$ *on the* $\left(\overline{GL}(\infty),q\right)$-oper *is in one-to-one correspondence with the elements of* $W = S_\infty$.

Given a Miura $\left(\overline{GL}(\infty),q\right)$-oper. By Theorem 3.4, the underlying (G,q)-connection can be written in the form of equation (3.5). As in the $SL(r+1)$ case, we obtain an $\bar{H}$-bundle $\mathcal{F}_{\overline{B}_+}/\bar{N}_+$, where $\bar{N}_+ = \left[\overline{B}_+,\overline{B}_+\right]$. The corresponding $(\bar{H},q)$-connection $A^{\bar{H}}(z)$ according to formula (8.3) is given by

$$A^H(z) = \prod_i g_i(z)^{\check{\alpha}_i}. \tag{8.7}$$

We call $A^{\bar{H}}(z)$ the *associated Cartan q-connection* of the Miura *q*-oper $A(z)$.

The same can be done in the infinite-dimensional case. If our Miura *q*-oper is *Z*-twisted (see Definition 8.6), then we also have $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_+(z)$. Notice that $v(z)$ can be written as

$$v(z) = \prod_i y_i(z)^{\check{\alpha}_i}n(z), \quad n(z) \in \bar{N}_+(z), \quad y_i(z) \in \mathbb{C}(z)^\times. \tag{8.8}$$

We refer to the associated Cartan q-connection $A^{\bar{H}}(z)$ as Z-twisted, so that the explicit realisation is given by the formula

$$A^H(z) = \prod_i \left[\zeta_i \frac{y_i(qz)}{y_i(z)} \right]^{\check{\alpha}_i}, \tag{8.9}$$

and we note that $A^H(z)$ determines the $y_i(z)$s uniquely up to a scalar.

9. Z-twisted Miura $\left(\overline{GL}(\infty), q\right)$-opers and QQ-systems

9.1. Definition and explicit realisation.

Let $\{\Lambda_i(z)\}_{i \in \mathbb{Z}}$ be a collection of nonconstant polynomials with accumulation points of roots at 0 or ∞ only.

Definition 9.1.

i) A $\left(\overline{GL}(\infty), q\right)$-*oper with regular singularities determined by* $\{\Lambda_i(z)\}_{i \in \mathbb{Z}}$ is a q-oper on $\mathbb{P}^1$ whose q-connection may be written in the form

$$A(z) = \left[\prod_{i=+\infty}^{k} g_i^{\check{\alpha}_i}(z) e^{\frac{\Lambda_i(z) t_i e_i}{g_i(z)}} \right] n'(z) \prod_{j=k-1}^{l+1} \left(\phi_j(z)^{\check{\alpha}_j} s_j \right) n(z) \left[\prod_{i=l}^{-\infty} g_i^{\check{\alpha}_i}(z) e^{\frac{\Lambda_i(z) t_i e_i}{g_i(z)}} \right], \tag{9.1}$$

for some $k, l \in \mathbb{Z}$, where $n(z), n'(z) \in N_-(z)$ and belong to the $SL(k+l)$ subgroup with H generated by $\{\check{\alpha}_j\}_{j=k-1}^{j=l+1}$.

ii) A *Miura* $\left(\overline{GL}(\infty), q\right)$-*oper with regular singularities determined by polynomials* $\{\Lambda_i(z)\}_{i=1,\ldots,r}$ is a Miura $\left(\overline{GL}(\infty), q\right)$-oper such that the underlying q-oper has regular singularities determined by $\{\Lambda_i(z)\}_{i=1,\ldots,r}$.

As in the $SL(r+1)$ case, from now on we set $t_i(z) = 1$, $i \in \mathbb{Z}$. Then we have an analogue of Theorem 3.4:

Corollary 9.2. *For every Miura* $\left(\overline{GL}(\infty), q\right)$-*oper with regular singularities determined by the polynomials* $\{\Lambda_i(z)\}_{i \in \mathbb{Z}}$, *the underlying q-connection can be written in the form*

$$A(z) = \prod_{i=+\infty}^{-\infty} g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \quad g_i(z) \in \mathbb{C}(z)^\times. \tag{9.2}$$

9.2. Fermionic realisation

Let F_i be the irreducible representation of $\overline{GL}(\infty)$ with the highest weight ω_i which we discussed in Section 7. Notice that the 1- and 2-dimensional subspaces L_i and W_i of F_i spanned by the weight vectors Ψ_i and $f_i \cdot \Psi_i$ are $\overline{B}_+$-invariant subspaces of V_i.

Now let $\left(\mathcal{F}_{\overline{GL}(\infty)}, A, \mathcal{F}_{\overline{B}_-}, \mathcal{F}_{\overline{B}_+} \right)$ be a Miura $\left(\overline{GL}(\infty), q\right)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i \in \mathbb{Z}}$ (see Definition 3.3). Recall that $\mathcal{F}_{\overline{B}_+}$ is a $\overline{B}_+$-reduction of an $i \in \mathbb{Z}$-bundle $\mathcal{F}_{(\overline{GL}(\infty)}$ on $\mathbb{P}^1$, preserved by the $\left(\overline{GL}(\infty), q\right)$-connection A.

Therefore for each $i \in \mathbb{Z}$, the vector bundle

$$\mathscr{F}_i = \mathcal{F}_{\overline{B}_+} \underset{\overline{B}_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} F_i.$$

Thus we have the following proposition:

Proposition 9.3. *For every Miura $\left(\overline{GL}(\infty), q\right)$-oper with regular singularities determined by the polynomials $\{\Lambda_j(z)\}_{j \in \mathbb{Z}}$, the underlying q-connection $\phi_i(A)$ in the associated bundle $\mathscr{F}_i$ for any $i \in \mathbb{Z}$ can be written in the form*

$$\phi_i(A)(z) = \prod_{j=+\infty}^{1} e^{\Lambda_j(z)\psi_j\psi_{j+1}^*} \left[\frac{g_j(z)}{g_{j-1}(z)} \right]^{\psi_j\psi_j^*} \cdot \prod_{j=0}^{-\infty} e^{\Lambda_j(z)\psi_j\psi_{j+1}^*} \left[\frac{g_j(z)}{g_{j-1}(z)} \right]^{-\psi_j^*\psi_j}. \quad (9.3)$$

We will discuss the Miura–Plücker condition. As we noted in the beginning of Section 8, the key difference between the finite- and infinite-dimensional cases is that we do not have a luxury of having a line bundle such that the q-Wronskian matrix of the corresponding section will produce minors describing the complete QQ-system. We must rely exclusively on the properties of the infinite QQ-system, which will allow us to construct the trivialising group element $v(z)$ so that $A(z) = v(qz)Zv(z)^{-1}$ for any Z-twisted Miura–Plücker $\left(\overline{GL}(\infty), q\right)$-oper (see Section 9.5).

9.3. Miura–Plücker $\left(\overline{GL}(\infty), q\right)$-opers

For all $i \in \mathbb{Z}$, the infinite-rank bundle $\mathscr{F}_i$ contains a rank 2 subbundle

$$\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$$

associated to $W_i \subset F_i$, and $\mathcal{W}_i$ in turn contains a line subbundle

$$\hat{\mathscr{L}}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$$

associated to $L_i \subset W_i$.

Note that $\phi_i(A)$ preserves subbundles $\mathcal{L}_i$ and $\mathcal{W}_i$ of $\mathscr{F}_i$ and thus produces a $(GL(2), q)$-oper on $\mathcal{W}_i$. We denote such a q-oper by A_i as in Section 4.1.

Notice that $\mathcal{W}_i$ decomposes into a direct sum of two subbundles–$\hat{\mathscr{L}}_i$, preserved by B_+, and $\mathscr{L}_i$–with respect to which it satisfies the $(GL(2), q)$-oper condition. We can unify all that in the following proposition:

Proposition 9.4. *The quadruple $\left(A_i, \mathcal{W}_i, \mathscr{L}_i, \hat{\mathscr{L}}_i\right)$ for any $i \in \mathbb{Z}$ forms a Miura $(GL(2), q)$-oper, so that explicitly*

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z)g_{i-1}(z) \\ 0 & g_i^{-1}(z)g_{i+1}(z)g_{i-1}(z) \end{pmatrix}, \quad i \neq 0, \quad A_0(z) = \begin{pmatrix} 1 & \Lambda_0(z)g_{-1}(z) \\ 0 & g_0^{-2}(z)g_1(z)g_{-1}(z) \end{pmatrix},$$

$$(9.4)$$

where we use the ordering of the simple roots determined by the Coxeter element c.

We can see that the expression for $A_0(z)$ looks slightly different from the rest of $A_i(z)$ in equation (9.4). However, if we multiply $A_0(z)$ by the diagonal matrix proportional to the identity $\mathrm{diag}(g_0(z), g_0(z))$, then it will be of the same form as the rest of the matrices. This is due to the central extension in the $\bar{\mathfrak{a}}_\infty$-algebra and a shift of the Chevalley generator $\check{\alpha}_0$.

Now we impose the Z-twisted condition on the corresponding A^H-connection, namely $g_i = \zeta_i \frac{y_i(qz)}{y_i(z)}$.

Let $G_i \cong SL(2)$ be the subgroup of $\overline{GL}(\infty)$ corresponding to the $\mathfrak{sl}(2)$-triple spanned by $\{e_i, f_i, \check{\alpha}_i\}$, which preserves W_i, using a diagonal gauge transformation as in Section 4.1. We associate to the connection A_i a (G_i, q)-oper with the explicit form

$$\mathcal{A}_i(z) = g_i^{\check{\alpha}_i}(z) e^{\frac{\beta_i(z)}{g_i(z)} e_i}, \quad \text{where } \beta_i(z) = \Lambda_i(z) \zeta_{i-1} y_{i+1}(z) y_{i-1}(qz). \tag{9.5}$$

Note that the diagonal transformation for $\mathcal{A}_0(z)$ looks a bit different than for other $\mathcal{A}_i(z)$ because of the aforementioned shift.

Now we are ready to define Miura–Plücker $\big(\overline{GL}(\infty), q\big)$-opers.

Definition 9.5. A *Z-twisted Miura–Plücker $\big(\overline{GL}(\infty), q\big)$-oper* is a meromorphic Miura $\big(\overline{GL}(\infty), q\big)$-oper on $\mathbb{P}^1$ with the underlying q-connection $A(z)$ such that there exists $v(z) \in \overline{B}_+(z)$ such that for all $i \in \mathbb{Z}$, the Miura $(GL(2), q)$-opers $A_i(z)$ associated to $A(z)$ by equation (9.4) can be written in the form

$$A_i(z) = v(zq) Z v(z)^{-1}|_{W_i} = v_i(zq) Z_i v_i(z)^{-1}, \tag{9.6}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$.

9.4. Nondegeneracy conditions

In this subsection we will generalise two nondegeneracy conditions we had in Section 4.2 for Z-twisted Miura–Plücker $\big(\overline{GL}(\infty), q\big)$-opers.

The first nondegeneracy condition deals with the associated H-connection.

Definition 9.6. A Miura $\big(\overline{GL}(\infty), q\big)$-oper $A(z)$ of the form (3.5) is called *H-nondegenerate* if the corresponding $(\bar{H}, q)$-connection $A^{\bar{H}}(z)$ can be written in the form (8.9), where zeroes and poles $y_i(z)$ and $y_{i\pm 1}(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$.

The second nondegeneracy condition addresses the associated (G_i, q)-opers.

Definition 9.7. A Z-twisted Miura-Plücker $\big(\overline{GL}(\infty), q\big)$-oper $A(z)$ is called *nondegenerate* if its associated Cartan q-connection $A^H(z)$ is nondegenerate and each associated Z_i-twisted Miura $(SL(2), q)$-oper $\mathcal{A}_i(z)$ is nondegenerate.

We arrive at the analogue of Proposition 4.7, which is proven in exactly the same way:

Proposition 9.8. *Let $A(z)$ be a Z-twisted Miura–Plücker (G, q)-oper. The following statements are equivalent:*

(1) $A(z)$ is nondegenerate.

(2) The Cartan q-connection $A^H(z)$ is nondegenerate, and each $A_i(z)$ has regular singularities–that is, $\rho_i(z)$ given by equation (4.3) is in $\mathbb{C}[z]$.

(3) Each $y_i(z)$ from equation (3.8) is a polynomial, and for all $i \in \mathbb{Z}$ the zeros of $y_i(z)$ and $y_{i\pm1}(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$.

9.5. Z-twisted Miura–Plücker $\left(\overline{GL}(\infty),q\right)$-oper is Z-twisted

From the previous section we see that the q-connection of the nondegenerate Miura–Plücker $\left(\overline{GL}(\infty),q\right)$-oper with regular singularities defined by polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ reads as follows:

$$A(z) = \prod_{i=+\infty}^{-\infty} g_i(z)^{\tilde{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)}e_i}, \quad g_i(z) = \zeta_i \frac{Q_i^+(qz)}{Q_i^+(z)}. \tag{9.7}$$

Let us assume as in the $SL(r+1)$ case (see equation (5.2)) that ξ_i is q-distinct from ξ_{i+1}. In particular, this means Z is regular semisimple.

First we define the *QQ-system* for $\overline{GL}(\infty)$ as an infinite generalisation of equation (5.3):

$$\xi_{i+1}Q_i^+(qz)Q_i^-(z) - \xi_i Q_i^+(z)Q_i^-(qz) = \Lambda_i(z)Q_{i-1}^+(qz)Q_{i+1}^+(z), \quad i \in \mathbb{Z}. \tag{9.8}$$

We say that a polynomial solution $\left\{Q_i^+(z), Q_i^-(z)\right\}$ of equation (9.8) is nondegenerate if for $i \neq j$, the zeros of $Q_i^+(z)$ and $Q_j^-(z)$ are q-distinct from each other and from the zeros of $\Lambda_k(z)$ for $|i-k|=1$, $|j-k|=1$.

The following theorem is a direct analogue of Theorem (5.1):

Theorem 9.9. *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura–Plücker $\left(\overline{GL}(\infty),q\right)$-opers and the set of nondegenerate polynomial solutions of the $\overline{GL}(\infty)QQ$-system (9.8).*

This theorem can proved the same way as in [24] with the help of Proposition 9.4. The following theorem serves as an infinite-dimensional generalisation of Theorem 5.2:

Theorem 9.10. *Let $A(z)$ be as in equation (9.7) and $Z = \prod_i \zeta_i^{\tilde{\alpha}_i}$. Then the q-gauge transformation $v(z)$ which diagonalises the q-connection*

$$A(z) = v(qz)Zv(z)^{-1}$$

reads

$$v(z) = \prod_{i=-\infty}^{+\infty} Q_i^+(z)^{\tilde{\alpha}_i} \cdot \prod_{i=-\infty}^{+\infty} V_i(z), \tag{9.9}$$

where

$$V_i(z) = \exp\left(-\sum_{j>i} \phi_{i,\dots,j}(z)\,e_{i,\dots,j}\right), \tag{9.10}$$

in which $e_{i,\ldots,j} = [\ldots[[e_i, e_{i+1}], e_{i+2}] \ldots e_j]$ *and functions* $\phi_{i,\ldots,j}(z)$ *satisfy the following relations:*

$$
\begin{aligned}
\xi_{i+1}\phi_i(z) - \xi_i\phi_i(qz) &= \rho_i(z), \\
\xi_{i+2}\phi_{i,i+1}(z) - \xi_i\phi_{i,i+1}(qz) &= \rho_{i+1}(z)\phi_i(z), \\
&\vdots \\
\xi_{i+j+1}\phi_{i,\ldots,i+j}(z) - \xi_i\phi_{i,\ldots,i+j}(qz) &= \rho_{i+j}(z)\phi_{i,\ldots,i+j-1}(z), \\
&\vdots
\end{aligned}
\tag{9.11}
$$

where $i \in \mathbb{Z}, j \in \mathbb{Z}_+$ *and we use the same notation as in Section* 5.

The set of equations (9.11) is called the *extended QQ-system* for $\overline{GL}(\infty)$, which can also be presented as

$$
\begin{aligned}
\xi_{i+1}Q_i^+(qz)Q_i^-(z) - \xi_i Q_i^+(z)Q_i^-(qz) &= \Lambda_i(z)Q_{i-1}^+(qz)Q_{i+1}^+(z), \\
\xi_{i+2}Q_{i+1}^+(qz)Q_{i,i+1}^-(z) - \xi_i Q_{i+1}^+(z)Q_{i,i+1}^-(qz) &= \Lambda_{i+1}(z)Q_i^-(qz)Q_{i+1}^+(z), \\
&\vdots
\end{aligned}
\tag{9.12}
$$

$$
\xi_{i+j+1}Q_{i+j}^+(qz)Q_{i,\ldots,i+j}^-(z) - \xi_i Q_{i+j}^+(z)Q_{i,\ldots,i+j}^-(qz) = \Lambda_{i+j}(z)Q_{i,\ldots,i+j-1}^-(qz)Q_{i+j+1}^+(z),
$$
$$
\vdots
$$

Proof. Let us first rewrite the diagonalisation condition as

$$
v(qz)^{-1}A(z) = Zv(z)^{-1},
\tag{9.14}
$$

as it will be easier to compute the left- and right-hand sides of and compare them. We can make a statement similar to Lemma 5.4 and write the $\left(\overline{GL}(\infty), q\right)$-oper as

$$
A(z) = \prod_{i=+\infty}^{-\infty} Q_i^+(qz)^{\check{\alpha}_i} \cdot \prod_{i=+\infty}^{-\infty} e^{\frac{\zeta_i}{\zeta_{i+1}}\rho_i(z)e_i} \cdot \prod_{i=+\infty}^{-\infty} \zeta_i^{\check{\alpha}} Q_i^+(z)^{-\check{\alpha}_i}.
\tag{9.15}
$$

Then the left-hand side of equation (9.14) reads

$$
v(qz)^{-1}A(z) = \prod_{i=+\infty}^{-\infty} \exp\left(\sum_{j>i} \phi_{i,\ldots,j}(qz)e_{i,\ldots,j}\right) \cdot \prod_{i=+\infty}^{-\infty} e^{\frac{\zeta_i}{\zeta_{i+1}}\rho_i(z)e_i} \cdot \prod_{i=+\infty}^{-\infty} \zeta_i^{\check{\alpha}_i} Q_i^+(z)^{-\check{\alpha}_i}.
\tag{9.16}
$$

We now need to move the ith element from the middle product to the left until it combines with the corresponding $e^{-\phi_i(qz)e_i}$ term. In this way, $e^{\frac{\zeta_i}{\zeta_{i+1}}\rho_i(z)e_i}$ will need to be carried over to $V_{i+1}(qz)^{-1}$ inside the first product. Each term $e^{\frac{\zeta_i}{\zeta_{i+1}}\rho_i(z)e_i}$ will have nontrivial commutators with exponentials containing e_{i+1}–the others will vanish due to

Serre relations. Thus equation (9.16) reads

$$
v(qz)^{-1}A(z) = \cdots \exp\left(\frac{\zeta_1}{\zeta_2}\rho_1(z) + \phi_1(qz)\right)e_1 \cdots \exp\left(\frac{\zeta_i}{\zeta_{i+1}}\rho_r(z)\phi_{1,\ldots,i-1}(qz) + \phi_{1,\ldots,i}(qz)\right)e_{1,\ldots,i}
$$
$$
\cdots \exp\left(\frac{\zeta_2}{\zeta_3}\rho_2(z) + \phi_2(qz)\right)e_2 \cdots \exp\left(\frac{\zeta_j}{\zeta_{j+1}}\rho_j(z)\phi_{2,\ldots,j-1}(qz) + \phi_{2,\ldots,j}(qz)\right)e_{2,\ldots,j}
$$
(9.17)
$$
\cdots \prod_{i=+\infty}^{-\infty}\zeta_i^{\check{\alpha}_i}Q_i^+(z)^{-\check{\alpha}_i}.
$$

This expression needs to be compared against the right-hand side of equation (9.14), which is given by

$$
Zv(z)^{-1} = \prod_i \zeta^{\check{\alpha}_i} \cdot \prod_{i=+\infty}^{-\infty}\exp\left(\sum_{j>i}\phi_{i,\ldots,j}(qz)e_{i,\ldots,j}\right) \cdot \prod_{i=+\infty}^{-\infty}Q_i^+(z)^{-\check{\alpha}_i}. \tag{9.18}
$$

To make the comparison manifest, one needs to move the Cartan terms from the end to the front using equation (5.13):

$$
Zv(z)^{-1} = \cdots \exp\left(\frac{\zeta_1^2}{\zeta_0\zeta_2}\phi_1(z)e_1\right) \cdots \exp\left(\frac{\zeta_1\zeta_j}{\zeta_0}\phi_{1,\ldots,j}(z)e_{1,\ldots,j}\right)
$$
$$
\cdots \exp\left(\frac{\zeta_2^2}{\zeta_1\zeta_3}\phi_2(z)e_2\right) \cdots \exp\left(\frac{\zeta_2\zeta_l}{\zeta_1}\phi_{2,\ldots,l}(z)e_{2,\ldots,l}\right)
$$
(9.19)
$$
\cdots \prod_{i=+\infty}^{-\infty}\zeta_i^{\check{\alpha}_i}Q_i^+(z)^{-\check{\alpha}_i}.
$$

Here we used the following fact about nested commutators in Chevalley basis:

$$
[\check{\alpha}_a, e_{i,\ldots,j}] = \begin{cases} e_{i,\ldots,j}, & a = i \text{ or } a = j, \\ -e_{i,\ldots,j}, & a = i-1 \text{ or } a = j+1, \\ 0, & \text{otherwise.} \end{cases} \tag{9.20}
$$

Comparing equations (9.17) and (9.19) leads to equation (9.11).

Corollary 9.11. *Theorem 5.2 follows.*

Proof. In the proof of Theorem 9.10, one needs to replace all infinite products with products ranging between 1 and r and put $\zeta_0 = \zeta_{r+1} = 1$.

Analogously to Theorem 5.6, we can make the following statement:

Theorem 9.12.

(1) *Every nondegenerate solution of the QQ-system for $GL(\infty)$ (9.8) is also a nondegenerate solution of the extended QQ-system for $\overline{GL}(\infty)$ (9.12).*

(2) *There is a one-to-one correspondence between the set of nondegenerate solutions of the the QQ-system for $GL(\infty)$ and the set of solutions of Bethe ansatz equations*

for $GL(\infty)$:

$$\frac{Q_i^+\left(qw_k^i\right)}{Q_i^+\left(q^{-1}w_i^k\right)}\frac{\xi_i}{\xi_{i+1}} = -\frac{\Lambda_i\left(w_k^i\right)Q_{i+1}^+\left(qw_k^i\right)Q_{i-1}^+\left(w_k^i\right)}{\Lambda_i\left(q^{-1}w_k^i\right)Q_{i+1}^+\left(w_k^i\right)Q_{i-1}^+\left(q^{-1}w_k^i\right)}, \qquad (9.21)$$

where $i \in \mathbb{Z}$ *and* $k = 1,\ldots,m_i$.

10. Toroidal q-opers

10.1. Quantum toroidal algebras, Bethe equations and the QQ-system of $\widehat{\widehat{A}}_0$ type

The quantum toroidal algebra $U_{t_1,t_2}\left(\widehat{\widehat{\mathfrak{gl}}}(1)\right)$ has attracted a lot of attention in recent years.[4] On one hand it has an explicit geometric realisation: there is a natural action of this algebra on equivariant K-theory of ADHM instanton spaces [49] which corresponds to the simplest framed quiver varieties with one loop and one vertex.

One can describe such moduli spaces $\mathcal{M}_N$ of rank N torsion-free sheaves F on $\mathbb{P}^2$ with framing at infinity (also known as the moduli space of $U(N)$ instantons on $\mathbb{R}^4$). The framing condition forces the first Chern class to vanish; however, the second Chern class can range over the nonpositive integers $c_2(F) = k$. The moduli space can be represented as a disjoint sum $\mathcal{M}_N = \sqcup_k \mathcal{M}_{k,N}$. Each $\mathcal{M}_{k,N}$ can be described as the moduli spaces of stable representations of the following ADHM quiver, where $\mathscr{W}$ is a trivial bundle of rank N and $\mathscr{V}$ is a bundle of rank k. For $N = 1$ this quiver variety describes a Hilbert scheme of k points on $\mathbb{C}^2$. For details and the equivalence of various descriptions of ADHM moduli spaces, we refer the reader to [47].

Let us denote $G = A \times (\mathbb{C}^\times)^2$, where A is the *framing* torus–that is, the maximal torus of $GL(N)$–and $(\mathbb{C}^\times)^2$ is the torus acting on $\mathbb{C}^2 \subset \mathbb{P}^2$. We denote equivariant parameters corresponding to A and G/A as $a_1,\ldots,a_N$ and t_1,t_2, correspondingly.

The G-equivariant K-theory of $\mathcal{M}_{k,N}$ is generated by the equivariant vector bundle $\mathscr{V}$ of rank k over $\mathcal{M}_{k,N}$ as in the case of the cotangent bundles to Grassmannians discussed in the introduction. The space of the localised $K_G(\mathcal{M}_{k,N})$ is a module spanned by the fixed points of $U_{t_1,t_2}\left(\widehat{\widehat{\mathfrak{gl}}}(1)\right)$.

---

[4]Sometimes in the literature a different notation is used, $U_{q_1,q_2,q_3}\left(\widehat{\widehat{\mathfrak{gl}}}(1)\right)$, where $q_1 = (t_1 t_2)^{-1}$, $q_2 = t_2$, $q_1 q_2 q_3 = 1$.

This module has the structure of the analogue of an XXZ-module for the toroidal algebra. Namely, the physical space

$$\mathcal{H} = \mathscr{F}(a_1) \otimes \cdots \otimes \mathscr{F}(a_N)$$

is the product of Fock-space representations of the toroidal algebra $\{\mathscr{F}(a_i)\}$, where the parameters $\{a_i\}$, which have the meaning of evaluation parameters, correspond to the zero mode value of the infinite-dimensional Heisenberg algebra. We refer to [52] for more details.

As we described in the introduction, the quantum-equivariant K-theory based on quasimaps is described by difference equations which coincide with quantum Knizhnik–Zamolodchikov equations and the related dynamical equations. The solutions to these difference equations can be computed as certain Euler characteristics on the moduli spaces of quasimaps. They are given by a certain integral formula with the asymptotics given by the Yang–Yang function Y, which can be described as follows.

Let $\ell(x)$ be a multivalued function, which can be written in dilogarithm terms [27], such that

$$\exp 2\pi \frac{\partial \ell(x)}{\partial x} = 2 \sinh \pi x.$$

The Yang–Yang function for the ADHM space $\mathcal{M}_{k,N}$ is given by [2]

$$
\begin{aligned}
Y_{ADHM}(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \epsilon_1, \epsilon_2) = &\sum_{a=1}^{k} \sum_{m=1}^{N} \ell(\sigma_a - \alpha_m) + \ell(-\sigma_a + \alpha_m - \epsilon_1 - \epsilon_2) \\
&+ \sum_{a \neq b}^{k} \ell(\sigma_a - \sigma_b + \epsilon_1) + \ell(\sigma_a - \sigma_b + \epsilon_2) + \ell(\sigma_a - \sigma_b - \epsilon_1 - \epsilon_2) \\
&- \kappa \sum_{a=1}^{k} \sigma_a,
\end{aligned}
\tag{10.1}
$$

where

$$s_b = e^{2\pi \sigma_b}, \qquad a_b = e^{2\pi \alpha_b}, \qquad t_1 = e^{2\pi \epsilon_1}, \qquad t_2 = e^{2\pi \epsilon_2}, \qquad \kappa = e^{2\pi \kappa}, \qquad \kappa = (t_1 t_2)^{-\frac{N}{2}} \mathfrak{z},$$

so that $\mathfrak{z}$ is a Kähler parameter of $\mathcal{M}_{k,N}$.

Then the Bethe equations in the case of $\mathcal{M}_{k,N}$ can be computed as critical points for Y_{ADHM}:

Lemma 10.1. *The equations*

$$\exp 2\pi \frac{\partial Y_{ADHM}}{\partial \sigma_a} = 1, \quad a = 1, \ldots, k, \tag{10.2}$$

are equivalent to the following Bethe equations:

$$\prod_{l=1}^{N} \frac{s_a - a_l}{t_1 t_2 s_a - a_l} \cdot \prod_{\substack{b=1 \\ b \neq a}}^{k} \frac{s_a - t_1 s_b}{t_1 s_a - s_b} \frac{s_a - t_2 s_b}{t_2 s_a - s_b} \frac{s_a - t_1 t_2 s_b}{t_1 t_2 s_a - s_b} = \mathfrak{z}, \quad a = 1, \ldots, k. \tag{10.3}$$

Recall that equation (10.3) describes relations in the quantum-equivariant K-theory of $\mathcal{M}_{k,N}$. Generalising the results of [53] with the help of [57], one can prove the following:

Proposition 10.2. *The eigenvalues of quantum multiplication operators by bundles $\Lambda^l \mathcal{V}$, $1 \leq l \leq k$, in the localised quantum G-equivariant K-theory of $\mathcal{M}_{N,k}$ are given by elementary symmetric polynomials $e_l(s_1,\ldots,s_k)$ of Bethe roots which satisfy the Bethe equations (10.3).*

The quantum-equivariant K-theory ring of $\mathcal{M}_{N,k}$ can be described by the symmetric functions of variables $s_1,\ldots,s_k$ subject to Bethe equations. We refer to that as the Bethe algebra of the XXZ model for a quantum toroidal algebra.

On the other hand, Feigin, Jimbo, Miwa and Mukhin [13, 17] studied such an XXZ model explicitly and derived such Bethe equations for the corresponding transfer matrices. Another important issue, featured in [17], is the explicit construction of the Q-operator. We recall that this is the operator in the Bethe algebra whose eigenvalues form a generating function of the elementary symmetric functions of Bethe roots–that is, it is a generating function of operators from Proposition (10.2).

Frenkel and Hernandez [22] wrote down the QQ-system leading to the Bethe ansatz equations for a quantum toroidal $\mathfrak{gl}_1$-algebra:

$$\xi \mathcal{Q}^+ \left((t_1 t_2)^{-1} z \right) \mathcal{Q}^-(z) - \mathcal{Q}^+(z) \mathcal{Q}^- \left((t_1 t_2)^{-1} z \right) = \mathcal{L}(z) \mathcal{Q}^+ \left(t_1^{-1} z \right) \mathcal{Q}^+ \left(t_2^{-1} z \right), \quad (10.4)$$

where we have altered the authors' notation slightly and introduced a 'framing polynomial' $\mathcal{L}(z)$. We will refer to this functional equation as the $\hat{A}_0 QQ$-system. Its equations can also be written as

$$\xi \boldsymbol{\phi}(z) - \boldsymbol{\phi} \left((t_1 t_2)^{-1} z \right) = \boldsymbol{\rho}(z), \quad (10.5)$$

where

$$\boldsymbol{\phi}(z) = \frac{\mathcal{Q}^-(z)}{\mathcal{Q}^+ \left((t_1 t_2)^{-1} z \right)}, \qquad \boldsymbol{\rho}(z) = \mathcal{L}(z) \frac{\mathcal{Q}^+ \left(t_1^{-1} z \right) \mathcal{Q}^+ \left(t_1^{-2} z \right)}{\mathcal{Q}^+(z) \mathcal{Q}^+ \left((t_1 t_2)^{-1} z \right)}. \quad (10.6)$$

We call the solutions for such a system *nondegenerate* if $\mathcal{Q}^+(z)$, $\mathcal{Q}^-(z)$ and $\mathcal{L}$ are t_1- and t_2-distinct and $\xi \neq 1$.

That leads to the following lemma:

Lemma 10.3. *There is a one-to-one correspondence between the set of nondegenerate solutions of equations (10.4) and (10.3).*

Proof. Since

$$\mathcal{Q}(u) = \prod_{a=1}^{k} (z - s_a), \qquad \mathcal{L}(u) = \prod_{i=1}^{N} (z - \mathrm{a}_i),$$

we can first evaluate equation (10.4) at $u = s_a$, then shift u by $t_1 t_2$ and evaluate the equation again at $z = s_a$. This leads us to

$$\frac{\mathcal{L}(s_a)}{\mathcal{L}(t_1 t_2 s_a)} \cdot \frac{\mathcal{Q} \left(t_1^{-1} s_a \right)}{\mathcal{Q}(t_1 s_a)} \frac{\mathcal{Q} \left(t_2^{-1} s_a \right)}{\mathcal{Q}(t_2 s_a)} \frac{\mathcal{Q}(t_1 t_2 s_a)}{\mathcal{Q} \left((t_1 t_2)^{-1} s_a \right)} = -\xi.$$

This shows the implication in one direction–from the QQ-system to the Bethe equations. The opposite statement can be proved analogously to Theorem 5.6. We leave it to the reader.

There exists a generalisation of this construction to a higher-rank quantum toroidal algebra $U_{t_1,t_2}\left(\widehat{\widehat{\mathfrak{gl}}}(N)\right)$ for cyclic quiver varieties with N vertices. It is easy to write the Yang–Yang function in this case as well as the Bethe equations (see [2] for a universal treatment). It is also easy to present the analogue of the QQ-system (see later). However, the representation-theoretic approach along the lines of [22, 17]–that is, construction of the Q-operator as a transfer matrix for a special auxiliary representation of $U_{t_1,t_2}\left(\widehat{\widehat{\mathfrak{gl}}}(N)\right)$– has not yet been developed.

10.2. Miura 1-toroidal q-opers

We can now define toroidal opers. Let us consider the automorphism of the Dynkin diagram of $\mathfrak{a}_\infty$ which corresponds to a shift by one vertex. This automorphism can be realised by the transformation corresponding to the conjugation via the infinite Coxeter element $\prod_{i=+\infty}^{-\infty} s_i$. In matrix notation, such an infinite Coxeter element can be realised via $\mathcal{V}_1 = \sum_{i\in\mathbb{Z}} E_{i,i-1}$.

Definition 10.4. Set $p,\xi \in \mathbb{C}^\times$. We refer to a Z-twisted Miura $\left(\overline{GL}(\infty),q\right)$-oper (3.4) satisfying

$$\mathcal{V}_1 A(z)\mathcal{V}_1^{-1} = \xi A(pz) \tag{10.7}$$

as the *Z-twisted 1-toroidal Miura q-oper*. We call it *nondegnerate* if it is nondegenerate as a Z-twisted Miura $\left(\overline{GL}(\infty),q\right)$-oper.

Equation (10.7) translates to the following conditions on polynomials which appear in the QQ-system:

$$\frac{g_{i+1}(z)}{g_i(z)} = \xi \frac{g_i(pz)}{g_{i-1}(pz)}, \qquad \Lambda_{i+1}(z) = \xi\Lambda_i(pz). \tag{10.8}$$

The first condition (recall that $\xi_i = \frac{\zeta_i}{\zeta_{i-1}}$) becomes

$$\xi_{i+1}\frac{Q_{i+1}^+(qz)Q_i^+(z)}{Q_{i+1}^+(z)Q_i^+(qz)} = \xi\xi_i\frac{Q_i^+(qz)Q_{i-1}^+(pz)}{Q_i^+(z)Q_{i-1}^+(pqz)}, \tag{10.9}$$

which can be satisfied provided that

$$\xi_i = \xi^i, \qquad Q_i^+(z) = \mathcal{Q}^+\left(p^i z\right), \qquad \Lambda_i(z) = \xi^i\Lambda\left(p^i z\right). \tag{10.10}$$

Let us now study how these periodic conditions affect the QQ-system for $GL(\infty)$. The QQ-equations (9.8) can be rewritten as

$$\xi^{i+1}\phi\left(p^i z\right) - \xi^i\phi\left(p^i qz\right) = \xi^i\Lambda\left(p^i z\right)\frac{\mathcal{Q}^+\left(p^{i-1}z\right)\mathcal{Q}^+\left(p^{i+1}qz\right)}{\mathcal{Q}^+\left(p^i z\right)\mathcal{Q}^+\left(p^i qz\right)}, \tag{10.11}$$

where $\phi_i(z)$ is replaced by $\phi\left(p^i z\right)$ and thus

$$Q_i^-(z) = Q^-\left(p^i z\right). \tag{10.12}$$

By shifting the variable $z \mapsto p^{-i}z$, we get

$$\xi\phi(z) - \phi(qz) = \Lambda(z)\frac{Q^+\left(p^{-1}z\right)Q^+(pqz)}{Q^+(z)Q^+(qz)}. \tag{10.13}$$

Equivalently, we can impose $\rho_i(z) = \boldsymbol{\rho}\left(p^i z\right)$.

Notice that these equations coincide with equations (10.5) and (10.6); therefore we recover the $\widehat{A}_0QQ$-system, provided that

$$\mathcal{L}(z) = \Lambda(z), \qquad p = t_1, \qquad q = (t_1 t_2)^{-1}. \tag{10.14}$$

This brings us to the following:

Theorem 10.5. *The space of nondegenerate Z-twisted Miura 1-toroidal q-opers with regular singularities at $a_1, \ldots, a_N$ is isomorphic to the space of solutions of the nondegenerate $\widehat{A}_0QQ$-system (10.6) or equivalently to*

$$\xi Q^+(qz)Q^-(z) - Q^+(z)Q^-(qz) = \Lambda(z)Q^+\left(p^{-1}z\right)Q^+(pqz). \tag{10.15}$$

The full set of equations for the extended QQ-system for $\overline{GL}(\infty)$ (9.12) reads

$$Q^+(qz)Q^-(z) - \xi Q^+(z)Q^-(qz) = \Lambda(z)Q^+\left(p^{-1}qz\right)Q^+(pz),$$
$$Q^+(qz)Q_{(1)}^-(z) - \xi^2 Q^+(z)Q_{(1)}^-(qz) = \xi\Lambda(z)Q^-\left(p^{-1}qz\right)Q^+(pz),$$
$$\vdots \tag{10.16}$$
$$Q^+(qz)Q_{(j)}^-(z) - \xi^{j+1} Q^+(z)Q_{(1)}^-(qz) = \xi^j\Lambda(z)Q_{(j-1)}^-\left(p^{-1}qz\right)Q^+(pz),$$
$$\vdots$$

where

$$Q_{(j)}^-\left(p^i z\right) = Q_{i-j+1,i-j+2,\ldots,i}^-(z). \tag{10.17}$$

The gauge transformation which brings a toroidal q-oper to the diagonal form can be directly generalised from equation (5.11) using equations (10.10), (10.12) and (10.17):

$$v(z)^{-1} = \begin{pmatrix} \ddots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \dfrac{\mathcal{Q}^+\left(p^{i-1}z\right)}{\mathcal{Q}^+\left(p^iz\right)} & \dfrac{\mathcal{Q}^-\left(p^{i-1}z\right)}{\mathcal{Q}^+\left(p^{i+1}z\right)} & \cdots & \cdots & \dfrac{\mathcal{Q}^-_{(j)}\left(p^{i-1}z\right)}{\mathcal{Q}^+\left(p^{i+j}z\right)} & \vdots \\ 0 & 0 & \dfrac{\mathcal{Q}^+\left(p^iz\right)}{\mathcal{Q}^+\left(p^{i+1}z\right)} & \cdots & \cdots & \dfrac{\mathcal{Q}^-_{(j-1)}\left(p^iz\right)}{\mathcal{Q}^+\left(p^{i+j}z\right)} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \dfrac{\mathcal{Q}^+\left(p^{i+j-2}z\right)}{\mathcal{Q}^+\left(p^{i+j-1}z\right)} & \dfrac{\mathcal{Q}^-\left(p^{i+j-2}z\right)}{\mathcal{Q}^+\left(p^{i+j}z\right)} & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \dfrac{\mathcal{Q}^+\left(p^{i+j-1}z\right)}{\mathcal{Q}^+\left(p^{i+j}z\right)} & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \ddots \end{pmatrix}. \tag{10.18}$$

We would like to mention that a similar folding procedure on the level of TQ-systems for $SL(\infty)$ was performed by Hernandez [30]. For further developments and applications of the QQ-system (10.15), see [40].

10.3. Miura $\mathscr{N}$-toroidal q-Opers

Here we briefly show how the foregoing construction can be immediately generalised to higher rank. Namely, one has to generalise the periodicity conditions.

Definition 10.6. Set $p, \xi \in \mathbb{C}^\times$. The Z-twisted $\mathscr{N}$-toroidal Miura q-oper is a $(GL(\infty), q)$-oper (3.4) satisfying

$$\mathcal{V}_{\mathscr{N}} A(z) \mathcal{V}_{\mathscr{N}}^{-1} = \xi^{\mathscr{N}} A\left(p^{\mathscr{N}} z\right), \tag{10.19}$$

where $\mathcal{V}_N = \mathcal{V}_1^{\mathscr{N}}$. We call it *nondegenerate* if it is nondegenerate as a Z-twisted Miura $\left(\overline{GL}(\infty), q\right)$-oper.

If we impose equation (10.19) on the q-connection, we get the following family of equations for $i \geq j$:

$$\frac{g_{i+\mathscr{N}}(z)}{g_{j+\mathscr{N}}(z)} = \xi^{\mathscr{N}} \frac{g_i\left(p^{\mathscr{N}} z\right)}{g_j\left(p^{\mathscr{N}} z\right)}, \qquad \Lambda_{i+\mathscr{N}}(z) = \xi^{\mathscr{N}} \Lambda_i\left(p^{\mathscr{N}} z\right), \tag{10.20}$$

which imposes $\mathscr{N}$-periodicity on all functions

$$Q^\pm_{i+\mathscr{N}}(z) = Q^\pm_i\left(p^{\mathscr{N}} z\right), \qquad \xi_{i+\mathscr{N}} = \xi^{\mathscr{N}} \xi_i,$$

for all i.

Thus we arrive at the generalisation of Theorem 10.5:

Theorem 10.7. *The nondegenerate Z-twisted Miura N-toroidal q-opers with regular singularities given by* $\Lambda_i(u) = \prod_{j=1}^{\mathscr{N}}\left(z - a_j^{(i)}\right)$ *are in one-to-one correspondence with the*

nondegenerate solutions of the following $\widehat{A}_{\mathcal{N}-1}QQ$-system:

$$\xi_1 Q_1^+(qz)Q_1^-(z) - \xi_2 Q_1^+(z)Q_1^-(qz) = \Lambda_1(z)Q_{\mathcal{N}}^+(qz)Q_2^+(z), \tag{10.21}$$

$$\xi_i Q_i^+(qz)Q_i^-(z) - \xi_{i+1}Q_i^+(z)Q_i^-(qz) = \Lambda_i(z)Q_{i-1}^+(qz)Q_{i+1}^+(z), \qquad i = 2,\ldots,\mathcal{N}-1,$$

$$\xi_N Q_N^+(qz)Q_N^-(z) - \xi_1 Q_{\mathcal{N}}^+(z)Q_{\mathcal{N}}^-(qz) = \Lambda_{\mathcal{N}}(z)Q_{\mathcal{N}-1}^+(qz)Q_1^+(z),$$

with the nondegeneracy conditions induced from the original $\overline{GL}(\infty)QQ$-system.

References

[1] M. AGANAGIC, E. FRENKEL AND A. OKOUNKOV, Quantum q-Langlands correspondence, *Trans. Moscow Math. Soc.* **79** (2018), 1–83.

[2] M. AGANAGIC AND A. OKOUNKOV, Quasimap counts and Bethe eigenfunctions, *Mosc. Math. J.* **17**(4) (2017), 565–600.

[3] V. BAZHANOV, R. FRASSEK, T. LUKOWSKI, C. MENEGHELLI AND M. STAUDACHER, Baxter Q-operators and representations of Yangians, *Nuclear Phys.* **B850** (2011), 148–174.

[4] V. BAZHANOV, S. L. LUKYANOV AND A. B. ZAMOLODCHIKOV, Integrable structure of conformal field theory. 3. The Yang-Baxter relation, *Comm. Math. Phys.* **200** (1999), 297–324.

[5] V. BAZHANOV, S. L. LUKYANOV AND A. B. ZAMOLODCHIKOV, Spectral determinants for Schroedinger equation and Q-operators of conformal field theory, *J. Stat. Phys.* **102**(3/4) (2001), 567–576.

[6] A. BERENSTEIN AND A. ZELEVINSKY, Total positivity in Schubert varieties, *Comment. Math. Helvet.* **72**(1) (1997), 128–166.

[7] A. BERENSTEIN, S. FOMIN AND A. ZELEVINSKY, Parametrizations of canonical bases and totally positive matrices, *Adv. Math.* **122**(1) (1996), 49–149.

[8] A. BRAVERMAN, D. MAULIK AND A. OKOUNKOV, 'Quantum cohomology of the Springer resolution', Preprint, 2010, 1001.0056.

[9] V. CHARI AND A. PRESSLEY, *A Guide to Quantum Groups* (Cambridge University Press, Location, 1994).

[10] D. CHERNYAK, S. LEURENT AND V. DMYTRO, 'Completeness of Wronskian Bethe equations for rational $gl(m|n)$ spin chain', Preprint, 2020, https://arxiv.org/abs/2004.02865.

[11] P. DOREY AND R. TATEO, Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations, *J. Phys. A* **32**(38) (1999), L419–L425.

[12] V. DRINFELD AND V. SOKOLOV, Lie algebras and equations of Korteweg-de Vries type, *J. Sov. Math.* **30**(2) (1985), 1975–2036.

[13] B. FEIGIN, Integrable systems, shuffle algebras, and Bethe equations, *Trans. Moscow Math. Soc.* **77** (2016), 203–246.

[14] B. FEIGIN AND E. FRENKEL, Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras, *Internat. J. Modern Phys. A* **07**(supp01a) (1992), 197–215.

[15] B. FEIGIN, E. FRENKEL AND N. RESHETIKHIN, Gaudin model, Bethe ansatz and critical level, *Comm. Math. Phys.* **166** (1994), 27–62.

[16] B. FEIGIN, E. FRENKEL AND V. TOLEDANO LAREDO, Gaudin models with irregular singularities, *Adv. Math.* **223** (2010), 873–948.

[17] B. FEIGIN, M. JIMBO, T. MIWA AND E. MUKHIN, Finite type modules and Bethe ansatz for quantum toroidal *gl*1, *Comm. Math. Phys.* **356**(1) (2017), 285–327.

[18] S. FOMIN AND A. ZELEVINSKY, Double Bruhat cells and total positivity, *J. Amer. Math. Soc.* **12**(2) (1999), 335–380.

[19] E. FRENKEL, Opers on the projective line, flag manifolds and Bethe Ansatz, *Mosc. Math. J.* **4** (2003), 655–705.

[20] E. FRENKEL, Lectures on Wakimoto modules, opers and the center at the critical level, *Adv. Math.* **195** (2005), 297–404.

[21] E. FRENKEL AND D. HERNANDEZ, Baxter's relations and spectra of quantum integrable models, *Duke Math. J.* **164**(12) (2015), 2407–2460.

[22] E. FRENKEL AND D. HERNANDEZ, Spectra of quantum KdV Hamiltonians, Langlands duality, and affine opers, *Comm. Math. Phys.* **362** (2018), 361–414.

[23] E. FRENKEL, D. HERNANDEZ AND N. RESHETIKHIN, to appear.

[24] E. FRENKEL, P. KOROTEEV, D. S. SAGE AND A. M. ZEITLIN, *q*-Opers, *QQ*-systems, and Bethe ansatz, *J. Europ. Math. Soc.*, to appear, 2002.07344.

[25] E. FRENKEL AND N. RESHETIKHIN, Quantum affine algebras and holonomic difference equations, *Comm. Math. Phys.* **146** (1992), 1–60.

[26] I. FRENKEL, Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory, *J. Funct. Anal.* **44**(3) (1981), 259–327.

[27] D. GAIOTTO AND P. KOROTEEV, On three dimensional quiver gauge theories and integrability, *J. High Energy Phys.* **1305** (2013), 126.

[28] V. GINZBURG, 'Lectures on Nakajima's quiver varieties', Preprint, 2009, 1703. https://arxiv.org/abs/0905.0686.

[29] R. GOPAKUMAR AND C. VAFA, On the gauge theory/geometry correspondence, *Adv. Theor. Math. Phys.* **3**(5) (1999), 1415–1443.

[30] D. HERNANDEZ, The algebra $U_q\left(\hat{\mathrm{sl}}_\infty\right)$ and applications, *J. Algebra* **329**(1) (2010), 147–162.

[31] D. HERNANDEZ AND M. JIMBO, Asymptotic representations and Drinfeld rational fractions, *Compos. Math.* **148**(5) (2012), 1593–1623.

[32] D. HERNANDEZ AND B. LECLERC, Cluster algebras and category O for representations of Borel subalgebras of quantum affine algebras, *Algebra Number Theory* **10**(9) (2016), 2015–2052.

[33] V. KAC, A. RAINA AND N. ROZHKOVSKAYA, *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, 2nd ed. (World Scientific Publishing, Location, 2013).

[34] V. KAZAKOV, S. LEURENT AND D. VOLIN, T-system on T-hook: Grassmannian solution and twisted quantum spectral curve, *J. High Energy Phys.* **2016**, 44(2016).

[35] V. KOREPIN, N. BOGOLIUBOV AND A. IZERGIN, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).

[36] P. KOROTEEV, A-type quiver varieties and ADHM moduli spaces, *Comm. Math. Phys.* **381**(1) (2021), 175–207.

[37] P. KOROTEEV, P. PUSHKAR, A. SMIRNOV AND A. ZEITLIN, 'Quantum K-theory of quiver varieties and many-body systems', Preprint, 2017, https://arxiv.org/abs/1705.10419.

[38] P. KOROTEEV, D. SAGE AND A. ZEITLIN, $(SL(N),q)$-opers, the q-Langlands correspondence, and quantum/classical duality, *Comm. Math. Phys.* **381**(2) (2021), 641–672.

[39] P. KOROTEEV AND A. SCIARAPPA, 'Quantum hydrodynamics from large-n supersymmetric gauge theories', *Lett. Math. Phys.* **108**, (2018), 45–95.

[40] P. KOROTEEV AND A. M. ZEITLIN, '3d mirror symmetry for instanton moduli spaces', Preprint, 2021, https://arxiv.org/abs/2105.00588.

[41] P. KOROTEEV AND A. M. ZEITLIN, qKZ/tRS duality via quantum K-theoretic counts, *Math. Res. Lett.* **28**(2) (2021), 435–470.

[42] I. KRICHEVER, O. LIPAN, P. WIEGMANN AND A. ZABRODIN, Quantum integrable models and discrete classical Hirota equations, *Comm. Math. Phys.* **188**(2) (1997), 267–304.

[43] D. MASOERO, A. RAIMONDO AND D. VALERI, Bethe ansatz and the spectral theory of affine Lie algebra-valued connections I. The simply-laced case, *Comm. Math. Phys.* **344**(3) (2016), 719–750.

[44] D. MASOERO, A. RAIMONDO AND D. VALERI, Bethe ansatz and the spectral theory of affine Lie algebra–valued connections II: The non simply–laced case, *Comm. Math. Phys.* **349**(3) (2017), 1063–1105.

[45] D. MAULIK AND A. OKOUNKOV, 'Quantum groups and quantum cohomology', Preprint, 2012, 1211.1287.

[46] E. MUKHIN AND A. VARCHENKO, Discrete Miura opers and solutions of the Bethe ansatz equations, *Comm. Math. Phys.* **256** (2005), 565–588.

[47] H. NAKAJIMA, *Lectures on Hilbert Schemes of Points on Surfaces* (American Mathematical Society, Providence, RI, 1999).

[48] A. NEGUT, 'The R-matrix of the quantum toroidal algebra', Preprint, 2020, 2005.14182.

[49] A. NEGUT, 'Quantum algebras and cyclic quiver varieties', Preprint, 2015, 1504.06525.

[50] N. NEKRASOV AND S. SHATASHVILI, Supersymmetric vacua and Bethe ansatz, *Nuclear Phys. Proc. Suppl.* **192-193** (2009), 91–112.

[51] A. OKOUNKOV, 'Lectures on K-theoretic computations in enumerative geometry', Preprint, 2015, https://arxiv.org/abs/1512.07363.

[52] A. OKOUNKOV AND A. SMIRNOV, 'Quantum difference equation for Nakajima varieties', Preprint, 2016, 1602.09007.

[53] P. PUSHKAR, A. SMIRNOV AND A. ZEITLIN, Baxter Q-operator from quantum K-theory, *Adv. Math.* **360** (2020), 106919.

[54] N. RESHETIKHIN, 'Lectures on the integrability of the 6-vertex model', Preprint, 2010, 1010.5031.

[55] O. SCHIFFMANN AND E. VASSEROT, The elliptic Hall algebra and the K-theory of the Hilbert scheme of A^2, *Duke Math. J.* **162**(2) (2013), 279–366.

[56] M. SEMENOV-TIAN-SHANSKY AND A. SEVOSTYANOV, Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras II. The general semisimple case, *Comm. Math. Phys.* **192** (1998), 631–647.

[57] A. SMIRNOV, 'Rationality of capped descendent vertex in K-theory', Preprint, 2016, https://arxiv.org/abs/1612.01048.

[58] V. TARASOV AND A. VARCHENKO, Duality for Knizhnik–Zamolodchikov and dynamical equations, *Acta Appl. Math.* **73**(1/2) (2002), 141–154.

[59] V. TARASOV AND A. VARCHENKO, Dynamical differential equations compatible with rational qKZ equations, *Lett. Math. Phys.* **71**(2) (2005), 101–108.