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Flexible periodic points

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Abstract. We define the notion of ε-flexible periodic point: it is a periodic point with stable index equal to two whose dynamics restricted to the stable direction admits ε-perturbations both to a homothety and a saddle having an eigenvalue equal to one. We show that an ε-perturbation to an ε-flexible point allows us to change it to a stable index one periodic point whose (one-dimensional) stable manifold is an arbitrarily chosen C^1-curve. We also show that the existence of flexible points is a general phenomenon among systems with a robustly non-hyperbolic two-dimensional center-stable bundle.

1. *Introduction*

Since Poincaré's discovery of the transverse homoclinic intersections and the complex behaviors near them, the search for the transverse homoclinic intersections, in other words, the control of the invariant manifolds of the systems, has been one of the central problems in dynamical systems. In the late 1990s, a breakthrough was achieved in the form of Hayashi's connecting lemma (see [**H**]). This allows us to control the effect of perturbations on the invariant manifolds and enables us to create new intersections. This perturbation technique provides one basis for the recent active development of the study of non-uniformly hyperbolic dynamical systems. For example, [**CP**] uses the connecting lemma and its generalizations for building heterodimensional cycles in order to characterize the robust non-hyperbolic behaviors.

In the uniformly hyperbolic context, the invariant manifold theorem tells us their rigidity. The local stable/unstable manifolds of a uniformly hyperbolic set are embedded discs varying continuously with respect to the points and the variation of the diffeomorphisms. A consequence of this fact is the great constraint on the geometric behavior of invariant manifolds under small perturbations. Such rigidities are paradoxically exploited for the construction of robustly non-hyperbolic systems such as Newhouse's example of robust tangencies (see [**N**]) or Abraham and Smale's example of robust heterodimensional cycles (see [**AS**]).

Meanwhile, the problem of the control of the variations of the invariant manifolds of periodic orbits under small perturbations in non-uniformly hyperbolic systems remains an important issue. On the one hand, non-hyperbolic systems in general contain plenty of regions where the local dynamics exhibits hyperbolic behaviors, which also give us some rigidity of the invariant manifolds. On the other hand, we may expect that the absence of uniform hyperbolicity implies the existence of periodic orbits whose invariant manifolds have less rigidity so that they can be altered considerably by small perturbations.

A prototype of arguments of this kind can be found in [**BD1**]. The rescaling invariant nature of C^1-distance gives us strong freedom for the change of relative position of objects in the homothetic regions, that is, contracting or repelling regions where the diffeomorphism is smoothly conjugated to a homothety. Furthermore, [**BD1**] gave an example of non-uniformly hyperbolic systems in which the homothetic behaviors are quite abundant. This enables us to construct interesting examples of dynamical systems, such as the construction of *universal dynamics* by the first author with Díaz, or the construction of heterodimensional cycles near the wild homoclinic classes by the second author [**S**].

The fact that non-hyperbolic dynamics may exhibit homothetic behaviors was first exploited by Mañé in [**M**] for surface diffeomorphism (see also [**PS**]), and has been generalized to higher-dimensional cases in [**BB**, **BDP**, **BGV**]. These works are essentially within the scope of perturbations of the derivative of periodic orbits, and their conclusions provide us only local information on perturbed systems. In this paper we pursue the possibility of such a strategy further and propose a new, semi-local technique for the control of invariant manifolds.

In many applications, we need to control the effect of the perturbation on the invariant manifold such as keeping a heteroclinic connection while the periodic orbit is changing its index. This is the aim of [**G1**] where Gourmelon uses invariant cone fields for keeping the strong stable manifold almost unchanged along the perturbation. Here we follow a completely opposite strategy: we will use the homothetic region (where there is no strictly invariant cone field) to obtain great freedom of choice of the position of the invariant manifolds.

The aim of this paper is twofold. First, we consider diffeomorphisms of two-dimensional manifolds and define the notion of flexible periodic points, which is an abstract sufficient condition for a periodic point which guarantees that the above strategy is available. We investigate the possible perturbations on such points. We extend the concept of flexible points to diffeomorphisms in higher dimension with stable index two periodic point and see that the perturbation technique proved in the two-dimensional setting is valid also in higher-dimensional situations. Second, we show that flexible points are quite abundant in some higher-dimensional partially hyperbolic dynamical systems.

Let us now state our main results.

1.1. *Flexible points of surface diffeomorphisms.* First, we briefly review the notion of linear cocycles. Let X be a topological space, $f : X \to X$ be a homeomorphism of X and $\mathcal{E}$ be a Riemannian vector bundle over X. A *linear cocycle* on $\mathcal{E}$ is a bundle isomorphism

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$\mathcal{A} : \mathcal{E} \to \mathcal{E}$ which is compatible with f. In this paper we are mainly interested in the situation where f is a diffeomorphism of some manifold, X is a periodic orbit of such dynamical systems and $\mathcal{A}$ is the restriction of the differential map acting on the restriction of the tangent bundle over the orbit. By taking coordinates, such a system can be identified with the situation where $X = \mathbb{Z}/n\mathbb{Z}$ (n is the period of the orbit), $f(x) := x + 1$ and $\mathcal{A}$ is a sequence of regular matrices. We call such system *a linear cocycle over a periodic orbit of period n*.

Let $\mathcal{A}$, $\mathcal{B}$ be linear cocycles on $\mathcal{E}$ over (X, f). We put

$$\operatorname{dist}(\mathcal{A}, \mathcal{B}) := \sup \|\mathcal{A}(x) - \mathcal{B}(x)\|$$

and call it the *distance* between $\mathcal{A}$ and $\mathcal{B}$ (where x ranges over all unit vectors in all fibers). This defines a topology on the space of linear cocycles on $\mathcal{E}$. Let $\mathcal{A}_t$ denote a continuous one-parameter family of cocycles, that is, a continuous map from some interval to the space of cocycles. We put

$$\operatorname{diam}(\mathcal{A}_t) := \sup_{s<u} \operatorname{dist}(\mathcal{A}_s, \mathcal{A}_u)$$

and call it the *diameter* of $\mathcal{A}_t$.

Let us now give a precise definition of *flexible cocycles*.

Definition 1.1. Let $\mathcal{A} = \{A_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, $A_i \in \mathrm{GL}(2, \mathbb{R})$, be a linear cocycle over a periodic orbit of period $n > 0$. Fix $\varepsilon > 0$. We say that $\mathcal{A}$ is *ε-flexible* if there is a continuous one-parameter family of linear cocycles $\mathcal{A}_t = \{A_{i,t}\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ defined over $t \in [-1, 1]$ such that the following conditions hold:

- $\operatorname{diam}(\mathcal{A}_t) < \varepsilon$;
- $A_{i,0} = A_i$, for every $i \in \{0, \ldots, n-1\}$;
- A_{-1} is a homothety;
- for every $t \in (-1, 1)$, the product $A_t := (A_{n-1,t}) \cdots (A_{0,t})$ has two distinct positive contracting eigenvalues;
- if we denote by λ_t the smallest eigenvalue of the product A_t, then $\max_{-1 \le t \le 1} \lambda_t < 1$;
- A_1 has a real positive eigenvalue equal to 1.

Definition 1.2. A periodic orbit of a diffeomorphism on a smooth two-dimensional manifold is called ε-flexible if the linear cocycle of the derivatives along its orbit is ε-flexible.

The interesting feature of ε-flexible points is the great freedom for changing the position of their strong stable manifold by an ε-perturbation supported in an arbitrarily small neighborhood of the orbit. More precisely, we can choose the fundamental domain of this strong stable manifold to be any prescribed curve subject to the unique limitation that it should remain in the same isotopy class in the orbit space.

Let us explain that. Let $\mathcal{N}$ be a compact neighborhood of some attracting periodic orbit $\mathcal{O}(x)$ of some diffeomorphism F on a two-dimensional manifold. Suppose that:

- the derivative of F on $\mathcal{O}(x)$ in the period has two distinct contracting eigenvalues;
- $\mathcal{N}$ is F-invariant and is contained in the basin of $\mathcal{O}(x)$.

Let us consider the *punctured neighborhood* $\mathcal{N} \setminus \mathcal{O}(x)$ and take the quotient space by the orbit equivalence (that is, $x \sim y$ if and only if $F^m(x) = F^n(y)$ for some $m, n \ge 0$).

We denote it by T_F^∞ and call it the *orbit space*. Then T_F^∞ is naturally identified with the two-dimensional torus $\mathbb{T}^2$, This torus is naturally endowed with a homotopy class of a *parallel*, which consists of the class of an essential circle around $\mathcal{O}(x)$. Moreover, the strong stable manifold $W^{ss}(x)\backslash\{x\}$ projects to the quotient space as two parallel circles, cutting the parallel with intersection number one. We call these two simple closed curves in T_F^∞ *meridians*.

Consider another diffeomorphism G which is a perturbation of F whose support is contained in a small neighborhood of $\mathcal{O}(x)$ and preserving the orbit of $\mathcal{O}(x)$. Let $\Lambda_G := \bigcap_{i\geq 0} G^i(\mathcal{N})$ denote the locally maximal invariant set in $\mathcal{N}$. Consider the set $\mathcal{N}\backslash\Lambda_G$ and take the quotient by the orbit equivalence under G. We denote the orbit space by T_G^∞. We identify T_F^∞ and T_G^∞ as follows. First, note that the restriction of $G|_{\mathcal{N}\backslash\Lambda_G}$ can be conjugated to $F|_{\mathcal{N}\backslash\mathcal{O}(x)}$ by a unique homeomorphism h which coincides with the identity map outside the support of the perturbation and is C^1 on $\mathcal{N}\backslash\mathcal{O}(x)$. Let us call this homeomorphism *standard conjugacy*. It gives us an identification between T_F^∞ and T_G^∞

Under this identification, the freedom of flexible points mentioned above can be formulated as follows.

THEOREM 1.1. *Let f be a C^1-diffeomorphism of a surface and D be an attracting periodic disc of period π, that is, D, $f(D), \ldots, f^{\pi-1}(D)$ are pairwise disjoint and $f^\pi(D)$ is contained in the interior of D. Assume that D is contained in the stable manifold of an ε-flexible periodic point p contained in D. Let $\gamma = \gamma_1 \cup \gamma_2 \subset T_f^\infty$ be the two simple closed curves which $W^{ss}(p)$ projects to.*

Then, for any pair of C^1-curves $\sigma = \sigma_1 \cup \sigma_2$ embedded in T_f^∞ isotopic to $\gamma_1 \cup \gamma_2$, there is an ε-perturbation g, supported in an arbitrarily small neighborhood of p (which is also sufficiently small so that we can define the standard conjugacy) such that g satisfies the following conditions:

- *p is a periodic attracting point having an eigenvalue $\lambda_1 \in \,]0, 1[$ and a eigenvalue $\lambda_2 = 1$;*
- *D is contained in the basin of p;*
- *$W^{ss}(p, g)$ projects to $(\sigma_1 \cup \sigma_2) \subset T_g^\infty \simeq T_f^\infty$.*

The orbit of p for g is a non-hyperbolic attracting point, having an eigenvalue equal to one. By another, arbitrarily small, perturbation one can change the index of p so that the strong stable manifold becomes the new stable manifold. Therefore we have the following corollary.

COROLLARY 1.1. *Under the hypotheses of Theorem 1.1, there is an ε-perturbation h of f, supported in an arbitrarily small neighborhood of p, such that:*

- *p is a periodic saddle point having two real eigenvalues $0 < \lambda_1 < 1 < \lambda_2$;*
- *$W^s(p)\backslash\mathcal{O}(p)$ is disjoint from the maximal invariant set Λ_H;*
- *$W^s(p, h)$ projects to $(\sigma_1 \cup \sigma_2) \subset T_h^\infty \simeq T_f^\infty$.*

The proof of Theorem 1.1 is given in §§2–4.

1.2. *Stable index two flexible points.* The existence of flexible points is certainly unusual for surface diffeomorphisms. However, it appears very naturally in higher-dimensional partially hyperbolic systems with two-dimensional stable directions. To

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explain that, first we extend the definition of flexible points to a higher-dimensional setting and see the direct consequence of Theorem 1.1 about them. Let f be a diffeomorphism of a smooth manifold, and p be a (not necessarily hyperbolic) periodic point of it. Then the *stable index* of p is the number of eigenvalues of the differential of the first return map with absolute value strictly less than one, counted with multiplicity.

Definition 1.3. Let f be a diffeomorphism of a smooth closed manifold with a metric and $\varepsilon > 0$. A stable index two hyperbolic period point x is called ε-flexible if the restriction of Df to the stable direction over $\mathcal{O}(x)$ is an ε-flexible cocycle.

Remark 1.1. The notion of flexibility is a robust property in the following sense: if q_f is an ε-flexible periodic orbit of f then there is a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a well-defined continuation q_g of q_f and q_g is 2ε-flexible.

Let x be a stable index two periodic point of period n, with two distinct real positive contracting eigenvalues. For flexible points in this setting, we can also define the notion of orbit space, parallel, and meridians as follows. Consider the neighborhood $\mathcal{N}$ in the local stable manifold of x which is strictly positively invariant. The orbit space of $\mathcal{N} \backslash \mathcal{O}(x)$ under f is again diffeomorphic to the torus $\mathbb{T}^2$ and we denote it by T_f^∞. As is in the previous case, this torus is naturally endowed with a parallel, and the strong stable manifold of x induces by projection on T_f^∞ two disjoint simple curved called meridians.

We consider the perturbation g of f whose support is contained in a small neighborhood of $\mathcal{O}(x)$ and which preserves the forward invariant property of $\mathcal{N}$. Then the space of g-orbits in $D\backslash(\Lambda_g)$ (where Λ_g is the locally maximal invariant set of g in $\mathcal{N}$), denoted by T_g^∞ , is naturally identified with T_f^∞ via the standard conjugacy.

The following remark is straightforward.

Remark 1.2. Any ε-perturbation of the restriction of f to the forward orbit of $\mathcal{N}$, in a sufficiently small neighborhood of $\mathcal{O}(x)$, can be realized as an ε-perturbation of f. Furthermore, if the perturbation preserves the periodic orbit $\mathcal{O}(x)$ then one may require that the eigenvalues of x transverse to $\mathcal{N}$ are kept unchanged.

Therefore, as a direct corollary of Theorem 1.1 we obtain the following result.

COROLLARY 1.2. *Let f be a diffeomorphism of a compact manifold, $\varepsilon > 0$, and x be an ε-flexible stable index two hyperbolic periodic point. Fix a strictly forward invariant neighborhood $\mathcal{N}$ of $\mathcal{O}(x)$ contained in the local stable manifold of x. Let T_f^∞, endowed with the meridians γ_1, γ_2, denote the orbit space of the punctured stable manifold of x.*

Let σ_1 and σ_2 be two disjoint simple curves isotopic to the meridians. Then there is an ε-perturbation g of f, supported in an arbitrarily small neighborhood of $\mathcal{O}(x)$, preserving the forward invariance of $\mathcal{N}$ and the orbit $\mathcal{O}(x)$ such that the following statements hold.

- *The eigenvalues in the directions transversal to $\mathcal{N}$ at x of g are the same as for f.*
- *$\mathcal{O}(x)$ is a stable index one saddle point: there is a contracting eigenvalue tangent to $\mathcal{N}$.*
- *The punctured stable manifold of x is disjoint from the maximal invariant in $\mathcal{N}$, and the projection of $W^s(x)\backslash\{x\}$ on T_f^∞ is precisely the two curves $\sigma_1 \cup \sigma_2$.*

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As a conclusion, for flexible points in this setting, we also have great freedom to choose the position of strong stable manifolds. However, in higher-dimensional settings, we have a priori little control over the effect of the perturbation to the local unstable manifold of x, and therefore over the position of the local strong unstable manifold of g.

More precisely, if the angle between the unstable bundle over $\mathcal{O}(x)$ and the orthogonal complement to the stable bundle is very small all along $\mathcal{O}(x)$, then every ε perturbation of f in the local stable manifold of $\mathcal{O}(x)$, supported in a very small neighborhood of $\mathcal{O}(x)$ and preserving $\mathcal{O}(x)$, may be realized as an ε-perturbation g of f, supported in a small neighborhood of $\mathcal{O}(x)$, and which coincides with f on the local unstable manifold of $\mathcal{O}(x)$. On the other hand, if this angle is large, that is, if the angle between the stable and unstable bundles is very small at some point of $\mathcal{O}(x)$, then perturbing f in the stable manifold without perturbing the unstable one can be very costly.

This can be a serious problem if one applies this technique to the problem of (non-)existence of homoclinic intersections. However, if we have a priori estimates on the angles mentioned above, the perturbation technique suggested in Corollary 1.2 works well. Note that such a priori estimates are available in the case where the system admits partially hyperbolic splitting, or more generally, dominated splitting.

1.3. *Abundance of flexible points.* At first glance, the definition of flexible points may appear strange, as it claims the existence of perturbations to two completely different situations. However, it is quite common in the context of non-uniformly hyperbolic situations, as we will see below.

First let us give a precise statement in the form of a C^1-generic property.

THEOREM 1.2. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{G}$, for every $\varepsilon > 0$, and for any chain recurrence class C containing*

- *a periodic point p of stable index two with complex (non-real) contracting eigenvalue*
- *and a periodic point q of stable index one,*

there are ε-periodic points $\{p_n\}$ homoclinically related to p whose orbits γ_n converge to the chain recurrence class C in the Hausdorff topology.

We observe there is a large class of diffeomorphisms satisfying the hypotheses of Theorem 1.2. To explain that, let us briefly review the notion of robust heterodimensional cycles. We say that two hyperbolic basic sets K and L of a diffeomorphism f form a *C^1-robust heterodimensional cycle* if:

- the stable-indices of K and L are different;
- for any g sufficiently C^1-close to f, the continuations K_g and L_g of K and L satisfy

$$W^u(K_g) \cap W^s(L_g) \neq \emptyset \quad \text{and} \quad W^s(K_g) \cap W^u(L_g) \neq \emptyset.$$

Robust heterodimensional cycles are very important mechanisms for the study of robustly non-hyperbolic behaviors of diffeomorphisms, as they are the mechanisms that account for the birth of robust non-hyperbolicity in the large class of C^1 non-hyperbolic diffeomorphisms. Indeed, up to now, all the known examples of robustly non-hyperbolic behaviors have been ascribed to robust heterodimensional cycles, and it is conjectured by the first author in [**B**] that every robustly non-hyperbolic diffeomorphism can be approximated by one that has a robust heterodimensional cycle. Furthermore, it is worth mentioning that the creation of robust heterodimensional cycles is a quite general phenomenon from the bifurcation of heterodimensional cycles between saddles of different indices (see [**BD2**, **BDK**]).

Let us now consider diffeomorphisms satisfying the following conditions.

- They have a robust heterodimensional cycle between two hyperbolic basic sets K and L.
- K has stable index one and L has stable index two.
- L has a periodic point with complex eigenvalues in the stable direction.

The set of such diffeomorphisms forms, by definition, an open set in $\mathrm{Diff}^1(M)$ which is non-empty if $\dim M \geq 3$. Then, every diffeomorphism contained in the intersection of this open set and the residual set in Theorem 1.2 serves as an example (for the chain recurrence set take the one that contains K and L).

To observe the largeness of the class of diffeomorphisms which are within the range of hypotheses of Theorem 1.2, let us briefly discuss the relationship between the hypotheses of Theorem 1.2 and the notion of *homoclinic tangencies*. The hypotheses of Theorem 1.2 require the existence of a periodic point which has complex eigenvalues in the stable direction. This implies the indecomposability in the stable direction. The other condition guarantees that the stable direction is not uniformly contracting. By the work of Gourmelon [**G2**] and Wen [**W**], this hypothesis is equivalent to saying that this diffeomorphism can be approximated by the one with homoclinic tangency of stable index one. Remember that the results in [**BD2**] (see Theorem 1.5 for example) say that the class of diffeomorphisms satisfying such conditions is quite large.

The proof of Theorem 1.2 is a consequence of Theorem 1.3 below, a 'local' result, combined with the generic property that C^1-generically a homoclinic class coincides with the chain recurrence class which contains it (see [**BC**]) and that the coexistence of the periodic point of different indices implies the existence of robust heterodimensional cycles [**BD2**]. To state Theorem 1.3, we require some definitions. Given a periodic point p and a neighborhood U of p, the *relative homoclinic class* $H(p, U)$ of p in U is the closure of the set of transverse homoclinic points whose whole orbit is contained in U. A periodic point q is *homoclinically related with* p *in* U if there are points of transverse intersection between $W^s(p)$ and $W^u(q)$, $W^s(q)$ and $W^u(p)$, such that their entire orbits are contained in U.

THEOREM 1.3. *Let f be a diffeomorphism of a compact smooth manifold. Suppose that f admits a hyperbolic periodic point p and an open neighborhood U of the orbit $\mathcal{O}(p)$ with the following properties.*

- *p has stable index two.*
- *There is periodic point p_1 homoclinically related with p in U, such that the p_1 has a complex (non-real) contracting eigenvalue.*
- *There is a periodic point q with $\mathcal{O}(q) \subset U$ with stable index one.*
- *There are hyperbolic transitive basic sets $K \subset H(p, U)$ and $L \subset H(q, U)$ containing p and q, respectively, such that K and L form a C^1-robust heterodimensional cycle in U.*

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Then, for any $\varepsilon > 0$, there is an arbitrarily small C^1-perturbation g of f having an ε-flexible point homoclinically related with p_g in U and whose orbit is ε-dense in the relative homoclinic class $H(p_g, U)$.

We give the proof of Theorem 1.3 (thus also of Theorem 1.2) in §5.

1.4. *Possible dynamical consequences and generalizations.* The notions of flexible points and their abundance are interesting in themselves. At the same time, we think that they would be a powerful tool for the study of C^1-generic systems in many ways. Let us explain in more detail.

The first possible application is to the investigation of *tame/wild* properties in $\text{Diff}^1(M)$ (see [**B**]). In a future work, the authors will use them as a mechanism for producing new examples of wild diffeomorphisms, that is, C^1-generic diffeomorphisms with infinitely many chain recurrence classes. The idea is very simple: if p is a flexible point of stable index two, then one can transform p into a stable index one periodic point whose stable manifold is an arbitrarily chosen curve in the old two-dimensional stable manifold. If we can choose this curve to be disjoint from the initial chain recurrence class, this implies that the point has been ejected from the class. Repeating this procedure, we obtain infinitely many saddles with trivial homoclinic classes in a neighborhood of any classes satisfying the hypotheses of Theorem 1.2.

However, this strategy is not complete: there are C^1-robustly transitive diffeomorphisms which satisfy the hypotheses of Theorem 1.2 (see, for example, [**BV**]). They have plenty of flexible points but cannot be expelled from the class! The reason is that, since the class is the whole manifold, there is no space to escape from the original classes. Thus, to carry out this strategy completely, we need to investigate the *topology of the chain recurrence class* in the center stable directions, which will be one of the central topics in [**BS**].

We suggest another possible application. The control of the position of the stable manifold may open the way to study the difference between C^1-generic diffeomorphisms and C^1-open diffeomorphisms. For example, it is known that for C^1-generic robustly transitive diffeomorphisms, the homoclinic class of every hyperbolic periodic point coincides with the whole manifold (this is a consequence of Hayashi's connecting lemma). One interesting question is whether this is an open property. A priori, there is no reason for it to be so. However, obtaining a rigorous conclusion is no simple matter. For that, what we need to understand is the position of homoclinic intersections. In the situation where we have abundance of flexible points, obtaining a better understanding of the position of homoclinic intersections sounds quite feasible, since the perturbation technique of Theorem 1.1 provides us with considerable freedom to control the (un)stable manifolds.

In this paper, we define the notion of flexible points of stable index two. We can define similar notions for larger stable index cases, for example, as points whose derivative in the stable direction can be perturbed both to a contracting homothety and to a saddle having at least one eigenvalue equal to unity. It would be interesting to establish similar perturbation techniques to control the position of stable manifolds for them, and to study possible topology of flags of strong stable manifolds in the orbit spaces. However, describing a deformation of a linear cocycle in higher dimensions is much more difficult and technical than in dimension two. Therefore, we think it is better to restrict our attention to the two-dimensional situation, deferring the generalization to higher-dimensional cases to future work.

2. *Flexibility and the control of the stable manifold*

The purpose of §§2–4 is the proof of Theorem 1.1.

Let us give a rough idea of the proof. First, we see that the flexibility property allows us to perform an ε-perturbation of the flexible linear cocycle, among the cocycle of diffeomorphisms, which inserts a region where the dynamic in the period is a homothety. Here, the important thing is that the number of fundamental domains in the homothethic region may be chosen arbitrarily large, keeping the smallness of the perturbation. In some sense, we require the orbit to spend an arbitrarily large time in the homothetic region. For this reason we call them *retardable cocycles*.

Iterating a homothetic dynamic does not introduce any distortion: it is therefore easy to control the effect of perturbations performed inside the homothetic region. The fact that we may use an arbitrarily large number of fundamental domains gives us the time to slowly deform the strong unstable manifold to the a priori chosen curve.

2.1. *Retardable cocycles.*

To explain what is meant by 'inserting a lot of homothetic regions' we first define the notion of retardable cocycles. That requires the notion of diffeomorphism cocycles over a finite orbit. We consider cocycles of diffeomorphisms on $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, for which $\{\mathbf{0}_i\}$ (where we put $\mathbf{0}_i := (0, i)$) is a periodic sink attracting all the points. More precisely, a *diffeomorphism cocycle* is a set of diffeomorphisms $\mathcal{F} = \{f_i \mid \mathbb{R}^2 \times \{i\} \to \mathbb{R}^2 \times \{i+1\}\}$. We denote the map induced on the total space $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ from the cocycle $\mathcal{F} = \{f_i\}$ also by $\mathcal{F}$.

In this paper, we assume that every diffeomorphism cocycle fixes the origin, that is, we always assume that $f_i(\mathbf{0}_i) = \mathbf{0}_{i+1}$ for every i. A diffeomorphism cocycle is called *contracting* if the zero section is an attracting periodic orbit and every point is contained in its basin. Given a linear cocycle $\mathcal{A} = \{A_i\}$, we regard it as a diffeomorphism cocycle in the obvious way, that is, we consider A_i to be the diffeomorphism from $\mathbb{R}^2 \times \{i\}$ to $\mathbb{R}^2 \times \{i+1\}$. For a diffeomorphism cocycle $\mathcal{F} = \{f_i\}$, we denote its first return map on $\mathbb{R}^2 \times \{0\}$ by F (dropping the subscript in $\mathbb{Z}/n\mathbb{Z}$ and capitalizing the symbol.) Note that a linear cocycle $\mathcal{A} = \{A_i\}$ is contracting if and only if all eigenvalues of A have absolute value strictly less than one.

In the following, we denote the two-dimensional disc of radius r centered at $\mathbf{0}_i$ by $B_i(r) \subset \mathbb{R}^2 \times \{i\}$, and, for any $0 < r < s$, we denote by $\Gamma_{r,s}$ the round closed annulus in $\mathbb{R}^2 \times \{0\}$ bounded by circles of radii r and s, that is, $\Gamma_{r,s} := \overline{B_0(s) \setminus B_0(r)}$.

Definition. A contracting cocycle of diffeomorphisms $\mathcal{F} = \{f_i\}$ over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ is called a *retardable cocycle* if there exist R_1, R_2, R_3 satisfying $R_1 > R_2 > R_3 > 0$ such that:

- $f_i|_{B_i(R_1)\setminus B_i(R_3)} = A_i$, where A_i is a linear map such that $A = \prod_{j=0}^{n-1} A_j = \lambda \mathrm{Id}$ where $0 < \lambda < 1$ (in other words, A is a contracting homothety);
- for every $x \in B_0(R_2)$ and i satisfying $0 \leq i < n$, we have $(\prod_{j=0}^{i-1} f_j)(x) \in B_i(R_1)$;
- $A(B_0(R_2)) = B_0(\lambda R_2)$ contains $B_0(R_3)$ in its interior—we call $\Gamma_{\lambda R_2, R_2}$ the *homothetic region* of $\mathcal{F}$.

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The main property of retardable cocycles is that one may insert arbitrarily many fundamental domains of homothety as follows.

PROPOSITION 2.1. *Let* $\mathcal{F} = \{f_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ *be a retardable cocycle. We define new cocycles* $\mathcal{F}_m = \{f_{i,m} \mid \mathbb{R}^2 \times \{i\} \to \mathbb{R}^2 \times \{i+1\}\}$, $(m \geq 0)$ *as follows:*

- *for* $x \in \mathbb{R}^2 \times \{i\}$ *with* $|x| > R_3$, $f_{i,m}(x) := f_i(x)$;
- *for* $x \in \mathbb{R}^2 \times \{i\}$ *with* $\lambda^m R_3 < |x| < R_3$, $f_{i,m}(x) := A_i(x)$;
- *for* $x \in \mathbb{R}^2 \times \{i\}$ *with* $|x| \leq \lambda^m R_3$, $f_{i,m}(x) := (A^m \circ f_i \circ A^{-m})(x)$.

Then these maps define a C^1*-diffeomorphism contracting cocycle. We call* $\{f_{i,m}\}$ *the* m-retard *of* $\{f_i\}$.

The proof of above proposition is obvious, so we omit it.

Roughly speaking, $\{f_{i,m}\}$ is a cocycle that is obtained by *the insertion of* m*-homothetic fundamental domains* into $\{f_i\}$. This insertion does not change the main dynamical properties of the cocycle: the orbits will just spend more time in the newly added homothetic region. More precisely, on the homothetic region, the relative position of objects such as the strong stable manifold are kept intact under the iteration. For example, we have the following properties of retarded cocycles.

Remark 2.1.

- $\{f_{i,0}\} = \{f_i\}$.
- Let $\Gamma_{\lambda R, R}$ be the homothetic region of $\mathcal{F}$. Then, $F_m|_{\Gamma_{\lambda^{m+1} R, R}}$ is a homothety of rate of contraction λ. We call $\Gamma_{\lambda^{m+1} R, R}$ the *homothetic region* of $\mathcal{F}_m$.
- Suppose that the origin $\{\mathbf{0}_i\}$ has strong stable manifold $W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F})$ of $\{f_i\}$. Then
$$W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F}_m) \cap \Gamma_{\lambda^{l+1} R, \lambda^l R} = A^l(W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F}) \cap \Gamma_{\lambda R, R})$$
for l satisfying $0 \leq l \leq m$.
- The item above can be stated in a more sophisticated way in the language of orbit spaces. Note that for every $m \geq 0$, $\mathcal{F}_m = \{f_{i,m}\}$ coincides with $\mathcal{F} = \{f_i\}$ outside some compact neighborhood of the origin. Thus the standard conjugacy gives the natural identification between $T^{\infty}_{\mathcal{F}_m}$ and $T^{\infty}_{\mathcal{F}}$. Then the above item is paraphrased as follows: $W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F}_m)$ and $W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F})$ project to the same curve in $T^{\infty}_{\mathcal{F}_m} = T^{\infty}_{\mathcal{F}}$.

Furthermore, in some special circumstances, the operation of retarding does not change the dynamics so much. To explain that, we introduce the notion of distance on diffeomorphism cocycles. Let $\{f_i\}$, $\{g_i\}$ be two diffeomorphism cocycles on $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$. We say that $\{g_i\}$ is a *perturbation of* $\{f_i\}$ if the support, that is, the set $\{x \in \mathbb{R}^2 \times \{i\} \mid f_i(x) \neq g_i(x)\}$, is relatively compact for all i. Suppose that $\{g_i\}$ is a perturbation of $\{f_i\}$ and $\varepsilon > 0$. Then $\{g_i\}$ is called an ε-perturbation of $\{f_i\}$ if
$$\max_{i \in \mathbb{Z}/n\mathbb{Z}, x \in \mathbb{R}^2 \times \{i\}} \|Df_i(x) - Dg_i(x)\| < \varepsilon.$$

For the notion of ε-perturbations of diffeomorphism cocycles, see Remark 2.2 below. Then, for the retarded cocycle, we have the following lemma.

LEMMA 2.1. *Suppose that* $\{A_i\}$ *is a linear cocycle and* $\{f_i\}$ *is a retardable diffeomorphism cocycle which is also an* ε*-perturbation of* $\{A_i\}$ *such that the support of the perturbation and the homothetic region is contained in a neighborhood* $\mathcal{N}$ *of* $\{\mathbf{0}_i\}$. *Then* $\{f_{i,m}\}$ *is also a* ε*-perturbation of* $\{A_i\}$ *whose support is contained in* $\mathcal{N}$.

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Remark 2.2. The notion of ε-perturbation gives a sense of the closeness between a cocycle and its perturbation. A priori, this is different from the usual notion of C^1-distance, since our notion does not take the contribution of C^0-distance into consideration. However, this difference is negligible for the following reason: in the following, we establish a perturbation technique which provides us an ε-perturbation with very small (indeed, arbitrarily small) support. This smallness of support combined with the smallness of the ε implies the smallness of C^0-distance, and it implies the C^1-smallness of the perturbation in the usual sense.

By inserting a lot of homothetic regions, combining the fragmentation lemma, we will see that we can obtain considerable freedom to change the relative positions of the objects.

2.2. *Proof of Theorem 1.1.* The aim of this section is to prove Theorem 1.1 as a consequence of the following propositions.

The first proposition gives the relation between flexible and retardable cocycles: retardable cocycles may be obtained as small perturbations of flexible cocycles.

PROPOSITION 2.2. *Let $\varepsilon > 0$ and $\mathcal{A} = \{A_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}$, $A_i \in \mathrm{GL}(2, \mathbb{R})$ be an ε-flexible linear cocycle over a periodic orbit of period $n > 0$. Then there is a contracting retardable diffeomorphism cocycle $\mathcal{F} = \{f_i\}$ with the following properties.*

- *For any $m \in \mathbb{N}$, the retarded cocycle $\mathcal{F}_m = \{f_{m,i}\}$ is an ε-perturbation of $\mathcal{A}$.*
- *There is an isotopy of contracting diffeomorphism cocycle connecting $\mathcal{F}$ and $\mathcal{A}$ such that for every moment the periodic orbit $\{\mathbf{0}_i\}$ has two different real eigenvalues.*
- *For every $i \in \mathbb{Z}/n\mathbb{Z}$, the map f_i coincides with A_i outside the unit balls $B_i(1) \subset \mathbb{R}^2 \times \{i\}$.*
- *The derivative DF at the origin $\{\mathbf{0}_0\}$ has a contracting eigenvalue and one eigenvalue equal to unity.*

The second proposition explores the effect of perturbations to retardable cocycles on the position of the strong stable manifold.

PROPOSITION 2.3. *Let $\mathcal{F} = \{f_i\}$ be a retardable diffeomorphism cocycle over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ whose origin has two distinct real positives eigenvalues. For any pair of disjoint simple curves σ_1, σ_2 in $T^\infty_{\mathcal{F}}$, isotopic to the meridians, and for any $\varepsilon_0 > 0$, there is $N > 0$ such that, for every $m \geq N$, there is an ε_0-perturbation $\mathcal{G}$ of the m-retarded cocycle $\mathcal{F}_m$ such that the following conditions hold.*

- *$\mathcal{G}$ is a perturbation of $\mathcal{F}_m$ with support in the homothetic region.*
- *$\mathcal{G}$ is a contracting cocycle.*
- *The strong stable manifold of the origin $\{\mathbf{0}_i\}$ induces $\sigma_1 \cup \sigma_2$ on the orbit space $T^\infty_{\mathcal{G}}$.*

We now give the proof of Theorem 1.1 assuming these two perturbation results.

Proof of Theorem 1.1 using Propositions 2.2 and 2.3. Let f be a C^1-diffeomorphism of some surface and D be an attracting periodic disc, in the basin of an ε-flexible hyperbolic periodic point p of period n. Remember that the orbit space in the punctured stable manifold of p is a torus T^∞_f endowed with a parallel and a meridian (isotopy class of the projection of the strong stable manifold of p).

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First, we perform a perturbation along the orbit of p so that we can reduce the problem to the linear cocycle case, which enables us to use Propositions 2.2 and 2.3. In the following, all the perturbations we give are tacitly assumed to be supported in a sufficiently small neighborhood of the orbit of p so that the orbits entering in D can always be identified with a point in T_f^∞ by the standard conjugacy. We fix a pair (σ_1, σ_2) of disjoint simple curves isotopic to a meridian. Then, by an arbitrarily C^1-small perturbation of f supported in an arbitrarily small neighborhood of the orbit of p, one can obtain a diffeomorphism f_0 whose expression in local charts around the orbit of p is linear, and coincides with the differential of f along the orbit of p. Furthermore, we can take f_0 such that it is isotopic to f through cocycles with the same eigenvalues. Hence, the continuous dependence of the strong stable manifolds implies that the meridians of f_0 in T_f^∞ are isotopic to the meridians of f.

Therefore, by changing f with f_0, we can assume that f is linear in a neighborhood of the orbit of p. Let $\mathcal{A} = \{A_i = Df(f^i(p))\}$ be the corresponding linear cocycle. As f is linear near the orbit of p and is a contraction, the space of orbits of the punctured linear cocycle $T_{\mathcal{A}}^\infty$ is canonically identified with T_f^∞, via the standard conjugacy (see the Introduction for the definition). Therefore, the circles σ_1 and σ_2 of T_f^∞ induce circles α_1 and α_2 of $T_{\mathcal{A}}^\infty$. Now the problem is translated to the perturbation problem of linear cocycles $\mathcal{A}$ over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, that is, to prove Theorem 1.1, we only need to show that there is an ε-perturbation of $\mathcal{A}$ in an arbitrary small neighborhood of the orbit of $\{\mathbf{0}_i\}$ which satisfies the conclusion of the theorem for the curves for α_1 and α_2 in $T_{\mathcal{A}}^\infty$. Let us construct such a perturbation by using Propositions 2.2 and 2.3.

Proposition 2.2 allows us to perform an ε-perturbation of $\mathcal{A}$ in order to get a retardable cocycle $\mathcal{F}$ which coincides with $\mathcal{A}$ outside the unit ball. By conjugating this perturbation by a homothety (which does not change the C^1-size of the perturbation), we can assume that the support of the perturbation given by Proposition 2.2 is contained in an arbitrarily small neighborhood of the orbit of the origin. Remember that the orbit of the origin for $\mathcal{F}$ and hence for $\mathcal{F}_m$ is a non-hyperbolic attracting orbit having exactly one eigenvalue equal to one, as stated in Theorem 1.1. Thus now the notion of the 'meridians in $T_{\mathcal{F}}^\infty$' makes sense. Since $\mathcal{F}$ is isotopic to $\mathcal{A}$ through a contracting cocycle having distinct real eigenvalues supported in the small ball, we have that the meridians of $\mathcal{F}$ in $T_{\mathcal{F}}^\infty$ are isotopic to the meridians of $\mathcal{A}$. Remark 2.1 tells us that the same holds for the m-retarded cocycles $\mathcal{F}_m$.

Now we apply Proposition 2.3: for m large enough, $\mathcal{F}_m$ admits an arbitrarily C^1-small perturbation supported in the homothetic region, so that the strong stable manifold of the periodic orbit induces the circles α_1 and α_2 on the orbit space $T_A^\infty = T_{\mathcal{F}}^\infty = T_{\mathcal{F}_m}^\infty$. □

It remains to prove Propositions 2.2 and 2.3. In §3 we prove Proposition 2.3 and in §4 we prove Proposition 2.2.

3. *Perturbation of retardable cocycles in the homothetic region*

In this section we will prove Proposition 2.3. As in [**BCVW**, **BD1**], we combine the following two simple ideas.

- The fragmentation lemma, which asserts that every diffeomorphism of a closed manifold isotopic to the identity map can be written as a finite product of

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diffeomorphisms arbitrarily close to the identity map, supported in balls of arbitrarily small size.

- Conjugating a diffeomorphism supported in a small ball by a contracting homothety does not change its C^1 distance to the identity. Therefore if one considers an ε-perturbation of a retarded cocycle $\mathcal{F}_m$ supported in some fundamental domain in the homothetic region, and if we put this perturbation in another fundamental domain by conjugating it by a homothety, it is still an ε-perturbation.

Note that the second item is one of the main ideas of Franks' lemma (linearization of local dynamics near the periodic point by arbitrarily small perturbation; see [**F**]), which is frequently used in the study of C^1-generic dynamical systems.

Proof of Proposition 2.3. Let $\mathcal{F} = \{f_i\}$ be a retardable contracting cocycle over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, and $T^\infty_{\mathcal{F}}$ the space of orbit of the punctured cocycle. Let $\gamma_1, \gamma_2 \subset T^\infty_{\mathcal{F}}$ be the meridian induced by the strong stable manifold of $\mathcal{F}$. Remember that for any contracting cocycle $\mathcal{G}$ which coincides with $\mathcal{F}$ outside a small neighborhood of the orbit of the origin, the punctured orbit space $T^\infty_{\mathcal{G}}$ is identified with $T^\infty_{\mathcal{F}}$ through standard conjugacy. In particular, according to Remark 2.1, we have that $W^{\mathrm{ss}}(\mathbf{0}_0, \mathcal{F}_m)$ (remember that $\mathcal{F}_m = \{f_{i,m}\}$ is m-retarded cocycle of $\mathcal{F}$) projects to the same meridian in $T^\infty_{\mathcal{F}} = T^\infty_{\mathcal{F}_m}$ for every $m \geq 0$.

Let $\sigma_1, \sigma_2 \subset T^\infty_{\mathcal{F}}$ be two disjoint simple curves which are isotopic to meridians. We take a diffeomorphism $\psi\colon T^\infty_{\mathcal{F}} \to T^\infty_{\mathcal{F}}$ which is isotopic to the identity and satisfies $\psi(\gamma_i) = \sigma_i$ for $i = 1, 2$. Fix some $\varepsilon_0 > 0$. To prove Proposition 2.3 one has to show that there is N such that every $\mathcal{F}_m$ with $m \geq N$ admits an ε_0-perturbation supported in the homothetic region, and such that the corresponding meridians are σ_1 and σ_2.

We want to perturb $\mathcal{F}_m$ in the homothetic region to realize the behavior of ψ. For this purpose let us first consider the relation between the diffeomorphism on the orbit space and that of the original space. Consider a diffeomorphism $\varphi\colon \mathbb{R}^2 \times \{0\} \to \mathbb{R}^2 \times \{0\}$ whose support is contained in a fundamental domain of the return map $F_m\colon \mathbb{R}^2 \times \{0\} \to \mathbb{R}^2 \times \{0\}$ (remember that F_m denotes the first return map of diffeomorphism cocycle $\mathcal{F}_m$). Then φ projects to a diffeomorphism of $T^\infty_{\mathcal{F}_m}$. Let us denote this projection by $\tilde{\varphi}$.

In some special cases, we can also define the lift of the diffeomorphism; more precisely, we can find a diffeomorphism defined on $\mathbb{R}^2 \times \{0\}$ which projects to the initial one. We observe that the round circles centered at the origin in $\mathbb{R}^2 \times \{0\}$ contained in the homothetic region of $\mathcal{F}_m$ induce a foliation by parallels on $T^\infty_{\mathcal{F}_m}$. We call each leaf of this foliation a *round parallel*. Then we make the following claim (remember that $\Gamma_{s,t}$ in the claim denotes the annulus bounded by two circles centered at the origin with radii $0 < s < t$).

CLAIM 1. *Given any $\eta > 0$, there is $\mu > 0$ satisfying the following condition. Let $\tilde{\varphi}$ be a diffeomorphism of $T^\infty_{\mathcal{F}_m}$ such that:*

- *the C^1-distance between $\tilde{\varphi}$ and the identity map is less than μ;*
- *there is a round parallel disjoint from the support of $\tilde{\varphi}$.*

Then for any $m > 0$ and any r such that $\Gamma_{\lambda^2 r, r}$ is contained in the homothetic region of F_m, there exists a diffeomorphism φ, supported in a round fundamental domain contained in $\Gamma_{\lambda^2 r, r}$, whose projection on $T^\infty_{\mathcal{F}_m}$ is $\tilde{\varphi}$ and is an η-perturbation of the identity map (for the definition of C^1-distance on diffeomorphisms of $T^\infty_{\mathcal{F}_m}$, see Remark 3.1).

Proof. The fact that the support of $\tilde{\varphi}$ is disjoint from one round parallel implies that it admits a lift on some round fundamental domain $\Gamma = \Gamma_{\lambda r_0, r_0} \subset \mathbb{R}^2 \times \{0\}$ of F_m in a homothetic region. Up to some homothetic conjugacy we can assume that $\Gamma \subset \Gamma_{\lambda^2 r, r}$.

In this situation one can easily see that there exists a (unique) lift φ of $\tilde{\varphi}$ supported in Γ. Let us consider the C^1-distance between the identity map and φ. Note that the C^1-distance of φ to the identity does not depend on the choice of the lift in the homothetic region. Thus we only need consider a specific lift in a $\Gamma_{\lambda^2 r, r}$. We can see that this correspondence $\tilde{\varphi} \mapsto \varphi$ is continuous and sends the identity map on T_F to the identity map on $\mathbb{R}^2 \times \{0\}$. Therefore, the choice of small μ guarantees the closeness of the lifted diffeomorphism to the identity map. □

We perform a perturbation by composing such lifted maps with $\mathcal{F}_m$. Let us see the effect of such a perturbation. First, for the C^1-distance, we have the following: given φ supported in a round fundamental domain contained in the homothetic region of F_m, we denote by $\mathcal{F}_{m,\varphi} := \{f_{i,m,\varphi}\}$ the perturbation of the cocycle F_m defined by $f_{i,m,\varphi} := f_{i,m}$ if $i \neq n-1$ and $f_{n-1,m,\varphi} := \varphi \circ f_{n-1,m}$. Then there is C (depending only on $f_{n-1,m}$) such that for every m, every $\eta > 0$ and every φ which is an η-perturbation of the identity map, the cocycle $\mathcal{F}_{m,\varphi}$ is a $C\eta$-perturbation of $\mathcal{F}_m$.

For the behavior of the strong stable manifold, we have the following lemma which follows immediately from the definition).

LEMMA 3.1. *Let $0 < r_1 < \lambda r_2 < r_2 < \lambda r_3 \cdots < r_k$ and $m > 0$ be given such that the round annulus $\Gamma_{\lambda r_1, r_k}$ is contained in the homothetic region of $\mathcal{F}_m$. Let $\{\varphi_i\}$ $(i = 1, \ldots, k)$ be diffeomorphisms on $\mathbb{R}^2 \times \{0\}$ such that φ_i is supported in $\Gamma_i = \Gamma_{\lambda r_i, r_i}$, and let Φ be the diffeomorphism which coincides with φ_i on Γ_i and equal to the identity outside $\Gamma_{\lambda r_1, r_k}$. Then:*

- *$\mathcal{F}_{m,\Phi}$ is a contracting cocycle which coincides with $\mathcal{F}_m$ outside the homothetic region;*
- *the meridians of $\mathcal{F}_{m,\Phi}$ are $(\tilde{\varphi}_k)^{-1} \circ \cdots \circ (\tilde{\varphi}_1)^{-1}(\gamma_i)$ $(i = 1, 2)$.*

Now let us perform the perturbation. Consider $\eta < \varepsilon_1 / C$ and μ associated to η by Claim 1. The fragmentation lemma ensures that the diffeomorphism ψ (for which $\psi(\gamma_i) = \sigma_i$) can be written as

$$\psi = (\tilde{\varphi}_k)^{-1} \circ \cdots \circ (\tilde{\varphi}_1)^{-1}$$

where $k > 0$ and $\tilde{\varphi}_i$ are diffeomorphisms of T_F^∞ supported in small discs such that each φ_i has at least one round parallel disjoint from its support and the C^1-distance from the identity less that μ.

Then we fix $m > 3(k+1)$ such that there is a round annulus $\Gamma_{\lambda^m r, r}$ contained in the homothetic regions of $\mathcal{F}_m$. Then each $\tilde{\varphi}_i$ admits a lift φ_i supported in an annulus $\Gamma_i = \Gamma_{\lambda r_i, r_i}$ for some $\lambda^{m-3i+1} r \leq r_i \leq \lambda^{m-3i} r$ such that the sequence $\{r_i\}$ satisfies the hypotheses of Lemma 3.1. Therefore, $\mathcal{F}_{m,\Phi}$ is the announced ε_1-perturbation where Φ is the diffeomorphism which coincides with φ_i on Γ_i and is equal to the identity outside Γ_i. □

Remark 3.1. In Claim 1, we did not specify the definition of C^1-distance put on the space of C^1-diffeomorphisms on $T_{\mathcal{F}}^\infty$. In fact, as was elucidated in the proof, such a choice is not important for Claim 1 and the whole proof.

4. *Construction of retardable cocycles*

The goal of this section is the proof of Proposition 2.2. Let $\mathcal{A} = \{A_i\}$ be an ε-flexible contracting linear cocycle, $i \in \mathbb{Z}/n\mathbb{Z}$. This gives us, by definition, a path $\{A_{i,t}\}$ of contracting linear cocycles. We will use this path to construct a retardable contracting cocycle isotopic to $\mathcal{A}$ with several other properties. Our main tool is Proposition 4.1 below, which realizes paths of contracting linear cocycles as diffeomorphism contracting cocycles. Note that Proposition 4.1 is independent of the notion of flexibility.

Let

$$\mathcal{C}_n := \mathrm{GL}(2, \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}} = \{\mathcal{A} = \{A_i\} \mid A_i \in \mathrm{GL}(2, \mathbb{R}), i \in \mathbb{Z}/n\mathbb{Z}\}$$

be the space of linear cocycles of period n. Remember that a cocycle is called contracting if the total space $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ is contained in the basin of the orbit of the origin. We denote by $\mathcal{C}_{n,\mathrm{con}} \subset \mathcal{C}_n$ the (open) subset of contracting cocycles.

PROPOSITION 4.1. *Let $\mathcal{O} \subset \mathcal{C}_{n,\mathrm{con}}$ be a relatively compact open subset. Then, for any $\varepsilon_1 > 0$, there is $\delta > 0$ with the following property. Consider any C^1-path $\mathcal{A}_t : [0, +\infty[\to \mathcal{O}$, $t \mapsto \{A_{i,t} \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ which is constant near $t = 0$. Assume that*

$$\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t}.$$

Then the cocycle of maps $\mathcal{F} = \{f_i \mid \mathbb{R}^2 \times \{i\} \to \mathbb{R}^2 \times \{i\}, i \in \mathbb{Z}/n\mathbb{Z}\}$ defined as

$$f_i(x) := A_{i,\|x\|}(x)$$

satisfies the following conditions:

- *$\mathcal{F}$ is a contracting diffeomorphism cocycle;*
- *at each point (x, i),*
$$\|Df_i(x) - A_{i,\|x\|}\| < \varepsilon_1.$$

Let us first show how to derive Proposition 2.2 from Proposition 4.1.

4.1. *Gradual realization of the path: Proof of Proposition 2.2.*

Let $\mathcal{A} = \{A_i\}$ be an ε-flexible cocycle and $\mathcal{A}_t = \{A_{i,t} \mid t \in [-1, 1]\}$ be the path of cocycles in the definition of ε-flexibility. Proposition 2.2 claims the existence of a contracting diffeomorphism cocycle $\mathcal{F}$ which coincides with $\mathcal{A}$ outside the unit ball and with the homothety on n consecutive round fundamental domains, and the origin has one eigenvalue equal to unity. Furthermore, $\mathcal{F}$ needs to be isotopic to $\mathcal{A}$ through contracting cocycles.

Recall that $\mathcal{A}_t$ has two contracting eigenvalues for $t \neq 1$, but $\mathcal{A}_1$ has an eigenvalue equal to one. In order to deal with the contracting property we will show the proposition by replacing $\mathcal{A}_1$ by $\mathcal{A}_{1-\eta}$ for some very small η: the corresponding diffeomorphism cocycle will coincide with $\mathcal{A}_{1-\eta}$ in a neighborhood of the periodic orbit and an extra small perturbation will change $\mathcal{A}_{1-\eta}$ in $\mathcal{A}_1$.

The path $\mathcal{A}_t$, $t \in [-1, 1 - \eta]$ is a compact segment in the open set of contracting linear cocycle. Therefore we can approximate it by a smooth path with the same properties. Thus we assume that $t \mapsto \mathcal{A}_t$ is smooth. Recall that $\mathcal{A}_{-1}$ is a homothety of ratio $\lambda < 1$ in the period, $\mathcal{A}_0 = \mathcal{A}$, and $\mathcal{A}_t$ has two different real eigenvalues for $t \neq -1$. First, we reparametrize $\mathcal{A}_t$ in order to apply Proposition 4.1.

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LEMMA 4.1. *Let $a(t) : [0, 1] \to V$ be a smooth path in an Euclidean space V. Then, for every $\delta > 0$ there exists a smooth non-decreasing function $\theta : [0, +\infty) \to [0, 1]$ such that:*

- $\theta(t) \equiv 0$ *near* $t = 0$;
- $\theta(t) \equiv 1$ *for* $t > 1$;
- *for every* $t \in (0, +\infty)$,

$$\left\| \frac{d(a \circ \theta)}{dt}(t) \right\| < \frac{\delta}{t}.$$

Proof. Note that the length of the path $a(t)$ is finite while the integral $\int_0^1 \delta t^{-1}\, dt$ is infinite for $\delta > 0$. This ensures that we can take the desired reparametrization $\theta(t)$. The explicit construction of $\theta(t)$ is left to the reader. □

We are now in a position to embark on the proof of Proposition 2.2.

Proof of Proposition 2.2. Let $\mathcal{A} = \{A_i\}$ be an ε-flexible cocycle. First, we fix ε_1 sufficiently small so that $\varepsilon_1 + \operatorname{diam}(\mathcal{A}_t) < \varepsilon$ holds. We also fix small $\eta > 0$. The precise choice of η is fixed at the end of the proof. By applying Lemma 4.1, we reparametrize the path $\mathcal{A}_t$, $t \in [-1, 1 - \eta]$, by a function $\Theta\colon [0, +\infty[\to [-1, 1 - \eta]$ (note that this Θ does not need to be monotone) such that we have the following consequences.

- For $t > 0$,

$$\left\| \frac{\partial A_{i,\Theta(t)}}{\partial t} \right\| \leq \frac{\delta}{t}.$$

- $\Theta(t) = 0$, for $t \geq 1$.
- $\Theta(t) = 1 - \eta$ near $t = 0$.
- There are $0 < t_3 < t_2 < t_1 < t_0 < 1$ such that:
  - $\Theta(t) = -1$ for $t \in [t_3, t_0]$;
  - for $\{t_i\}$,

$$t_2 < \lambda t_1, \quad t_3 < K^{-n} t_2 < t_1 K^n < t_0,$$

  where n is the period of the cocycle, $K := \max\{\|A_{i,t}^{\pm 1}\|\} \geq 1$ and λ is the rate of the contraction of the homothety A_{-1}.

Then the announced cocycle $\mathcal{F} = \{f_i\}$ is defined by $f_i(x) := A_{i,\Theta(\|x\|)}(x)$. Indeed, Proposition 4.1, together with the first condition on Θ, implies the contraction property of $\mathcal{F}$. The last condition of Θ ensures that this cocycle (in the period) is a homothety of ratio λ on at least one fundamental domain, implying the retardable property. More precisely, by choosing $R_1 = t_3$, $R_2 = t_1$ and $R_3 = t_0$, we can check the retardable property. Note that by the choice of ε_1, we can deduce that the diffeomorphism cocycle $\mathcal{F}$ itself is an ε-perturbation of $\{A_i\}$ and Lemma 2.1 implies that its retarded cocycles are also ε-perturbation.

Furthermore, the cocycle $\mathcal{F}$ is isotopic to $\mathcal{A}$ through contracting cocycles which coincide with $\mathcal{A}$ outside the unit ball, and whose periodic orbit has two distinct real positive eigenvalues: for that it is enough to change Θ to $\Theta_s(t) = s\Theta(t)$ where $s \in [0, 1]$ (note that, for every $s \in [0, 1]$, we can apply Proposition 4.1).

To finish the proof, it remains to perform an extra perturbation in a very small neighborhood of the periodic point in order to make the weakest eigenvalue of $\mathcal{A}_{1-\eta}$ equal to unity, preserving the contracting property of the cocycle. We can see that if η is sufficiently small, then such a perturbation can be attained in the form of isotopy. More

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precisely, first we perform an perturbation so that the local dynamics along the periodic orbit exhibit an eigenvalue-one direction. Then we add another perturbation so that the central direction is topologically attracting, keeping the eigenvalue.

Thus the proof is complete. □

4.2. *Realizing a path of linear cocycles as a diffeomorphism cocycle: proof of Proposition 4.1.* Let us start the proof of Proposition 4.1. We consider a relatively compact open subset $\mathcal{O} \subset \mathcal{O}_{n,\mathrm{con}}$ (remember that $\mathcal{O}_{n,\mathrm{con}}$ is the space of contracting linear cocycles of period n). We start with some auxiliary observations. We put

$$K_{\mathcal{O}} := \max\{\|A_i^{\pm 1}\| \mid \mathcal{A} = \{A_i\}_{i\in\mathbb{Z}/n\mathbb{Z}} \in \mathcal{O}\} \geq 1,$$

that is, the bound of matrices and their inverse for the cocycles in $\mathcal{O}$.

The relative compactness of $\mathcal{O}$ (compactness of the closure) implies that this bound is finite. The relative compactness of $\mathcal{O}$, together with the fact that each cocycle in $\mathcal{O}$ is contracting, implies that they are uniformly contracting in the following sense.

LEMMA 4.2. *Let $\mathcal{O}$ be a relatively compact set of contracting linear cocycles. Then there is $k_{\mathcal{O}} > 0$ such that, for every $\mathcal{A} = \{A_i\} \in \mathcal{O}$ and for every $i \in \mathbb{Z}/n\mathbb{Z}$,*

$$\|A_{i+k_{\mathcal{O}}-1} \circ \cdots \circ A_i\| < \tfrac{1}{2}.$$

Remark 4.1. In fact, we will prove that the number δ that appears in Proposition 4.1 depends only on ε, $K_{\mathcal{O}}$ and $k_{\mathcal{O}}$ and is independent of the period n and of the relatively compact set $\mathcal{O}$.

Note that the relative compactness of $\mathcal{O}$ also implies that $\overline{\mathcal{O}}$ does not contain any singular matrices. This fact, combined with a compactness argument, yields the following lemma.

LEMMA 4.3. *Given a relatively compact set $\mathcal{O} \subset \mathcal{C}_{n,\mathrm{con}}$, there exists $\mu_{\mathcal{O}} > 0$ such that, for every $\{A_i\} \in \mathcal{O}$, if $\{B_i\} \in M(2, \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}}$ (where $M(2, \mathbb{R})$ is a set of square matrices of size 2) satisfies $\|A_i - B_i\| < \mu_{\mathcal{O}}$ then $\{B_i\} \in \mathrm{GL}(2, \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}}$.*

Remark 4.2. Let $K > 1$. Then

$$\mathcal{B}_K := \{\{A_i\} \in \mathrm{GL}(2, \mathbb{R})^{\mathbb{Z}/n\mathbb{Z}} \mid \max\{\|A_i^{\pm 1}\|\} \leq K\}$$

is a compact set. Thus, we can apply Lemma 4.3 to $\mathcal{B}_K$. We denote the corresponding μ by μ_K.

The following proposition implies Proposition 4.1.

PROPOSITION 4.2. *Given $K \geq 1$, $\varepsilon_1 > 0$ and an integer $k > 0$, there exists $\delta > 0$ such that the following result holds. Let $n > 0$ and $\mathcal{A}_t = \{A_{i,t}\}_{i\in\mathbb{Z}/n\mathbb{Z}}$ ($t \in [0, 1]$) be a path satisfying:*

- $\|A_{i,t}^{\pm 1}\| < K$, *for every i and t;*
- $\|A_{i+k-1,t} \circ \cdots \circ A_{i,t}\| < 1/2$ *for every i and t;*
- $$\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t}.$$

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Then the diffeomorphism cocycle $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$*, where* $f_i : \mathbb{R}^2 \times \{i\} \to \mathbb{R}^2 \times \{i+1\}$ *is defined as*

$$f_i(x) := A_{i,\|x\|}(x),$$

such that:

- $\mathcal{F}$ *is a contracting diffeomorphisms cocycle;*
- *for each point* $(x, i) \in \mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ *and for every* $0 \le j \le k-1$*,*

$$\|D\mathcal{F}^j(x, i) - A_{i+j-1,\|x\|} \circ \cdots \circ A_{i,\|x\|}\| < \varepsilon_1$$

(remember that $\mathcal{F}$ *is the diffeomorphism of the total space* $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$*).*

Proof. First, note that the inequality in the second item of the conclusion implies the contracting property: it implies that $\|D\mathcal{F}^k\| < 1/2 + \varepsilon_1$, thus by exchanging ε_1 with a number smaller than $1/2$ if necessary, we obtain $\|DF^k\| < 1$.

Let us start the proof of this inequality for $j = 1$. By a direct calculation, for every i,

$$Df_i(x) = D(A_{i,\|x\|})(x, i) = A_{i,\|x\|} + \left(\frac{dA_{i,t}}{dt}\bigg|_{t=\|x\|}\right) \otimes \left(\frac{\partial(\|x\|)}{\partial x}\right)(x).$$

Therefore,

$$\|Df_i(x) - A_{i,\|x\|}\| \le C\|x\| \cdot \left\|\frac{dA_{i,\|x\|}}{dt}\right\|,$$

where C is a constant that does not depend on particular choices of other constants.

Then we fix δ so that $\delta < \min\{\varepsilon_1, \mu_K\}/C$ holds (for the definition of μ_K, see Lemma 4.3 and Remark 4.2). This guarantees that

$$\left\|\frac{\partial A_{i,t}}{\partial t}\right\| \le \frac{\delta}{t} < \min\{\varepsilon_1, \mu_K\}/(Ct).$$

Therefore,

$$\|Df_i(x) - A_{i,\|x\|}\| \le \|x\| \cdot \frac{C\delta}{\|x\|} = C\delta < \min\{\varepsilon_1, \mu_K\},$$

which implies the desired inequality for $j = 1$. Furthermore, this inequality, together with Lemma 4.3, shows that $Df_i : \mathbb{R}^2 \times \{i\} \to \mathbb{R}^2 \times \{i+1\}$ is a local diffeomorphism. This fact, combined with some topological observations, leads to the conclusion that f_i is a diffeomorphism for every i.

Now we start the proof of the inequality above for general $j < k$. The difficulty is as follows. Remember that the differential $D\mathcal{F}^j(x, i)$ is the product

$$D\mathcal{F}^j(x, i) = D\mathcal{F}(\mathcal{F}^{j-1}(x, i)) \circ \cdots \circ D\mathcal{F}(x, i).$$

A priori, the distance between $\mathcal{F}^{j-1}(x, i)$ and $(x, i + j)$ can be very large. Thus the corresponding differentiations can be very different. Our strategy is to choose sufficiently small δ so that the matrix $A_{i+\ell,\|F^\ell(x,i)\|}$ remains almost equal to $A_{i+\ell,\|x\|}$, for $0 \le \ell \le k-1$.

We start by bounding $\|\mathcal{F}^\ell(x, i)\|$, for $0 \le \ell < k-1$, as follows:

$$\|x\|/K^k \le \|F^\ell(x, i)\| \le K^k\|x\|,$$

where K is the uniform bound of the norms $\|A_{i,t}^{\pm 1}\|$. Then, a simple argument involving the mean value theorem implies the following claim.

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CLAIM 2. *For any $\nu > 0$, there is $\delta > 0$ such that, if $\|\partial A_{i,t}/\partial t\| \leq \delta/t$ for every $t > 0$, then, for every $t > 0$, $i \in \mathbb{Z}/n\mathbb{Z}$, ℓ satisfying $0 \leq \ell < k$ and $s \in [K^{-k}t, K^k t]$,*

$$\|A_{i+\ell,t} - A_{i+\ell,s}\| < \nu.$$

Also a simple compactness argument, together with the continuity of products of matrices, shows the following claim.

CLAIM 3. *There is $\nu > 0$ such that, for any matrices $\{B_{i,t}\}_{i\in\mathbb{Z}/n\mathbb{Z},t\in[0,1]}$ with $\|B_{i,t} - A_{i,t}\| < 2\nu$, any $0 \leq j < k$ and for any i, t, $\varepsilon_1 > 0$,*

$$\|B_{i+j-1,t} \circ \cdots \circ B_{i,t} - A_{i+j-1,t} \circ \cdots \circ A_{i,t}\| < \varepsilon_1.$$

We are now in a position to prove the inequality. We fix $\nu > 0$ given by Claim 3 and δ given by Claim 2. Then, for every $0 \leq \ell < k$,

$$\begin{aligned}
&\|D\mathcal{F}(\mathcal{F}^\ell(x, i)) - A_{i+\ell,\|x\|}\| \\
&\quad \leq \|D\mathcal{F}(\mathcal{F}^\ell(x, i)) - A_{i+\ell,\|\mathcal{F}^\ell(x,i)\|}\| + \|A_{i+\ell,\|\mathcal{F}^\ell(x,i)\|} - A_{i+\ell,\|x\|}\| \\
&\quad \leq \nu + \nu = 2\nu.
\end{aligned}$$

Now the choice of ν implies the inequality

$$\|D\mathcal{F}^j(x, i) - A_{i+j-1,\|x\|} \circ \cdots \circ A_{i,\|x\|}\| < \varepsilon_1.$$

Thus the proof is complete. □

5. *Abundance of flexible periodic points*

The purpose of this section is the proof of Theorem 1.3 (and therefore of Theorem 1.2). The proof contains two steps. The first step is to show that the hypotheses of Theorem 1.3 lead (up to arbitrarily small perturbations) to the coexistence of points with complex stable eigenvalues and points with a stable eigenvalue arbitrarily close to unity in the same basic set. The second step is to show that such basic sets contain flexible points.

Throughout this section, M denotes a smooth closed manifold endowed with a Riemannian metric.

5.1. *Periodic points with an arbitrarily weak stable eigenvalue.*

Let us start the proof of the first step. We will show the following proposition.

PROPOSITION 5.1. *Let $f \in \mathrm{Diff}^1(M)$ and let p be a hyperbolic periodic point of f. Suppose that U is an open neighborhood of the orbit $\mathcal{O}(p)$ with the following properties.*

- *p has stable index two.*
- *There is a stable index one periodic point q with $\mathcal{O}(q) \subset U$.*
- *There are hyperbolic transitive basic sets $K \subset H(p, U)$ and $L \subset H(q, U)$ containing p and q, respectively, so that K and L form a C^1-robust heterodimensional cycle in U (that is, the orbits of K and L are contained in U and there exist heteroclinic points between $W^u(K)$ and $W^s(L)$, $W^u(L)$ and $W^s(K)$ whose orbits are contained in U).*

Then, for every $\nu > 0$, there is $g \in \mathrm{Diff}^1(M)$ arbitrarily C^1-close to f having a periodic point p_2 with the following properties:

- *p_2 has stable index two and is homoclinically related with p in U;*
- *p_2 has a real stable eigenvalue $\lambda^{cs}(p_2)$ with $|\lambda^{cs}| \in [1-\nu, 1)$;*
- *p_2 has the smallest Lyapunov exponent such that*

$$\chi^{ss}(p_2) \in [\inf\{\chi^{ss}(p), \chi^{ss}(q)\} - \nu, \sup\{\chi^{ss}(p), \chi^{ss}(q)\} + \nu];$$

- *the orbit of p_2 is ν-dense in the relative homoclinic class $H(p, U, f)$.*

Sketch of proof of Proposition 5.1. The creation of periodic orbits with eigenvalues arbitrarily close to unity inside a homoclinic class containing a robust heterodimensional cycle has been already done in [**ABCDW**]. The proof of Proposition 5.1 can be done in a similar fashion. So we only show the sketch the proof.

We fix $\nu > 0$. First, note that either K or L is non-trivial, since otherwise they cannot form a robust heterodimensional cycle. Thus by performing a perturbation by Hayashi's connecting lemma if necessary, we can assume that both of them are non-trivial.

Recall that the homoclinic class of p is the closure of transverse homoclinic intersection, hence is the Hausdorff limit of an increasing sequence of hyperbolic basic sets. The corresponding fact is also true for relative homoclinic classes. Therefore one can choose a hyperbolic basic set $\tilde{K} \subset U$ whose Hausdorff distance from $H(p, U, f)$ is less than $\nu/10$. We can also find an arbitrarily small perturbation f_0 of f and a periodic point $\tilde{p} \in \tilde{K}(f_0)$, where $\tilde{K}(f_0)$ is the hyperbolic continuation of $\tilde{K}$ such that:

- the Hausdorff distance between the orbit $\mathcal{O}(\tilde{p})$ and $H(p, U, f_0)$ is less that $\nu/5$;
- $|\chi^{ss}(\tilde{p}, f_0) - \chi^{ss}(p, f)| < \nu/10$;
- the Lyapunov exponent $\chi^{ss}(\tilde{p}, f_0)$ has multiplicity 1, that is, the restriction of the derivative to the stable plane has two distinct real eigenvalues (see [**ABCDW**, §2] or [**BCDG**, §4]).

If the perturbation f_0 is sufficiently close to f, then one still has a C^1-robust heterodimensional cycle associated with $K(f_0)$ and $L(f_0)$ (and therefore $\tilde{K}(f_0)$ and $L(g_0)$). In other words, by replacing f with f_0 and ν with $\nu/2$, one may assume that:

- the orbit of p is $\nu/2$ dense in $H(p, U, f)$;
- p has two real distinct eigenvalues.

In the same way, by changing q to another periodic point in L and performing an arbitrarily small perturbation of f, one may assume that q has the real weakest unstable eigenvalue.

Then we perform a second perturbation to construct a heterodimensional cycle between p and q in U as follows (see [**ABCDW**, §2]).

- As p and q belong to the same chain recurrence class, an arbitrarily small perturbation (using, for instance, the connecting lemma in [**BC**]) allows us to create a transverse intersection between $W^u(q)$ and $W^s(p)$;
- As the C^1-robust cycle persists under the first perturbation, p and q still belong to the same class. Hence, an arbitrarily small perturbation, preserving the first intersection, allows us to create a transverse intersection between $W^s(q)$ and $W^u(p)$.

Now [**ABCDW**, §3] (see also [**BD2**, **BDK**]) tells us that by performing an arbitrarily small perturbation to the heterodimensional cycle as above, we can create a periodic point p_2 with the following properties (we denote the perturbed diffeomorphism by g):

- p_2 has a stable index two;
- p_2 has a weakest stable eigenvalue λ^{cs} with absolute value $|\lambda^{cs}| = 1 - \nu < 1$;
- the orbit of p_2 passes arbitrarily close to p_g (as a consequence $\mathcal{O}(p_2)$ will be $\nu/2$ dense in $H(p, U, f)$);
- $\chi^{ss}(p_2)$ is arbitrarily close to a convex sum of $\chi^{ss}(p, f)$ and $\chi^{ss}(q, f)$;
- the unstable manifold of p_2 cuts transversely the stable manifold of p and the stable manifold of p_2 cuts transversely the unstable manifold of q.

The last item implies that p_2 and p are robustly in the same chain recurrence class (since we can always find a pseudo-orbit from q to p following the robust heterodimensional cycle between K and L): a new arbitrarily small perturbation by the connecting lemma in [**BC**] creates a transverse intersection between the stable manifold of p with the unstable manifold of q, which completes the proof. □

5.2. *Weak eigenvalues, complex eigenvalues, and flexible points.* The aim of the rest of this section is the proof of the next proposition.

PROPOSITION 5.2. *Given $C > 1$, $\chi < 0$ and $\varepsilon > 0$, there exists $\nu \in (0, 1)$ with the following property. Let $f \in \mathrm{Diff}^1(M)$ be a diffeomorphism and Λ be a compact invariant hyperbolic basic set of f with stable index two such that $\|Df\|$ and $\|Df^{-1}\|$ are bounded by C from above over Λ. Suppose that Λ contains a hyperbolic periodic point q (of stable index two) with complex (non-real) stable eigenvalues and a point p having two distinct real stable eigenvalues such that:*

- *the smallest Lyapunov exponent of p is less than $\chi < 0$;*
- *the stable eigenvalue with largest absolute value λ^{cs} satisfies $|\lambda^{cs}| \in (1 - \nu, 1)$.*

Then f admits an arbitrarily small perturbation which creates an ε-flexible point x_L containing the continuation of Λ, and the ε-neighborhood of $\mathcal{O}(x_L)$ contains $\mathcal{O}(p)$.

This proposition, together with Proposition 5.1 and a standard genericity argument (involving the generic continuity of the homoclinic classes with respect to the Hausdorff distance), implies Theorem 1.3.

The main ingredient of the proof is that, in a basic set, given a finite set of periodic points, one may choose a periodic point which travels around these periodic points with the predetermined itinerary. By choosing a convenient itinerary, we can find a periodic point whose differential behaves in a way very close to what we want. Thus, by adding some perturbation, we can obtain the desired orbit. This technique has been formalized in [**BDP**] by the notion of *transition*.

5.3. *Transitions on the periodic points.* In this subsection we will extract a consequence from [**BDP**]. For the proof of Lemma 5.1, see [**BDP**, Lemma 1.9] (indeed, Lemma 5.1 is just a special case of [**BDP**, Lemma 1.9]).

LEMMA 5.1. *Suppose that $f \in \mathrm{Diff}^1(M)$ has a hyperbolic basic set Λ with stable index two. Let $T\Lambda = E^s \oplus E^u$ be the hyperbolic splitting (thus $\dim E^s = 2$). We fix a coordinate on $\Lambda|_{E^s}$ and take the matrix representation of df. Let $x_1, x_2 \in \Lambda$ be two hyperbolic periodic saddle points of period π_i ($i = 1, 2$), respectively.*

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Then, given $\varepsilon_2 > 0$*, there exist two finite sequences of matrices* (T_i^j) $(j = 0, \ldots, j_i - 1, i = 1, 2)$ *in* $\mathrm{GL}(2, \mathbb{R})$ *(where* j *denotes the superscript, not the power of the matrix) with the following property. For any* $L = (l_1, l_2, l_3, l_4) \in \mathbb{N}^4$ *satisfying* $(l_1, l_2) \neq (l_3, l_4)$*, there exists a periodic point* $x_L \in \Lambda$ *such that:*

- *the period of* x_L *is* $(l_1 + l_3)\pi_1 + (l_2 + l_4)\pi_2 + 2(j_1 + j_2)$*;*
- *if* $k = K$ *with* $0 \leq K < \pi_1 l_1$ *or* $k = l_1\pi_1 + l_2\pi_2 + j_1 + j_2 + K$ *with* $0 \leq K < \pi_1 l_3$*, then* $Df|_{E^s}(f^k(x_L))$ *is* ε_2*-close to* $Df|_{E^s}(f^k(x_1))$*;*
- *if* $k = l_1\pi_1 + j_1 + K$ *with* $0 \leq K < \pi_2 l_2$ *or* $k = (l_1 + l_3)\pi_1 + l_2\pi_2 + 2j_1 + K$ *with* $0 \leq K < \pi_2 l_4$*, then* $Df|_{E^s}$ *is* ε_2*-close to* $Df|_{E^s}(f^K(x_2))$*;*
- *if* $k = l_1\pi_1 + K$ *or* $k = (l_1 + l_3)\pi_1 + l_2\pi_2 + K$ *with* $0 \leq K < j_1$*, then* $Df|_{E^s}(f^k(x_L))$ *is* ε_2*-close to* T_1^K*;*
- *if* $k = l_1\pi_1 + l_2\pi_2 + j_1 + K$ *or* $k = (l_1 + l_3)\pi_1 + (l_2 + l_4)\pi_2 + 2j_1 + j_2 + K$ *with* $0 \leq K < j_2$*, then* $Df|_{E^s}(f^k(x_L))$ *is* ε_2*-close to* T_2^K*.*

We put $T_i := \prod_{j=0}^{j_i-1} T_i^j$ $(i = 1, 2)$ *and call them* transition matrices.

Remark 5.1. In the above lemma, by adjusting L we can control the position of the orbit of x_L. More precisely, if we take l_1 or l_3 (respectively, l_2 or l_4) very large, then x_L passes arbitrarily close to x_1 (respectively, x_2). See **[BDP]** for details.

5.4. *Rudimentary results from linear algebra.* We collect two results from linear algebra, which will be used in the proof of Proposition 5.2.

First, we prove the following lemma.

LEMMA 5.2. *Let* $T \in \mathrm{GL}_+(2, \mathbb{R})$ *and* Q *be a contracting homothety (that is,* $Q = \lambda\mathrm{Id}$ *where* λ *satisfies* $0 < \lambda < 1$*). Then, given* $\varepsilon > 0$*, there exist* $h > 0$ *and a sequence of matrices* (J_i) *(respectively,* (L_i)*)* $(i = 0, \ldots, h - 1)$ *such that:*

- *each* J_i *(respectively,* L_i*) is* ε*-close to* Q*;*
- *the product* $T(\prod_{i=0}^{h-1} J_i)$ *(respectively,* $(\prod_{i=0}^{h-1} L_i)T$*) is a contracting homothety.*

Proof. We only give the proof of the existence of (J_i). The proof of (L_i) is similar.

Let T, Q, and ε be given. First we fix $\delta > 0$ such that if $X \in \mathrm{GL}_+(2, \mathbb{R})$ is δ-close to Id, then Q and XQ are ε-close. We can fix such δ because of the continuity of the multiplication.

For every $T \in \mathrm{GL}_+(2, \mathbb{R})$, there is a continuous path $I(t)$ in $\mathrm{GL}_+(2, \mathbb{R})$ such that $I(0) = T$ and $I(1) = \mathrm{Id}$ (since $\mathrm{GL}_+(2, \mathbb{R})$ is path-connected). Then, because of the compactness of the path, we can take a sufficiently large integer $m > 0$ such that, for every k $(0 \leq k < m - 1)$, $I((k + 1)/m)(I(k/m))^{-1}$ is δ-close to the identity. We put $h = m$ and $J_k = I((k + 1)/m) \cdot (I(k/m))^{-1}Q$. Since Q is a homothety,

$$\begin{aligned} T\prod_{k=0}^{h-1}(J_k) &= T\prod_{k=0}^{h-1}(I((k + 1)/h) \cdot (I(k/h))^{-1}Q) \\ &= TQ^h \prod_{k=0}^{h-1} I((k + 1)/h) \cdot (I(k/h))^{-1} \\ &= TQ^h \cdot I(1) \cdot (I(0))^{-1} = Q^h. \end{aligned}$$

This completes the proof. □

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Denote by $R(\theta)$ the rotation matrix of angle θ; more precisely, put

$$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We prove the following lemma.

LEMMA 5.3. *Let* $1 > \lambda_1 > \lambda_2 > 0$. *Then, for* $0 < t < 1$, *the matrix*

$$M(t) := R(-t) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R(t) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfies the following properties:

- *for* $0 \le t < \pi/2$, $M(t)$ *has two distinct positive contracting eigenvalues;*
- *for* $t = \pi/2$, $M(t)$ *is a homothetic contraction.*

Proof. Both items can be checked by calculating the characteristic polynomial of $M(t)$ directly. Indeed, it is given by $x^2 - \mathrm{tr}(M(t))x + \det(M(t))$. Note that $\det(M(t))$ is equal to $(\lambda_1\lambda_2)^2$ (independent of t). By a direct calculation, we can see that $\mathrm{tr}(M(t))$ is equal to

$$(\lambda_1^2 + \lambda_2^2)\cos^2 t + 2\lambda_1\lambda_2 \sin^2 t.$$

One can check that this value is monotone decreasing on $[0, \pi/2]$. Then, by a direct calculation we can check the desired properties of the path $M(t)$. □

5.5. *Proof of Proposition 5.2.* To prove Proposition 5.2 we use Lemma 5.3, which explains the behavior of products of rotation matrices and diagonal matrices. To use the lemma we need to take convenient bases on tangent spaces of our basic set. First let us see how to fix such bases.

5.5.1. *Diagonalizing coordinates.* In the assumption of Proposition 5.2, the basic set Λ contains a periodic point p with distinct real eigenvalues: one eigenvalue is very close to ± 1 but the smallest Lyapunov exponent is bounded away from 0. In particular, the restriction of the differential $Df^{\pi}(p)$, where π is the period of p, to the stable plane is diagonalizable. We want to take the pair of eigenvectors as the basis of stable tangent directions along the orbit of p. However, such a coordinate change may be large (with respect to the metric induced from the original Riemannian metric) and the seemingly small perturbations observed through the diagonalized coordinate can be very large. Thus, we need to estimate the size of such a coordinate change.

To clarify our argument, we generalize the situation as follows. Let $A_i : V_i \to V_{i+1}$ ($i \in \mathbb{Z}/\pi\mathbb{Z}$) be a sequence of isomorphisms between the sequence of two-dimensional Euclidean vector spaces $\{V_i\}$. Suppose that the product $A_{\pi-1} \cdots A_0$ is diagonalizable. Thus, on each V_i, there are images of two eigenspaces U_i and W_i of dimension one. We assume that U_i is the strong contracting direction.

For each V_i, we fix an orthonormal basis $\langle e_{1,i}, e_{2,i}\rangle$ so that $U_i \wedge W_i$ and $(e_{1,i}) \wedge (e_{2,i})$ define the same orientation. Then let $B_i \in \mathrm{SL}(2, \mathbb{R})$ be the matrix representing the change of basis from the orthonormal one to the one having one unit vector on the most contracted eigendirection. More precisely, $B_i : V_i \to V_i$ is the (unique) matrix in

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$SL(2, \mathbb{R})$ satisfying $B_i(e_{1,i}) \in U_i$, $B_i(e_{2,i}) \in W_i$ and $\|B_i(e_{1,i})\| = 1$ (we take the matrix representation regarding $\langle e_{1,i}, e_{2,i}\rangle$ as the standard basis). Up to a multiplication by an orthogonal matrix (deriving from the ambiguity of the choice of initial orthonormal basis), B_i is well defined. In particular, $\|B_i\|$ is well defined.

The value $\|B_i\|$ measures the angle between two eigendirections of V_i in the sense that it is a strictly decreasing function of the angle. The norm $\|B_i\|$ is equal to unity when two eigendirections are orthogonal and diverges to $+\infty$ as the angle tends to zero (the proof is easy, so we omit it).

We now use the following lemma from linear algebra.

LEMMA 5.4. *For every $C_1 > 1$ and $\chi < 0$, there exists $\alpha > 0$ such that the following result holds. Let $A_i : V_i \to V_{i+1}$ ($i \in \mathbb{Z}/\pi\mathbb{Z}$) be a sequence of isomorphisms between two-dimensional Euclidean vector spaces with norms $\|A_i^{\pm}\| < C_1$. Suppose that the product $A_{\pi-1} \cdots A_0$ is diagonalizable and its Lyapunov exponents are such that:*

- *the smaller one is less than χ;*
- *the larger one is greater than $\chi/2$.*

Then there is $i \in \mathbb{Z}/\pi\mathbb{Z}$ on which the matrix of coordinate change B_i (defined as above) has a norm $\|B_i\|$ smaller than α.

We apply Lemma 5.4 to our situation, letting $C_1 = C$, χ as in the hypotheses of Proposition 5.2, $V_i = T_{f^i(p)}\Lambda|_{E^s}$ and $A_i = Df(f^i(x))|_{E^s}$. It implies that there is a constant α depending only on C_1, χ such that there is at least a point $f^i(p)$ of the orbit of x where $\|B_i\| < \alpha$. Thus, by replacing p with $f^i(p)$, we can assume that $\|B_0\| < \alpha$. In other words, up to a conjugacy by matrices in $SL(2, \mathbb{R})$ whose norm is bounded by α from above, one may assume that $Df^{\pi}(x)$ is diagonal. This bounded change of coordinates induces a bounded change of the notion of perturbations (remember that, for matrices in $SL(2, \mathbb{R})$, the norm of the matrix and of its inverse are the same).

Thus, up to a multiplication by some constant, we can assume that a δ-perturbation with respect to this coordinate is also a δ-perturbation to the orthonormal coordinate. So we fix some coordinate around $T_p\Lambda|_{E^s}$ explained as above and continue the proof.

Let us prove Lemma 5.4. First, note that a simple compactness argument shows the following lemma.

LEMMA 5.5. *Let $C_1 > 1$. Then, for every $\kappa > 1$, there exists $\tau > 0$ such that, for every $A \in GL(2, \mathbb{R})$ satisfying $\|A\|, \|A^{-1}\| < C_1$, if u, v are unit vectors such that the angle between them is less than τ then $1/\kappa < \|Au\|/\|Av\| < \kappa$.*

Proof of Lemma 5.4. First, we fix $C_1 > 0$ and $\chi < 0$. Then, apply Lemma 5.5 for this C_1, letting $\kappa = \exp(-\chi/4)$ and fix τ. We then fix α sufficiently large so that if $B \in SL(2, \mathbb{R})$ is the matrix representing the change of the basis with $\|B\| > \alpha$, then the corresponding angle is less than τ.

For such choice of α, we show the existence of good i where the corresponding matrix B_i has norm less than α. Suppose not; that is, every B_i has norm greater than α. This implies that on each $T_{f^i(x)}|_{E^s}$, the image of eigenvectors has angle less than τ. Then Lemma 5.5 (used inductively) implies that if u and v are eigenvectors in V_0, then for each i we have $\exp((\chi/4)i) < \|(A_{i-1} \cdots A_0)(v)\|/\|(A_{i-1} \cdots A_0)(u)\| < \exp(-(\chi/4)i)$, but this contradicts the hypotheses on the Lyapnov exponents of the first return map $A_{n-1} \cdots A_0$. □

5.5.2. *Creating the homothety.* The next lemma, inspired by [**BDP**] or [**S**], provides the announced homothety which is used to realize the division of perturbations.

LEMMA 5.6. *Let $f \in \mathrm{Diff}^1(M)$ and Λ be a non-trivial basic set of stable index two. Suppose that there exists a hyperbolic periodic point $q \in \Lambda$ which has a contracting complex eigenvalue. Then, C^1-arbitrarily close to f, there exists $g \in \mathrm{Diff}^1(M)$ which has a periodic point r whose differential $dg^{\mathrm{per}(r)}$ restricted to the stable direction is a contracting homothety. Furthermore, the support of the perturbation from f to g can be taken arbitrarily close to an orbit of some periodic point of f in Λ.*

The proof of Lemma 5.6 is essentially done in [**BDP**] (see [**BDP**, Proposition 2.5]) or [**S**] (see [**S**, Lemma 3.3]). Thus we only sketch the proof.

Sketch of the proof. The proof is a direct consequence of Lemma 5.2 and a variant of Lemma 5.1. The idea of Lemma 5.1 is that there are periodic orbits whose differential restricted to the stable direction is an arbitrarily small perturbation of a product of a fixed transition matrix T with an arbitrarily large power of the differential of the point q, which has a complex stable eigenvalue. Large powers of a matrix in $\mathrm{GL}(2, \mathbb{R})$ with a complex eigenvalue admit perturbations so that the product is a homothety. Therefore, Λ contains periodic orbits whose stable differential is, up to an arbitrarily small perturbation, arbitrarily close to the product of the transition matrix T and an arbitrarily large power of a homothety. Now Lemma 5.2 allows us to perform a small perturbation along the orbit to cancel the fixed intermediate differentials. As a result, such a periodic orbit has a contracting homothety in the stable direction. □

5.5.3. *Creating flexible points.* We are now ready for the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $C > 1$ and $\chi < 0$ be given, and let Λ be the hyperbolic basic of stable index two containing periodic points p and q as in the hypothesis of Proposition 5.2.

According to Lemma 5.4, we can fix coordinates on $T\mathcal{O}(p)|_{E^s}$ so that we may assume that:

- at the periodic point p of period $\pi(p)$ the first return map has the form of diagonal matrix
$$Df^{\pi(p)}|_{E^s}(p) = P := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
with $|\lambda_1| \in [1-\nu, 1)$, and the smaller Lyapunov exponent $(1/\pi(p)) \log |\lambda_2|$ is less than χ.

Furthermore, according to Lemma 5.6, by giving an arbitrarily small perturbation whose support is away from some neighborhood of $\mathcal{O}(p)$ and changing r to q, we can assume that:

- at the periodic point q, the first return map is a contracting homothety, that is,

$$Df^{\pi(q)}|_{E^s}(q) = Q := \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

where $\pi(q)$ denotes the period of q and $r \in (0, 1)$.

We choose ν so that multiplying Df by the homothety of ratio $(1-\nu)^{-2}$ is an $\varepsilon/2$-perturbation. Note that the choice of ν can be determined only from the value of C and χ.

To prove Proposition 5.2, it remains to show that an arbitrarily small perturbation of f may create an ε-flexible point in Λ. We apply Lemma 5.1: there are transition matrices T_1, T_2 so that given any $L = (l_1, l_2, l_3, l_4)$ with $(l_1, l_2) \neq (l_3, l_4)$, we can find a periodic point x_L whose first return map of differential cocycle restricted to the stable direction admits a small perturbation so that the first return map is equal to

$$T_2 Q^{l_4} T_1 P^{l_3} T_2 Q^{l_2} T_1 P^{l_1}.$$

Indeed, by fixing ε_2 in the statement of Lemma 5.1 (which can be taken arbitrarily small), we can assume, by performing an ε_2-perturbation, that the differential along x_L is indeed given by this matrix. Now, let us choose L so that the point x_L can be perturbed to an ε-flexible point.

5.5.4. *First case:* $T_i \in \mathrm{GL}_+(2, \mathbb{R})$ *and* $\lambda_i > 0$. From here the proof bifurcates depending on the signature of $\det(T_i)$ and λ_i. First, we consider the case $T_1, T_2 \in \mathrm{GL}_+(2, \mathbb{R})$ and $\lambda_1, \lambda_2 > 0$. The case where some of the T_i are in $\mathrm{GL}_-(2, \mathbb{R})$ or $\lambda_i < 0$ can be treated similarly. We will explain how one can modify the proof for the other cases after the end of the proof of this case.

Let us continue the proof. First, we apply Lemma 5.2 (letting $Q = Q$ and $T_1 = T$). Then we can find a sequence of matrices (L_i) $(i = 0, \ldots, i_1 - 1)$ such that each L_i is arbitrarily close to the homothety Q and $T_2(\prod_{i=0}^{i_1-1} L_i) = c_1\mathrm{Id}$. Similarly, we take (J_i) $(i = 0, \ldots, i_2 - 1)$ so that each J_i is arbitrarily close to Q and $(\prod_{i=0}^{i_2-1} J_i)T_1 = c_2\mathrm{Id}$ holds. We also fix an integer i_3 such that the multiplication of the rotation $R(\pi t/2i_3)$ by the matrix Q is a small perturbation of Q for every $t \in [-1, 1]$.

Then let us fix $L = (l_1, l_2, l_3, l_4)$ as follows:

- $l_1 = l_3$ are sufficiently large integers that $\mathcal{O}(x_L)$ contains $\mathcal{O}(\gamma_n)$ in its $\varepsilon/2$-neighborhood with respect to the Hausdorff distance (see Remark 5.1);
- $l_2 \neq l_4$ and $l_2, l_4 > i_1 + i_2 + i_3$;
- $$\mu_L := \exp\left(\frac{1}{\pi(x_L)} \cdot \log((c_1c_2)^2 r^{l_2+l_4-2(i_1+i_2)} \lambda_1^{l_1+l_3})\right) > 1 - \nu.$$

We can take such L as follows: first take l_2, l_4 satisfying the second condition and then take l_1 and l_3 sufficiently large that the rest of the conditions are satisfied (remember that $\pi(x_L) = (l_1 + l_3)\pi(p) + (l_2 + l_4)\pi(q) + 2(j_1 + j_2)$, thus just by taking large l_1, l_3 we can obtain the inequality in the third condition).

First, we perform a preliminary perturbation along x_L so that (denoting the perturbed map also by f):

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- for $k = K\pi(q) + l_1\pi(p) + j_1$ or $k = K\pi(q) + (l_1 + l_3)\pi(p) + l_2\pi(q) + 2j_1 + j_2$ $(K = 0, \ldots, i_1 - 1)$,
$$\prod_{i=0}^{\pi(q)-1} Df(f^{k+i}(x_L)) = L_k;$$
- for $k = K\pi(q) + l_1\pi(p) + j_1$ or $k = K\pi(q) + (l_1 + l_3)\pi(p) + l_2\pi(q) + 2j_1 + j_2$ $(K = i_1, \ldots, i_1 + i_2 - 1)$,
$$\prod_{i=0}^{\pi(q)-1} Df(f^i(x)) = J_k.$$

Now, by using the equalities $(\prod J_i)T_1 = c_2\mathrm{Id}$, $T_2(\prod L_i) = c_1\mathrm{Id}$ and the commutativity of homothetic transformations, we can check that the first return map of the derivative of f along x_L has the form
$$(c_1c_2)^2 Q^{l_4-i_1-i_2} P^{l_3} Q^{l_2-i_1-i_2} P^{l_1} = (c_1c_2)^2 Q^{l_2+l_4-2(i_1+i_2)} P^{l_1+l_3}; \qquad (\dagger)$$
in particular, x_L has two real contracting eigenvalues $(c_1c_2)^2 r^{l_2+l_4-2(i_1+i_2)}\lambda_j^{l_1+l_3}$, $j = 1, 2$.

We show that x_L is an ε-flexible point by constructing the path of linear cocycles in the definition of the flexibility. Recall that the path $\mathcal{A}_t$ we need to construct is the one which joins the cocycle $\mathcal{A}_0$ induced by Df on the stable bundle to a cocycle $\mathcal{A}_{-1}$ whose return map is a homothety, and to a cocycle $\mathcal{A}_1$ having an eigenvalue equal to unity.

First we construct the path from $\mathcal{A}_0$ to $\mathcal{A}_1$. For that we only need to multiply $\mathcal{A}_0$ along the orbit of x_L by a homothety of ratio μ_L^{-t}. We put $\mathcal{A}_t := \mu_L^{-t}\mathcal{A}_0$. By construction, there exists a moment $t = t_0$ for which $\mathcal{A}_{t_0}$ has an eigenvalue equal to unity. This gives a path of contracting cocycles between the original one and the one with a neutral saddle. Indeed, our second condition on the choice of L implies that the ratio of this homothety is always less than $(1 - \nu)^{-2}$ (and greater than one). Furthermore, our choice of ν implies that the multiplication by such a homothety remains an ε-perturbation of $\mathcal{A}_0$.

Now let us construct the path $\mathcal{A}_t$ for $t \in [-1, 0]$. For that purpose, we rewrite the product of the differential as
$$T_2\left(\prod_{k=0}^{i_1-1} L_k\right) Q^{l_4-i_1-i_2}\left(\prod_{k=0}^{i_2-1} J_k\right) T_1 P^{l_3} T_2\left(\prod_{k=0}^{i_1-1} L_k\right) Q^{l_2-i_1-i-2}\left(\prod_{k=0}^{i_2-1} J_k\right) T_1 P^{l_1}$$
$$= (c_1\mathrm{Id})\,\underbrace{(Q^{i_3})}_{(**)}\, Q^{l_4-i_1-i_2-i_3}(c_2\mathrm{Id})P^{l_1}(c_1\mathrm{Id})\,\underbrace{(Q^{i_3})}_{(*)}\, Q^{l_2-i_1-i_2-i_3}(c_2\mathrm{Id})P^{l_1}.$$

In this product, we replace each homothetic matrix Q in the second Q^{i_3} (indicated by $(*)$) by $R(\pi t/2i_3) \circ Q$ and each Q in the first one (indicated by $(**)$) by $R(-\pi t/2i_3) \circ Q$. By the choice of i_3, this perturbation can be made very small for every $t \in [-1, 0]$; in particular, it can be made with size less than ε. The effect of this perturbation on the product matrix is to replace the whole product $(\dagger)$ with
$$(c_1c_2)^2 Q^{l_2+l_4-2(i_1+i_2)} R\left(-\frac{\pi}{2}t\right) P^{l_1} R\left(\frac{\pi}{2}t\right) P^{l_3}.$$

Now Lemma 5.3 ensures (remember that $l_1 = l_3$) that the product matrix of $\mathcal{A}_t$ has two different real contracting positive eigenvalues for $t \neq -1$ and is a homothety for $t = -1$.

Thus combining these two paths, we obtain the desired path, which completes the proof. □

5.5.5. *Other cases: matrices in* $\mathrm{GL}_-(2, \mathbb{R})$ *or negative eigenvalues.* Finally, let us consider the case where some of the signs of $\det(T_i)$, λ_i are negative.

First, in the case where λ_1 or λ_2 is negative, we just need to take l_1, l_3 to be even numbers: this replaces the matrix P by P^2 everywhere in the proof, and the proof works identically.

For the case where one and only one of the transitions T_i, say T_1, reverses the orientation, the argument in [**BDP**] allows us to consider the matrix $\tilde{T}_1 = T_1 P^i T_2 Q^j T_1$ as a new transition substituting T_1 (keeping T_2 unchanged as the other transition matrix), and now $\tilde{T}_1$ and T_2 are both in $\mathrm{GL}_+(2, \mathbb{R})$.

Let us consider the case where T_1 and T_2 are both orientation reversing. In this case, we apply Lemma 5.2 to the matrix

$$T_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which provides a sequence of matrices (L_i) $(i = 0, \ldots, i_1 - 1)$ arbitrarily close to the homothety Q so that

$$T_2\left(\prod_{i=0}^{i_1-1} L_i\right) = (c_1 \mathrm{Id}) \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly, we take (J_i) $(i = 0, \ldots, i_2 - 1)$ arbitrarily close to Q so that

$$\left(\prod_{i=0}^{i_2-1} J_i\right) T_1 = (c_2 \mathrm{Id}) \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the proof works identically, just noticing that the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is an involution which commutes with P and Q.

Remark 5.2. In the proof, we can assume that the differential of the first return map restricted to the unstable direction is also orientation preserving, by adjusting the number l_i appropriately.

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