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Relaxed states in electron-depleted electronegative dusty plasmas with two-negative ion species

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The relaxation of an electron-depleted electronegative dusty plasma with two-negative ions is investigated. When the ratio of canonical vorticities to corresponding flows of all the plasma species is the same and all inertial and non-inertial forces are present, the relaxed state appears as a double Beltrami magnetic field which is the superposition of two force-free relaxed states. The numerical results show that highly diamagnetic relaxed magnetic fields can be obtained by controlling the flow and vorticities through a single Beltrami parameter. The study is useful to investigate the creation of diamagnetic plasma configurations which are considered to be very important in the context of nuclear fusion.

1. Introduction

A magnetized plasma, in spite of the complexity introduced by interaction of magnetic field and flow, has been observed to show an ordered behaviour. The creation of the Beltrami magnetic field **B** is a familiar example of this phenomenon. The Beltrami magnetic field expressed as $\nabla \times \mathbf{B} = \lambda \mathbf{B}$, where λ is a scalar function, represents the equilibrium state when the flow energy can be neglected (Ortolani and Schnack 1993). The Beltrami magnetic field also shows the force-free macroscopic state of the magnetoplasma. For a perfectly conducting plasma, the Beltrami field characterized by a constant scale parameter λ can be derived through variational principle by minimizing the magnetic energy under the constraint that local magnetic helicity, a measure of structural complexity of field lines, remains constant (Woltjer 1958). The Beltrami magnetic field is also known as Taylor's relaxed state because Taylor derived the equilibrium state by conjecturing that magnetic energy dissipates faster than magnetic helicity in the presence of a small amount of resistivity in real plasmas (Taylor 1974). The non-force-free relaxed states characterized by coupling of two Beltrami fields with a strong flow are also shown to be accessible in a magnetized plasma by constrained minimization of magnetofluid energy using the variational principle or invoking the Hall magnetohydrodynamics (HMHD) (Sudan 1979; Avinash and Taylor 1991; Steinhauer and Ishida 1997; Mahajan and Yoshida 1998; Shukla 2004; Shukla and Mahajan 2004a, b; Shukla 2005). It is pointed out by Yoshida and Mahajan (2002) that a two-fluid plasma system relaxes when its canonical enstrophy which is a measure of dissipation and turbulence minimizes while the magnetofluid energy, magnetic and generalized helicities remain constant. The relaxed

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states governed by the superposition of two Beltrami fields could explain a variety of physical phenomena occurring in space and laboratory plasmas. For example, the high-confinement (H-mode) boundary layer developed in tokamak discharges could be viewed as a self-organized double Beltrami field structure (Mahajan and Yoshida 2000). The double Beltrami equilibrium predicts the creation of high beta (ratio of kinetic to magnetic pressures) relaxed states in different plasma devices (Yoshida et al. 2001; Iqbal 2005) and can be employed to study the eruptive events in solar corona (Ohsaki et al. 2001).

It has also been shown that inertia of plasma species plays an important role to create relaxed equilibrium composed of multi Beltrami fields (Bhattacharyya et al. 2003; Iqbal and Shukla 2011, 2012). In this paper, it is shown that the flow and canonical vorticities of plasma components also affect the creation of Beltrami fields. For this purpose, we have investigated the relaxation of two-negative ion species dusty plasma. The dusty plasmas have been studied to investigate the dust-acoustic solitary structures (Mamun 1999) and nonlinear ion-acoustic structures (El-Tantawy et al. 2011). We have assumed the dusty plasma so that the electron number density sufficiently depletes due to attachment of background electrons onto the negative dust grains. The two-negative ion species plasmas could be found in space (Coates et al. 2007; Vuitton et al. 2009) and laboratory (Ichiki et al. 2001). It is shown that it is possible for a two-negative ion plasma to relax to a double Beltrami field taking into account all the inertial and non-inertial forces.

The paper is arranged as follows: The steady-state equilibrium equation is derived in Sec. 2. Equations of motion of plasma species are given and the Beltrami condition (alignment of fluid flow to vorticity) is employed. A single Beltrami parameter is considered and the inertial effects are taken into account. The analytical structure given in Sec. 3 shows that equilibrium equation can be written as a sum of two Beltrami fields. The radial profiles of relaxed magnetic fields as a function of scale parameters are displayed in Sec. 4 and it is shown that by varying a single scale parameter, one could obtain paramagnetic as well as diamagnetic field structures. The results are summarized in Sec. 5.

2. Model

A multicomponent electronegative dusty plasma comprising of positive ions and two-negative ions in addition to charged dust grains is investigated to look for the self-organization towards double Beltrami fields. The dust grains are immobile and only contribute to preserve charge neutrality. The density of electrons (n_e) is assumed negligibly small as compared to the density of dust particles (n_d) in the background i.e. $z_d n_d \gg n_e$, where z_d represents the number of electrons residing on the surface of dust particles. In this scenario, the electrons are highly depleted because of their attachments to extremely massive dust grains (Mamun et al. 1996; Tribeche et al. 2008). The equation of motion for s-species of ions is given by

$$\frac{\partial}{\partial t}\left(\mathbf{V}_\mathrm{s}+\frac{z_s e_\mathrm{s}\mathbf{A}}{m_\mathrm{s}c}\right)=\mathbf{V}_\mathrm{s}\times\left(\nabla\times\mathbf{V}_\mathrm{s}+\frac{z_s e_\mathrm{s}}{m_\mathrm{s}c}\mathbf{B}\right)-\nabla\left(\frac{e_\mathrm{s}\phi}{m_\mathrm{s}}+\frac{\mathbf{V}_\mathrm{s}^2}{2}+h_s\right), \tag{1}$$

where $\mathbf{V}_\mathrm{s}$, m_s, z_s and e_s represent the velocity, mass, charge number and charge of s ions ($s = i, 1,$ and 2 for positive ions, first- and second-negative ion, respectively). $\mathbf{B}$, $\mathbf{A}$ and ϕ, respectively, represent magnetic field, vector and scalar potentials. c is the speed of light in vacuum, $\nabla h_s = (\nabla p_\mathrm{s})/\rho_\mathrm{s}$, h_s is the enthalpy, $\rho_\mathrm{s} = n_\mathrm{s} m_\mathrm{s}$ is the constant mass density of s-species, $p_\mathrm{s} = n_\mathrm{s} T_\mathrm{s}$ is the pressure and n_s and T_s are the number densities and temperature of the ion species, respectively. The identity $(\mathbf{V_s}\cdot\nabla)\mathbf{V_s} = \frac{1}{2}\nabla\mathbf{V_s}^2 - \mathbf{V_s}\times(\nabla\times\mathbf{V_s})$, is used and $\mathbf{E}$ in favour of scalar potential ϕ and vector potential $\mathbf{A}$ by relation $\mathbf{E} = -\nabla\phi - c^{-1}\partial\mathbf{A}/\partial t$, is eliminated to obtain (1). It is convenient to normalize the variables with $\mathbf{B}$ to some arbitrary B_0, length to skin depth of positive ions $\lambda_i = (c\sqrt{m_i})/\sqrt{4\pi n_i z_i^2 e^2}$, time to gyroperiod of positive ions $m_i c/z_i e B_0$, velocities to Alfvén speed $V_A = B_0/(4\pi n_i m_i)^{1/2}$, pressure to $B_0^2/4\pi$ and the potentials ϕ and $\mathbf{A}$ to $B_0^2/4\pi n_i e$ and $V_A m_i c/z_i e$, respectively. The normalized macroscopic evolution equations are

$$\frac{\partial}{\partial\hat{t}}(\hat{\mathbf{V}}_i + \hat{\mathbf{A}}) = \hat{\mathbf{V}}_i \times (\hat{\nabla}\times\hat{\mathbf{V}}_i + \hat{\mathbf{B}}) - \hat{\nabla}\left(\hat{\phi} + \frac{\hat{\mathbf{V}}_i^2}{2} + \hat{p}_i\right), \tag{2}$$

$$\frac{\partial}{\partial\hat{t}}(\hat{\mathbf{V}}_1 - \mathscr{Z}_1\mathscr{M}_1\hat{\mathbf{A}}) = \hat{\mathbf{V}}_1 \times (\hat{\nabla}\times\hat{\mathbf{V}}_1 - \mathscr{Z}_1\mathscr{M}_1\hat{\mathbf{B}}) - \hat{\nabla}\left(-\mathscr{M}_1\hat{\phi} + \frac{\hat{\mathbf{V}}_1^2}{2} + \mathscr{D}_1\hat{p}_1\right), \tag{3}$$

and

$$\frac{\partial}{\partial\hat{t}}(\hat{\mathbf{V}}_2 - \mathscr{Z}_2\mathscr{M}_2\hat{\mathbf{A}}) = \hat{\mathbf{V}}_2 \times (\hat{\nabla}\times\hat{\mathbf{V}}_2 - \mathscr{Z}_2\mathscr{M}_2\hat{\mathbf{B}}) - \hat{\nabla}\left(-\mathscr{M}_2\hat{\phi} + \frac{\hat{\mathbf{V}}_2^2}{2} + \mathscr{D}_2\hat{p}_2\right), \tag{4}$$

where $\mathscr{Z}_j = z_j/z_i$, $\mathscr{M}_j = m_i/m_j$, and $\mathscr{D}_j = \rho_i/\rho_j$ (the subscript i stands for ions and $j = 1, 2$ represent the two-negative ions). In what follows, for convenience, we will not use the normalization sign above the variables. Taking curl of above equations, we obtain

$$\frac{\partial}{\partial t}(\nabla\times\mathbf{V}_i + \mathbf{B}) = \nabla\times[\mathbf{V}_i \times (\nabla\times\mathbf{V}_i + \mathbf{B})], \tag{5}$$

$$\frac{\partial}{\partial t}\left(\nabla\times\mathbf{V}_j - \mathscr{Z}_j\mathscr{M}_j\mathbf{B}\right) = \nabla\times\left[\mathbf{V}_j \times \left(\nabla\times\mathbf{V}_j - \mathscr{Z}_j\mathscr{M}_j\mathbf{B}\right)\right], \tag{6}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$. The simplest steady-state solution of above equations is given by the Beltrami condition which demands the alignment of vorticities to the corresponding flow. The Beltrami conditions for positive and negative ions, respectively, read as

$$\nabla\times\mathbf{V}_i + \mathbf{B} = a\mathbf{V}_i, \tag{7}$$

$$\nabla\times\mathbf{V}_1 - \mathscr{Z}_1\mathscr{M}_1\mathbf{B} = a\mathbf{V}_1, \tag{8}$$

$$\nabla\times\mathbf{V}_2 - \mathscr{Z}_2\mathscr{M}_2\mathbf{B} = a\mathbf{V}_2, \tag{9}$$

where a is a constant and called the Beltrami parameter. The above equations show that canonical (generalized) vorticities become parallel to the corresponding velocities to achieve the relaxation.

The quasi neutrality condition is $z_i n_i = z_d n_d + \sum_{j=1,2} z_j n_j$, where n_d is the number density of dust grains. Using Ampere's Law and the definition of current density ($\mathbf{J} = z_i e n_i \mathbf{V}_i - \sum_{j=1,2} z_j e n_j \mathbf{V}_j$), the normalized velocity of positive ions is given by

$$\nabla\times\mathbf{B} = \mathbf{V}_i - \sum_{j=1,2}\mathscr{Z}_j N_j \mathbf{V}_j, \tag{10}$$

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where $N_j = n_j/n_i$. Multiplying (8) and (9) by $\mathscr{Z}_1 N_1$ and $\mathscr{Z}_2 N_2$, respectively and adding, we obtain

$$\nabla \times \sum_{j=1,2} \mathscr{Z}_j N_j \mathbf{V}_j - \sum_{j=1,2}^{2} \mathscr{Z}_j^2 N_j \mathscr{M}_j \mathbf{B} = a \sum_{j=1,2} \mathscr{Z}_j N_j \mathbf{V}_j. \tag{11}$$

Eliminating $\sum_{j=1,2} \mathscr{Z}_j N_j \mathbf{V}_j$ from (10) and (11), we have

$$\nabla \times \mathbf{V}_i - a\mathbf{V}_i = \nabla \times \nabla \times \mathbf{B} - a\nabla \times \mathbf{B} + \sum_{j=1,2} \mathscr{Z}_j^2 N_j \mathscr{M}_j \mathbf{B}. \tag{12}$$

Using (7) into above equation, we obtain

$$\nabla \times \nabla \times \mathbf{B} - a\nabla \times \mathbf{B} + k\mathbf{B} = 0, \tag{13}$$

where $k = 1 + \sum_{j=1,2} \mathscr{Z}_j^2 N_j \mathscr{M}_j$. This is double curl Beltrami equation and can be cast as a linear sum of two different Beltrami fields.

3. Analytical structure

The Beltrami magnetic field $\mathbf{B}_j$ (where $j = 1, 2$) is defined as follows:

$$\begin{cases} \nabla \times \mathbf{B}_j = \lambda_j \mathbf{B}_j & (\text{in } \boldsymbol{\Omega}), \\ \mathbf{n} \cdot \mathbf{B}_j = 0 & (\text{on } \partial\boldsymbol{\Omega}), \end{cases} \tag{14}$$

where $\mathbf{n}$ is the unit normal vector and λ_j are scalar constants. The magnetic field $\mathbf{B}$ in (13), therefore, can be written as

$$\mathbf{B} = C_1 \mathbf{B}_1 + C_2 \mathbf{B}_2, \tag{15}$$

where C_j $(j = 1, 2)$ are arbitrary constants. In cylindrical coordinates, the eigenfunctions of the curl operator $\mathbf{B}_j$ may be represented by Chandrasekhar–Kendall functions (Chandrasekhar and Kendall 1957). If we take

$$\lambda_1 + \lambda_2 = a, \tag{16}$$

$$\lambda_1 \lambda_2 = k, \tag{17}$$

then (13) can be expressed as

$$(\nabla \times -\lambda_1)(\nabla \times -\lambda_2)\mathbf{B} = 0. \tag{18}$$

The eigenvalues of the curl operator (λ_j) may be arbitrary real (and even complex) number, if the domain is multiply connected (Yoshida and Giga 1990) and represents the solutions of the quadratic equation

$$\lambda^2 - a\lambda + k = 0. \tag{19}$$

The eigenvalues, therefore, can be expressed as

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 - 4k}}{2}. \tag{20}$$

For $a^2 > 4k$, the roots will be real and if $a^2 < 4k$, roots are complex. The roots will be purely imaginary for $a = 0$, and for $a^2 = 4k$, the roots are degenerate. It is worth noting that k strongly depends on mass and density of plasma species.

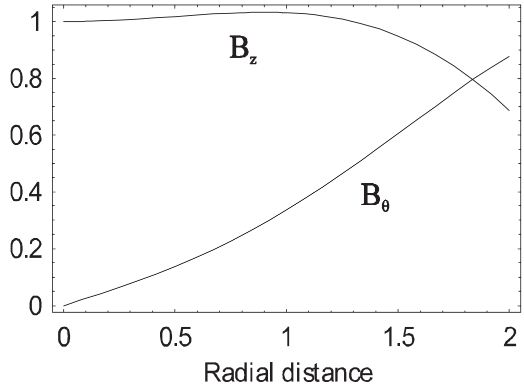

FIGURE 1. Plot of magnetic fields for $a = 2.53$. The scale parameters are $\lambda_1 = 1.3386$ and $\lambda_2 = 1.1914$.

4. Diamagnetic field structures

The magnetic fields given in (18) depend on two characteristic length scales ($\lambda_{1,2}^{-1}$). In order to show the role of two scales, we consider the explicit solution of (18) in one-dimensional cylindrical system. Taking the boundary conditions $\mathbf{B}_z(r = 0) = \mathbf{B}_0 \equiv 1$, and $\mathbf{J}_z = (\nabla \times \mathbf{B})_z(r = 0) = h$, the solution of (18) reads as

$$\mathbf{B}_z = C_1 J_0(\lambda_1 r) + C_2 J_0(\lambda_2 r), \tag{21}$$

$$\mathbf{B}_\theta = C_1 J_1(\lambda_1 r) + C_2 J_1(\lambda_2 r), \tag{22}$$

where $C_1 = (\lambda_2 - h)/(\lambda_2 - \lambda_1)$ and $C_2 = (\lambda_1 - h)/(\lambda_1 - \lambda_2)$. To evaluate the value of k, we consider Ar^+-F^--SF_6^- plasma system (El-Tantawy and Moslem 2012). As $\mathscr{M}_j = m_i/m_j$, therefore $\mathscr{M}_1 = m_i/m_1 = 40/19$ and $\mathscr{M}_2 = m_i/m_2 = 40/146$, where m_i, m_1, and m_2 are the masses of positive-ion Ar^+, first-negative ion F^-, and second-negative ion SF_6^-, respectively. The ratio of densities of first- and second-negative ions to positive ions $N_j = n_j/n_i$ is taken to be 0.25. The value of k depends on charge, mass and densities of plasma components. For this particular system, $k = 1.5948$ remains constant. A constant boundary value of $h = 0.5$ is taken to show the magnetic field structures by varying only the values of a - ratio of canonical vorticities to flows. Figure 1 shows the radial profiles of magnetic fields for Beltrami parameters $a = 2.53$. The eigenvalues of the curl operator are real and read as $\lambda_1 = 1.3386$ and $\lambda_2 = 1.1914$. It is observed that $\mathbf{B}_z$ decreases towards the edge whereas $\mathbf{B}_\theta$ increases towards edge of plasma. In this case, canonical vorticities of all the plasma components are greater than the respective flows. Figure 2 shows the magnetic field profiles for a complex conjugate pair of eigenvalues ($\lambda_{1,2} = 0.25 \pm 1.2579i$) and a Beltrami parameter $a = 0.5$. The flows are greater than the canonical vorticities and magnetic fields show the diamagnetic character. For $a = 0$, the canonical vorticities become zero and flow vorticities become parallel to magnetic field. The eigenvalues are purely imaginary and given by $\lambda_{1,2} = \pm 1.2629i$. The plasma behaves as a perfectly diamagnetic medium as shown in Fig. 3. The plots of magnetic fields illustrate the possibility of creating highly diamagnetic relaxed states by varying a single Beltrami parameter a. It is also evident that the eigenvalues ($\lambda_{1,2}$) defining the length scales of relaxed structures explicitly depend on charge, mass and density of flowing plasmas.

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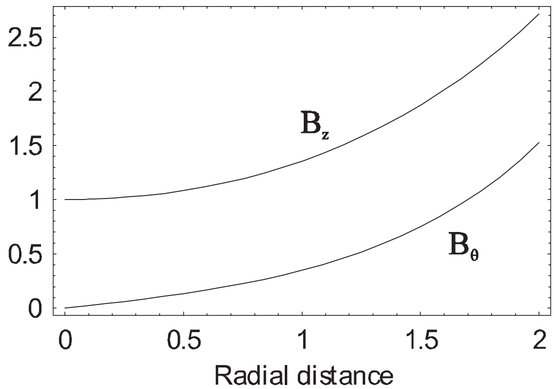

FIGURE 2. Plot of magnetic fields for $a = 0.5$. The scale parameters are $\lambda_{1,2} = 0.25 \pm 1.2579i$.

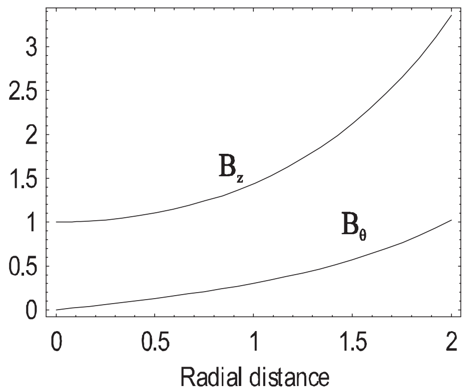

FIGURE 3. Plot of magnetic fields for $a = 0$. The scale parameters are $\lambda_{1,2} = \pm 1.2629i$.

5. Summary

The role of inertia, canonical vorticities and flows in relaxation of multifluid plasma is described. It is shown that in contrast to HMHD plasma relaxation, an electron-depleted two-negative ion dusty plasma relaxes to a double Beltrami field even if the inertia of all the plasma species is taken into account provided the ratio of canonical vorticities to flows of different species is same. The relaxed state involves two scale parameters and it is possible to get two relaxed vortices of different scale lengths. One relaxed structure characterized by small-scale parameter represents the macroscale (of the order of system size) while the other one corresponding to large eigenvalue could be of the order of ion skin depth. The diamagnetic as well as the paramagnetic plasma configurations can also be obtained by changing the canonical vorticities and flows. This work is important to study relaxation and creation of different structures in electron-depleted dusty plasmas having two negative ions in addition to a positive ion. The present results could be applied to understand the magnetic field structures formed in planetary rings and Titan's upper atmosphere, where positive ions, multifarious negative ions are the major constituents of plasmas with dust impurities of negative charge. In the present analysis, we have considered only a single Beltrami parameter a. If we take different Beltrami parameters for different plasma components, the relaxed state will be changed and appear as a superposition of more than two Beltrami fields. The effect of multi Beltrami parameters on formation of relaxed structures will be investigated and discussed somewhere else.

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