DIFFERENTIAL EQUATIONS DRIVEN BY Π-ROUGH PATHS

LAJOS GERGELY GYURKÓ[1,2]

[1]*Mathematical Institute, University of Oxford, Andrew Wiles Building,*
Radcliffe Observatory Quarter, Woodstock Road,
Oxford OX2 6GG, UK
[2]*Oxford-Man Institute of Quantitative Finance, University of Oxford,*
Eagle House, Walton Well Road, Oxford OX6 2ED, UK
(gyurko@maths.ox.ac.uk)

Abstract   This paper revisits the concept of rough paths of inhomogeneous degree of smoothness (geometric Π-rough paths in our terminology) sketched by Lyons in 1998. Although geometric Π-rough paths can be treated as p-rough paths for a sufficiently large p, and the theory of integration of Lip^γ one-forms ($\gamma > p-1$) along geometric p-rough paths applies, we prove the existence of integrals of one-forms under weaker conditions. Moreover, we consider differential equations driven by geometric Π-rough paths and give sufficient conditions for existence and uniqueness of solution.

1. Introduction

The theory of rough paths due to Lyons [**8**, **9**] enables the definition of a wide class of stochastic differential equations in the pathwise sense. In particular, the rough paths representation of Brownian motion (enhanced or lifted Brownian motion) was considered in [**9**] and has been extensively studied by Friz and Victoir [**2**, **4**] and many others. Coutin and Qian [**1**] proved the existence of a geometric-rough path associated with the fractional Brownian motion with Hurst parameter greater than $\frac{1}{4}$. A different approximation of the enhanced fractional Brownian motion was studied by Millet and Sanz-Solé [**11**]. The rough path representation of an even larger class of Gaussian processes was explored by Friz and Victoir [**3**, **4**] and others. In all these works, the roughness degree of the driving noise is described by a single real number p, and hence it is assumed to be homogeneous in all directions.

In §2 we revisit the definition of geometric-rough paths of inhomogeneous degree of smoothness (geometric Π-rough paths in our terminology) sketched by Lyons [**8**]. Our sharpened extension theorem of Π-rough paths specifies which terms of the signature of a rough path of inhomogeneous degree of smoothness determine the whole signature. In

particular, the extension theorem determines which terms are to be specified in order to lift stochastic processes with mixed components such as Brownian and fractional Brownian components to Π-rough paths.

We note that geometric Π-rough paths can be treated as p-rough paths for a sufficiently large p, and that the theory of integration of Lip^γ one-forms ($\gamma > p-1$) along geometric p-rough paths (see [8,9]) can be applied. In §3 we show that the Lip^γ condition on the one-form can be weakened if we exploit the fact that the underlying Π-rough path has components with roughness parameter smaller than p. In particular, we introduce the definition of $\mathrm{Lip}^{\Gamma,\Pi}$ one-forms. Moreover, as the main result of the paper, we define and prove the existence of integrals of $\mathrm{Lip}^{\Gamma,\Pi}$ one-forms along Π-rough paths (Theorem 3.9).

In §4, as a particular application of the main result, we consider differential equations of the form

$$\mathrm{d}Y_t = f(X_t, Y_t)\,\mathrm{d}X_t, \quad Y_0 = \xi \in W, \tag{1.1}$$

where X is a geometric Π-rough path defined on some Banach space V, $f\colon V \oplus W \to L(V, W)$ and W is some Banach space. This equation can be rewritten as follows:

$$\mathrm{d}\tilde{Y}_t = \tilde{f}(\tilde{Y}_t)\,\mathrm{d}X_t, \quad \tilde{Y}_0 = (X_0, \xi) \in V \oplus W, \tag{1.2}$$

where $\tilde{f}\colon V \oplus W \to L(V, V \oplus W)$ is defined by

$$\tilde{f}(v, w)(u) = (u, f(v, w)(u)).$$

From Lyons's universal limit theorem (see [8–10]) we know that a solution to (1.2) exists and is unique if f is Lip^γ with $\gamma > p$. We adapt the proof of [10] and show the existence and uniqueness of solution for our case under sufficient conditions that are weaker than that required by Lyons's universal limit theorem in the homogeneous case.

The results of §4 are derived for geometric-rough paths. However, due to [6], Theorem 4.3 can be extended to non-geometric-rough paths in a somewhat weaker form. The particular extension is yet to be explored.

We note that the Π-rough paths form a more general class than the class of (p, q)-rough paths as defined in [7]; in the case of Π-rough paths, apart from assuming the parameters that specify the roughness in the various directions to be at least 1, we do not make any further restrictions on them.

Moreover, throughout the paper, we assume the spatial structure of the inhomogeneity to be static. However, in general applications, the directions of the inhomogeneity of the driving noise, and hence those of the solution, may change in time and/or in space. Such applications fall beyond the scope of the paper, although our results may be relevant to them.

2. Π-rough paths

Throughout this section k denotes a fixed positive integer and $\Pi = (p_1, \ldots, p_k)$ is a *real k-tuple* such that $p_i \geqslant 1$ is a real number for all $i \in \{1, \ldots, k\}$. Furthermore, let a Banach space V of the form $V = V^1 \oplus \cdots \oplus V^k$ be given for some Banach spaces $V^1, \ldots, V^k$.

Definition 2.1. We say that $R = (r_1, \ldots, r_l)$ is a *k-multi-index* if $1 \leqslant r_j \leqslant k$ is an integer for all $j \in \{1, \ldots, l\}$. The empty multi-index is denoted by ϵ and the set of all k-multi-indexes of finite length is denoted by $\mathcal{A}^k$.

Given the multi-index $R = (r_1, \ldots, r_l)$, we define the k-multi-index $R-$ by

$$R- = (r_1, r_2, \ldots, r_{l-1}, r_l)- = (r_1, r_2, \ldots, r_{l-1}).$$

The concatenation of the multi-indices $R = (r_1, \ldots, r_l)$ and $Q = (q_1, \ldots, q_m)$ is denoted by

$$R * Q = (r_1, \ldots, r_l) * (q_1, \ldots, q_m) = (r_1, \ldots, r_l, q_1, \ldots, q_m).$$

Definition 2.2. For the k-multi-index $R = (r_1, \ldots, r_l)$ we denote the *length* by $\|R\| = l$. Furthermore, we define the function n_j for $j \in \{1, \ldots, k\}$ by

$$n_j(R) := \text{card}\{i \mid r_i = j, \ r_i \in R\}.$$

We introduce the Π-degree of R as

$$\deg_\Pi(R) = \sum_{j=1}^k \frac{n_j(R)}{p_j}.$$

Note that $\deg_\Pi(\epsilon) = 0$. We also introduce the function $\Gamma_\Pi : \mathcal{A}^k \to [0, \infty)$:

$$\Gamma_\Pi(R) = \left(\frac{n_1(R)}{p_1}\right)! \cdots \left(\frac{n_k(R)}{p_k}\right)! \quad \text{for } R \in \mathcal{A}^k,$$

where $(\cdot)!$ denotes the Γ-function.

Let $s \geqslant 0$ be real. We introduce a family of subsets of the set of k-multi-indices:

$$\mathcal{A}_s^\Pi := \{R = (r_1, \ldots, r_l) \mid l \geqslant 1, \ \deg_\Pi(R) \leqslant s\}.$$

Let S^Π denote the set

$$S^\Pi = \{s = \deg_\Pi(R) \mid R \in \mathcal{A}^k\}.$$

Note that S^Π is unbounded from above and closed under addition. Also note that since, for any $R \in \mathcal{A}^k$,

$$\deg_\Pi(R) \geqslant \frac{\|R\|}{\max_{1 \leqslant i \leqslant k} p_i},$$

the sublevel set $\{\deg_\Pi(R) \mid R \in \mathcal{A}^k, \ \deg_\Pi(R) \leqslant s\}$ of S^Π is finite for all $s \geqslant 0$. This implies that the elements of S^Π can be listed in ascending order. The mth element in the ordered S^Π will be denoted by s_m.

Definition 2.3. The space of formal series of tensors of V is equivalently represented by

$$T(V) = \bigoplus_{n=0}^\infty V^{\otimes n} = \bigoplus_{(r_1, \ldots, r_l) \in \mathcal{A}^k} V^{r_1} \otimes \cdots \otimes V^{r_l},$$

where $V^{\otimes 0} := \mathbb{R}$.

For a k-multi-index $R = (r_1, \ldots, r_l)$ we introduce the notation

$$V^{\otimes R} = V^{r_1} \otimes \cdots \otimes V^{r_l} \quad \text{and} \quad V^{(\Pi, s)} = \bigoplus_{\deg_\Pi(R) = s} V^{\otimes R} \quad \text{for } s \in S^\Pi.$$

In general, for a vector space $U = A \oplus B$, π_A and π_B denote the canonical projection onto A and B, respectively, i.e. for $u = a + b \in U$ such that $a \in A$ and $b \in B$, $\pi_A u = a$ and $\pi_B u = b$. We extensively use the projection π_V onto the V component of $T(V)$.

Let $\pi_R := \pi_{V^{r_1} \otimes \cdots \otimes V^{r_l}}$ and $\pi_{T(V^i)}$ for $i \in \{1, \ldots, k\}$ denote the canonical projections

$$\pi_R := \pi_{V^{r_1} \otimes \cdots \otimes V^{r_l}} \colon T(V) \to V^{\otimes R},$$
$$\pi_{T(V^i)} \colon T(V) \to T(V^i).$$

Given an element $v \in V$ and a multi-index $R = (r_1, \ldots, r_l)$, we introduce the element v_R as follows:

$$v_R := (\pi_{(r_1)} v) \otimes \cdots \otimes (\pi_{(r_l)} v) \in V^{\otimes R}.$$

The set B_s^Π defined by

$$B_s^\Pi := \{a \in T(V) \mid \forall R \in \mathcal{A}_s^\Pi, \ \pi_R(a) = 0\}$$

is an ideal in $T(V)$.

The *truncated tensor algebra of order* (Π, s) is defined as the quotient algebra

$$T^{(\Pi, s)}(V) := T(V)/B_s^\Pi.$$

We chose the tensor norms $\|\cdot\|_R$ for all $R \in \mathcal{A}^k$ to satisfy

$$\|a \otimes b\|_{R*Q} \leqslant \|a\|_R \|b\|_Q \quad \forall a \in V^{\otimes R}, \ \forall b \in V^{\otimes Q}.$$

We will drop the multi-index from the notation of the norm if it does not result in any ambiguity.

Definition 2.4 (control function). Let T be a positive real number and let Δ_T denote the set $\{(s, t) \in [0, T] \times [0, T] \mid s \leqslant t\}$. A *control function*, or *control*, on $[0, T]$ is a uniformly continuous non-negative function $\omega \colon \Delta_T \to [0, +\infty)$ that is superadditive, i.e.

$$\omega(s, u) + \omega(u, t) \leqslant \omega(s, t) \quad \forall s, u, t \in [0, T], \ s \leqslant u \leqslant t,$$

and for which $\omega(t, t) = 0$ for all $t \in [0, T]$.

Definition 2.5 (finite Π-variation). Let ω be a control on $[0, T]$. For a positive real q the map $X \colon \Delta_T \to T^{(\Pi, q)}$ is *multiplicative* if for all $0 \leqslant s \leqslant t \leqslant T$, $\pi_\varepsilon X_{s,t} = 1$ and for all $0 \leqslant s \leqslant t \leqslant u \leqslant T$,

$$X_{s,u} = X_{s,t} \otimes X_{t,u}.$$

Furthermore, X is said to have finite Π-variation controlled by ω if there exists a positive β such that

$$\|\pi_R(X_{s,t})\| \leqslant \frac{\omega(s, t)^{\deg_\Pi(R)}}{\beta^k \Gamma_\Pi(R)}$$

for all $(s, t) \in \Delta_T$ and for all k-multi-index $R \in \mathcal{A}_q^\Pi$.

The extension theorem states that a $\Delta_T \to T^{(\Pi,1)}(V)$ multiplicative functional of finite Π-variation can be uniquely extended to a $\Delta_T \to T^{(\Pi,q)}(V)$ multiplicative functional of finite Π-variation for any positive q. This will allow us to define Π-rough paths as $\Delta_T \to T^{(\Pi,1)}(V)$ multiplicative functionals satisfying certain properties (see Definition 2.7).

Theorem 2.6 (extension theorem of multiplicative functionals of finite Π-variation). *Let $X \colon \Delta_T \to T^{(\Pi,1)}(V)$ be a multiplicative functional of finite Π-variation controlled by ω. Then for every k-multi-index $R \in \mathcal{A}^k \setminus \mathcal{A}_1^\Pi$ there exists a unique continuous function $X^R \colon \Delta_T \to V^{\otimes R}$ such that*

$$(s,t) \mapsto X_{s,t} = \sum_{R \in \mathcal{A}^k} X_{s,t}^R \in T(V)$$

is a multiplicative functional of finite Π-variation controlled by ω in the following sense:

$$\|X_{s,t}^R\| \leqslant \frac{\omega(s,t)^{n_1(R)/p_1 + \cdots + n_k(R)/p_k}}{\beta^k (n_1(R)/p_1)! \cdots (n_k(R)/p_k)!} = \frac{\omega(s,t)^{\deg_\Pi(R)}}{\beta^k \Gamma_\Pi(R)}$$

for all $R \in \mathcal{A}^k$, where

$$\beta \geqslant \left(p_1^2 \cdots p_k^2 \left(1 + \sum_{r=3}^\infty \left(\frac{2}{r-2} \right)^{s_{m^*}+1} \right) \right)^{1/k}$$

and s_{m^*} and s_{m^*+1} are the unique pair of adjacent elements of the ordered S^Π for which $s_{m^*} \leqslant 1 < s_{m^*+1}$.

A proof of this theorem based on the proof of the extension theorem of p-rough paths (see [**8**]) is derived in [**5**].

Definition 2.7 (Π-rough paths). A *Π-rough path* in V is a continuous $\Delta_T \to T^{(\Pi,1)}(V)$ multiplicative functional X with finite Π-variation controlled by some control ω.

The space of Π-rough paths is denoted by $\Omega_\Pi(V)$.

Definition 2.8. Let $C_{0,\Pi}(\Delta_T, T^{(\Pi,1)}(V))$ denote the space of all continuous functions from the simplex Δ_T into the truncated tensor algebra $T^{(\Pi,1)}(V)$ with finite Π-variation. The Π-variation metric $d_{\Pi\text{-var}}$ on this linear space is defined as

$$d_{\Pi\text{-var}}(X,Y) := \max_{R \in \mathcal{A}_1^\Pi} \sup_{\mathcal{D} \in \mathcal{P}([0,T])} \left(\sum_{\mathcal{D}} \|\pi_R(X_{t_{l-1},t_l} - Y_{t_{l-1},t_l})\|^{1/\deg_\Pi(R)} \right)^{\deg_\Pi(R)},$$

where $\mathcal{P}([0,T])$ denotes the set of all the finite partitions of the interval $[0,T]$.

The following subset of the space of Π-rough paths is crucial for our further analysis.

Definition 2.9 (geometric Π-rough path). A *geometric Π-rough path* is a Π-rough path that can be expressed as a limit of (1)-rough paths* (or *smooth* rough paths) in the Π-variation distance. The space of geometric Π-rough paths in V is denoted by $G\Omega_\Pi(V)$.

---

* Here (1) denotes the 1-tuple with the single element 1, and hence (1)-rough paths in this paper coincide with 1-rough paths in the sense of [**8**].

Remark 2.10. In the special case, when $k = 1$ and $\Pi = (p)$ for some $p \geqslant 1$, we will use the simplified terminology: 'finite p-variation', 'p-rough paths', '$d_{p\text{-var}}$-distance' and 'geometric p-rough paths'. Direct definitions for these terms can be found in [**8**, **10**]. Furthermore, for a 1-multi-index R with length j, we will use the notation $\pi_j = \pi_R$, and we will write $T^i(V)$ for $T^{(\Pi, i/p)}(V)$.

3. Integration with respect to Π-rough paths

Lyons [**8**] introduced integrals of Lip^γ one-forms along p-rough paths for $\gamma > p - 1$. In this section we introduce $\mathrm{Lip}^{\Gamma, \Pi}$ one-forms (Definition 3.2) and integrals of $\mathrm{Lip}^{\Gamma, \Pi}$ one-forms along Π rough paths (Theorem 3.9).

First we define the s-symmetric maps for $s \in S^\Pi$.

Definition 3.1 (s-symmetric maps). Let i be a positive integer and let $x \in V^{\otimes i}$. Let $x^{(i)}$ denote the i-symmetric part of x. Let $q \in (0, \infty]$, let $X \in T^{(\Pi, q)}(V)$ and let $s \in S^\Pi$. Then the s-symmetric part of X is defined as

$$X^{(s)} := \sum_{\deg_\Pi(R) = s} \pi_R\left(\sum_{i \in \mathbb{N}} \pi_{(i)}(X)^{(i)}\right).$$

A map f defined on $T^{(\Pi, q)}(V)$ is *s-symmetric* if for all $X \in T^{(\Pi, q)}(V)$,

$$f(X) = f(X^{(s)}).$$

Now we can give the definition of $\mathrm{Lip}^{\Gamma, \Pi}$ one-forms.

Definition 3.2 ($\mathrm{Lip}^{\Gamma, \Pi}$ one-forms). Let $\Pi = (p_1, \ldots, p_k)$, let $\Gamma = (\gamma_1, \ldots, \gamma_k)$ be k-tuples with positive components, let F be a closed subset of V and let W be a Banach space. The function $\alpha \colon F \to L(V, W)$ is a (Π, Γ)-Lipschitz one-form on F if $\alpha(u) = \sum_{i=1}^k \alpha_i(u) \circ \pi_{V_i}$ such that $\alpha_i \colon F \to L(V^i, W)$ and for each i and $s_m < \gamma_i$ (where s_m is the mth element in the ordered set S^Π) there exist functions $\alpha_i^{s_m} \colon F \to L(V^{(\Pi, s_m)}, L(V^i, W))$ taking values in the space of s_m-symmetric maps satisfying

$$\alpha_i^{s_m}(y)(v) = \sum_{s_m \leqslant s_n < \gamma_i} \alpha_i^{s_n}(x)\left(v \otimes \sum_{\deg_\Pi(R) = s_n - s_m} \frac{(x - y)_R}{\|R\|!}\right) + R_i^{s_m}(x, y)(v)$$

for all $x, y \in F$ and $v \in V^{\otimes(\Pi, s_m)}$, where $R_i^{s_m} \colon F \times F \to L(V^{(\Pi, s_m)}, L(V^i, W))$ with

$$\|R_i^{s_m}(x, y)\| \leqslant M \sum_{j=1}^k \|\pi_{V^j}(x - y)\|^{(\gamma_i - s_m) p_j}. \tag{3.1}$$

In addition to the above definition, and for practical reasons, we introduce the functions $\alpha^{s_m} \colon F \to L(V^{(\Pi, s_m)}, L(V, W))$ for $s_m < \max_{1 \leqslant i \leqslant k} \gamma_i = \gamma_{\max}$ defined by

$$\alpha^{s_m}(v)(u) = \sum_{i, s_m < \gamma_i} \alpha_i^{s_m}(v)(u) \circ \pi_{V^i} \quad \forall v \in F, \ \forall u \in V^{(\Pi, s_m)}.$$

Note that α^{s_m} takes s_m-symmetric linear maps as values. Furthermore, we introduce the functions $R_{s_m} : F \times F \to L(V^{\otimes(\Pi, s_m)}, L(V, W))$ defined by

$$R^{s_m}(x, y)(u) = \sum_{i, s_m < \gamma_i} R_i^{s_m}(x, y)(u) \circ \pi_{V^i} \quad \forall x, y \in F, \ \forall u \in V^{(\Pi, s_m)}. \tag{3.2}$$

Remark 3.3. Note that for $k = 1$, $\Pi = (p)$ and $\Gamma = (\hat{\gamma})$, Definition 3.2 simplifies to the classical definition of Lip^γ functions (see [**8**]) for $\gamma = \hat{\gamma}p$, although the notation is slightly different.

Remark 3.4. The above definition can be interpreted as a 'decomposition and partial reconstruction' of the classical Lip^γ functions. In the classical definition f^j (i.e. the jth term in the expansion) is an $F \to L(V^{\otimes j}, L(V, W))$-valued function, which can be decomposed as

$$f^j(x) = \sum_{\|R\| = j} f^{j,R}(x) \circ \pi_{V^{\otimes R}}$$

for $x \in F$, where $f^{j,R}(x)$ is a linear function in $L(V^{\otimes R}, L(V, W))$. In the above definition these $f^{j,R}$ functions are grouped and summed by the degree of R, leaving out those that have degree corresponding to multi-index with degree greater than a certain value (γ_i). Similar decomposition and partial reconstruction is done with the remainder terms. Note that the condition (3.1) on the remainder terms is weak compared to the homogeneous case ($k = 1$, $\Pi = (p_{\max})$, $\Gamma = (\hat{\gamma})$).

Remark 3.5. In the special case when all the multi-indices R of degree less than γ_i are of the form $(j, \ldots, j)$ for some $1 \leqslant j \leqslant k$ (for example, $k = 2$ and $1/p_1 + 1/p_2 > \gamma_j$ for $j = 1$ or 2), the term

$$v \otimes \sum_{\deg_\Pi(R) = s_n - s_m} \frac{(x - y)_R}{\|R\|!}$$

for $v \in V^{\otimes(\Pi, s_m)}$ lies in $(V^j)^{\otimes n}$, where $n = s_m p_j$. In this case the condition (in the above definition) on α_i is equivalent to the following. For each $j = 1, \ldots, k$ the function

$$x \mapsto \alpha_i((y^1, \ldots, y^{j-1}, x, y^{j+1}, \ldots, y^k)), \quad x \in V^j,$$

for fixed

$$(y^1, \ldots, y^{j-1}, j^{j+1}, \ldots, j^k) \in V^1 \oplus \cdots \oplus V^{j-1} \oplus V^{j+1} \oplus \cdots V^k$$

is $\mathrm{Lip}^{\gamma_i p_j}$ (in the classical sense) with Lipschitz norm uniform in

$$(y^1, \ldots, y^{j-1}, j^{j+1}, \ldots, j^k).$$

In the case in which $1/p_j > \gamma_i$, the $\mathrm{Lip}^{\gamma_i p_j}$ condition is equivalent to $\gamma_i p_j$-Hölder continuity.

In Theorem 3.9 an integral approximation formula is introduced and we prove the existence of a unique rough path associated with the integral approximating formula. This unique rough path is referred to as the integral of a one-form along a Π-rough path. To prove the existence and uniqueness of the rough path associated with the integral approximating formula we reformulate the problem in terms of almost p-rough paths.

Definition 3.6 (almost *p*-rough path). Let $p \geqslant 1$ be a real number and let ω be a control. A function $Y \colon \Delta_T \to T^{(\lfloor p \rfloor)}(V)$ is an *almost p-rough path* if

(i) Y has finite *p*-variation controlled by ω, i.e.

$$\|\pi_{(i)}(Y_{s,t})\| \leqslant \frac{\omega(s,t)^{i/p}}{\beta(i/p)!} \quad \forall i = 1, \ldots, \lfloor p \rfloor, \ \forall (s,t) \in \Delta_T;$$

(ii) Y is *almost multiplicative* in the sense that

$$\|\pi_{(i)}(Y_{s,u} \otimes Y_{u,t} - Y_{s,t})\| \leqslant \omega(s,t)^{\theta} \quad \forall i = 1, \ldots, \lfloor p \rfloor, \ \forall s, u, t \in [0,T], \ s \leqslant u \leqslant t,$$

and for some $\theta > 1$.

Almost rough paths have the crucial property that each one of them determines a rough path in the sense of Theorem 3.7. This property is exploited when we derive the existence and uniqueness of integrals along Π-rough paths.

Theorem 3.7. *Let $p \geqslant 1$ be a real number and let ω be a control. Let $Y \colon \Delta_T \to T^{(\lfloor p \rfloor)}(V)$ be an almost p-rough path with p-variation controlled by ω as in Definition 3.6. Then there exists a unique p-rough path $X \colon \Delta_T \to T^{(\lfloor p \rfloor)}(V)$ such that*

$$\sup_{\substack{0 \leqslant s < t \leqslant T, \\ i=0,\ldots,\lfloor p \rfloor}} \frac{\|\pi_i(X_{s,t} - Y_{s,t})\|}{\omega(s,t)^{\theta}} < +\infty.$$

Moreover, there exists a constant K depending only on p, θ and $\omega(0,T)$ such that the supremum is smaller than K and the p-variation of X is controlled by $K\omega$.

The reader is referred to [**8**, **9**] for a proof.

Although it is not required for the main result of this section, Theorem 3.7 can be extended for general $k > 1$ and k-tuple Π as follows.

Theorem 3.8. *Let the Banach space V be of the form $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1, \ldots, V^k$. Let $\Pi = (p_1, \ldots, p_k)$ denote a k-tuple as in Definition 2.1 and let ω be a control. Let the functional $Y \colon \Delta_T \to T^{(\Pi,1)}(V)$ be a θ-almost Π-rough path controlled by ω, i.e. the following hold.*

(i) *It has a finite Π-variation controlled by ω:*

$$\|X_{s,t}^R\| \leqslant \frac{\omega(s,t)^{\deg_{\Pi}(R)}}{\beta^k \Gamma_{\Pi}(R)}$$

for all $(s,t) \in \Delta_T$ and for all multi-index $R \in \mathcal{A}_1^{\Pi}$.

(ii) *It is almost-multiplicative, i.e. there exists $\theta > 1$ such that*

$$\|\pi_R(X_{s,u} \otimes X_{u,t} - X_{s,t})\| \leqslant \omega(s,t)^{\theta} \quad \forall s < u < t \in [0,T], \ \forall R \in \mathcal{A}_1^{\Pi}.$$

There then exists a unique Π-rough path $X\colon \Delta_T \to T^{(\Pi,1)}(V)$ such that

$$\sup_{\substack{0\leqslant s<t\leqslant T,\\ R\in\mathcal{A}_1^\Pi}} \frac{\|\pi_R(X_{s,t}-Y_{s,t})\|}{\omega(s,t)^\theta} < +\infty. \tag{3.3}$$

Moreover, there exists a constant K that depends only on Π, θ and $\omega(0,T)$ such that the supremum (3.3) is smaller than K and the Π-variation of X is controlled by $K\omega$.

The proof of Theorem 3.8 is sketched in [**5**] and based on the proof of Theorem 3.7 as derived in [**10**].

Finally, we can state the main theorem of the section.

Theorem 3.9 (integration of $\mathrm{Lip}^{\Gamma,\Pi}$ one-forms). *Let V and W be Banach spaces such that $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1,\ldots,V^k$. Let $\Pi = (p_1,\ldots,p_k)$ denote a k-tuple as in Definition 2.1 with $p_{\max} = \max_{1\leqslant i\leqslant k} p_i$, and let ω be a control. Let $Z\colon \Delta_T \to T^{(\Pi,1)}(V)$ be a geometric Π-rough path controlled by ω. Let $\Gamma = (\gamma_1,\ldots,\gamma_k)$ be a real k-tuple such that $\gamma_i > 1-1/p_i$ for $i=1,\ldots,k$ and $\gamma_{\max} = \max_{1\leqslant i\leqslant k}\gamma_i$. Finally, let $\alpha\colon V \to L(V,W)$ be a $\mathrm{Lip}^{\Gamma,\Pi}$ function as in Definition 3.2.*

Then $Y\colon \Delta_T \to T^{((p_{\max}),1)}(W)$ is defined for all $(s,t) \in \Delta_T$ by

$$\begin{aligned}
Y_{s,t}^n &:= \pi_{W^{\otimes n}}(Y_{s,t}) \\
&= \sum_{s_{m_1}+\cdots+s_{m_n}<\gamma_{\max}} \alpha^{s_{m_1}}\big(\pi_V(Z_{0,s})\big) \otimes \cdots \otimes \alpha^{s_{m_n}}\big(\pi_V(Z_{0,s})\big) \\
&\quad \times \sum_{\substack{R_1,\ldots,R_n\in\mathcal{A}^k,\\ \deg_\Pi(R_i-)=s_{m_i},\,i=1,\ldots,n,\\ \sigma\in\mathrm{OS}(\|R_1\|,\ldots,\|R_n\|)}} \sigma^{-1}\pi_{R_1*\cdots*R_n}\big(Z_{s,t}\big)
\end{aligned} \tag{3.4}$$

and is an almost $p_{\max}$-rough path, where $\mathrm{OS}(k_1,\ldots,k_n)$ denotes the subset of the symmetric group S_K for $K = k_1 + \cdots + k_n$, such that for all $\sigma \in \mathrm{OS}(k_1,\ldots,k_n)$, we have

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k_1), \quad \sigma(k_1+1) < \cdots < \sigma(k_1+k_2),\ldots,$$

$$\sigma(K-k_n+1) < \cdots < \sigma(K) \quad \text{and} \quad \sigma(k_1) < \sigma(k_2) < \cdots < \sigma(k_n).$$

Moreover, with a slight abuse of notation for $x_1 \otimes \cdots \otimes x_K \in V^{R_1*\cdots*R_n}$ and $\sigma \in \mathrm{OS}(\|R_1\|,\ldots,\|R_n\|)$, we define

$$\sigma(x_1 \otimes \cdots \otimes x_K) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(K)}.$$

Theorem 3.9 leads to the following definition.

Definition 3.10 (integration of $\mathrm{Lip}^{\Gamma,\Pi}$ one-forms). *Let V and W be Banach spaces such that $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1,\ldots,V^k$. Let $\Pi = (p_1,\ldots,p_k)$ denote a k-tuple as in Definition 2.1 with $p_{\max} = \max_{1\leqslant i\leqslant k} p_i$ and let ω be a control. Let $Z\colon \Delta_T \to T^{(\Pi,1)}(V)$ be a geometric Π-rough path controlled by ω. Let $\Gamma = (\gamma_1,\ldots,\gamma_k)$ be a real k-tuple such that $\gamma_i > 1-1/p_i$ for $i=1,\ldots,k$. Finally, let $\alpha\colon V \to L(V,W)$ be a $\mathrm{Lip}^{\Gamma,\Pi}$ function.*

Let $Y\colon \Delta_T \to T^{((p_{\max}),1)}(W)$ be the almost $p_{\max}$-rough path defined by Theorem 3.9. The unique $(p_{\max})$-rough path associated with Y by Theorem 3.7 is called the integral of α along Z and it is denoted by

$$\int_{\cdot}^{\cdot} \alpha(Z)\,\mathrm{d}Z \colon \Delta_T \to T^{((p_{\max}),1)}(W).$$

Remark 3.11. In the general case the integral is a $p_{\max}$-rough path in the sense of [**8**]. However, for special forms of the $\mathrm{Lip}^{\Gamma,\Pi}$ one-form α, the integral itself is a Π-rough path. The reader is referred to [**5**] for examples.

In the remaining part of the section, we present a proof of Theorem 3.9.

Equation (3.4) describes an integral approximating formula projected onto $W^{\otimes n}$. The intuition behind this formula comes from integrals with respect to paths of finite length. In particular, let $Z\colon [0,T] \to V$ be a path of finite variation. For a multi-index $R = (r_1,\ldots,r_l)$ let $Z_{s,t}^R \in V^{\otimes R}$ be defined as

$$Z_{s,t}^R = \int_{s<u_1<\cdots<u_l<t} \mathrm{d}\pi_{(r_1)}(Z_{u_1}) \otimes \cdots \otimes \mathrm{d}\pi_{(r_l)}(Z_{u_l}). \tag{3.5}$$

Furthermore, let the function $Y^1\colon \Delta_T \to W$ be defined for all $(s,t) \in \Delta_T$ by

$$Y_{s,t}^1 := \sum_{s_m<\gamma_{\max}} \alpha^{s_m}(Z_s) \sum_{\substack{R\in\mathcal{A}^k,\\ \deg_\Pi(R-)=s_m}} Z_{s,t}^R = \int_s^t \alpha(Z_u)\,\mathrm{d}Z_u - \int_s^t R^0(Z_s,Z_u)\,\mathrm{d}Z_u. \tag{3.6}$$

Then

$$
\begin{aligned}
Y_{s,t}^n &:= \int_{s<u_1<\cdots<u_n<t} \mathrm{d}Y_{s,u_1}^1 \otimes \cdots \otimes \mathrm{d}Y_{s,u_n}^1 \\
&= \int_{s<u_1<\cdots<u_n<t} \sum_{s_{m_1}<\gamma_{\max}} \alpha^{s_{m_1}}(Z_s) \\
&\qquad \times \sum_{\substack{R\in\mathcal{A}^k,\\ \deg_\Pi(R-)=s_{m_1}}} \mathrm{d}Z_{s,u_1}^R \otimes \cdots \otimes \sum_{s_{m_n}<\gamma_{\max}} \alpha^{s_{m_n}}(Z_s) \sum_{\substack{R\in\mathcal{A}^k,\\ \deg_\Pi(R-)=s_{m_n}}} \mathrm{d}Z_{s,u_n}^R \\
&= \sum_{s_{m_1},\ldots,s_{m_n}<\gamma_{\max}} \alpha^{s_{m_1}}(Z_s) \otimes \cdots \otimes \alpha^{s_{m_n}}(Z_s) \\
&\qquad \times \sum_{\substack{R_1,\ldots,R_n\in\mathcal{A}^k,\\ \deg_\Pi(R_i-)=s_{m_i},\,i=1,\ldots,n}} \int_{s<u_1<\cdots<u_n<t} \mathrm{d}Z_{s,u_1}^{R_1} \otimes \cdots \otimes \mathrm{d}Z_{s,u_n}^{R_n} \\
&= \sum_{s_{m_1},\ldots,s_{m_n}<\gamma_{\max}} \alpha^{s_{m_1}}(Z_s) \otimes \cdots \otimes \alpha^{s_{m_n}}(Z_s) \sum_{\substack{R_1,\ldots,R_n\in\mathcal{A}^k,\\ \deg_\Pi(R_i-)=s_{m_i},\,i=1,\ldots,n,\\ \sigma\in\mathrm{OS}(\|R_1\|,\ldots,\|R_n\|)}} \sigma^{-1} Z_{s,t}^{R_1*\cdots*R_n}
\end{aligned}
$$
$$\tag{3.7}$$

for $n = 2,\ldots,\lfloor p_{\max}\rfloor$.

Equation (3.7) is an adaptation of the results of [**10**, § 4.2]. The crucial ingredient of the adaptation is the fact that for all multi-indices $R_1, \ldots, R_n$ in $\mathcal{A}^k$ we have

$$\int_{s < u_1 < \cdots < u_n < t} \mathrm{d}Z^{R_1}_{s,u_1} \otimes \cdots \otimes \mathrm{d}Z^{R_n}_{s,u_n} = \sum_{\sigma \in \mathrm{OS}(\|R_1\|, \ldots, \|R_n\|)} \sigma^{-1} Z^{R_1 * \cdots * R_n}_{s,t}.$$

Lemma 3.12. *If* $Z \colon [0, T] \to V$ *is a path of finite variation and is piecewise differentiable, then for any* $s < u < t$ *in* $[0, T]$,

$$\sum_{s_m < \gamma_{\max}} \alpha^{s_m}(Z_s) \left(\sum_{\deg_\Pi(R) = s_m} Z^R_{s,t} \right)(\dot{Z}_t)$$
$$= \sum_{s_m < \gamma_{\max}} (\alpha^{s_m}(Z_u) - R^{s_m}(Z_s, Z_u)) \left(\sum_{\deg_\Pi(R) = s_m} Z^R_{u,t} \right)(\dot{Z}_t). \quad (3.8)$$

The proof of the lemma is analogous to the proof of Lemma 5.5.2 in [**9**].

Lemma 3.13. *Let* $Z \colon [0, T] \to V$ *be a path of finite variation. Let the map*

$$Y = (1, Y^1, \ldots, Y^{\lfloor p_{\max} \rfloor}) \colon \Delta_T \to \mathbb{R} \oplus W \oplus W^{\otimes 2} \otimes \cdots \otimes W^{\otimes \lfloor p_{\max} \rfloor}$$

be defined by (3.6) and (3.7).
Then, for all $s < u < t$ *in* $[0, T]$,

$$Y_{s,u} \otimes Y_{u,t} - Y_{s,t} = Y_{s,u} \otimes N_{s,u,t}, \quad (3.9)$$

where

$$N^i_{s,u,t} = \pi_{W^{\otimes i}} N_{s,u,t} := \sum_{\substack{s_{m_1}, \ldots, s_{m_i} < \gamma_{\max}, \\ \varepsilon_1, \ldots, \varepsilon_i \in \{0,1\}, \\ \varepsilon_1 \cdots \varepsilon_i = 0}} \beta^{\varepsilon_1}_{s_{m_1}}(Z_s, Z_u) \cdots \beta^{\varepsilon_i}_{s_{m_i}}(Z_s, Z_u)$$
$$\times \sum_{\substack{R_1, \ldots, R_i \in \mathcal{A}^k, \\ \deg_\Pi(R_j -) = s_{m_j}, \, j = 1, \ldots, i, \\ \sigma \in \mathrm{OS}(\|R_1\|, \ldots, \|R_i\|)}} \sigma^{-1} Z^{R_1 * \cdots * R_i}_{s,t} \quad (3.10)$$

with

$$\beta^\varepsilon_{s_m}(Z_s, Z_u) = \begin{cases} R^{s_m}(Z_s, Z_u) & \text{if } \varepsilon = 0, \\ -\alpha^{s_m}(Z_s) & \text{if } \varepsilon = 1. \end{cases}$$

The proof is based on Lemma 3.12 and (3.7), and is analogous to the proof of Lemma 5.5.3 in [**9**].

Remark 3.14. Equation (3.7) and Lemmas 3.12 and 3.13 are stated for a smooth rough path Z. However, for each of (3.7)–(3.9) both the right-hand side and the left-hand side are continuous in the Π-variation metric. This fact extends the lemmas for geometric Π-rough paths and this is the key to the next proof.

We now prove Theorem 3.9.

Proof of Theorem 3.9. First we prove that $\hat{Y} \colon \Delta_T \to T^{((p_{\max}),1)}(W)$, defined by

$$\hat{Y}^n_{s,t} = \sum_{s_{m_1},\ldots,s_{m_n} < \gamma_{\max}} \alpha^{s_{m_1}}(\pi_V(Z_{0,s})) \otimes \cdots \otimes \alpha^{s_{m_n}}(\pi_V(Z_{0,s}))$$

$$\times \sum_{\substack{R_1,\ldots,R_n \in \mathcal{A}^k, \\ \deg_\Pi(R_i-)=s_{m_i},\, i=1,\ldots,n, \\ \sigma \in \mathrm{OS}(\|R_1\|,\ldots,\|R_n\|)}} \sigma^{-1} \pi_{R_1 * \cdots * R_n}(Z_{s,t}),$$

is an almost $p_{\max}$-rough path. Each term in the above sum is of the form

$$\alpha^{s_{m_1}}(\pi_V(Z_{0,s})) \otimes \cdots \otimes \alpha^{s_{m_n}}(\pi_V(Z_{0,s}))\sigma^{-1}\pi_{R_1 * \cdots * R_n}(Z_{s,t}), \qquad (3.11)$$

where $\deg_\Pi(R_i-) = s_{m_i}$. Since such a term is bounded by $C_0\|\alpha\|^n_{\mathrm{Lip}\,\Gamma,\Pi}\omega(s,t)^{n/p_{\max}}$, where C_0 only depends on Γ, Π and $\omega(0,T)$, this implies that Definition 3.6 (i) is satisfied.

We prove condition (ii) by giving a bound on the norm of

$$(\hat{Y}_{s,u} \otimes \hat{Y}_{u,t})^n - \hat{Y}^n_{s,t} = \sum_{i=0}^{n} \hat{Y}^i_{s,u} \otimes N^{n-i}_{s,u,t}.$$

The representation of $N^{n-i}_{s,u,t}$ in (3.10) implies that there is at least one factor of the form $R^{s_m}(\pi_V(Z_{0,s}), \pi_V(Z_{0,u}))$. Considering the representation (3.2) of R^{s_m} and the error bound (3.1) on $R^{s_m}_i$, the following bound is implied:

$$\|R^{s_m}(\pi_V(Z_{0,s}), \pi_V(Z_{0,u}))\| \leqslant M \sum_{i,s_m < \gamma_i} \sum_{j=1}^{k} \|\pi_{V_j}(Z_{s,u})\|^{(\gamma_i - s_m)p_j}$$

$$\leqslant M \sum_{i,s_m < \gamma_i} \sum_{j=1}^{k} \omega(s,t)^{\gamma_i - s_m}.$$

Moreover, given that $R^{s_m}_i(x,y)(u)$ only acts on elements of V^i, there exists a constant C_1 depending only on $\|\alpha\|_{\mathrm{Lip}\,\Gamma,\Pi}$, Γ, Π and $\omega(0,T)$ such that

$$\|(\hat{Y}_{s,u} \otimes \hat{Y}_{u,t})^n - \hat{Y}^n_{s,t}\| \leqslant C_1 \sum_{i=1}^{k} \omega(s,t)^{\gamma_i + (1/p_i)}.$$

By the choice of Γ, $\theta := \min_{1 \leqslant i \leqslant k}(\gamma_i + (1/p_i)) \geqslant 1$, which implies that there exists a constant C depending only on $\|\alpha\|_{\mathrm{Lip}\,\Gamma,\Pi}$, Γ, Π and $\omega(0,T)$ such that

$$\|(\hat{Y}_{s,u} \otimes \hat{Y}_{u,t})^n - \hat{Y}^n_{s,t}\| \leqslant C\omega(s,t)^\theta,$$

and hence Y is a θ-almost $p_{\max}$-rough path.

Arguments analogous to [**10**, Proposition 4.10] prove that Y is also a θ-almost $p_{\max}$-rough path, and furthermore that the $p_{\max}$-rough path associated with Y, by Theorem 3.7, coincides with the $p_{\max}$-rough path associated with $\hat{Y}$. $\qquad \square$

Theorem 3.15. *Under the conditions of Definition 3.10 there exists a constant K depending only on Γ, Π and $\omega(0,T)$ such that*

$$\left\| \pi_{W^{\otimes i}} \left(\int_s^t \alpha(Z)\,\mathrm{d}Z \right) \right\| \leqslant K \|\alpha\|_{\mathrm{Lip}(\Pi,\Gamma)}^i \, \omega(s,t)^{i/p_{\max}}.$$

The proof is analogous to the proof of Theorem 4.12 of [**10**].

4. Differential equations driven by Π-rough paths

When stating and proving the slightly generalized version of Lyons's universal limit theorem we will refer to (linear) images of Π-rough paths in the following sense.

Definition 4.1 (image by a function). Let $Z\colon \Delta_T \to T^{(\Pi,1)}(V)$ be a geometric Π-rough path as in §2. Let $f\colon V \to W$ be a $\mathrm{Lip}^{\Gamma,\Pi}$ function for some k-tuple $\Gamma = (\gamma_1,\ldots,\gamma_k)$ satisfying $\gamma_i > 1 - 1/p_i$ for $i = 1,\ldots,k$. Then the integral $\int \mathrm{d}f(Z)\,\mathrm{d}Z$ is by definition a rough path in $\Omega_{(p_{\max})}(W)$. We will denote this rough path by $\hat{f}(Z)$.

We make use of linear images of rough paths and in particular projections of rough paths. For example, if X is a rough path in $\Omega_\Pi(V)$, then the image of X under the projection π_{V^i} will be denoted by $\hat{\pi}_{V^i}(X)$.

Now we can formally introduce differential equations driven by geometric Π-rough paths.

Definition 4.2 (differential equations driven by Π-rough paths). Let $k \geqslant 1$ be an integer and let V and W be Banach spaces such that $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1,\ldots,V^k$. Let $\Pi = (p_1,\ldots,p_k)$ denote a k-tuple and let $\Pi^* = (p_1,\ldots,p_k,p_{\max})$ denote a $(k+1)$-tuple, both as in Definition 2.1. Let $f\colon V \oplus W \to L(V,W)$ be a function. Finally, let $X \in G\Omega_\Pi(V)$ be a geometric Π-rough path and let ξ be an element in W.

We will say that $Z \in G\Omega_{\Pi^*}(V \oplus W)$ is a solution of the differential equation

$$\mathrm{d}Y_t = f(X_t, Y_t)\,\mathrm{d}X_t, \quad Y_0 = \xi, \tag{4.1}$$

if $\hat{\pi}_V(Z) = X$ and

$$Z = \int h_0(Z)\,\mathrm{d}Z, \tag{4.2}$$

where $h_0\colon V \oplus W \to \mathrm{End}(V \oplus W)$ is defined by

$$h_0(x,y) = \begin{pmatrix} \mathrm{Id}_V & 0 \\ f(x, y+\xi) & 0 \end{pmatrix}$$

provided that the integral (4.2) is well defined.

In the remainder of the section we give a sufficient condition for the existence and uniqueness of solution to (4.1). We will assume the existence of the function $g_\xi\colon V \times W \times W \to L(W, L(V,W))$ such that

$$f(x, y_1 + \xi) - f(x, y_2 + \xi) = g_\xi(x, y_1, y_2)(y_1 - y_2) \quad \text{for all } x \in V, \ y_1, y_2 \in W.$$

We introduce the one-forms $h_1\colon V \oplus W \oplus W \to \mathrm{End}(V \oplus W \oplus W)$ and $h_2\colon V \oplus W \oplus W \oplus W \to \mathrm{End}(V \oplus W \oplus W \oplus W)$ as follows:

$$h_1(x, y_1, y_2) = \begin{pmatrix} \mathrm{Id}_V & 0 & 0 \\ 0 & 0 & \mathrm{Id}_W \\ f(x, y_2 + \xi) & 0 & 0 \end{pmatrix},$$

$$h_2(x, y_1, y_2, d) = \begin{pmatrix} \mathrm{Id}_V & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_W & 0 \\ f(x, y_2 + \xi) & 0 & 0 & 0 \\ \rho g_\xi(x, y_1, y_2)(d) & 0 & 0 & 0 \end{pmatrix},$$

where ρ is an arbitrary real number greater than 1, fixed for the remainder of the section.

Theorem 4.3 (universal limit theorem, inhomogeneous case). *Let $k \geqslant 1$ be an integer and let V and W be Banach spaces such that $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1, \ldots, V^k$. Let $\Pi = (p_1, \ldots, p_k)$ denote a k-tuple, let $X \in G\Omega_\Pi(V)$ be a geometric Π-rough path and let ξ be an element in W.*

Suppose that there exist real numbers $\gamma_1, \ldots, \gamma_{k+3}$ such that $\gamma_i > 1 - 1/p_i$ for $i = 1, \ldots, k$ and $\gamma_{k+j} > 1 - 1/p_{\max}$ for $j = 1, 2, 3$. Furthermore, the functions h_0, h_1 and h_2 are $\mathrm{Lip}^{\Gamma_0, \Pi_0}$, $\mathrm{Lip}^{\Gamma_1, \Pi_1}$ and $\mathrm{Lip}^{\Gamma_2, \Pi_2}$ one-forms, respectively, for $\Gamma_0 = (\gamma_1, \ldots, \gamma_{k+1})$, $\Pi_0 = (p_1, \ldots, p_k, p_{\max})$, $\Gamma_1 = (\gamma_1, \ldots, \gamma_{k+2})$, $\Pi_1 = (p_1, \ldots, p_k, p_{\max}, p_{\max})$, and $\Gamma_2 = (\gamma_1, \ldots, \gamma_{k+3})$, $\Pi_2 = (p_1, \ldots, p_k, p_{\max}, p_{\max}, p_{\max})$.

Then

$$\mathrm{d}Y_t = f(X_t, Y_t)\,\mathrm{d}X_t, \quad Y_0 = \xi, \tag{4.3}$$

has a unique solution.

The proof of Theorem 4.3 is based on the proof of Lyons's universal limit theorem in [**10**]. We start by adapting some lemmas used in the original proof.

Lemma 4.4. *Let the Banach space V be of the form $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1, \ldots, V^k$. Let $\Pi = (p_1, \ldots, p_k)$ denote a k-tuple, let $\varepsilon > 0$ and let ω be a control function.*

Consider $Z = (X, Y) \in G\Omega_{\Pi * \Pi}(V \oplus V)$ and let $W \in G\Omega_{\Pi * \Pi}(V \oplus V)$ be the image of Z under the linear map $(x, y) \to (x, (y - x)/\varepsilon)$. Assume that the $\Pi * \Pi$-variation of W is controlled by ω. Then there exists a constant C depending only on Π, $\omega(0, T)$ and β such that

$$\|\pi_R(X_{s,t} - Y_{s,t})\| \leqslant C(\varepsilon + \varepsilon^{\|R\|})\omega(s,t)^{\|R\|/p_{max}} \quad \forall (s,t) \in \Delta_T, \ \forall R \in \mathcal{A}_1^\Pi.$$

Proof. The claim is equivalent to [**10**, Lemma 5.6] adapted to the inhomogeneous smoothness case, and the proof is analogous.

Let $R = (r_1, \ldots, r_l) \in \mathcal{A}_1^\Pi$ and let $(s, t) \in \Delta_T$. First, assuming that $Z = (X, Y) \in V \oplus V$ has bounded variation, using the notation introduced in (3.5) and writing $Y = X + \varepsilon(Y -$

$X)/\varepsilon$, we obtain

$$Y_{s,t}^R = X_{s,t}^R + \sum_{\substack{k_1,\ldots,k_l \in \{0,1\}, \\ k_1+\cdots+k_l > 0}} \varepsilon^{k_1+\cdots+k_l} W_{s,t}^{(r_1+k_1*l,\ldots,r_l+k_l*l)}.$$

The assertion is implied by the continuity in the $\Pi * \Pi$-variation topology and by the control on W. $\qquad\square$

Lemma 4.5 (scaling lemma, inhomogeneous version). *Let the Banach space V be of the form $V = V^1 \oplus \cdots \oplus V^k$ for some Banach spaces $V^1,\ldots,V^k$. Let $\Pi = (p_1,\ldots,p_k)$ denote a k-tuple, let ω be a control function and let $M \geqslant 1$ be a real number. Let $E = V^1 \oplus \cdots \oplus V^l$ and let $F = V^{l+1} \oplus \cdots \oplus V^k$ be Banach spaces. Let $\Pi_1 = (p_1,\ldots,p_l)$ and $\Pi_2 = (p_{l+1},\ldots,p_k)$ denote the corresponding l and $(k-l)$-tuples.*
Let $Z = (X,Y)\colon \Delta_T \to T^{(\Pi,1)}(V)$ be a geometric Π-rough path such that

(i) *the Π-variation of Z is controlled by $M\omega$,*

(ii) *the Π_1-variation of $X = \hat{\pi}_E(Z)$ is controlled by ω,*

(iii) *$Y = \hat{\pi}_F(Z)$.*

Then, for all $0 \leqslant \varepsilon \leqslant M^{-s_{m^*}}$, the Π-variation of $(X,\varepsilon Y)$ is controlled by ω, where

$$s_{m^*} = \max_{s_m \leqslant 1}\{s_m \in S^\Pi\} = \max_{R \in \mathcal{A}_1^\Pi} \deg_\Pi(R).$$

Proof. This lemma is analogous to [**10**, Lemma 5.8], adapted to the inhomogeneous smoothness case.

Let $W \in G\Omega_\Pi(V)$ denote the image of Z under the linear map $(x,y) \to (x,\varepsilon y)$. For a multi-index $R = (r_1,\ldots,r_m)$, let $|R|_F$ denote the cardinality of the set $\{r \mid r \in R, \ r > l\}$. Then, if Z has bounded variation, by simple rescaling arguments, we obtain

$$W_{s,t}^R = \varepsilon^{|R|_F} Z_{s,t}^R.$$

By continuity, the last equality holds for a general geometric Π-rough path Z. The following inequality is now implied and completes the proof:

$$\|\pi_R(W_{s,t})\| \leqslant \varepsilon^{|R|_F} M^{\deg_\Pi(R)} \frac{\omega(s,t)^{\deg_\Pi(R)}}{\beta^k \Gamma_\Pi(R)}.$$

$\qquad\square$

Given the one-forms h_i, $i = 1,2,3$, we define the following sequences of rough paths

$$Z_0(0) = (X,0) \qquad \text{and} \quad Z_0(n+1) = \int h_0(Z_0(n))\,\mathrm{d}Z_0(n),$$

$$Z_1(0) = (X,0,Y(1)) \qquad \text{and} \quad Z_1(n+1) = \int h_1(Z_1(n))\,\mathrm{d}Z_1(n),$$

$$Z_2(0) = (X,0,Y(1),Y(1)) \quad \text{and} \quad Z_2(n+1) = \int h_2(Z_2(n))\,\mathrm{d}Z_2(n)$$

for $n = 0,1,\ldots$, where $Y(n) = \hat{\pi}_W(Z_0(n))$.

The definition of the above iterations imply the following lemma.

Lemma 4.6. *For all $n \geqslant 0$,*

$$Z_0(n) = (X, Y(n)),$$
$$Z_1(n) = (X, Y(n), Y(n+1)),$$
$$Z_2(n) = (X, Y(n), Y(n+1), \rho^n(Y(n+1) - Y(n))).$$

Furthermore, if the Π-variation of X is controlled by ω, then the Π_i-variation of $Z_i(0)$ is controlled by $M\omega$ for $i = 1, 2$ on $[0, T_\rho]$, where M and T_ρ are defined below.

Recall the definitions $\Gamma_0 = (\gamma_1, \ldots, \gamma_{k+1})$, $\Pi_0 = (p_1, \ldots, p_k, p_{\max})$, $\Gamma_2 = (\gamma_1, \ldots, \gamma_{k+3})$, $\Pi_2 = (p_1, \ldots, p_k, p_{\max}, p_{\max}, p_{\max})$. Furthermore, we define $\Gamma_1 = (\gamma_1, \ldots, \gamma_{k+2})$, $\Pi_1 = (p_1, \ldots, p_k, p_{\max}, p_{\max})$. By Theorem 3.15, there exists a constant M_i depending only on Π_i, Γ_i, $\hat{\Gamma}_i$ and polynomially on the $\mathrm{Lip}^{\Gamma_i, \Pi_i}$-norm of h_i such that if Z_i is a rough path in the appropriate space with Π_i-variation controlled by some control ω such that $\omega(0, T) < 1$, then the Π_i-variation of $\int h_i(Z_i) \, \mathrm{d}Z_i$ is controlled by ω for $i = 0, 1, 2$. We define $M = \max(M_0, M_1, M_2)$, and without loss of generality we assume that $M \geqslant 1$. We choose $\varepsilon = M^{-s_{m^*}}$.

Let ω_0 be a control of the Π-variation of X. Let $T_\rho > 0$ be chosen to satisfy $\omega_0(0, T_\rho) = \varepsilon^{p_{\max}}$. Note that for $R \in \mathcal{A}_1^\Pi$,

$$1 \geqslant \deg_\Pi(R) = \sum_{i=1}^k \frac{n_j(R)}{p_i} \geqslant \sum_{i=1}^k \frac{n_j(R)}{p_{\max}} = \frac{\|R\|}{p_{\max}}.$$

This implies that by setting $\omega = \varepsilon^{-p_{\max}}\omega_0$, $\varepsilon^{-1}X$ is controlled by ω and $\omega(0, T_\rho) \leqslant 1$.

Lemma 4.7. *For all $n \geqslant 0$ the Π_0-, Π_1- and Π_2-variations of the rough paths*

$$(\varepsilon^{-1}X, Y(n)),$$
$$(\varepsilon^{-1}X, Y(n), Y(n+1))$$

and

$$(\varepsilon^{-1}X, Y(n), Y(n+1), \rho^n(Y(n+1) - Y(n))),$$

respectively are controlled by ω on $[0, T_\rho]$.

The proof is based on the scaling lemma (Lemma 4.5) and is analogous to the proof of Proposition 5.9 in [**10**].

Now we prove the main theorem. We follow the proof of the universal limit theorem corresponding to the homogeneous case presented in [**10**].

Proof of Theorem 4.3. By Lemma 4.7, the Π_2-variation of $Z_2(n)$ for all $n \geqslant 0$ is controlled by ω on $[0, T_\rho]$. We define the linear map $A\colon V \oplus W \oplus W \oplus W \to (V \oplus W) \oplus (V \oplus W)$ by

$$A(x, y_1, y_2, d) = ((x, y_1), (0, d)).$$

This linear map has norm 1. Note that

$$A(Z_2(n)) = ((X, Y(n)), \rho^n(0, Y(n+1) - Y(n)))$$
$$= ((X, Y(n)), \rho^n[(X, Y(n+1)) - (X, Y(n))])$$

is controlled by ω on $[0, T_\rho]$. Then Lemmas 4.4 and 4.7 imply the existence of a constant C depending only on Π, $\omega(0, T)$ and β such that for all $(s, t) \in \Delta_T$,

$$\|\pi_R((X, Y(n))_{s,t} - (X, Y(n+1))_{s,t})\| \leqslant C\rho^{-n}\omega(s, t)^{\|R\|/p_{\max}} \quad \forall R \in \mathcal{A}_1^{\hat{\Pi}}. \qquad (4.4)$$

The inequality implies that $(X, Y(n))$ converges in the Π_0-variational topology on the interval $[0, T_\rho]$ to a rough path $(X, Y) \in G\Omega_{\Pi_0}$, which is also a solution to the rough differential equation (RDE) (4.3).

Note that once ρ is chosen, T_ρ is bounded from below, where the bound only depends on the Lip-norm of h_0, h_2, Π, Γ_2 and the modulus of continuity of ω on $[0, T]$. This implies that one can paste together local solutions in order to get a solution on the whole interval $[0, T]$.

In order to prove uniqueness, we assume that $\hat{Z} = (X, \hat{Y})$ is also a solution to the RDE (4.3). We compare $Y(n)$ and $\hat{Y}$ by defining the function $h_3 \colon V \oplus W \oplus W \oplus W \to \text{End}(V \oplus W \oplus W \oplus W)$ by

$$h_3(x, y, \hat{y}, \hat{d}) = \begin{pmatrix} \text{Id}_V & 0 & 0 & 0 \\ f(y + \xi) & 0 & 0 & 0 \\ 0 & 0 & \text{Id}_W & 0 \\ \rho g_\xi(y, \hat{y})(\hat{d}) & 0 & 0 & 0 \end{pmatrix}$$

and defining $Z_3(n)$ by

$$Z_3(0) = (X, 0, \hat{Y}, \hat{Y}) \quad \text{and} \quad Z_3(n+1) = \int h_3(Z_3(n)) \, dZ_3(n).$$

Arguments analogous to the proof of Lemma 4.6 (see [**10**]) imply that

$$Z_3(n) = (X, Y(n), \hat{Y}, \rho^n(\hat{Y} - Y(n))).$$

Now, analogously to Lemma 4.7, the Π_2-variation of $Z_3(n)$ is controlled by ω on a small enough interval. Then, by Lemma 4.4, $Y = \hat{Y}$ on the same interval. The uniqueness of Y is implied by the uniform continuity of ω.

Define $I_f(X, \xi) = (X, Y)$. Analogous arguments to the proof of the universal limit theorem in [**10**] imply that I_f is continuous from $G\Omega_\Pi(V) \times W \to G\Omega_{\Pi_0}(V \oplus W)$ in the Π-Π_0-variation topology. □

References

1. L. Coutin and Z. Qian, Stochastic analysis, rough paths analysis and fractional Brownian motion, *Prob. Theory Relat. Fields* **122** (2002), 108–140.
2. P. Friz and N. Victoir, Approximations of the Brownian rough path with applications to stochastic analysis, *Annales Inst. H. Poincaré Prob. Statist.* **41** (2005), 703–724.
3. P. Friz and N. Victoir, Differential equations driven by Gaussian signals, *Annales Inst. H. Poincaré Prob. Statist.* **46** (2010), 369–413.
4. P. Friz and N. Victoir, *Multidimensional stochastic processes as rough paths* (Cambridge University Press, 2010).
5. L. G. Gyurkó, Numerical methods for approximating solutions to rough differential equations, PhD thesis, University of Oxford (2009) (available at http://ora.ox.ac.uk/objects/uuid:d977be17–76c6–46d6–8691–6d3b7bd51f7a).
6. M. Hairer and D. Kelly, Geometric versus non-geometric rough paths, *Annales Inst. H. Poincaré Prob. Statist.* **51**(1) (2015), 207–251.
7. A. Lejay and N. Victoir, On (p, q)-rough paths, *J. Diff. Eqns* **225**(1) (2006), 103–133.
8. T. J. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iber.* **14**(2) (1998), 215–310.
9. T. J. Lyons and Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs (Clarendon Press, Oxford, 2002).
10. T. J. Lyons, M. Caruana and T. Lévy, *Differential equations driven by rough paths: École d'Eté de Probabilités de Saint-Flour XXXIV – 2004*, Lecture Notes in Mathematics, Volume 1908 (Springer, 2007).
11. A. Mille and M. Sanz-Solé, Approximation of rough paths of fractional Brownian motion, in *Seminar on stochastic analysis, random fields and applications V*, Progress in Probability, Volume 59, pp. 275–303 (Birkhäuser, 2008).