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HOMOGENEOUS FUNCTIONALLY ALEXANDROFF SPACES

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Abstract

A function $f : X \to X$ determines a topology $P(f)$ on X by taking the closed sets to be those sets $A \subseteq X$ with $f(A) \subseteq A$. The topological space $(X, P(f))$ is called a functionally Alexandroff space. We completely characterise the homogeneous functionally Alexandroff spaces.

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1. Introduction

Alexandroff spaces [1] are topological spaces in which arbitrary intersections of open sets are open. Equivalently, a topological space is Alexandroff if and only if every point has a least neighbourhood. Clearly, every finite topological space is an Alexandroff space. Since computer arithmetic and computer graphics are based on finite sets of machine numbers or pixels, computer applications have driven much interest in Alexandroff spaces. Elementary properties of Alexandroff spaces are given in [2, 3, 7]. From a different perspective, Uzcátegui and Vielma [9] studied Alexandroff spaces viewed as closed sets of the Cantor cube 2^X.

Suppose that (X, τ) is an Alexandroff space. The associated specialisation quasi-order on X is given by $x \leq_\tau y$ if and only if $x \in cl\{y\}$ (the closure of $\{y\}$). If (Y, σ) is also an Alexandroff topological space, $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if $f : (X, \leq_\tau) \to (Y, \leq_\sigma)$ is increasing (that is, order preserving). Thus, f is a homeomorphism if and only if it is bijective and an order isomorphism (that is, $a \leq_\tau b$ in X if and only if $f(a) \leq_\sigma f(b)$ in Y). When the topology τ is understood, we may simply say that X is an Alexandroff space with associated specialisation quasi-order $\leq$.

Suppose that $(X, \leq)$ is a quasi-ordered set. The *upper hull* of $A \subseteq X$ is the set $\uparrow A$ defined by $\uparrow A = \{x \in X : \exists a \in A, a \leq x\}$, and A is an *upper set* if $A = \uparrow A$. *Lower hulls* and *lower sets* are defined dually. We say that x *covers* y and y *is covered by* x if $y \prec x$

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and $y \leq z \leq x$ implies that $z \in \{x, y\}$. If (X, τ) is an Alexandroff space, then the τ-open sets are the $\leq_\tau$-upper sets, and the τ-closed sets are the $\leq_\tau$-lower sets.

If X is a set and $f : X \to X$ is a function, the set $\{A \subseteq X : f(A) \subseteq A\}$ of f-invariant subsets of X forms the closed sets of an Alexandroff topology $P(f)$ on X. Alexandroff topologies which are realised as $P(f)$ for some function $f : X \to X$ are *functionally Alexandroff spaces*. These were introduced independently by Shirazi and Golestani [8] in 2011 and by Echi [4] in 2012. Shirazi and Golestani called such spaces *functional Alexandroff spaces* and Echi called them *primal spaces*. Further properties of functionally Alexandroff spaces are given in [6].

In a functionally Alexandroff space $(X, P(f))$, we denote by $(X, \leq_f)$ the corresponding set equipped with the specialisation quasi-order. Then, for any $x \in X$, the closure of $\{x\}$ is the orbit $\downarrow\{x\} = \{f^n(x), n \in \mathbb{N}\}$ and the smallest open neighbourhood $V_f(x)$ of x is given by

$$V_f(x) = \uparrow\{x\} = \{y \in X : \text{there exists } n \in \mathbb{N} \text{ such that } f^n(y) = x\},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ (see [4, 8]).

A topological space X is called *homogeneous* if for any points $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$. Homogeneous spaces are those spaces X on which $\mathrm{Aut}(X)$ acts transitively, that is, for every x and y there exists $\phi \in \mathrm{Aut}(X)$ with $\phi(x) = y$.

An ordered set X is said to be *k-homogeneous* if every order isomorphism between two k-element sets can be extended to an order isomorphism from X to X. An ordered set X is *homogeneous* if it is k-homogeneous for all cardinal numbers k. Thus, a functionally Alexandroff space $(X, P(f))$ is topologically homogeneous if and only if the corresponding quasi-ordered set $(X, \leq_f)$ is 1-homogeneous.

In this paper, we characterise functionally Alexandroff spaces which are homogeneous.

2. Homogeneity of functionally Alexandroff spaces

If $(X, P(f))$ is a functionally Alexandroff space generated by f and $p \in \mathbb{N} - \{0\}$, we say that $x \in X$ has *period p* if $x, f(x), f^2(x), \ldots, f^{p-1}(x)$ are distinct and $f^p(x) = x$, and we say that f (or the associated quasi-order) has a *cycle of length p* or a *p-cycle*. A point with period 1 is a *fixed point* of f and is easily seen to be a minimal point in $(X, \leq_f)$.

Proposition 2.1. *If $(X, P(f))$ is homogeneous, then f is a surjective map.*

Proof. If f is not surjective, then there exists a point $x \in X$ with an empty inverse image. Hence, $V_f(x) = \{x\}$ and by the homogeneity of $(X, P(f))$, for any $y \in X$, $V_f(y) = \{y\}$. Since $x \in V_f(f(x)) = \{f(x)\}$, it follows that $x = f(x)$, which contradicts the assumption that $\{x\}$ has empty preimage. □

If $(X, P(f))$ is homogeneous and x is an element of a p-cycle of f, then $cl\{x\}$ has p elements and, by homogeneity, every element belongs to a p-cycle. Thus, the

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specialisation quasi-order consists of unrelated p-cycles, whose T_0 reflection (formed by identifying the equivalence classes from the relation $x \approx y$ if and only if $x \leq y$ and $y \leq x$, resulting in a partial order on the equivalence classes) is an antichain. The topology $P(f)$ is a partition topology (that is, there is a partition of X which is a basis for $P(f)$) in which every equivalence class has p elements.

Thus, we may now focus on functionally Alexandroff spaces with no cycles.

Proposition 2.2. *Suppose $(X, P(f))$ is a functionally Alexandroff space and $f : X \to X$ has no cycles. Let $q : X \to X$ be a bijection. Then q is a homeomorphism if and only if $q \circ f = f \circ q$.*

Proof. Suppose that q is a homeomorphism. Since f has no cycles, then, for each $x \in X$, $f(x)$ is the only point covered by x and $f(q(x))$ is the only point covered by $q(x)$. On the other hand, since $q : (X, \leq_f) \to (X, \leq_f)$ is an order isomorphism, then $q(f(x))$ is the only point covered by $q(x)$. Thus, $q(f(x)) = f(q(x))$ for every $x \in X$.

Conversely, if $q \circ f = f \circ q$, then, for any $n \in \mathbb{N}$, we have $q(f^n(x)) = f^n(q(x))$. Therefore, $q(cl\{x\}) = cl\{q(x)\}$ and, since X is an Alexandroff space, q is a closed map. Applying the same argument to q^{-1} shows that q^{-1} is a closed map and thus q is a homeomorphism. □

If q is a homeomorphism on $(X, P(f))$ with $q(x) = y$, then q maps $V_f(x)$ to $V_f(y)$ and thus is an order isomorphism on $(X, \leq_f)$ which maps $V_f(x) - \{x\}$ to $V_f(y) - \{y\}$. Thus, the minimal elements in $V_f(x) - \{x\}$ are in one-to-one correspondence with the minimal elements of $V_f(y) - \{y\}$. That is, the elements in $(X, \leq_f)$ which cover x are in one-to-one correspondence with the elements which cover y. In a functionally Alexandroff space, we have $x \geq f(x) \geq f(f(x)) \geq \cdots$ and, if x is not a cyclic point, x covers precisely one element $f(x)$. It follows that in a homogeneous functionally Alexandroff space X, there is a fixed cardinal number κ such that every element in X covers exactly one element $f(x)$, and is covered by exactly κ other elements. This is the basis for the characterisation of specialisation orders which represent functionally Alexandroff spaces [4, 8] as those in which every interval $[x, y] = \{z : x \leq y \leq z\}$ is finite (that is, the quasi-ordered set $(X, \leq)$ is *causal*), $\downarrow\{x\}$ is a chain for every $x \in X$ and $[x, x] = \{x\}$ unless x is minimal.

Furthermore, these observations show that if a functionally Alexandroff space $(X, P(f))$ is homogeneous, then there exists a cardinal number κ such that the function $f : X \to X$ is κ-to-one. Our main result (Theorem 2.10) will show that the converse holds if f has no cycles. Specifically, we will show that for any cardinal number κ, there exists a homogeneous functionally Alexandroff space in which each element is covered by exactly κ points, and every homogeneous functionally Alexandroff space $(X, P(f))$ has this form if f has no periodic points.

Let $(X, P(f))$ be a functionally Alexandroff space. For any $a \in X$, we write $f^{-1}[a] = f^{-1}(\{a\})$ and, for any natural number $n \geq 2$, $f^{-n}[a] = f^{-1}(f^{-(n-1)}[a])$. Then $f^0[a] = \{a\}$, $f^{-1}[a] = \{y : y \text{ covers } a\}$ and $V_f(a) = \bigcup_{n \geq 0} f^{-n}[a]$.

Proposition 2.3. *Every functionally Alexandroff space $(X, P(f))$ is locally connected and the component of $a \in X$ is given by $C_a = \bigcup_{n \geq 0} V_f(f^n(a))$.*

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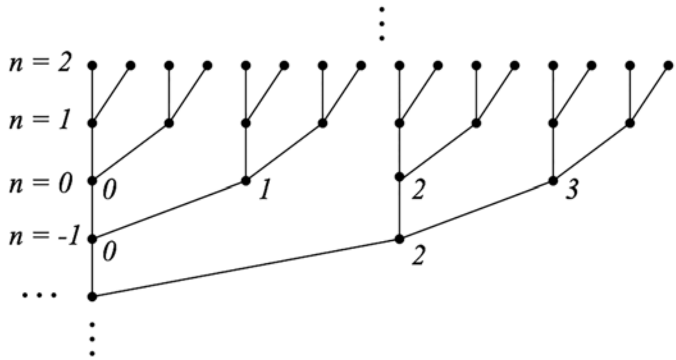

FIGURE 1. A homogeneous functionally Alexandroff space in which every point is covered by two points.

PROOF. It is easy to see that for any point $x \in X$, the smallest neighbourhood $V_f(x)$ of x is connected. Thus, X is locally connected. As the union of connected sets containing the point a, the set $C_a = \bigcup_{n\geq 0} V_f(f^n(a))$ is connected. Suppose that $x \notin C_a$. We will show that $V_f(x) \cap C_a = \emptyset$. If not, there exists $y \in V_f(x) \cap C_a$ so that $f^k(y) = x$ and $f^j(y) = f^n(a)$ for some integers $j, k, n \geq 0$. Now $f^{j+k}(y) = f^j(x) = f^{n+k}(a)$, contrary to $x \notin C_a$. Since $V_f(x)$ and C_a are disjoint and open, their restriction to $C_a \cup \{x\}$ shows that $C_a \cup \{x\}$ is not connected, so C_a is the maximal connected set containing a. □

Since $(X, P(f))$ is locally connected, X is homeomorphic to the disjoint union of its components, that is, $X \simeq \coprod_{i\in I} C_i$, where $\{C_i\}_{i\in I}$ is the collection of distinct components of X. If C_i is homeomorphic to C for each $i \in I$, then $\coprod_{i\in I} C_i \simeq C \times I$, where I is equipped with the discrete topology. Hence, we conclude that $(X, P(f))$ is homogeneous if and only if C is homogeneous. In particular, in a disconnected homogeneous functionally Alexandroff space, every component C_a of X must be homeomorphic to every other component C_b and thus X is (homeomorphic to) the disjoint union $\coprod_{i\in I} C$ of copies of the component C. Thus, we may focus on the description of connected homeomorphic functionally Alexandroff spaces.

We begin by considering κ-to-one functions for finite cardinals $\kappa \in \mathbb{N}$.

If $(X, P(f))$ is a connected functionally Alexandroff space with no minimal points (that is, f has no fixed points) and each point is covered by $\kappa = 1$ points, then the specialisation order is a countably infinite chain and is order isomorphic to $\mathbb{Z}$.

The following example constructs a countable homogeneous functionally Alexandroff space with $|f^{-1}[x]| = 2$ for every $x \in X$.

EXAMPLE 2.4. Let $X = \{(r, n) \in \mathbb{Q} \times \mathbb{Z} : n \in \mathbb{Z}, r = 2^{-n}k$ for some $k \in \mathbb{Z}\}$. Given $(2^{-n}k, n) \in X$, define $f(2^{-n}k, n) = (2^{-n}(k - j), n - 1)$, where $j = k \pmod 2$. The associated quasi-order $\preceq$ is $(r, n) \preceq (s, m)$ if and only if $n \leq m$ and $s \in [r, r + 2^{-n}[$, as depicted in Figure 1.

Given $(2^{-n}k, n)$ and $(2^{-m}j, m) \in X$, the function $h : X \to X$ defined by $h(x, y) = (2^{-m}(2^n x + j - k), y + m - n)$ maps $(2^{-n}k, n)$ to $(2^{-m}j, m)$. It is easily seen that h is one-to-one and an order isomorphism. Furthermore, h is onto, since given any $(2^t s, l) \in X$,

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$(2^t s, l) = h(2^{-n}(2^{m-t}s - j + k), l - m + n)$. Thus, h is a homeomorphism and this shows that X is homogeneous. In this example, each point of $(X, \leq_f)$ is covered by exactly two elements.

Note that if 2 is replaced by any finite cardinal κ, the construction remains valid, giving a homogeneous functionally Alexandroff space X in which every element is covered by exactly κ others.

For another concrete realisation in an uncountable space, let us consider the following example.

EXAMPLE 2.5. Let $X = \{x \in \mathbb{C} : |x| > 1\}$ and let $f : X \to X$ be defined by $f(x) = x^2$. Now f has no periodic or fixed points and since every $x \in X$ has exactly two square roots, $|f^{-1}[x]| = 2$ for every $x \in X$. First, we show that any connected component of X is homogeneous. Suppose that $a = r\exp(i\theta)$ and $b = s\exp(i\varphi)$ belong to the same component C_a of X. For every $x \in C_a$, there exist $m, n \in \mathbb{N}$ with $f^m(x) = f^n(a)$, and we may take m and n to be the smallest such integers. Now $x^{2^m} = a^{2^n} = r^{2^n}\exp(i2^n\theta)$, so $x \in \{\eta, \eta\omega, \eta\omega^2, \ldots, \eta\omega^{2^m-1}\}$, where $\eta = r^{2^{n-m}}\exp(i2^{n-m}\theta)$ and $1, \omega, \omega^2, \ldots, \omega^{p-1}$ are the pth roots of unity for $p = 2^m$. Define $h : C_a \to C_a$ by $h(x) = h(\eta\omega^k) = \eta'\omega^k$, where $\eta' = s^{2^{n-m}}\exp(i2^{n-m}\varphi)$. Noting that $f^n(a) = f^m(x) = f^{m-1}(x^2)$, it is easy to verify that h is a bijection with $h(a) = b$ and $h \circ f = f \circ h$. Thus, C_a is homogeneous.

Furthermore, it is intuitively clear that any two components are homeomorphic. More formally, the argument above remains valid if $b \in X - C_a$, showing that the function $h : C_a \to C_b$ is a homeomorphism. Thus, X is homogeneous.

With X as above and $f : X \to X$ defined by $f(x) = x^n$ for $n \geq 2$, one may similarly show that the n-to-one function f produces a homogeneous functionally Alexandroff space $(X, P(f))$.

Example 2.4 gives the prototype for an n-to-one function for $n \in \mathbb{N}$. For any given infinite cardinal number κ, we will use the next example as the prototype for a κ-to-one function. The theorems that follow will show that the associated functionally Alexandroff space is homogeneous.

EXAMPLE 2.6. Let κ be a nonzero cardinal number and let $\Bbbk$ be a set of cardinality κ. For an integer $p \in \mathbb{Z}$, let $\mathbb{Z}_p = \{n \in \mathbb{Z} : n \geq p\}$. Fix $\eta \in \Bbbk$. We define the space $Y_\Bbbk$ to be the set of all ultimately η-sequences in $\Bbbk$. That is,

$$Y_\Bbbk = \{g : \mathbb{Z}_p \to \Bbbk : p \in \mathbb{Z}\ (\exists n \in \mathbb{Z}_p, \forall i \geq n,\ g(i) = \eta)\}$$

is the set of sequences indexed by some $\mathbb{Z}_p$ which are eventually constantly η. We denote an element $(g : \mathbb{Z}_p \to \Bbbk)$ in $Y_\Bbbk$ by (g, p).

Define $f_\Bbbk : Y_\Bbbk \to Y_\Bbbk$ by $f_\Bbbk((g, p)) = (g|_{\mathbb{Z}_{p+1}}, p + 1)$, where $g|_{\mathbb{Z}_{p+1}}$ is the restriction of g to $\mathbb{Z}_{p+1}$. Since $f_\Bbbk$ is a κ-to-one function, every point of $(Y_\Bbbk, \preceq)$ is covered by exactly κ others.

For any two points (g, p) and $(h, q) \in Y_\Bbbk$, there are integers $n \geq p$ and $m \geq q$ with $g(\mathbb{Z}_n) = h(\mathbb{Z}_m) = \{\eta\}$. If $r \geq \max\{m - p, n - p\}$, then $f_\Bbbk^{r-p}((g, p)) = f_\Bbbk^{r-q}((h, q)) = (\overline{\eta}, r)$, where $\overline{\eta}$ is the constant function on $\mathbb{Z}_r$ with value η. Thus, $cl\{(g, p)\} \cap cl\{(h, q)\} = \downarrow\{(g, p)\} \cap \downarrow\{(h, q)\} \neq \emptyset$ and consequently $Y_\Bbbk$ is connected.

THEOREM 2.7. *Let $(X, P(f))$ and $(Y, P(g))$ be functionally Alexandroff spaces such that each element is covered by exactly κ others. Then, for each $a \in X$ and for each $b \in Y$, $V_f(a)$ and $V_g(b)$ are homeomorphic.*

PROOF. The result is trivial for $\kappa = 0$. Let $\kappa \geq 1$ and let $(\overline{\eta}, 0) \in Y_{\mathbb{k}}$ be as defined in Example 2.6. It suffices to show that the subspaces $V_f(a)$ of $(X, P(f))$ and $V_{f_{\mathbb{k}}}((\overline{\eta}, 0))$ of $(Y_{\mathbb{k}}, P(f_{\mathbb{k}}))$ are homeomorphic.

For each $b \in V_f(a)$, we denote by $\mathcal{A}_b$ the set of all bijective functions from $f^{-1}[b]$ to $\mathbb{k}$. By hypothesis, $\mathcal{A}_b \neq \emptyset$ for every $b \in V_f(a)$. So, by the Axiom of Choice, there is a choice function $\mu : V_f(a) \to \bigcup_{b \in V_f(a)} \mathcal{A}_b$, where $\mu(b) = \mu_b \in \mathcal{A}_b$.

Inductively, for any natural number n, we construct an order-isomorphism map

$$q_n : \bigcup_{i=0}^{n} f^{-i}[a] \to \bigcup_{i=0}^{n} f^{-i}[(\overline{\eta}, 0)]$$

such that q_n restricted to $\bigcup_{i=0}^{n-1} f^{-i}[a]$ is equal to q_{n-1}.

- For $n = 0$, let q_0 be the trivial order isomorphism which maps the point a to the point $(\overline{\eta}, 0)$.
- Suppose that $n \geq 1$ and that $q_0, \ldots, q_{n-1}$ have already been defined. As $f^{-n}[a] = \bigcup_{b \in f^{-n+1}[a]} f^{-1}[b]$, for any $x \in f^{-n}[a]$, there exists a unique $b_x \in f^{-n+1}[a]$ such that $x \in f^{-1}[b_x]$. Define the map q_n by

$$q_n(x) = \begin{cases} q_{n-1}(x) & \text{if } x \in \bigcup_{i=0}^{n-1} f^{-i}[a], \\ (g, n) \text{ where } \begin{cases} (g|_{\mathbb{Z}_{n+1}}, n+1) = q_{n-1}(b_x) \\ g(n) = \mu_{b_x}(x) \end{cases} & \text{if } x \in f^{-n}[a]. \end{cases}$$

By the construction itself, the map $q_n : \bigcup_{i=0}^{n} f^{-i}[a] \to \bigcup_{i=0}^{n} f^{-i}[(\overline{\eta}, 0)]$ is a well-defined order isomorphism (q_n is bijective and $q_n \circ f_{\mathbb{k}} = f_{\mathbb{k}} \circ q_n$) such that q_n restricted to $\bigcup_{i=0}^{n-1} f^{-i}[a]$ is equal to q_{n-1}.

Informally, we used the choice function μ to tag elements of $V_f(a)$ by elements of $V_{f_{\mathbb{k}}}((\overline{\eta}, 0))$ in an order-preserving manner. Thus, for $x \in f^{-n}[a]$, there exist unique $k_{-1}, k_{-2}, \ldots, k_{-n} \in \mathbb{k}$ such that x is represented by the sequence $\{k_{-n}, \ldots, k_{-2}, k_{-1}, \eta, \eta, \eta, \ldots\}$ in $Y_{\mathbb{k}}$ indexed by $\mathbb{Z}_{-n}$.

Since functions on X are relations on X (that is, subsets of $X \times X$), we may take the union of functions, viewing them as relations. Since $V_f(a) = \bigcup_{n \geq 0} f^{-n}[a]$, from our construction, the relation $q_a = \bigcup_{n \geq 0} q_n : V_f(a) \to V_{f_{\mathbb{k}}}((\overline{\eta}, 0))$ is a well-defined function and an order isomorphism. Thus, $V_f(a)$ is homeomorphic to $V_{f_{\mathbb{k}}}((\overline{\eta}, 0))$. □

COROLLARY 2.8. *Let $(X, P(f))$ and $(Y, P(g))$ be functionally Alexandroff spaces such that each element is covered by exactly κ others. If $a \in X$, $b \in Y$ and $q : V_f(a) \to V_g(b)$ is a homeomorphism, then there is an extension of q to a homeomorphism from $V_f(f(a))$ to $V_g(g(b))$.*

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PROOF. We notice that if each element is covered by exactly κ others, then the space has no cycles.

The result is trivial for $\kappa = 1$, because the space is order isomorphic to $\mathbb{Z}$.

Let $\kappa \geq 2$. In the posets $V_f(f(a)) - V_f(a)$ and $V_g(g(b)) - V_g(b)$, every point is covered by κ others if κ is an infinite cardinal number. If κ is finite, then every point is covered by κ others except for the points $f(a)$ and $g(b)$, which are covered by $\kappa - 1$ others.

Using the same order-isomorphism construction as in Theorem 2.7, there exists an order-isomorphism map h from $V_f(f(a)) - V_f(a)$ to $V_g(g(b)) - V_g(b)$. Hence, the map $h \cup q : V_f(f(a)) \to V_g(g(b))$ is a homeomorphism which extends q. □

COROLLARY 2.9. *Let $(X, P(f))$ be a functionally Alexandroff space such that each element is covered by exactly κ others. Let $\{C_i\}_{i\in I}$ be the collection of components of X. Then C_i and C_j are homeomorphic for every $i, j \in I$.*

PROOF. Let $a \in C_i$ and $b \in C_j$. By the previous results, there exists a homeomorphism $q_{f^0(a)} : V_f(a) \to V_f(b)$ and, for each integer $n \geq 1$, there exists a homeomorphic extension $q_{f^n(a)} : V_f(f^n(a)) \to V_f(f^n(b))$ of $q_{f^{n-1}(a)}$. Since C_i is connected, $C_i = \bigcup_{n\geq 0} V_f(f^n(a))$. Hence, the map $q = \bigcup_{n\geq 0} q_{f^n(a)} : C_i \to C_j$ is the desired homeomorphism. □

THEOREM 2.10. *Let $(X, P(f))$ be a functionally Alexandroff space with no cycles. Then $(X, P(f))$ is homogeneous if and only if f is a κ-to-one function for some cardinal number κ.*

PROOF. We have previously noted that if $(X, P(f))$ is homogeneous, then f is κ-to-one.

Suppose that f is κ-to-one. By Corollary 2.9, we may assume that the space X is connected. Suppose that $a \neq b \in X$. By Proposition 2.3, the components $C_a = \bigcup_{n\geq 0} V_f(f^n(a))$ and $C_b = \bigcup_{k\geq 0} V_f(f^k(b))$ are equal, so there exist $n, k \geq 0$ with $f^n(a) = f^k(b)$. In particular, $\downarrow\{a\} \cap \downarrow\{b\} \neq \emptyset$.

First, we consider the case $V_f(a) \cap V_f(b) \neq \emptyset$. Now, for $z \in V_f(a) \cap V_f(b)$, there exist distinct $m, n \in \mathbb{N}$ with $f^n(z) = a$ and $f^m(z) = b$. Without loss of generality, $n < m$, so, with $p = m - n$, we have $b = f^p(f^n(z)) = f^p(a) \in \downarrow\{a\}$.

By the Hausdorff maximal principle [5] (Kuratowski's lemma), there is a maximal chain $\mathcal{M}$ in $V_f(a)$ which contains $\{a\}$. By causality of X, $\mathcal{M}$ is isomorphic to the first infinite ordinal ω and can be denoted by $\mathcal{M} = \{a = a_0 > a_{-1} > \cdots > a_{-n} > \cdots\}$, where $a_{-n} \in f^{-n}[a]$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $a_n = f^n(a)$. Now $A = \{a_n\}_{n\in\mathbb{Z}}$ is order isomorphic to $\mathbb{Z}$. Define $K : A \to A$ by $K(a_n) = a_{n+p}$, where $f^p(a) = b$. Thus, K is an order isomorphism on A with $K(a) = b$.

For each $n \in \mathbb{Z}$,

$$(V_f(a_n) - V_f(a_{n-1})) \cap (V_f(K(a_n)) - V_f(K(a_{n-1}))) = \emptyset.$$

Then, using the construction of the map q_a in Theorem 2.7, we conclude that there is an order isomorphism

$$h_n : V_f(a_n) - V_f(a_{n-1}) \to V_f(K(a_n)) - V_f(K(a_{n-1}))$$

such that $h_n(a_n) = K(a_n)$. In particular, $h_0(a) = b$.

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For any $x \in X$, there is a unique $n_x \in \mathbb{Z}$ such that $x \in V_f(a_{n_x}) - V_f(a_{n_x-1})$ (n_x is the smallest integer n such that $x \in V_f(a_n)$). So, the map

$$h = \bigcup_{n\in\mathbb{Z}} h_n : X \to X$$
$$x \mapsto h_{n_x}(x)$$

is a well-defined homeomorphism from X to X with $h(a) = b$.

Now we consider the remaining case $V_f(a) \cap V_f(b) = \emptyset$. As mentioned above, $\downarrow\{a\} \cap \downarrow\{b\} \neq \emptyset$. Choose $z \in \downarrow\{a\} \cap \downarrow\{b\}$. Now $\uparrow\{z\} \cap \uparrow\{a\} = V_f(z) \cap V_f(a) \neq \emptyset$ and similarly $V_f(z) \cap V_f(b) \neq \emptyset$. By the previous argument, there exist homeomorphisms $q_1, q_2 : X \to X$ such that $q_1(a) = z$ and $q_2(z) = b$. Now $q = q_2 \circ q_1 : X \to X$ is the desired homeomorphism with $q(a) = b$. □

Theorem 2.10 and the remarks following Proposition 2.1 provide a complete characterisation of the homogeneous functionally Alexandroff spaces. Either they are of the form $(X, P(f))$ for a κ-to-one function f (if f has no cycles) or they have a basis $\mathcal{B} = \{B_i : i \in I\}$ of mutually disjoint sets B_i with $|B_i| = p$ for all $i \in I$, where p is a fixed finite integer.

We conclude with another application of Theorem 2.7. A topological space X is called *uniquely homogeneous* provided that for every $x, y \in X$ there is a unique homeomorphism of X taking x onto y.

Corollary 2.11. *If $(X, P(f))$ is a homogeneous functionally Alexandroff space with $|X| \geq 3$, then $(X, P(f))$ is uniquely homogeneous if and only if $(X, P(f))$ is homeomorphic to $(\mathbb{Z}, P(g))$, where $g(n) = n + 1$.*

Proof. Suppose that $(X, P(f))$ is a homogeneous functionally Alexandroff space with $|X| \geq 3$. Now f is injective if and only if $(X, P(f))$ is homeomorphic to the uniquely homogeneous functionally Alexandroff space $(\mathbb{Z}, P(g))$, where $g(n) = n + 1$. If f is not injective, then there exist $a \neq b$ in X with $f(a) = f(b)$. By Theorem 2.7, there exists a homeomorphism $h' : V_f(a) \to V_f(b)$ with $h'(a) = b$. Since $V_f(a)$ and $V_f(b)$ are disjoint, $h : X \to X$ defined by

$$h(x) = \begin{cases} h'(x) & \text{if } x \in V_f(a) \cup V_f(b), \\ x & \text{if } x \notin V_f(a) \cup V_f(b). \end{cases}$$

is a homeomorphism on X with $h(a) = b \neq a$. Thus, h and Id are two distinct homeomorphisms on X which map $f(a)$ to $f(a)$, so X is not uniquely homogeneous. □

Thus, the only uniquely homogeneous functionally Alexandroff spaces (up to homeomorphism) are $(\{0\}, P(Id))$, $(\{0, 1\}, P(Id))$, $(\{0, 1\}, P(f))$, where $f(0) = 1$ and $f(1) = 0$, and $(\mathbb{Z}, P(g))$, where $g(n) = n + 1$.

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