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Mixing length model of convection in stellar cores

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Abstract. The standard mixing length model of convection is ill behaved at the centre of a star since the pressure scale height $H_p = P/(\rho g) \to \infty$ as $r \to 0$, and the convective flux remains non zero at $r = 0$. We propose a modifcation of this model of convection that has the correct behaviour in the central regions of a star and smoothly changes to the standard MLT away from the centre.

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1. The standard mixing length model

For simplicity we neglect radiative cooling and take viscous drag to be half the buoyancy term, in which case the equations governing the mixing length model are (cf. Böhm-Vitense 1958)

$$\frac{\mathrm{d}(\delta T)}{\mathrm{d}r} = \left[\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{ad} - \frac{\mathrm{d}T}{\mathrm{d}r}\right], \quad \frac{\mathrm{d}v^2}{\mathrm{d}r} = g\frac{\delta T}{T}, \quad F = \frac{1}{2}\left[(c_p v\delta T)_{up} + (c_p v\delta T)_{down}\right] \tag{1}$$

Downwards moving eddies have $\delta T < 0, v < 0$ and so contribute a positive (upwards) energy flux. Upwards moving eddies have $\delta T > 0, v > 0$ and likewise contribute a positive energy flux.

Multiplying the first two equations in (1) by $P/(\mathrm{d}P/\mathrm{d}r) = -H_p = -P/(\rho g)$ gives

$$\frac{\mathrm{d}(\delta T)}{\mathrm{d}\log P} = -\frac{\Delta\nabla}{T}, \qquad \frac{\mathrm{d}v^2}{\mathrm{d}\log P} = -\frac{P}{\rho T}\delta T, \quad \text{where} \quad \Delta\nabla = \frac{\mathrm{d}\log T}{\mathrm{d}\log P} - \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{ad}. \tag{2}$$

Eddies are considered on average to have come from a distance $\ell/2$ away, where ℓ is the mixing length, so integrating from $r_0 = r + \ell/2$ to r, and from $r - \ell/2$ to r, gives

$$F_{down} = \frac{1}{2}c_p\rho T\sqrt{\frac{P}{2\rho}}\,(\Delta\nabla)^{3/2}\left[\log\left(\frac{P(r)}{P(r+\ell/2)}\right)\right]^2,$$

$$F_{up} = \frac{1}{2}c_p\rho T\sqrt{\frac{P}{2\rho}}\,(\Delta\nabla)^{3/2}\left[\log\left(\frac{P(r)}{P(r-\ell/2)}\right)\right]^2. \tag{3}$$

Taking the mixing length $\ell = \alpha H_p$ gives the standard result

$$F = \frac{1}{4}c_p\rho T\sqrt{\frac{P}{2\rho}}\,(\Delta\nabla)^{3/2}\,\alpha^2. \tag{4}$$

2. Solution near the centre

In the central regions this analysis is no longer valid since $\mathrm{d}P/\mathrm{d}r \to 0$, $H_p \to \infty$ as $r \to 0$, and the total flux must anyway go to zero whereas the expression in equation (4) remains non zero.

Consider a point X at small r: a downward moving eddy starting at $r_0 = r + \ell/2$ has a negative δT and negative v and so contributes a positive outward energy flux the same as F_{down} in Eq. (3). However the upwards eddy at X started a distance $\ell/2$ below X and hence at $r_1 = (\ell/2 - r)$ on the other side of the origin. It then accelerated downwards to the origin with negative δT and v then decelerated upwards to arrive at X with negative δT but now a positive v, and so contributes a negative upwards flux. Since the motion of the eddy is taken to be adiabatic, the magnitude of $|\delta T|$ and $|v|$ are given by taking an eddy starting at $\ell/2 - r$ and moving downwards to r. The two fluxes are then

$$F_{down} = \frac{1}{2} c_p \rho T \sqrt{\frac{P}{2\rho}} \, (\Delta\nabla)^{3/2} \left[\log\left(\frac{P(r)}{P(\ell/2 + r)}\right)\right]^2 ,$$

$$F_{up} = -\frac{1}{2} c_p \rho T \sqrt{\frac{P}{2\rho}} \, (\Delta\nabla)^{3/2} \left[\log\left(\frac{P(r)}{P(\ell/2 - r)}\right)\right]^2 . \tag{5}$$

In the neighbourhood of $r = 0$ the pressure P has the series expansion

$$P = P_c \left(1 - \frac{r^2}{H_c^2} + \ldots\right), \qquad H_c^2 = -\frac{P}{\mathrm{d}P/\mathrm{d}r^2} = 2\, r\, H_p = \frac{3P_c}{2\pi G \rho_c^2}. \tag{6}$$

Substituting for $P(r), P(\ell/2 + r), P(\ell/2 - r)$, taking the limit for small r and setting $\ell = \alpha H_c$ gives

$$F = \frac{1}{2} c_p \rho T \sqrt{\frac{P}{2\rho}} \, (\Delta\nabla)^{3/2} \left(\frac{r}{H_c}\right) \alpha^3 \qquad (7),$$

which goes to zero $\propto r$, as it should since the radiative flux F_{rad} and total flux F go to zero $\propto r$.

The results for $r >> \ell$ and for $r << \ell$ can be combined in the approximate expression

$$F = f \frac{1}{4} c_p \rho T \sqrt{\frac{P}{2\rho}} \, (\Delta\nabla)^{3/2} \, \alpha^2, \quad \text{where} \quad f = \min\left(\frac{2\,\alpha\, r}{H_c}, 1\right). \tag{8}$$

Reference

Böhm-Vitense, E. 1958, *ZfA* 46, 108