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INDETERMINACY AND PERIOD LENGTH UNDER BALANCED BUDGET RULES

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We analyze the importance of the frequency of decision making for macroeconomic dynamics, in the context of a simple, well-known business cycle model with balanced budget rules. We explain how the frequency of decision making (period length) and the measurement unit of time (calibration frequency) differ and examine how local stability is affected by changes in the period length. We find that as the period grows longer, indeterminacy occurs less often. This may have significant quantitative implications: for the model at hand, there is a wide range of economically relevant labor tax rates (from 30% to 38%) for which the continuous-time model gives indeterminacy, whereas the discrete-time model has determinate dynamics.

Keywords: Calibration, Period Length, Local Indeterminacy, Discounting, Depreciation

1. INTRODUCTION

Modern macroeconomic theory relies on the construction, parameterization, and solution of dynamic optimization problems. The interest in dynamic optimization problems is due to their close relation to dynamic general equilibrium models. In economies where the fundamental welfare theorems hold, one can find the equilibrium of an economy by focusing on the corresponding planner's optimization problem; but even when the welfare theorems fail to apply, the various agents in the model may have to solve dynamic optimization problems. As a result, the

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equilibrium conditions of the model are typically described by a set of differential or difference equations. To construct, parameterize, solve, and intuitively interpret the results of such settings, one needs to make an assumption on the frequency with which economic activities take place or decisions are made, i.e., the period length. In this paper we explore and analyze some issues arising from this assumption in the context of the business cycle model with a balanced budget rule of Schmitt-Grohé and Uribe (1997).

The contributions of our work are the following. First, we write up a general discrete-time version of the model, where the period length is explicitly modeled as a free parameter h. By letting h vary, we are able to disentangle two related but different concepts, namely the *period length* (determined by the frequency of decision making) and the *calibration frequency* (determined by convention via the measurement unit of time, e.g., a year). These two can easily be confused in discrete time, because they coincide in commonly used models, yet in continuous time the difference is clear: the period length is zero (economic agents make decisions continuously), whereas the calibration frequency can be whatever the modeler deems suitable (for example, years or quarters) and is of no true economic significance.

Second, the model we use provides a good platform for understanding the importance of the period length for real local indeterminacy, i.e., multiplicity of equilibria in dynamic macroeconomic models. Indeterminacy has been shown to be present in many popular models and is usually associated with the possibility of belief-driven sunspot fluctuations; it is therefore imperative (from a policy design perspective) to have a good understanding of the role of indeterminacy in business cycle fluctuations and economic growth.[1] With this context in mind, we show that standard discrete and continuous versions of the "same" model can lead to different conclusions regarding local stability properties. Our framework allows us to consider a whole range of intermediate cases and pinpoint the exact period length at which the switch from determinacy to indeterminacy occurs. We find an interesting regularity: the smaller the period length is (i.e., the more frequently decisions are made), the larger the ranges of indeterminacy are. This pattern follows from a simple intuitive explanation: indeterminacy arises when expectations about the future affect current investment decisions in such a way as to render the expectations self-fulfilling. Specifically, indeterminacy arises if the effect on current decisions is strong enough. The closer the future is (i.e., the shorter the period length and the sooner agents are allowed to make decisions again), the stronger the effect on today and therefore the easier it is for expectations to be self-fulfilling.[2] For the framework of Schmitt-Grohé and Uribe (1997), the implication of this result is quantitatively significant: there is a wide range of economically relevant labor tax rates (from 30% to 38%) for which the continuous-time model gives indeterminacy, whereas the discrete-time model has determinate dynamics. This reinforces the importance of carefully separating and determining the period length and the parameter values in such models, particularly when these are used for macroeconomic policy analysis.

Third, we find that the sensitivity of the indeterminacy region to the period length depends crucially on the rate of time preference and the capital depreciation rate. We find that as the rate of time preference increases (i.e., as agents become less patient), the ranges of indeterminacy decrease (i.e., indeterminacy occurs less often). This is because as agents become less patient, they value the future less; therefore expectations about the future have a smaller impact on today's decisions and are thus less likely to be self-fulfilling.[3]

There is a significant literature studying the relationship between discrete- and continuous-time versions of a model. Telser and Graves (1968) and Leung (1995) discuss how uncertainty can give rise to subtle differences between the two setups. Li (2003) and Carlstrom and Fuerst (2005) point out a reversal of the monetary policy prescription in the model of Dupor (2001) when one moves from a continuous- to a discrete-time specification. Mino et al. (2008) point out a similar change in the predictions of a two-sector endogenous growth model. Foley (1975), Turnovsky and Burmeister (1977), and Karni (1979) look into different specifications of asset market equilibrium and how those can be translated from a discrete- to a continuous-time setting. We differ from this literature in that we analyze the importance of period length as opposed to two extreme cases of discrete versus continuous time. This allows us to further refine some of the statements made in the discrete- versus continuous-time literature. Mercenier and Michel (1994) and Bosi and Ragot (2009) take a different approach: They start by assuming that continuous-time models provide a good description of the world and are interested in obtaining optimal approximations of such models in discrete time. Our approach is more agnostic: we simply recognize that discrete- and continuous-time versions of the same type of economies are essentially different models, and our aim is to clarify the importance of period length in such models.[4]

More recently, the idea of disentangling the frequency of decision making from the calibration frequency has also been explored by Oberfield and Trachter (2011). They consider the question in a search model of money and find that higher frequency reduces the multiplicity of equilibria.[5] The key difference is that, as the period length shrinks, the probability of being matched with a trading partner is reduced and, as a result, the current trading strategy exerts smaller effects on the future. In our framework, each individual decides and acts for sure within each period. The probability effect is thus absent and smaller period length implies larger intertemporal interactions. Finally, Medio (2011) considers a class of optimal growth models, explicitly models period length so that it determines the frequency of decision making, and examines its importance for bifurcation analysis and complex dynamics. He finds that high discount rates are more likely to generate cycles and chaotic dynamics, but such rates may only be relevant for unreasonably large calibrations of the measurement unit of time.

Closest to our work is the paper of Hintermaier (2005). He demonstrates that the stability properties of the model of increasing returns of Benhabib and Farmer (1994) depend on the period length. We show the same is also true in the (related) model of Schmitt-Grohé and Uribe (1997), which has the additional attractive feature of exhibiting local indeterminacy for empirically plausible parameter values. We also differ in that we explicitly model the period length as a free parameter and are able to distinguish between calibration frequency and decision-making frequency. This allows us to provide an intuitive economic interpretation for what he calls the *sunspot paradox*; i.e., we give an explanation for why some frequencies of decision making imply aggregate stability whereas others imply the existence of sunspot equilibria.

The rest of the paper is organized as follows. Section 2 briefly describes the model of Schmitt-Grohé and Uribe (1997) for a general period length. Section 3 addresses the differences between period length and calibration frequency and shows the implications of these differences for the local stability properties of the model. Section 4 concludes.

2. A MODEL WITH BALANCED BUDGET RULES

We present the model of Schmitt-Grohé and Uribe (1997) with the added feature that the period length is a free parameter denoted by h.[6] We maintain the assumption that time evolves continuously. A discrete-time model can be thought of as a continuous-time model where the time line $[0, \infty)$ has been partitioned into intervals of length h: $[0, h)$, $[h, 2h)$, etc. These intervals are called periods and can be indexed by $\frac{t}{h} \in \{0, 1, 2, \ldots\}$, where t is the time instant at the beginning of each period, so that $t \in I = \{0, h, 2h, \ldots\}$.[7] This continuous-time interpretation of discrete-time models also requires the following assumption: stock (state) variables can only be adjusted at the beginning of a period (at $t \in I$). We maintain this assumption throughout the rest of the paper.

The economic environment consists of households, firms, and a government. Households decide on capital and labor supply as well as demand for the single good produced. Firms decide on capital and labor demand and use these inputs to produce the single good and supply it in the goods market. Thus, there are three markets in this economy, namely the goods, capital, and labor markets. All markets are perfectly competitive. The government uses proportional labor taxes to finance exogenous spending.

To be more precise, households own the capital stock k_t and rent it to the firms at an *instantaneous* real rate r_t. The capital stock, being the state variable in this model, can only be adjusted at the beginning of a period. This is intended to capture the essence of a standard discrete-time model. We find it helpful to think of this as an assumption on the market structure: capital markets are closed most of the time and only open at discrete points in time $t \in I$. We also assume that the arrangement is one that stipulates a constant instantaneous rate of return r_t within the period $[t, t+h)$.[8] At any instant $t+s,\ s \in [0, h)$, households rent labor $n_t(s)$ to the firms and are compensated at a rate $w_t(s)$. The government taxes labor income at a rate $\tau_t(s)$. Thus the instantaneous flow of income for a household at any given instant $t+s$ is given by $r_t k_t + (1-\tau_t(s))w_t(s)n_t(s)$. This income can be used for consumption $c_t(s)$ or savings $S_t^k(s)$. The budget constraint at any instant

$t+s$ reads

$$c_t(s) + S_t^k(s) = r_t k_t + (1-\tau_t(s))\, w_t(s) n_t(s). \tag{1}$$

Within a period, savings are accumulated. At the end of the period, households receive the depreciated capital stock back from firms and can add their accumulated savings to this capital stock before renting it out again for the following period. The capital accumulation equation is thus

$$k_{t+h} = (1-\delta h)\, k_t + \int_0^h S_t^k(s)\, ds. \tag{2}$$

Here we have assumed that the instantaneous rate of depreciation is constant and equal to δ, so that the fraction of capital depreciating within a period is equal to δh. As is common in the literature, we have also assumed that depreciation affects the capital stock being used in production, whereas savings not yet put into production do not depreciate.

Households decide on optimal consumption and savings rates, taking prices as given, in order to maximize their utility, given by

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h u\left(c_t(s), n_t(s)\right) ds, \tag{3}$$

where $u(c_t(s), n_t(s))$ is the instantaneous flow of utility and $\int_0^h u(c_t(s), n_t(s))ds$ is the total flow of utility within a period.[9] Similarly to the depreciation rate, the time preference rate ρ is an instantaneous rate; the corresponding rate over a period of length h is ρh. It should therefore be clear that δ and ρ are independent of the period length. They do depend, however, on the calibration frequency. In other words, we distinguish between the period length and the calibration frequency, two concepts that coincide in a standard discrete model. We return to this subtle distinction at the beginning of the next section.

Firms use the rented inputs k_t and $n_t(s)$ in a constant–returns to scale production function $F(k_t, n_t(s))$ to maximize the sum of discounted profits

$$\sum_{t\in I} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h \frac{u_{c,t}(s)}{u_{c,0}(0)} \left[F(k_t, n_t(s)) - r_t k_t - w_t(s) n_t(s)\right] ds. \tag{4}$$

Households are assumed to be identical, which allows us to focus on the representative household and ensures that the firm's objective is well defined. In particular, the firm is owned by the representative household, which has a unique valuation of instantaneous profits at any point in time $t+s$. From the point of view of time 0, the value of one unit of profits at time $t+s$ is simply the price of a contingent claim in terms of time-0 consumption, given by

$$\left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \frac{u_{c,t}(s)}{u_{c,0}(0)}. \tag{5}$$

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Finally, the government uses the labor tax to finance a constant instantaneous flow rate of spending G and, crucially for indeterminacy, is required to maintain a balanced budget

$$G = \tau_t(s) w_t(s) n_t(s). \tag{6}$$

The model is closed by assuming that all markets clear. A precise definition of equilibrium is given in the Appendix.[10]

The first-order conditions of the household for consumption and labor are given by

$$\lambda_t(s) = u_{c,t}(s), \tag{7}$$

$$u_{n,t}(s) = -\lambda_t(s)\left(1 - \tau_t(s)\right) w_t(s) \tag{8}$$

for all $s \in [0, h)$, $t \in I$, where $\lambda_t(s)$ is the multiplier on the budget constraint. Condition (7) is the standard first-order condition for consumption, ensuring that, at an optimum, the marginal value of income equals the marginal utility of consumption. Condition (8) is the standard first-order condition for labor supply, ensuring that the marginal utility of leisure is equalized to the marginal value of income times the after-tax wage rate. The first-order condition with respect to $S_t^k(s)$ gives

$$\lambda_t(s) = \mu_t \tag{9}$$

for all $s \in [0, h)$, $t \in I$, where μ_t is the multiplier on the capital accumulation constraint. Optimality of the savings decision necessitates equating the marginal cost and benefit of savings. The marginal cost of savings at instant $t + s$, $\lambda_t(s)$, arises from reduced resources available for consumption. This cost depends on the specific instant $t + s$ because consumption at different points in time could, in principle, be valued differently. The marginal benefit comes from the fact that these savings will eventually become investment and be added to the capital stock, k_{t+h}. This will only happen the next time the capital market opens, so the specific instant within period t at which saving occurs is irrelevant. The implication is that the shadow value of income must be constant within a period, because savings can be costlessly reshuffled across the period leaving the overall accumulated savings $\int_0^h S_t^k(s)ds$ unaffected. Finally, the capital Euler equation is given by

$$\mu_t = \left(\frac{1}{1+\rho h}\right)\left[\mu_{t+h}\left(1 - \delta h\right) + \int_0^h \lambda_{t+h}(s) r_{t+h} ds\right] \tag{10}$$

$$\Rightarrow \mu_t = \frac{\mu_{t+h}}{1+\rho h}\left[1 + \left(r_{t+h} - \delta\right) h\right] \tag{11}$$

for all $t \in I$, where we have used (9) to substitute for $\lambda_{t+h}(s)$. Note that capital accumulation depends on the overall return hr_{t+h} earned over a period h. In

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addition, a transversality condition must hold:

$$\lim_{T\to\infty}\left(\frac{1}{1+\rho h}\right)^{\frac{T}{h}} u_{c,T}(0)k_{T+h}=0. \tag{12}$$

On the firm's side, optimality requires that the wage rate be equal to the marginal product of labor at every instant $t+s$:

$$w_t(s)=F_n\left(k_t,n_t(s)\right), \quad \text{for all } s\in[0,h) \text{ and } t\in I. \tag{13}$$

Optimal capital supply given r_t must satisfy

$$\int_0^h u_{c,t}(s)r_t ds=\int_0^h u_{c,t}(s)F_k\left(k_t,n_t(s)\right)ds, \quad \text{for all } t\in I. \tag{14}$$

Given (6)–(9) and (13) it is straightforward to show that $c_t(s)$, $n_t(s)$, and $u_{c,t}(s)$ are all constant within a period, so that the preceding condition can be more concisely written as

$$r_t=F_k\left(k_t,n_t\right) \quad \text{for all } t\in I. \tag{15}$$

This is the standard capital demand equation stating that the rental rate of capital must equal the marginal product of capital. Conditions (6)–(15), together with the goods market–clearing condition

$$c_t(s)+S_t^k(s)+G=F\left(k_t,n_t(s)\right) \ \text{ for all } s\in[0,h) \text{ and all } t\in I, \tag{16}$$

characterize the equilibrium in this model.

It is straightforward to establish that all flows are constant within a period. This is a direct outcome of the assumption of no within-period discounting and makes this model equivalent to a discrete-time model with period length h.

Our general model nests the discrete- and continuous-time versions of the model as special cases. The important condition is the Euler equation, which can be rearranged as

$$\frac{\mu_{t+h}-\mu_t}{h}=\frac{\rho-r_{t+h}+\delta}{1+r_{t+h}h-\delta h}\mu_t. \tag{17}$$

When $h=1$, this is the standard discrete-time Euler equation

$$\mu_{t+1}-\mu_t=\frac{\rho-r_{t+1}+\delta}{1+r_{t+1}-\delta}\mu_t. \tag{18}$$

As $h\to 0$, this converges to the standard continuous-time Euler equation

$$\dot{\mu}_t=\left(\rho-r_t+\delta\right)\mu_t. \tag{19}$$

For general h, capital k_t is rented out once at the beginning of the period; whatever is saved throughout the period remains inoperative in the possession of consumers. At the end of the period, the rented (depreciated) capital returns to the possession of the households and is added to the newly accumulated capital.

This new capital stock k_{t+h} remains in the possession of the households until the beginning of the next period, when it is rented out again. This model has an inherent delay (just like any discrete-time model), because at any point in time within the period there exists capital that is not used for production. This type of delay is not the same as what is commonly known as *time-to-build* delay. The classic example of time-to-build delay is given in Kydland and Prescott (1982). In that model, an h-period delay implies that investment at t will only produce capital at $t+h$, where h is an integer. This still allows new investment to take place at $t+1$, which will yield capital at $t+h+1$ and so on. The continuous-time counterpart of this, studied in Licandro and Puch (2006), is one where investment can take place continuously but productive capital is only created after an interval h, where now h is a real number. In our model, savings take place continuously within a period at every instant $t+s$, $s \in [0, h)$ but the accumulated savings are suddenly invested and produce capital at $t+h$, regardless of whether they were saved at the beginning of the period or right before the end. Put differently, in our case the delay in putting capital into production varies and depends on the instant within the period at which the capital is put aside. Here we have illustrated that this different arrangement is an implicit assumption of any standard discrete model, even in the absence of a time-to-build delay. We have also provided a market (equilibrium) interpretation of this assumption.

One could also think of this model in relation to the work of Turnovsky (1977). Turnovsky interprets a discrete-time model as a setting where time is continuous, but because of adjustment costs, firms can alter their capital only in a discrete manner. He then shows that the standard limiting continuous-time relation between capital and investment, $i = \dot{k}$, is true only under the restrictive assumption of no adjustment costs. In a continuous-time model with adjustment costs, it would not be true in general: the demand for investment i cannot be matched with a change in capital, because capital is not perfectly malleable. Thus, in our setting we can interpret the fact that the capital market is closed within a period as an infinite adjustment cost.

3. INDETERMINACY AND PERIOD LENGTH

In this section we investigate the effect of period length h on the incidence of indeterminacy in this model. We begin with a discussion that clarifies the distinction between period length and calibration frequency. Subsequently, we calibrate the model in a way that ensures that our calibration remains consistent, in the sense of Hintermaier (2005), as we change the period length h. Finally, we show that different, commonly used values for h can lead to economically large and relevant differences in the indeterminacy regions and provide an intuitive interpretation of this finding.

3.1. Period Length and Calibration Frequency

In any dynamic model, there are two important concepts that have to do with modeling time: the *unit of measurement* of time and the *frequency* with which

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activities take place or decisions are made. Here we explain how the first relates to the calibration frequency and the second relates to the period length.

Given a continuous timeline, the unit of measurement of time gives meaning to the quantity $\int_0^1 y_t(s)ds$, where $y_t(s)$ is an instantaneous flow rate. For example, if the measurement unit is a year, this quantity measures the total flow of y in one year, starting at t. The choice of units dictates the values for parameters such as the discount rate and the depreciation rate. When the unit is a year, the parameter δ will denote the fraction of capital that depreciates over a year. Therefore, once the unit of measurement is determined, we have a calibration frequency for the model.

On the other hand, the frequency with which decisions on stock variables are made or economic activities take place determines the period length h. For example, consider a model with calibration frequency equal to a year. If decisions are to be updated four times a year then the period length is $h = 1/4$ and the corresponding parameters $\rho h = \rho/4$ and $\delta h = \delta/4$ now give values for discount and depreciation rates over a quarter. If, in the same model, decisions are made only once a year, then $h = 1$ and the corresponding parameters $\rho h = \rho$ and $\delta h = \delta$ now denote yearly rates. As the frequency of decision making increases and $h \to 0$, we retrieve the standard continuous-time model in the limit, without losing the correct interpretation of δ and ρ.

In short, the measurement unit of time and the period length are not necessarily the same, but in order to obtain the *standard* discrete-time model we have to set $h = 1$. This is because in the standard discrete-time framework, the calibration frequency and the frequency of decision making necessarily coincide. Using our general setup as described in Section 2, we are able to disentangle the two concepts in a transparent way.

In our experiments, we fix the unit of measurement of time to a year. As a result, the parameters (e.g., ρ) will remain fixed as we change h. In this way we can obtain a consistent comparison of a model with a one-year period length, a model with a one-quarter period length, and a model with a zero period length (continuous time), all of them calibrated at yearly frequencies.

3.2. Calibration and Steady State

We follow Schmitt-Grohé and Uribe (1997) in choosing a Cobb–Douglas production function

$$F(k_t, n_t(s)) = k_t^{s_k} n_t(s)^{s_n}, \tag{20}$$

with $s_k + s_n = 1$ and $s_k, s_n > 0$ and utility that is separable in consumption and labor and linear in labor as in Hansen (1985),

$$u(c_t(s), n_t(s)) = \log c_t(s) - An_t(s). \tag{21}$$

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Using these specifications, we can derive the steady state output-to-capital ratio from the capital Euler equation

$$1 = \frac{1}{1+\rho h}\left[1 + \left(s_k \frac{\bar{y}}{\bar{k}} - \delta\right) h\right] \Rightarrow \frac{\bar{y}}{\bar{k}} = \frac{\rho+\delta}{s_k}, \tag{22}$$

where $\bar{k}$ is the steady state capital stock and $\bar{y} = \bar{k}^{s_k}\bar{n}^{s_n}$ is the instantaneous flow rate of output at steady state and $\bar{n}$ is the steady state flow of labor supply. For a model with period length h, the flow of output over the entire period is $\bar{y}h$ and thus the corresponding output-to-capital ratio is equal to $\frac{\rho+\delta}{s_k}h$. By construction of the model, the principle for consistent calibration used by Hintermaier (2005), who builds upon the ideas in Aadland and Huang (2004), is satisfied by our calibration strategy. We fix parameters using the yearly calibration of Schmitt-Grohé and Uribe, i.e., $s_k = 0.3$, $s_n = 0.7$, $\delta = 0.1$, and $\rho = 0.04$.

3.3. Dynamic Adjustment and Indeterminacy

Since Schmitt-Grohé and Uribe's (1997) work, it has been well known that the balanced budget requirement introduces the potential for indeterminacy into the real–business cycle framework. The reason is as follows. An increase in current taxes shifts labor supply downward (see equation (8)) and leads to a reduction in equilibrium employment. The dependence of labor supply on taxes and the balanced budget create the possibility of self-fulfilling expectations. In particular, if households expect high future labor taxes, they reduce future labor supply. This leads to a reduction in future equilibrium labor. The resulting fall in the expected marginal product of capital implies, through the Euler equation, that current marginal utility (λ_t) has to fall. This, in turn, leads to a combination of increased consumption (equation (7)) and leisure (equation (8)). The expectation of high future taxes thus leads to less work today, which, in turn, forces the government to increase current labor income taxes to maintain a balanced budget. If the resulting increase in current taxes is large enough, such a situation can be an equilibrium and expectations become self-fulfilling.

Schmitt-Grohé and Uribe (1997) consider the possibility of such indeterminacy as a function of steady state tax rates. We perform a similar analysis in our context and look at how the ranges of indeterminacy depend on the period length h. Strictly speaking, the labor tax rate in this model is an endogenous variable and G is an exogenous constant parameter. Because of the existence of a Laffer curve, there are two steady state labor taxes for a given G. However, like Schmitt-Grohé and Uribe (1997), we choose to treat the labor tax rate as a parameter and work out what the corresponding G is. The Laffer curve in the steady state is identical to the one derived by Schmitt-Grohé and Uribe (1997), it is independent of h, and its peak occurs at

$$\tau^* = \frac{\rho + \delta s_n - \sqrt{s_k \rho\left(\rho + \delta s_n\right)}}{s_n\left(\rho+\delta\right)}.$$

We provide a more detailed discussion of the shape and characteristics of the Laffer curve in Section 3.4, where we explain the importance of discounting and depreciation for indeterminacy.

We now examine the local stability properties of the model by log-linearizing the equilibrium conditions around the steady state (see Appendix A.2 for detailed derivations). Let

$$c_{11} = -(\rho+\delta)\frac{s_n(1-\tau)}{s_k-\tau},$$

$$c_{12} = -(\rho+\delta)\frac{s_n\tau}{s_k-\tau},$$

$$c_{21} = \frac{\rho+\delta}{s_k}\left[\frac{s_n(1-\tau)}{s_k-\tau}+s_c\right],$$

$$c_{22} = (\rho+\delta)\frac{1-\tau}{s_k-\tau}-\delta,$$

where $s_c = \bar{c}/\bar{y}$ is the steady state ratio of consumption to output. The Jacobian describing the dynamic adjustment for an arbitrary h is then given by

$$D(h) = \begin{pmatrix} \dfrac{c_{11}+hc_{12}c_{21}}{1+\rho h-hc_{11}} & \dfrac{c_{12}(1+hc_{22})}{1+\rho h-hc_{11}} \\ c_{21} & c_{22} \end{pmatrix}, \tag{23}$$

where the local dynamics is described by

$$\begin{pmatrix} \dfrac{\hat{\mu}_{t+h}-\hat{\mu}_t}{h} \\ \dfrac{\hat{k}_{t+h}-\hat{k}_t}{h} \end{pmatrix} = D(h)\begin{pmatrix} \hat{\mu}_t \\ \hat{k}_t \end{pmatrix}, \tag{24}$$

and a hat denotes a log deviation from steady state.

When $h \to 0$, the matrix that characterizes the dynamics is $C = (c_{ij})$, $i, j = 1, 2$, as in Schmitt-Grohé and Uribe (1997). In the case of continuous time, C has two real and negative eigenvalues (indeterminacy) whenever $s_k < \tau < \tau^*$, where τ^* gives the peak of the Laffer curve. For an arbitary period length h, it is possible show that indeterminacy can only occur for tax rates in the same range, i.e., for $s_k < \tau < \tau^*$ (see Appendix A.3). The upper bound τ^* is sharp; however, the lower bound is sharp only as $h \to 0$. Although we can show that indeterminacy occurs only for tax rates $\tau \in (\hat{\tau}(h), \tau^*)$, where $s_k < \hat{\tau}(h)$, further analytical characterization of the lower bound $\hat{\tau}(h)$ does not generate any meaningful insights; for this reason, we explore the properties of the lower bound of indeterminacy numerically.

Figure 1 shows the stability properties for the general discrete-time model when the two key parameters, namely the labor tax rates and the period length, are varied. We let $\tau \in [0, 1)$ and $h \in [0, 4]$. Gray areas indicate saddlepath stability and white

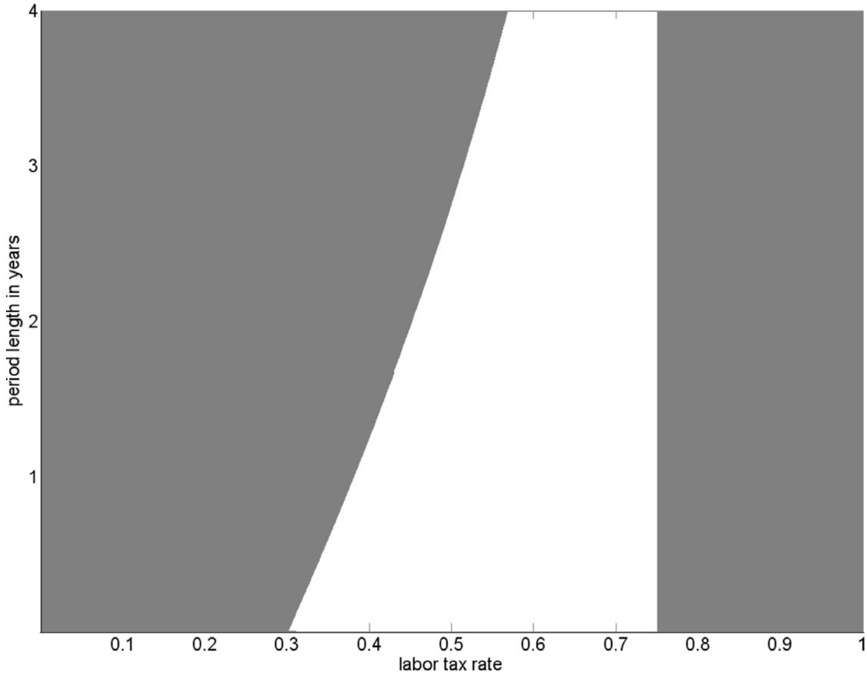

FIGURE 1. Stability properties for the general model. Gray areas show saddlepath stability and white areas show indeterminacy.

areas indicate indeterminate dynamics. As h increases, the indeterminacy regions become smaller overall, despite the fact that the upper bound remains constant and equal to 0.75. For $h \to 0$, the range of indeterminacy is $\tau \in (0.3, 0.75)$, just as in Schmitt-Grohé and Uribe (1997), whereas for $h = 1$, the range of indeterminacy is $\tau \in (0.38, 0.75)$. In other words, for a labor tax rate between 30% and 38%, the result of Schmitt-Grohé and Uribe is reversed when we move to the standard discrete-time setup.

To understand whether and how the period length h matters for local determinacy, consider the intuition for indeterminacy explained in the first paragraph of Section 3.3. The sequence of arguments that leads to indeterminacy can fail for two distinct reasons. The first is that an increase in taxes may not necessarily increase revenues for the government. This will only occur for high levels of taxes, i.e., when the economy finds itself on the right-hand side of the peak of the Laffer curve and $\tau > \tau^*$. With the current calibration, the peak of the Laffer curve is at $\tau^* = 0.75$, so indeterminacy can only arise for $\tau < 0.75$. This upper bound does not depend on h, because the steady state that gives the peak of the Laffer curve is invariant to changes in h.

The second reason that the argument may fail is that the increase in current taxes as a response to an expected increase in future taxes may not be large enough. This will occur for relatively low taxes: when taxes are low, the economy is on a steeply increasing part of the Laffer curve, so that revenues can be increased substantially with small changes in the tax rate. As we move along the Laffer curve, the increase in taxation required to raise a certain amount of extra revenues becomes larger. There is a threshold value for taxes $\hat{\tau}$ (h) at which the required increase in taxation is large enough to sustain an equilibrium with self-fulfilling expectations. As taxes are increased further, it becomes easier for expectations to become self-fulfilling, because balancing the budget in response to a reduction in labor requires ever larger increases in taxes (and this is abruptly stopped at τ^*, from where onward the government cannot raise more revenues by increasing taxes).

Our numerical exercise quantitatively pinpoints the lower bound of indeterminacy and shows that the period length does matter for this; as h increases, the range where indeterminacy occurs becomes smaller. The intuition is as follows. To get self-fulfilling expectations, it must be that an increase in expected future labor tax rates leads to a *large enough* increase in current labor tax rates. This is because, on an equilibrium path, tax rates must converge monotonically to their steady state. Therefore, the question is how much current choices are affected by changes in expectations about the future. Not surprisingly, because agents discount the future, if the future is one year ahead it has less of an impact on today's choices than if it is one quarter (or an instant) away. Put differently, as the frequency of decision making decreases (i.e., as h becomes larger), the response of current employment to higher expected future tax rates is milder, and therefore tax rates must be larger to make indeterminacy relevant. This is a direct result of the effects of the period length on optimal intertemporal decisions, as described by the Euler equation. An equivalent way of seeing this is by considering the corresponding Bellman equation for the household problem, where the discounting of the continuation value governs the importance of the future for current decisions.

Finally, we wish to emphasize that the range of taxes for which the different dynamic properties arise is both wide and empirically relevant. The result of Schmitt-Grohé and Uribe is particularly important because many OECD countries' tax rates fall within or very close to the range of indeterminacy they computed. Looking at the estimated effective labor income tax rates in Mendoza et al. (1994), in 1988 the United States, the United Kingdom, Canada, and Japan had rates only just below 30%. Italy, Germany, and France, on the other hand, fell within the range of indeterminacy with rates at 40% or more. Of course, these rates vary over time and one can find years where the U.K. rate was above 30% and European rates were less than 40%. Volkerink and De Haan (2001) provide updated estimates for 18 OECD countries in 1992. The labor income tax rates reported vary between 25% and 45%. Roughly speaking, this is the range of cross-sectional variation across developed countries. The range of taxes for which a standard discrete- and a standard continuous-time model produce opposite results (30–38%) lies exactly in the middle of this and covers almost half of the interval width. These observations reinforce the importance of understanding the difference between period length and calibration frequency.

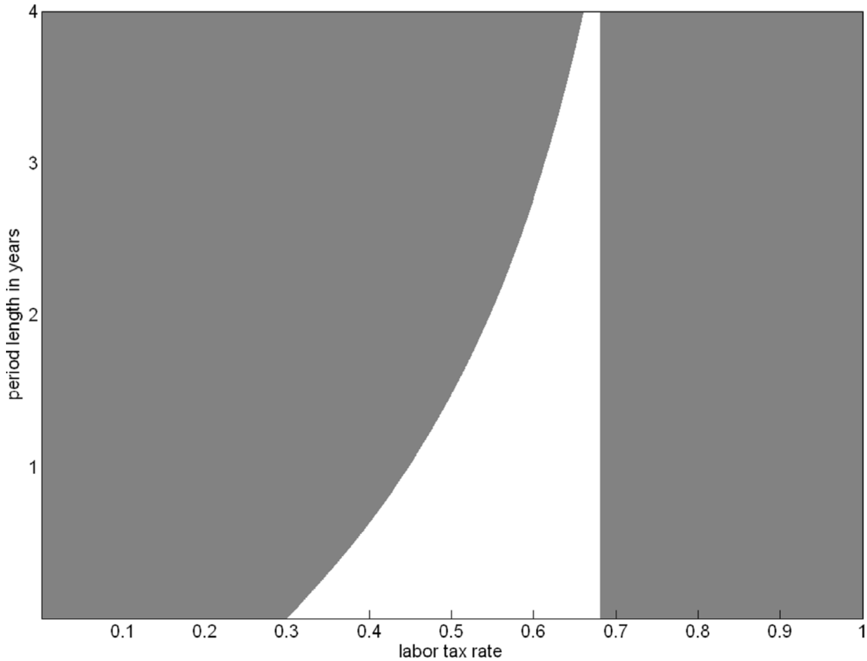

FIGURE 2. Stability properties for $\rho = 0.2$. Gray areas show saddlepath stability and white areas show indeterminacy.

3.4. Depreciation and Discounting

For a given period length, two parameters play an important role in indeterminacy in this model: the capital depreciation rate δ and the time preference rate ρ. Changes in ρ or δ will, in general, affect the regions of indeteminacy by shifting both the lower and the upper bounds. Although the two parameters have similar effects on the lower bound, their effects on the upper bound are opposite.

Starting with the time preference rate, comparing Figure 2 to Figure 1 illustrates that as ρ increases, i.e., as households become less patient, the range of indeterminacy becomes smaller for any given h. This is both because the upper bound moves to the left and because the lower bound moves to the right. The lower bound is, as explained, related to the strength of the effect the future has on today's actions. With more discounting, expectations about the future have a weaker effect on today's decisions and, as a result, these expectations are less likely to be self-fulfilling. The upper bound is given by τ^*, which can be shown to be decreasing in ρ. The intuition for why this is the case stems from the properties of the Laffer curve, specifically the condition determining the peak of the Laffer curve,

$$\frac{\partial G}{\partial \tau} = 0.$$

As is standard, this derivative involves a trade-off between the increase in revenues on the existing tax base and the decrease in revenues due to the endogenous reduction in the tax base. Using the fact that $G = \tau w\bar{n} = \tau s_n \kappa^{s_k}\bar{n}$, we obtain

$$\frac{\partial G}{\partial \tau} = s_n \bar{\kappa}^{s_k}\bar{n} + \tau s_n \bar{\kappa}^{s_k}\frac{\partial \bar{n}}{\partial \tau} = 0,$$

or equivalently

$$\frac{\partial \bar{n}}{\partial \tau}\frac{\tau}{\bar{n}} = -1.$$

The peak of the Laffer curve occurs when the elasticity of labor with respect to taxes is equal to -1. If the elasticity is larger than one (in absolute value), the endogenous reduction in the tax base from a marginal increase in taxes is too large and we are on the downward-sloping side of the Laffer curve. If the elasticity is less than one (in absolute value), then the reduction in the tax base is not very large and a marginal increase in taxes brings extra revenues; i.e., we are on the upward-sloping side. Straightforward algebra shows that this elasticity depends on ρ and δ:

$$\frac{\partial \bar{n}}{\partial \tau}\frac{\tau}{\bar{n}} = -\left(\frac{\tau}{1-\tau}\right)\left[\frac{s_k \rho}{\rho\left(1 - s_n \tau\right) + s_n \delta\left(1-\tau\right)}\right]$$

Crucially for our argument, its absolute value is increasing in ρ. Thus, as ρ increases, the tax base is more strongly affected by taxes and the peak of the Laffer curve shifts to the left. To summarize, because the lower bound of indeterminacy increases and the upper bound decreases, the indeterminacy region is squeezed unambiguously when ρ increases.

The effect of the depreciation rate on the possibility of indeterminacy is ambiguous. We present the regions of indeterminacy for the extreme case of complete depreciation of capital in Figure 3. From comparing Figure 3 to Figure 1, it becomes apparent that for any given h, both the upper bound and the lower bound of the indeterminacy range move to the right. Again, the upper bound is related to the peak of the Laffer curve, which can now be shown to be increasing in δ, because the elasticity of the tax base with respect to taxes decreases as δ increases (i.e., higher capital depreciation makes agents' labor choices less responsive to changes in taxes). Regarding the lower bound, an intuition similar to that for ρ applies. Specifically, recall that for any calibration frequency, δ represents the depreciation of capital over one unit of time. As δ increases, capital depreciates more overall, so when households make decisions today about future investment, future capital is less attractive for them and therefore it has a smaller impact on current decisions. The ranges of indeterminacy can therefore be larger or smaller when δ increases, depending on the frequency h. For example, the range of indeterminacy when δ increases is larger for $h \to 0$ and smaller for $h = 1$.

In some sense, it is not surprising that we observe these three results relating to h, δ, and ρ in all our examples. All three parameters are interrelated and reflect how

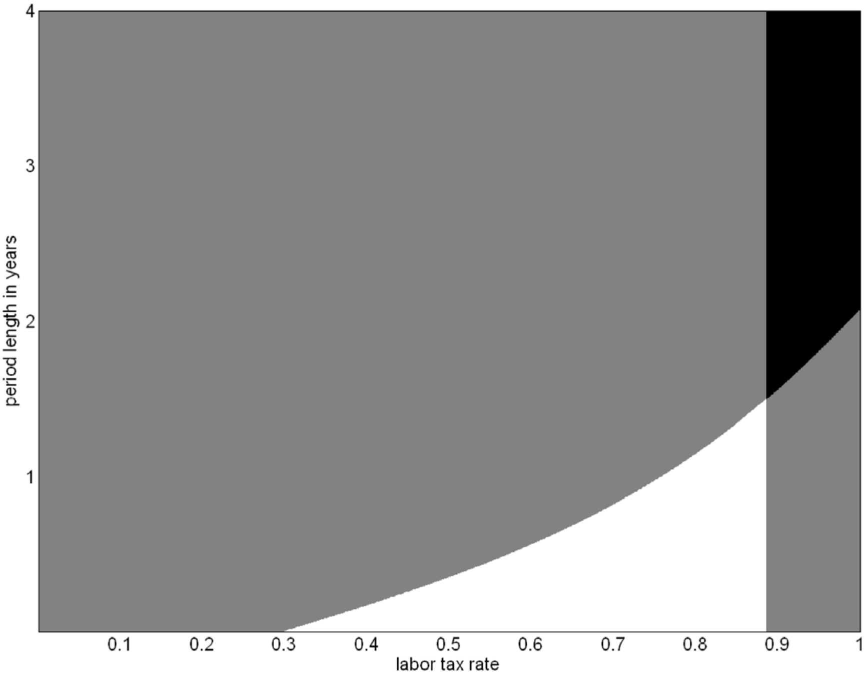

FIGURE 3. Stability properties for $\delta = 1$. Gray areas show saddlepath stability, white areas show indeterminacy, and black areas show parameters for which no solutions exist.

relevant the future is to making consumption/savings decisions today. For the cases of h and ρ we can clearly see that the more irrelevant the future becomes (i.e., the larger h and ρ are), the weaker is the intertemporal link that renders expectations self-fulfilling. The case of depreciation, where an increase in δ increases the upper bound of the indeterminacy range, serves as a manifestation of the importance of separating the period length h and the calibrated value of a parameter for a given unit of time; if we were to consider δh jointly, it would not be possible to have a clear message for the indeterminacy regions.

4. CONCLUDING REMARKS

This paper has brought to the fore the underlying assumptions inherent in discrete-time modeling and explored the often hidden consequences of such assumptions. We have shown, using the framework of Schmitt-Grohé and Uribe (1997), that the choice of period length is a choice of economic significance that is separate from, although related to, the issue of calibration.

We wish to close the paper with a word of caution to researchers who employ dynamic general equilibrium models for analysis of macroeconomic dynamics and policy design. Given our finding that the stability of such systems may be quite sensitive to the period length, quantitative results based on such models should be interpreted with care. We hope that our work will aid researchers in assessing the robustness of their results to different assumptions about the period length. Ultimately, we believe that policy prescriptions arising from dynamic macroeconomic modeling would be significantly strengthened if we could carefully estimate (or at least calibrate) the frequency of decision making (which determines h) in actual economies.

NOTES

1. The first paper to show the link between indeterminacy and sunspot fluctuations was by Azariadis (1981). Benhabib and Farmer (1999) provide an excellent overview of the topic.
2. A similar point has previously been noted by Guo (2004) in the context of a real business cycle model with increasing returns and variable capital utilization. We also provide additional examples where this is true in the longer working paper version of this article, Anagnostopoulos and Giannitsarou (2010). We conjecture that this is a quite general result in models with endogenous capital.
3. The importance of the rate of time preference for indeterminacy has been implicitly or explicitly pointed out by various authors in the literature [see Schmitt-Grohé (1997), Baierl et al. (1998), Mitra (1998), and Guo (2004)].
4. In related work, Bambi and Licandro (2004), Licandro and Puch (2006), and Bambi and Gori (2010) investigate the connection between a time-to-build model and its continuous-time counterpart, a model with delays. Benhabib (2004) mixes continuous and discrete dynamics in a model of inflation.
5. Abreu et al. (1991) also disentangle these two concepts in order to investigate the effect of period length on the possibilities of cooperation in a repeated Prisoner's Dilemma game with imperfect monitoring.
6. Obstfeld (1992) provides some optimal control results by also allowing for a general period length h and taking appropriate limits when $h \to 0$.
7. In what follows we will use t to index variables even though, strictly speaking, the index is $\frac{t}{h}$.
8. This assumption is innocuous, because all that matters is the average within-period return and not how it is distributed within the period. See Anagnostopoulos and Giannitsarou (2010) for a detailed discussion.
9. We assume throughout the paper that there is no discounting within the period. For a more general specification, see the working paper version [Anagnostopoulos and Giannitsarou (2010)].
10. Standard monotonicity and convexity assumptions are maintained throughout for the return and constraint functions. We also assume standard Inada conditions for utility and production and thus ignore any non-negativity constraints on the households' problem.

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APPENDIX

A.1. EQUILIBRIUM DEFINITION

In this Appendix, we provide a formal definition of competitive equilibrium. Let us first describe the set of admissible paths. This will consist of sequences of continuous functions $\{c_t(\cdot)\}_{t\in I}$, $\{n_t(\cdot)\}_{t\in I}$, $\{S_t^k(\cdot)\}_{t\in I}$, $\{w_t(\cdot)\}_{t\in I}$, $\{r_t(\cdot)\}_{t\in I}$, and $\{\pi_t(\cdot)\}_{t\in I}$ for consumption, labor, net savings, wages, rental rates, and profits, respectively, where each element of these sequences is a continuous function with domain $[0, h)$ and range R_+. It will also contain $\{k_{t+h}\}_{t\in I}$, i.e. a sequence of real numbers for capital stock.

DEFINITION A.1. *A competitive equilibrium with sequential trade consists of sequences of price functions* $\{w_t^*(\cdot)\}_{t\in I}$ *and* $\{r_t^*(\cdot)\}_{t\in I}$*, sequences of quantity functions* $\{c_t^*(\cdot)\}_{t\in I}$, $\{n_t^*(\cdot)\}_{t\in I}$, $\{S_t^{k*}(\cdot)\}_{t\in I}$*, and* $\{\pi_t^*(\cdot)\}_{t\in I}$*, a sequence of capital stocks* $\{k_{t+h}^*\}_{t\in I}$*, and a sequence of tax functions* $\{\tau_t^*(\cdot)\}_{t\in I}$ *such that*

(i) Given $\{\tau_t^*(\cdot)\}_{t\in I}$, $\{w_t^*(\cdot)\}_{t\in I}$, $\{r_t^*(\cdot)\}_{t\in I}$, and $\{\pi_t^*(\cdot)\}_{t\in I}$, the quantities $\{c_t^*(\cdot)\}_{t\in I}$, $\{n_t^*(\cdot)\}_{t\in I}$, $\{S_t^{k*}(\cdot)\}_{t\in I}$, and $\{k_{t+h}^*\}_{t\in I}$ are optimal for the households. That is,

$$\{c_t^*, n_t^*, i_t^*, k_{t+h}^*\}_{t\in I} = \max_{\{c_t, n_t, i_t, k_{t+h}\}_{t\in I}} \sum_{t\in I} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h u(c_t(s), n_t(s))ds, \quad \textbf{(A.1)}$$

$$\text{s.t. } c_t(s) + S_t^k(s) = r_t^*(s)k_t + \left(1 - \tau_t^*(s)\right) w_t^*(s)n_t(s) + \pi_t^*(s), \quad \textbf{(A.2)}$$

$$k_{t+h} - (1 - \delta h)\, k_t = \int_0^h S_t^k(s)ds, \quad \textbf{(A.3)}$$

$$c_t(s) \geq 0, n_t(s) \geq 0, k_{t+h} \geq 0 \quad \textbf{(A.4)}$$

$$k_0 \text{ given.} \quad \textbf{(A.5)}$$

(ii) Given $\{w_t^*(\cdot)\}_{t\in I}$ and $\{r_t^*(\cdot)\}_{t\in I}$, the quantities $\{n_t^*(\cdot)\}_{t\in I}$, $\{k_{t+h}^*\}_{t\in I}$, and $\{\pi_t^*(\cdot)\}_{t\in I}$ are optimal for the firms. That is,

$$\{n_t^*(s), \pi_t^*(s), k_{t+h}^*\}_{t\in I} = \max_{\{n_t^*, k_{t+h}^*\}_{t\in I}} \sum_{t\in I} \left(\frac{1}{1+\rho h}\right)^{\frac{t}{h}} \int_0^h \frac{u_{c,t}(s)}{u_{c,0}(0)} \pi_t(s)ds, \quad \textbf{(A.6)}$$

subject to

$$\pi_t(s) = k_t^{s_k} n_t(s)^{s_n} - r_t^*(s)k_t - w_t^*(s)n_t(s). \quad \textbf{(A.7)}$$

(iii) The government budget is balanced at every instant:

$$G = \tau_t^*(s)\, w_t^*(s)\, n_t^*(s) \quad \text{for all } s \in [0, h) \text{ and all } t \in I. \quad \textbf{(A.8)}$$

(iv) All markets clear at every instant. The market-clearing condition for the goods market is

$$c_t^*(s) + S_t^{k*}(s) + G = (k_t^*)^{s_k} (n_t^*(s))^{1-s_n} \quad \text{for all } s \in [0, h) \text{ and all } t \in I. \quad \textbf{(A.9)}$$

The labor and capital markets clear by definition of the sequences $\{n_t^*(\cdot)\}_{t\in I}$ and $\{k_{t+h}^*\}_{t\in I}$.

A.2. LOG-LINEARIZATION

In this Appendix we show how to obtain the log-linearized equations for the general discrete-time model of Schmitt-Grohé and Uribe (1997). We start from the conditions describing equilibrium in the economy and reduce them to the following three relations:

$$\mu_t = \frac{\mu_{t+h}}{1+\rho h}\left[1+\left(s_k k_{t+h}^{s_k-1} n_{t+h}^{s_n} - \delta\right) h\right], \qquad \textbf{(A.10)}$$

$$A n_t = s_n \mu_t k_t^{s_k} n_t^{s_n} - G\mu_t, \qquad \textbf{(A.11)}$$

$$h\mu_t^{-1} + k_{t+h} - (1-\delta h)\, k_t = h k_t^{s_k} n_t^{s_n} - Gh. \qquad \textbf{(A.12)}$$

We denote steady state levels of variables with upper bars and log deviations of variables from the steady state with circumflexes. Log-linearizing these around the beginning-of-period steady state values of the variables, we get

$$\hat{\mu}_t = \hat{\mu}_{t+h} + (s_k - 1)\frac{h}{1+\rho h} s_k \bar{k}^{s_k-1}\bar{n}^{s_n}\hat{k}_{t+h} + s_n \frac{h}{1+\rho h} s_k \bar{k}^{s_k-1}\bar{n}^{s_n}\hat{n}_{t+h}, \qquad \textbf{(A.13)}$$

$$\hat{n}_t = \frac{s_k}{\psi - s_n}\hat{k}_t + \frac{\psi}{\psi - s_n}\hat{\mu}_t, \qquad \textbf{(A.14)}$$

$$\hat{k}_{t+h} = \frac{1}{\bar{k}}\left[h s_k \bar{k}^{s_k}\bar{n}^{s_n}\hat{k}_t + h s_n \bar{k}^{s_k}\bar{n}^{s_n}\hat{n}_t + h\bar{\mu}^{-1}\hat{\mu}_t + (1-\delta h)\,\bar{k}\hat{k}_t\right], \qquad \textbf{(A.15)}$$

where

$$\psi = \frac{A\bar{n}^{1-s_n}}{s_n\bar{\mu}\bar{k}^{s_k}}. \qquad \textbf{(A.16)}$$

We eliminate $\hat{n}_t$ to end up with a dynamic system of equations in $\hat{\mu}_t$ and $\hat{k}_t$ given by

$$P\begin{pmatrix}\hat{\mu}_{t+h}\\ \hat{k}_{t+h}\end{pmatrix} = S\begin{pmatrix}\hat{\mu}_t\\ \hat{k}_t\end{pmatrix}, \qquad \textbf{(A.17)}$$

where

$$P = \begin{pmatrix} 1 + s_n \frac{h}{1+\rho h} s_k \bar{k}^{s_k-1}\bar{n}^{s_n}\frac{\psi}{\psi - s_n} & s_n\frac{h}{1+\rho h} s_k \bar{k}^{s_k-1}\bar{n}^{s_n}\left[\frac{s_k}{\psi - s_n} - 1\right] \\ 0 & 1 \end{pmatrix}, \qquad \textbf{(A.18)}$$

$$S = \begin{pmatrix} 1 & 0 \\ \left[\frac{\psi(s_n)\bar{k}^{s_k-1}\bar{n}^{s_n}}{\psi - s_n} + \frac{1}{\bar{k}}\frac{1}{\gamma}\bar{\mu}^{-\frac{1}{\gamma}}\right] h & \left[(1-\delta h) + h s_k \bar{k}^{s_k-1}\bar{n}^{s_n} + h\frac{s_n s_k \bar{k}^{s_k-1}\bar{n}^{s_n}}{\psi - s_n}\right] \end{pmatrix}. \qquad \textbf{(A.19)}$$

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Using the steady state relations and defining $s_c = \frac{\bar{c}}{\bar{y}}$, the elements of these matrices simplify to

$$p_{11} = 1 + \frac{s_n (\rho + \delta) h}{1 + \rho h} \frac{1 - \tau}{s_k - \tau}, \tag{A.20}$$

$$p_{12} = \frac{s_n (\rho + \delta) h}{1 + \rho h} \frac{\tau}{s_k - \tau}, \tag{A.21}$$

$$s_{21} = h \frac{\rho + \delta}{s_k} \left[\frac{(1 - \tau) s_n}{s_k - \tau} + s_c \right], \tag{A.22}$$

$$s_{22} = h (\rho + \delta) \frac{1 - \tau}{s_k - \tau} + 1 - \delta h. \tag{A.23}$$

Using the matrix C defined in Section 3.3, we can write

$$P = \begin{pmatrix} 1 + \dfrac{h}{1 + \rho h} c_{11} & \dfrac{h}{1 + \rho h} c_{12} \\ 0 & 1 \end{pmatrix}, \tag{A.24}$$

$$S = \begin{pmatrix} 1 & 0 \\ h c_{21} & 1 + h c_{22} \end{pmatrix}, \tag{A.25}$$

and so

$$D(h) = P^{-1} S, \tag{A.26}$$

which gives (23) and the linearized system (24).

A.3. BOUNDS FOR INDETERMINACY

The dynamics for the general discrete-time model can be written as

$$\begin{pmatrix} \hat{\mu}(t + h) \\ \hat{k}(t + h) \end{pmatrix} = \begin{pmatrix} 1 + h \dfrac{c_{11} + h c_{12} c_{21}}{1 + \rho h - h c_{11}} & \dfrac{h c_{12} (1 + h c_{22})}{1 + \rho h - h c_{11}} \\ h c_{21} & 1 + h c_{22} \end{pmatrix} \begin{pmatrix} \hat{\mu}(t) \\ \hat{k}(t) \end{pmatrix}.$$

We denote the matrix that describes the dynamics by $F(h)$. Indeterminacy occurs when both eigenvalues of $F(h)$ are inside the unit circle. A set of equivalent conditions [see Medio and Lines (2001)] is

$$1 - \det F(h) > 0,$$
$$1 + \operatorname{tr} F(h) + \det F(h) > 0,$$
$$1 - \operatorname{tr} F(h) + \det F(h) > 0,$$

which here reduce to

$$\frac{\operatorname{tr} C + h\rho c_{22}}{1 + h\rho - hc_{11}} < 0, \qquad \textbf{(A.27)}$$

$$2 + h\frac{\operatorname{tr} C + h\rho c_{22}}{1 + h\rho - hc_{11}} > \frac{h^2}{2}\left(\frac{\det C}{1 + \rho h - hc_{11}}\right), \qquad \textbf{(A.28)}$$

$$\frac{\det C}{1 + \rho h - hc_{11}} > 0. \qquad \textbf{(A.29)}$$

To prove our statement, we consider two cases; namely, we check if and when the conditions hold for $\tau > s_k$ and $\tau < s_k$. We can then show that (A.27) and (A.28) imply that for indeterminacy, we need to have $\tau < \tau^*$, irrespective of h. Using this, we can also show that (A.28) cannot be satisfied in the second case, i.e., when $\tau < s_k$. Combining these two statements, we conclude that indeterminacy can occur only for tax rates in the range (s_k, τ^*). The exact lower bound for indeterminacy can be found by combining expressions from (A.27) and (A.28); however, these do not yield elegant analytical expressions and are therefore omitted here.

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