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ESSENTIAL EXTENSIONS OF T_1-SPACES

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ABSTRACT. In the category $\mathcal{T}_1$ of T_1-spaces and continuous maps, every space X has a unique maximal essential extension. X has an injective hull iff card $X \leq 1$.

Let $\mathfrak{J}$ be a class of morphisms in a category $\mathscr{C}$ which is closed under composition in $\mathscr{C}$ and contains all $\mathscr{C}$-isomorphisms. Then a $\mathfrak{J}$-morphism $f: A \to B$ is called an essential ($\mathfrak{J}$-)morphism iff whenever $g = hf \in \mathfrak{J}$ for some $\mathscr{C}$-morphism $h: B \to C$, then $h \in \mathfrak{J}$. A $\mathscr{C}$-object X is called ($\mathfrak{J}$-)injective iff for every $f: A \to B$ in $\mathfrak{J}$ the map $\mathrm{Hom}_{\mathscr{C}}(f, X): \mathrm{Hom}_{\mathscr{C}}(B, X) \to \mathrm{Hom}_{\mathscr{C}}(A, X)$ is surjective, i.e. iff whenever we are given a diagram with $f \in \mathfrak{J}$ we can fill in the

$$\begin{tikzcd} X & \\ A \arrow[u, "v"] \arrow[r, "f"'] & B \arrow[ul, dashed, "u"'] \end{tikzcd}$$

dotted arrow u in order to make it into a commutative triangle; in other words: every $\mathscr{C}$-morphism $v: A \to X$ is "extendible" over every $\mathfrak{J}$-morphism $f: A \to B$. A $\mathscr{C}$-morphism $f: A \to C$ is a ($\mathfrak{J}$-)injective hull iff f is $\mathfrak{J}$-essential and C is $\mathfrak{J}$-injective. If A has an injective hull $f: A \to C$, then f is the unique (up to an isomorphism with domain C) maximal essential extension of A, provided there exists any maximal essential extension of A, Cf. [2, 3].

In [1] B. Banaschewski has discussed these concepts for $\mathscr{C} = T_0$-spaces and continuous maps, $\mathfrak{J} =$ (topological) embeddings. He has found the exceptional phenomenon that although every T_0-space has a unique maximal essential extension, this, however, need not be an injective hull, i.e. may fail to be injective. It is the purpose of this note to show that the same phenomenon is available in case $\mathscr{C} = T_1$-spaces and continuous maps, $\mathfrak{J} =$ (topological) embeddings. The verification in this case is much easier than the proofs in [1] for T_0-spaces.

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In the following we refer to the category $\mathscr{C}$ of T_1-spaces and continuous maps (to be denoted by $\mathscr{T}_1$) and to the class $\mathfrak{J}$ of (topological) embeddings in $\mathscr{C}$ without always explicitly mentioning this. The central notion for the characterization of $\mathscr{T}_1$-essentially complete spaces is that of a *degenerate* point.

DEFINITION. A point p of a T_1-space X is called "*degenerate*" iff every neighborhood of p is co-finite in X.

LEMMA 1. *Let $f: X \hookrightarrow Y$ be an essential embedding in $\mathscr{T}_1$, and let $p \in Y - X$. Then p is a degenerate point of Y.*

Proof. Let Z be a space with the same points as Y: A subset M of Z is declared to be open in Z iff it is open in Y and $p \notin M$ or $Z-M$ is finite. If $h: Y \to Z$ maps every point identically, then h is continuous and $g := hf: X \to Z$ is an embedding. Thus, by the essentiality of f, $h: Y \to Z$ is a homeomorphism.

LEMMA 2. *Let $f: X \hookrightarrow Y$ be an essential embedding in $\mathscr{T}_1$. Then $Y-X$ consists of at most one point.*

Proof. Suppose, on the contrary, $p, q \in Y - X$, $p \neq q$. Then $h: Y \to Y - \{q\}$ with $h(y) = y$ iff $y \neq q$, $h(q) = p$ is continuous by lemma 1. Since $hf: X \to Y - \{q\}$ is an embedding, this contradicts the essentiality of f.

For an arbitrary T_1-space X let $|X^{\bullet}| = |X| \cup \{\omega\}$ with $\omega \notin |X|$ ($|?|$ denotes the underlying set of ?). The open sets of X and the co-finite subsets of $|X^{\bullet}|$ are declared to be the open sets of $X^{\bullet}$. The embedding $X \hookrightarrow X^{\bullet}$ is, obviously, the unique extension of X in $\mathscr{T}_1$ by adjoining ω as a degenerate point.

LEMMA 3. *If a T_1-space X has a proper essential extension in $\mathscr{T}_1$, then this is, up to an isomorphism, the embedding $X \hookrightarrow X^{\bullet}$.*

Proof. Immediate from lemmas 1, 2.

LEMMA 4. *A T_1-space X is a retract of $X^{\bullet}$ iff X contains a degenerate point p.*

Proof. (a) Let $r: X^{\bullet} \to X$ be a retraction, i.e. $r \mid X = id_X$. Let V be a neighborhood of $p := r(\omega)$ in X. Since $r^{-1}[V] = V \cup \{\omega\}$ is co-finite in $X^{\bullet}$, V is co-finite in X.

(b) The map $h: X^{\bullet} \to X$ with $h \mid X = id_X$ and $h(\omega) = p$ is easily seen to be continuous.

THEOREM 1. (a) *If a T_1-space X has a degenerate point, then X is $\mathscr{T}_1$-essentially complete, i.e. does not admit any proper essential extension in $\mathscr{T}_1$.*

(b) *If a T_1-space X does not have a degenerate point, then the embedding $f: X \hookrightarrow X^{\bullet}$ is the unique (up to ...) proper essential extension of X in $\mathscr{T}_1$.*

Proof. (a) By lemma 4, X is a retract of $X^{\bullet}$, thus $X \hookrightarrow X^{\bullet}$ fails to be an essential extension. Thus—by lemma 3—there is no proper essential extension of X in $\mathscr{T}_1$.

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(b) Suppose $h: X^\bullet \to Y$ is a continuous map such that $g := hf: X \to Y$ is an embedding. It is convenient to consider g as an inclusion of the subspace X into Y. If X does not contain a degenerate point, then $h(\omega) \notin X \subseteq Y$ (otherwise we would obtain a retraction $X^\bullet \to X$). Let O be open in X, then $O = V \cap X$ with V open in Y, hence $0 = h^{-1}[V - \{h(\omega)\}]$, since $h(\omega) \notin h[0]$. If M is co-finite in $X^\bullet$, then $M = h^{-1}[h[M] \cup (Y - h[X^\bullet])]$ is a pre-image of a co-finite subset of Y. As a consequence, h is an embedding. For the uniqueness of $f: X \hookrightarrow X^\bullet$ see lemma 3.

Corollary 1. *In the category $\mathcal{T}_1$ every space has a unique (up to ...) maximal essential extension.*

Theorem 2. *A T_1-space X is injective in $\mathcal{T}_1$ iff X is a singleton (i.e. card $X = 1$).*

Proof. (1) The identity on the empty space $\varnothing$ is not extendible over any embedding $\varnothing \hookrightarrow X$ with $X \neq \varnothing$, hence $\varnothing$ is not injective.

(2) Suppose card $X \geq 2$. Let Y be a set containing X. Y is made into a topological space by declaring $\varnothing$ and all sets of the form $V \cup M$ with V open in X and $M \subseteq Y - X$ co-finite to be open in Y. Every fiber $f^{-1}[\{x\}]$ of a non-constant continuous map $f: Y \to X$ is a proper closed subset of Y, hence has cardinality at most $\aleph_0 + \text{card}\, X$. Since there are at most card X many fibers, Y cannot have cardinality strictly greater than $\aleph_0 \pm \text{card}\, X = (\aleph_0 + \text{card}\, X) \cdot \text{card}\, X$. A suitable choice of Y now guarantees that the identity on X is not extendible over $X \hookrightarrow Y$.

Corollary 2. *A T_1-space X has an injective hull in $\mathcal{T}_1$ iff it has cardinality at most one.*

Remarks. (a) It may be worth pointing out that in the proof of theorem 2, X is closed in Y, but no neighborhood of X in Y can be retracted onto X, so X is not even an "*absolute neighborhood retract*" for the category $\mathcal{T}_1$. The latter, generally weaker notion than injectivity, plays a great role in the classical literature (for metrizable spaces, and, resp., for compacta; cf. [4] chap. IV, V).

(b) The method of weakening of the neighborhood filter of a point p in a T_1-space X employed in the proof of lemma 1 goes back to an idea of W. J. Thron ([7] p. 675/676 for X = unit interval). It has been pointed out in the proof of [5] 2.10 that this weakening is also admissible in the category $\mathcal{S}ob$-$\mathcal{T}_1$ of sober T_1-spaces and continuous maps, i.e. it does not lead outside the category. By [5] 2.8 X is sober iff so is X. Thus all of the above results up to theorem 1 and corollary 1 carry over to the category $\mathcal{S}ob$-$\mathcal{T}_1$. The proof of theorem 2, however, does not work in $\mathcal{S}ob$-$\mathcal{T}_1$, since the space Y constructed there is irreducible without generic point (provided that Y is chosen sufficiently large), hence is not sober.

(*c*) For a *finitary variety* $\mathcal{K}$, i.e. a category given by all algebras of a given (finitary) type which satisfy a specified set of equations and all algebra homomorphisms between them, the "*boundedness*" of essential extensions is one crucial ingredient for the existence of enough injectives (with regard to all *monomorphisms* = *embeddings*), the other one being that in any *pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[u]{} & D \end{array}$$

the morphism u is an embedding, whenever f is ([3] prop. 5). These conditions also hold in $\mathcal{T}_1$, and thus $\mathcal{T}_1$ has no non-trivial injectives in the presence of conditions which *elsewhere* guarantee the existence of enough injectives.

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